An introduction to scalar conservation laws

I) Introduction:

<u>Definition:</u> A scalar conservation law is a partial differential equation (PDE) of the following type:

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x}\Big(f(u(x,t))\Big) = 0, \quad \forall x \in \mathbb{R}, \forall t \geq 0$$

where the unknown u is a function $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $(x,t) \mapsto u(x,t)$

and $f : \mathbb{R} \to \mathbb{R}$ is a given smooth function (C^1 at least)

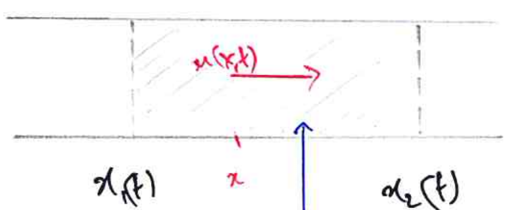
x : 1D space variable.

t : time variable.

Lighter notation: $\partial_t u + \partial_x f(u) = 0$.

<u>Example of derivation of a scalar conservation law (SCL):</u>

Compressible fluid with density $\rho(x,t)$ moving at speed $u(x,t)$.



control volume of fixed mass, moving with the flow velocity.

Mass of the control volume: $m(t) = \int_{x_1(t)}^{x_2(t)} \rho(x,t)\, dt$.

The mass is constant in time : $\frac{dm}{dt} = 0$.

$$0 = \frac{dm}{dt} = \int_{x_1(t)}^{x_2(t)} \frac{\partial \rho}{\partial t}(x,t)\,dx + \underbrace{\frac{dx_2}{dt}}_{\uparrow \; u(x_2(t),t)} \rho(x_2(t),t) - \underbrace{\frac{dx_1}{dt}}_{\uparrow \; u(x_1(t),t)} \rho(x_1(t),t)$$

$\Downarrow$

$$0 = \int_{x_1(t)}^{x_2(t)} \frac{\partial \rho}{\partial t}(x,t)\,dt + \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial x}\big(\rho(x,t)\,u(x,t)\big)\,dt = 0$$

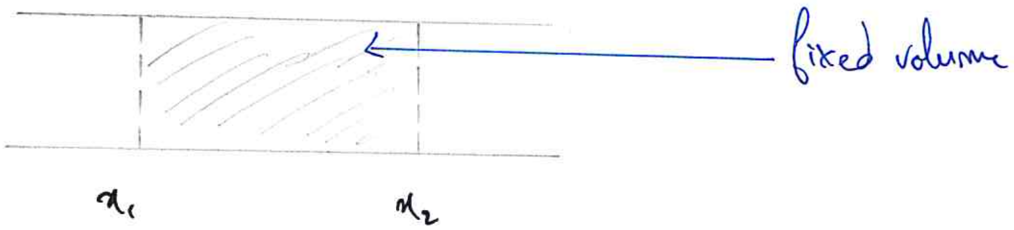
The interval $[x_1(t), x_2(t)]$ is arbitrary :

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0}$$

or $\partial_t \rho + \partial_x f(\rho) = 0$ with $f : \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ \rho \mapsto \rho u \end{array}$ where u is supposed to be known here.

This scalar conservation law expresses the conservation of mass of the fluid.

Other point of view:



Mass variation in the control volume :

what gets out of the volume at x_2

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = \rho(x_1,t)\,u(x_1,t) - \rho(x_2,t)\,u(x_2,t)$$

what gets into the volume at x_1

$$\Rightarrow \int_{x_1}^{x_2} \frac{\partial \rho}{\partial t}\, dx = -\int_{x_1}^{x_2} \frac{\partial (\rho u)}{\partial x}\, dx \Rightarrow \partial_t \rho + \partial_x (\rho u) = 0.$$

<u>Vocabulary:</u> $\partial_t u + \partial_x f(u) = 0 \quad (1)$

1) This equation is called a <u>scalar conservation law</u>.

2) u is called the <u>conservative variable</u>.

3) The function $u \mapsto f(u)$ is called the <u>flux</u>.

3) Eq (1) is called the <u>conservative form</u> of the SCL. when the solution u is smooth ($u \in \mathcal{C}^1(\mathbb{R}_x \times \mathbb{R}_t^+)$) eq (1) is equivalent to :

$$\partial_t u + f'(u)\, \partial_x u = 0 \qquad (2)$$

Eq (2) is the <u>non-conservative form</u> of the SCL.

<u>Remark:</u>

⚠ (1) $\Leftrightarrow$ (2) <u>only if</u> the solution u is <u>smooth</u>! In general the solutions are not smooth. The expression (3) hereunder which expresses the physical principal of conservation requires only the local boundedness of the solution: $u \in L^\infty_{loc}(\mathbb{R}_x \times \mathbb{R}_t^+)$

$$\int_{x_1}^{x_2} u(x,t_2)\,dx - \int_{x_1}^{x_2} u(x,t_1)\,dx = \int_{t_1}^{t_2} f(u(x_1,t))\,dt - \int_{t_1}^{t_2} f(u(x_2,t))\,dt \qquad (3)$$

(3) can be obtained by integrating (1) on $[x_1,x_2] \times [t_1,t_2]$.

Definition: A Cauchy problem is a problem with initial value:

Find $u : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ such that:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & x \in \mathbb{R},\ t>0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

Examples of SCL:

1) **Linear transport equation:**

Let $a \in \mathbb{R}$ with $a \neq 0$. Let $f : u \mapsto au$.

→ Flow with constant velocity a.

$\partial_t u + \partial_x (a u) = 0$ conservative form

$\partial_t u + a\,\partial_x u = 0$ non-conservative form

2) **Burgers equation:** $f(u) = \frac{u^2}{2}$.

$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0$ conservative form

$\partial_t u + u\,\partial_x u = 0$ non-conservative form

3) **A SCL modelling car traffic:**

Consider a flow of vehicles on a straight linear road.

* Local density of vehicles per unit length.
* $V(u)$ speed of the vehicles.

We assume $V(u) = V_m\left(1 - \frac{u}{u_m}\right)$ with:

V_m: maximum speed when traffic is fluid

u_m: maximum density of vehicles where the speed is zero.

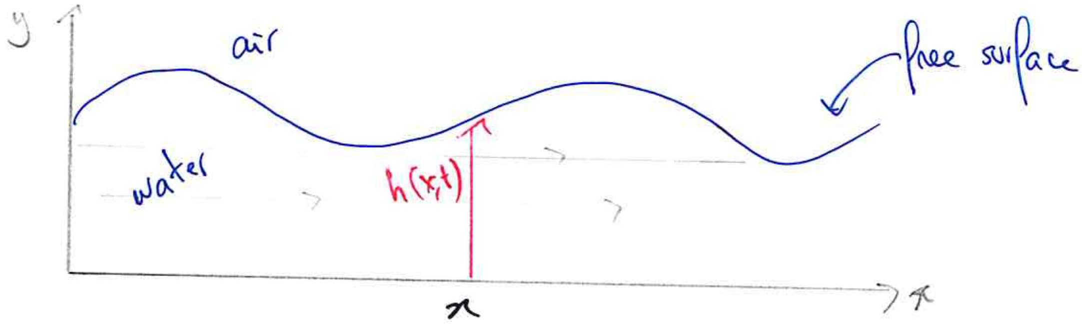
Conservation of vehicles: $\partial_t u + \partial_x (V(u) u) = 0$ conservative form

$$\partial_t u + \underbrace{V_m\left(1 - \frac{u}{2 u_m}\right)}_{= \frac{d}{du}(V(u) u)} \partial_x u = 0$$ non-conservative form

4) System of conservation laws:

One can consider conservation laws where the unknown V is a vector valued function: $V : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^n$ with $n \geq 2$.

Example of the shallow water equations:



$U(x, y, t)$: horizontal speed of the incompressible fluid.

Define the mean horizontal speed at x: $u(x,t) = \frac{1}{h(x,t)} \int_0^{h(x,t)} U(x, y, t)\, dy$.

The two unknowns are h and u.

Mass conservation + incompressibility imply:

$$\boxed{\partial_t h + \partial_x (h u) = 0.}$$

Upon neglecting the viscous dissipation and assuming a hydrostatic pressure : $p(x,y,t) = \rho g h(x,t) - y$, the averaging the Navier - Stokes equations gives:

$$\boxed{\partial_t (hu) + \partial_x \left(h u^2 + g \frac{h^2}{2}\right) = 0.}$$

Denoting $q = hu$ and $V = \begin{pmatrix} h \\ hu \end{pmatrix} = \begin{pmatrix} h \\ q \end{pmatrix}$.

We have $hu^2 + g\frac{h^2}{2} = \frac{q^2}{h} + g\frac{h^2}{2}$.

Define $F(V) = \begin{pmatrix} hu \\ hu^2 + g h^2/2 \end{pmatrix} = \begin{pmatrix} q \\ \frac{q^2}{h} + g h^2/2 \end{pmatrix}$.

We get $\boxed{\partial_t V + \partial_x F(V) = 0}$ System of two conservation laws (conservative form).

<u>To obtain the non-conservative form.</u>

1) $\boxed{\partial_t h + u \partial_x h + h \partial_x u = 0}$

2) $h \partial_t u + \underbrace{u \partial_t h}_{} + (hu)\partial_x u + \underbrace{u \partial_x (hu)}_{} + g h \partial_x h = 0$

$= 0$ by the first eq.

$\Rightarrow$ $\boxed{\partial_t u + u \partial_x u + g \partial_x h = 0}$

Denoting $W = \begin{pmatrix} h \\ u \end{pmatrix}$: $\partial_t W + A(W)\,\partial_x W = 0$ non-conservative form

where $A(W) = \begin{pmatrix} u & h \\ g & u \end{pmatrix}$.

Remark: with $F(W) = \begin{pmatrix} hu \\ hu^2 + g\frac{h^2}{2} \end{pmatrix}$, $A(W) \neq \mathrm{Jac}(F(W))$.

Semblable à

Definition: A system of conservation laws $\partial_t V + \partial_x F(V) = 0$ is said to be hyperbolic if the matrix $M = \mathrm{Jac}(F(V))$ is $\mathbb{R}$-diagonalisable.

The system of shallow-water eq. is hyperbolic since $A = \begin{pmatrix} u & h \\ g & u \end{pmatrix}$ has two distinct eigenvalues $u + \sqrt{gh}$ and $u - \sqrt{gh}$

flow velocity

speed of sound in the fluid.

II) Searching for smooth solutions: the method of characteristics:

Definition: A classical solution of the Cauchy problem

$$(C) \quad \begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $u_0 : \mathbb{R} \to \mathbb{R}$ are two given functions in $\mathcal{C}^1(\mathbb{R}, \mathbb{R})$, is a solution $u \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^+, \mathbb{R})$.

Notation: $a(u) := f'(u)$.

Definition: Let $(x,t) \mapsto u(x,t)$ be a classical solution of (C).
We call characteristic curve, the graph of the function $t \mapsto x(t)$ where x is the solution of the following ODE cauchy problem:

$$\begin{cases} x'(t) = a(u(x(t),t)), & t>0 \\ x(0) = X_0 \end{cases}$$

X_0 is called the "foot of the characteristics".

If u is a classical solution then $(x,t) \mapsto a(u(x,t)) \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R}^+, \mathbb{R})$.
the theorem of Cauchy-lipschitz ensure the existence of a unique maximal solution on an interval $[0,T^*[$.

Properties: Let $x: [0,T^*[\to \mathbb{R}$, $t \mapsto x(t)$ be a characteristic curve.
then:
1) The solution u is constant along the curve $t \mapsto x(t)$.
2) The curve $t \mapsto x(t)$ is a straight line.

Proof:
1) let us prove that $t \mapsto u(x(t),t)$ is constant:

$$\begin{aligned} \frac{d}{dt} u(x(t),t) &= \frac{dx}{dt}\partial_x u(x(t),t) + \partial_t u(x(t),t) \\ &= a(u(x(t),t))\, \partial_x u(x(t),t) + \partial_t u(x(t),t) \\ &= f'(u(x(t),t))\, \partial_x u(x(t),t) + \partial_t u(x(t),t) \\ &= \partial_x f(u(x(t),t)) + \partial_t u(x(t),t) = 0 \end{aligned}$$

2) $t \mapsto u(x(t), t)$ is constant. Hence:

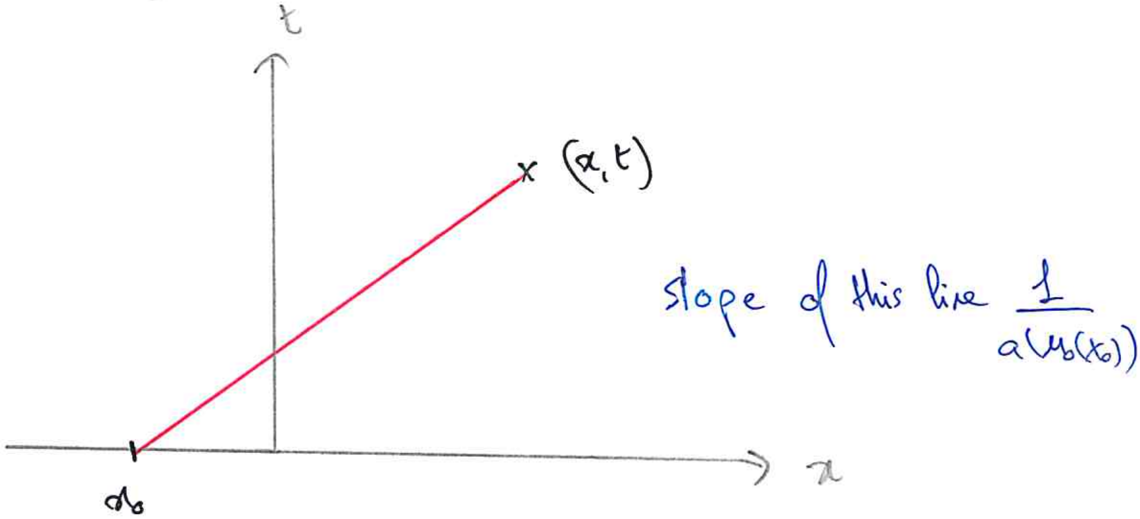
$$u(x(t), t) = u(x(0), 0) = u(x_0, 0) = u_0(x_0).$$ Hence:

$$\forall t > 0: \quad x'(t) = a(u_0(x_0))$$

$$\Rightarrow \boxed{\forall t > 0: \quad x(t) = a(u_0(x_0))\, t + x_0}$$

Consequence:

In order to determine $u(x,t)$ (u at the point $(x,t) \in \mathbb{R} \times \mathbb{R}_t^+$), one has to find x_0 the foot of the characteristic line which passes through (x,t):



Here: $u(x,t) = u_0(x_0)$.

Remark: the properties 1) and 2) are closely linked with the particular form of the equation $\partial_t u + \partial_x f(u) = 0$. They are no more true if there is a source term: $\partial_t u + \partial_x f(u) = g(u)$, or if the depends on time $\partial_t u + \partial_x f(u,t) = 0$. In theses cases, one must adapt the proofs: see the exercises.

1) Method of characteristics applied to the linear transport equation.

Let $a \in \mathbb{R}$, $a \neq 0$. We assume $a > 0$.

We take $f(u) = au$ (f is linear with respect to u).

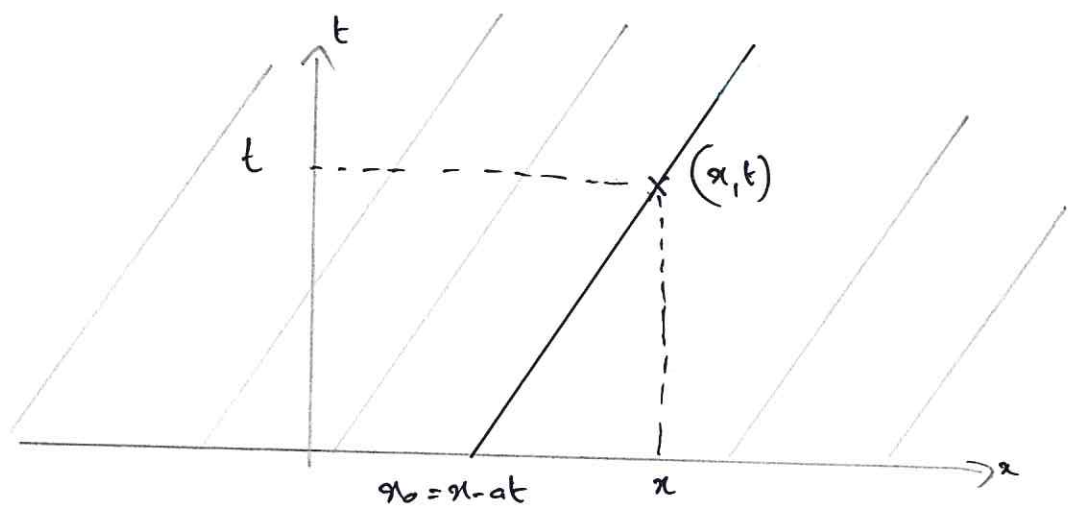
We have $a(u) = f'(u) = a$, $\forall u \in \mathbb{R}$: the speed is constant.

* Eq. of the characteristics: $x(t) = at + x_0$.

* Value of the solution u of $\begin{cases} \partial_t u + a\,\partial_x u = 0 \\ u(x,0) = u_0(x) \end{cases}$ at the point $(x,t) \in \mathbb{R} \times \mathbb{R}^+$? We want to know which characteristic goes through the point (x,t):

$(x,t) \in$ the characteristic $t \mapsto at + x_0 \iff x = at + x_0 \iff x_0 = x - at$.

Hence, the characteristic line wich goes through (x,t) is the line which starts at $x_0 = x - at$ with slope a in the (t,x) plan (ie slope $= \frac{1}{a}$ in (x,t) plan):



The solution is constant along the characteristics:

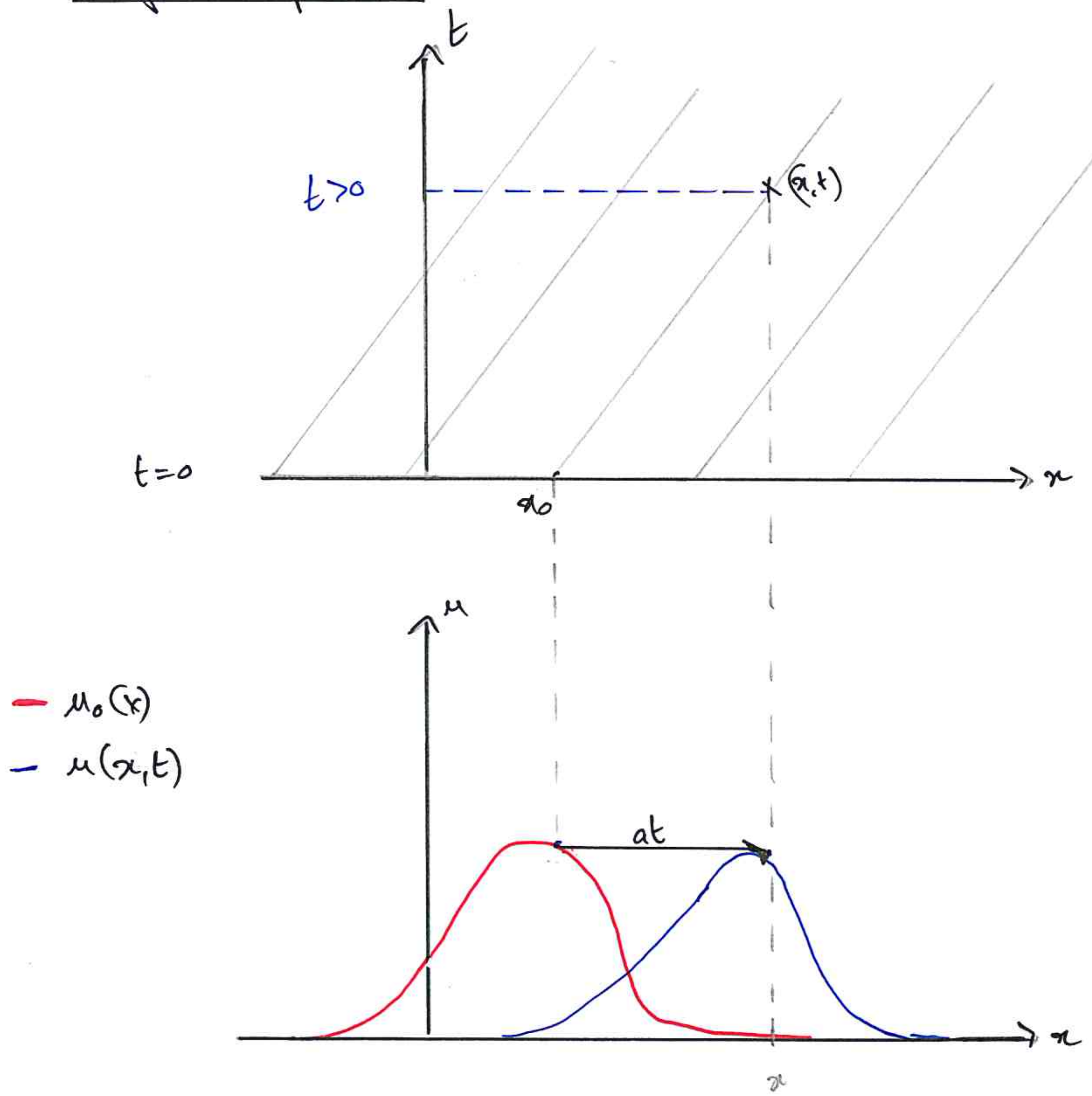
$$u(x,t) = u_0(x_0) = u_0(x - at).$$

The exact solution is given by $(x,t) \mapsto u_0(x-at)$.

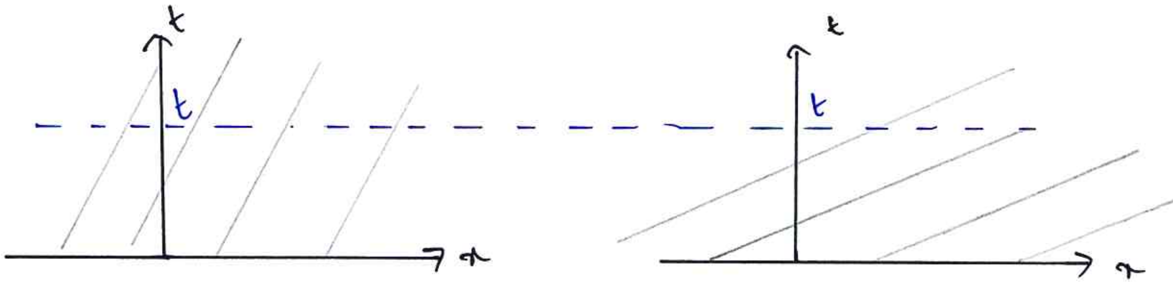
Verification: $\left.\begin{array}{l} \partial_t u(x,t) = -a\, u_0'(x-at) \\ \partial_x u(x,t) = u_0'(x-at) \end{array}\right\} \Rightarrow \partial_t u + a \partial_x u = 0.$

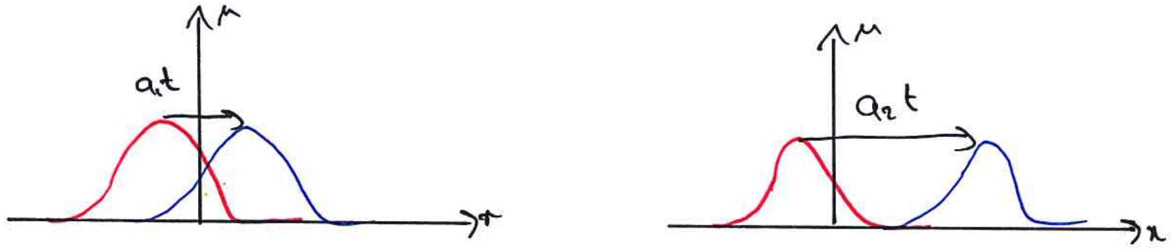
and $u(x,0) = u_0(x)$. ok!

Graphic interpretation:



The initial condition u_0 is simply transported at the speed a.
During the time t, it has travelled the distance at.
The larger is a, the "larger" is the slope of the characteristics, the faster the initial condition travels





$\partial_t u + a_1 \partial_x u = 0$

$\partial_t u + a_2 \partial_x u = 0$

with $a_2 > a_1$

2) The non-linear case: Burgers equation

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

with $f(u) = \frac{u^2}{2}$. $\quad$ NB: $f(u)$ is not linear.

$a(u) = f'(u) = u$.

* Eq of the characteristics.

$x(t) = a(u_0(x_0))\,t + x_0 \quad$ with $a(u) = u$.

Hence: $\quad x(t) = u_0(x_0)\,t + x_0$

Fundamental differences with the linear case:

1) The characteristics have different slopes depending on x_0. Hence, two characteristics starting from x_0 and y_0 can intersect at one point (x,t). At this intersection the classical solution is no more defined: it has two different values $u_0(x_0)$ and $u_0(y_0)$ which is impossible.

1) Here the propagation speed is not uniform.

An example:

$$\begin{cases} \partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0, & x \in \mathbb{R},\ t>0 \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases} \qquad \text{with } u_0(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 1-x & \text{if } x \in [0,1] \\ 0 & \text{if } x \geq 1 \end{cases}$$

Remark: $u_0 \notin \mathcal{C}^1$ but u_0 is continuous. One can still use the method of characteristics.

Eq. of characteristics:

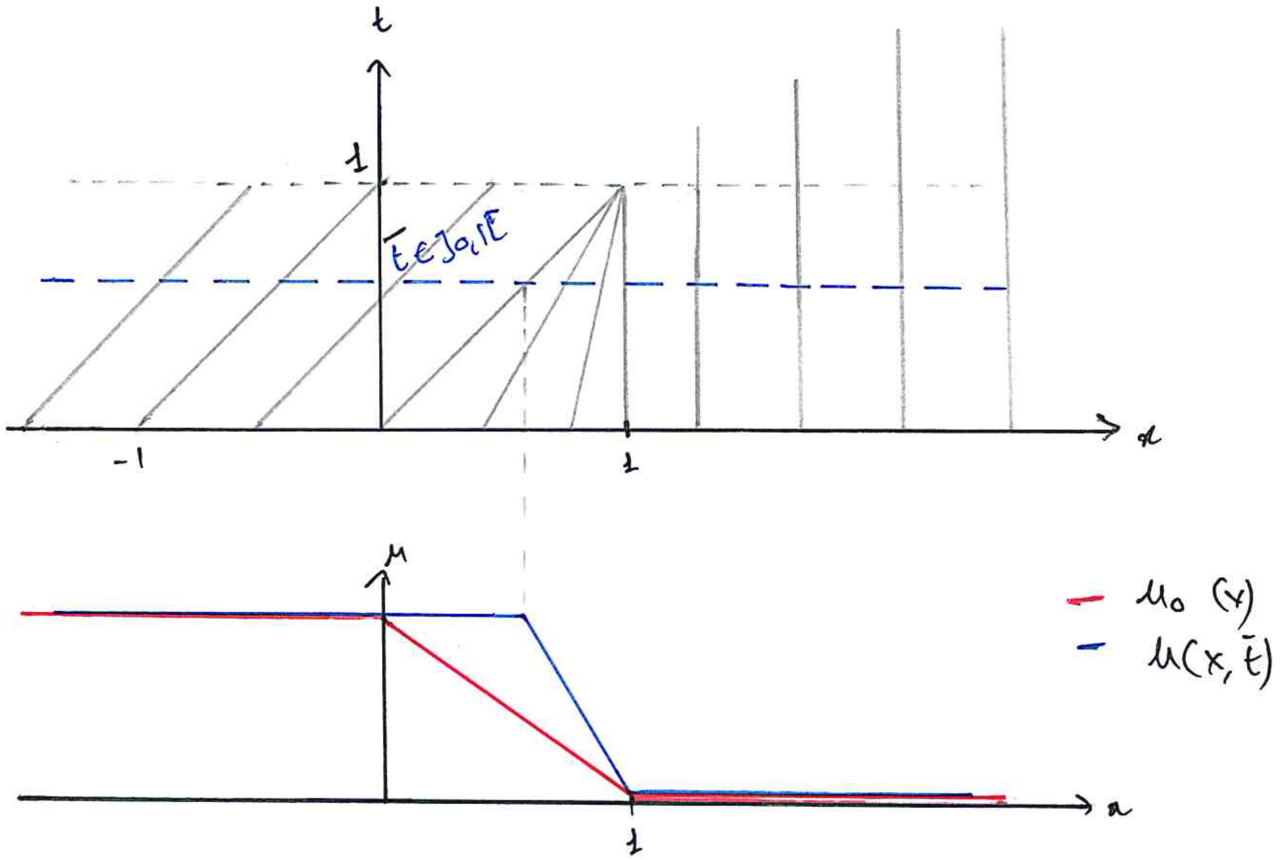
$$x(t) = \begin{cases} t + x_0 & \text{if } x_0 \leq 0 \\ (1-x_0)t + x_0 & \text{if } 0 \leq x_0 \leq 1 \\ x_0 & \text{if } x_0 \geq 1 \end{cases}$$

Hence: if $x_0 \leq 0$: constant slope $= 1$

if $x_0 \geq 1$: constant slope $= 0$

For each $x_0 \in [0,1]$ a different slope ranging from 1 to 0.

Graphic resolution:



Analytic resolution:

We observe that $x(t) = (1-x_0)t + x_0 = t + (1-t)x_0$

Hence in the space region where le slope depends on x_0, if $t=1$ we have $x = 1, \forall x_0 \in [0,1]$. This means that all the characteristic lines starting from $x_0 \in [0,1]$ intersect at the point $(x,t) = (1,1)$.

For $t \geq 1$, the classical solution no more exists.

For $t < 1$: 3 cases:

1) If $x - t \leq 0$ (recall $x = t + x_0$ if $x_0 \leq 0$) then
$x_0 = x - t \leq 0$ and $u(x,t) = u_0(x_0) = 1$ since $x_0 \leq 0$.

2) If $t \leq x \leq 1$ then $\dfrac{x-t}{1-t} = x_0 \in [0,1]$ and
$u(x,t) = u_0(x_0) = u_0\left(\dfrac{x-t}{1-t}\right) = 1 - \dfrac{x-t}{1-t} = \dfrac{1-x}{1-t}$ since $x_0 \in [0,1]$.

3) If $x \geq 1$, $u(x,t) = u_0(x_0) = u_0(x) = 0$ (since $x \geq 1$).

Conclusion: If $t < 1$, the classical solution is given by:

$$u(x,t) = \begin{cases} 1 & \text{if } x \leq t \\ \dfrac{1-x}{1-t} & \text{if } t \leq x \leq 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

Important remarks:

1) In the linear case, the regularity of the initial condition is preserved throughout time.

2) In the non-linear case, the solution may become discontinuous in finite time, even if the initial condition is very smooth.

3) When the solution becomes singular, the method of characteristics is no longer valid.

We have the following theorem.

Theorem. Let $u_0 \in \mathcal{C}^1(\mathbb{R})$ bounded and assume u_0' also bounded.

* If $x \mapsto a(u_0(x))$ is non-decreasing then $\exists!$ classical solution to (C) on $\mathbb{R}\times\mathbb{R}_+$

* Otherwise, $\exists x_0$ s.t $\frac{d}{dx} a(u_0(x))\big|_{x_0} < 0$. Defining

$$T^* = - \frac{1}{\inf\limits_{x_0 \in \mathbb{R}} \frac{d}{dx} a(u_0(x))\big|_{x_0}}, \quad \exists! \text{ classical solution to the}$$

Cauchy pb on $\mathbb{R}\times[0,T^*[$ and for $t \geq T^*$, there is no classical solutions.

Question: What can we do for $t > T^*$? $\Rightarrow$ define weak solutions.

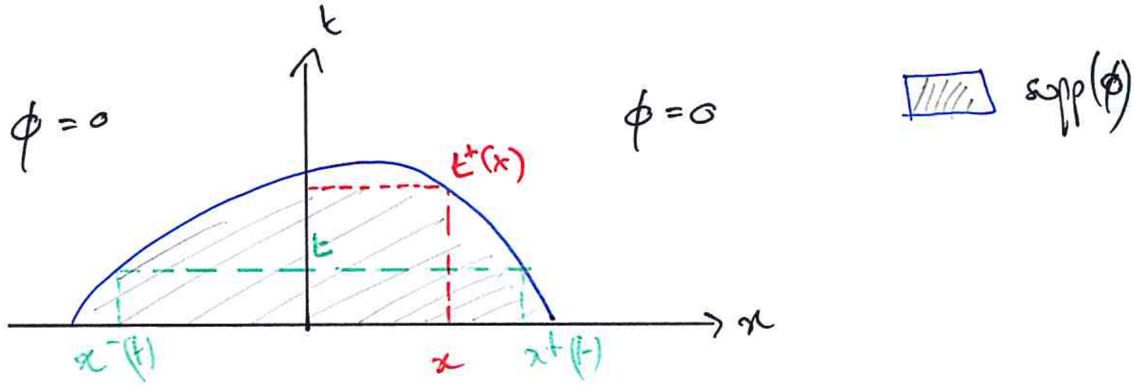
III) Weak solutions for SCL :

1) Definitions.

<u>Principle :</u> Multiply the PDE by a test function and integrate by parts.

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \quad x \in \mathbb{R},\ t \geq 0 \quad (*).$$

Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}\times[0,+\infty[)$ ie $\phi : \mathbb{R}\times[0,+\infty[\longrightarrow \mathbb{R}$
$$(x,t) \longmapsto \phi(x,t)$$
infinitely differentiable and equals zero outside a compact included in $\mathbb{R}\times[0,+\infty[$



$\iint (*)\ \phi(x,t)\,dx\,dt$:

$$\int_0^{+\infty}\int_{\mathbb{R}} \partial_t u(x,t)\phi(x,t)\,dx\,dt + \int_0^{+\infty}\int_{\mathbb{R}} \partial_x f(u(x,t))\phi(x,t)\,dx\,dt = 0$$

Since ϕ has compact support, these integrals are finite.

$$\Rightarrow \int_{\mathbb{R}}\left(\int_0^{+\infty} \partial_t u(x,t)\phi(x,t)\,dt\right)dx + \int_0^{+\infty}\left(\int_{\mathbb{R}} \partial_x f(u(x,t))\phi(x,t)\,dx\right)dt = 0$$

$$\Rightarrow \int_{\mathbb{R}} \Big(-\int_0^{+\infty} u(x,t)\,\partial_t \phi(x,t)\,dt + u(x, t^+(x))\underbrace{\phi(x, t^+(x))}_{=0} - u(x,0)\phi(x,0) \Big) dx$$

$$+ \int_0^{+\infty} \Big(-\int_{\mathbb{R}} f(u(x,t))\,\partial_x \phi(x,t)\,dx + f(u(x^+(t),t))\underbrace{\phi(x^+(t),t)}_{=0} - f(u(x^-(t),t))\underbrace{\phi(x^-(t),t)}_{=0} \Big) dt = 0$$

$$\Rightarrow \int_{\mathbb{R}} -\int_0^{+\infty} u(x,t)\,\partial_t \phi(x,t)\,dt\,dx + \int_{\mathbb{R}} -u_0(x)\phi(x,0)\,dx + \int_0^{+\infty} -\int_{\mathbb{R}} f(u(x,t))\phi(x,t)\,dx\,dt = 0$$

$$\Rightarrow \int_0^{+\infty}\int_{\mathbb{R}} \Big(u(x,t)\,\partial_t \phi(x,t) + f(u(x,t))\,\partial_x \phi(x,t) \Big) dx\,dt + \int_{\mathbb{R}} u_0(x)\,\phi(x,0)\,dx = 0 \qquad (**)$$

For a function u to satisfy $(*)$, it needs to be $\mathcal{C}^1$. But for a function u to satisfy $(**)$ for all test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}\times[0,+\infty[)$ it suffices that u be bounded on every compact set included in $\mathbb{R}\times[0,+\infty[$ ie $u \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

<u>Def</u>: A function $u \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^+)$ is said to be a <u>weak solution</u> of the Cauchy pb $\begin{cases} \partial_t u + \partial_x f(u) = 0, & x\in\mathbb{R}, t>0 \\ u(x,0) = u_0(x), & x\in\mathbb{R} \end{cases}$ if

$$\int_0^{+\infty}\int_{\mathbb{R}} \big(u\,\partial_t\phi + f(u)\,\partial_x\phi \big)\,dx\,dt + \int_{\mathbb{R}} u_0(x)\,\phi(x,0)\,dx = 0 \qquad (**)$$

for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}\times[0,+\infty[)$.

There are more functions that satisfy (**) than functions that satisfy (*). We have therefore enlarged the space in which we are searching for solutions, hoping to find new solutions for $t \geq T^*$.

<u>Proposition:</u> Let u be a smooth function defined on $\mathbb{R} \times \mathbb{R}^+$. Then u is a weak solution $\Leftrightarrow$ u is a classical solution.

<u>Remark:</u> We already know that a classical solution is a weak solution. If a weak solution is smooth, then it is a classical solution.

2) Particular weak solutions:

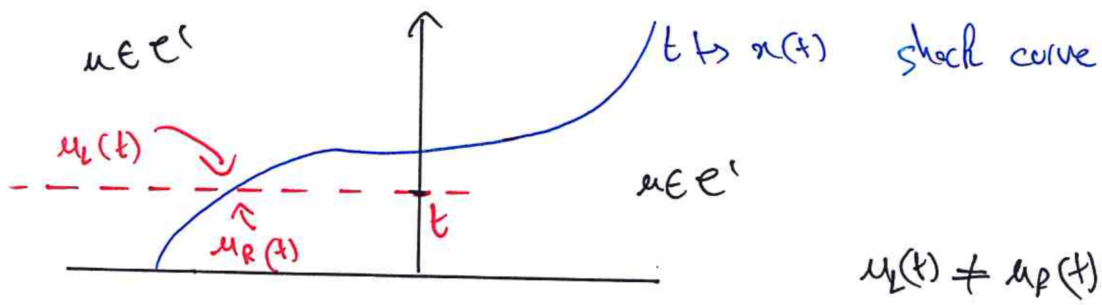
<u>Def:</u> A function $u \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R}^+)$ is said to be piecewise continuously differentiable (u piecewise $\mathcal{C}^1$) if:

1) u is $\mathcal{C}^1$ outside a finite number of curves $t \mapsto x(t)$ called shock curves.

2) On every shock curve, u admits left and right limits,

$$u_L(t) = \lim_{\varepsilon \to 0^+} u(x(t) - \varepsilon, t) \quad ; \quad u_R(t) = \lim_{\varepsilon \to 0^+} u(x(t) + \varepsilon, t)$$

where $u_L(t)$ and $u_R(t)$ are continuous.



$u_L(t) \neq u_R(t)$

The definition of weak solutions allows the existence of discontinuous solutions, but not any discontinuous solutions.

Theorem: (Rankin-Hugoniot)

Let u be a piecewise $\mathcal{C}^1$ function on $\mathbb{R}\times\mathbb{R}_+$. Then, u is a weak solution of $\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x,0) = u_0(x) \end{cases}$ if and only if:

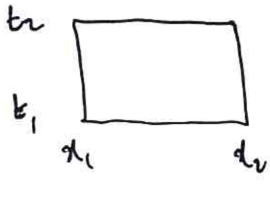
1) u is a classical solution outside the shock curves
2) On every shock curve, u satisfies the Rankin-Hugoniot jump relation:

$$f(u_R(t)) - f(u_L(t)) = s(t)\left(u_R(t) - u_L(t)\right)$$

where $s(t) = x'(t)$ is the shock speed.

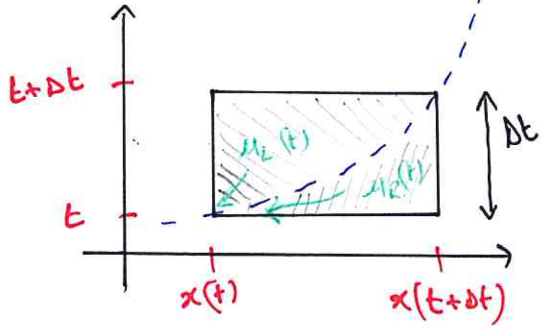
Sketch of proof:

For a weak solution, the integral formulation is still valid: $\forall [x_1, x_2]\times[t_1,t_2]$:



$$\int_{x_1}^{x_2} u(x,t_2)\,dx - \int_{x_1}^{x_2} u(x,t_1)\,dx = \int_{t_1}^{t_2} f(u(x_1,t))\,dt - \int_{t_1}^{t_2} f(u(x_2,t))\,dt$$

We choose the following volume cut in two by the shock:



$$x(t+\Delta t) = x(t) + s(t)\Delta t + O(\Delta t^2)$$

$$\Rightarrow \Delta x = x(t+\Delta t) - x(t) = s(t)\Delta t + O(\Delta t^2)$$

$$\Rightarrow \frac{\Delta x}{\Delta t} = s(t) + O(\Delta t)$$

In the region, $u(x,t) = u_L(t) + O(\Delta t) + O(\Delta x) = u_L(t) + O(\Delta t)$

In the region, $u(x,t) = u_R(t) + O(\Delta t) + O(\Delta x) = u_R(t) + O(\Delta t)$

since u is smooth in these regions and $\frac{\Delta x}{\Delta t} = s(t) + O(\Delta t) = O(1)$

Hence, the integral formulation on $[x(t), x(t+\Delta t)] \times [t, t+\Delta t]$ gives:

$$\Delta x\left(u_L(t) + O(\Delta t)\right) - \Delta x\left(u_R(t) + O(\Delta t)\right) = \Delta t\left(f(u_L(t)) + O(\Delta t)\right) - \Delta t\left(f(u_R(t)) + O(\Delta t)\right)$$

Divide by Δt:

$$[s(t) + O(\Delta t)]\left(u_R(t) - u_L(t)\right) = f(u_R(t)) - f(u_L(t)) + O(\Delta t)$$

Let $\Delta t \to 0$:

$$s(t)\left(u_R(t) - u_L(t)\right) = f(u_R(t)) - f(u_L(t)) \qquad \blacksquare$$

Application:

Back to Burgers eq. $\begin{cases} \partial_t u + \partial_x(u^2/2) = 0 \\ u(x,0) = u_0(x) \end{cases}$ with $f(u) = u^2/2$

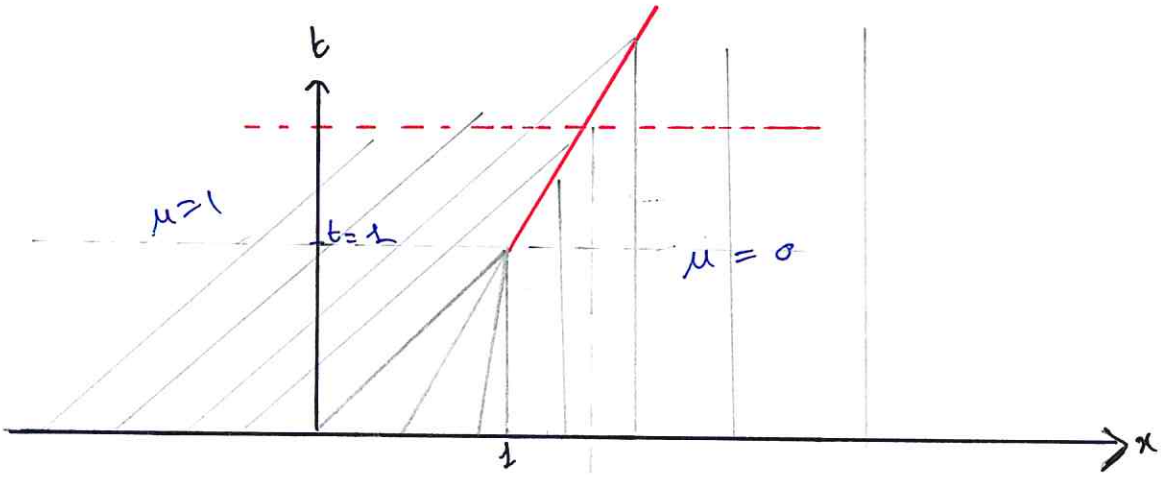
$$u_0(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 1-x & \text{if } x \in (0,1) \\ 0 & \text{if } x \ge 1 \end{cases}$$

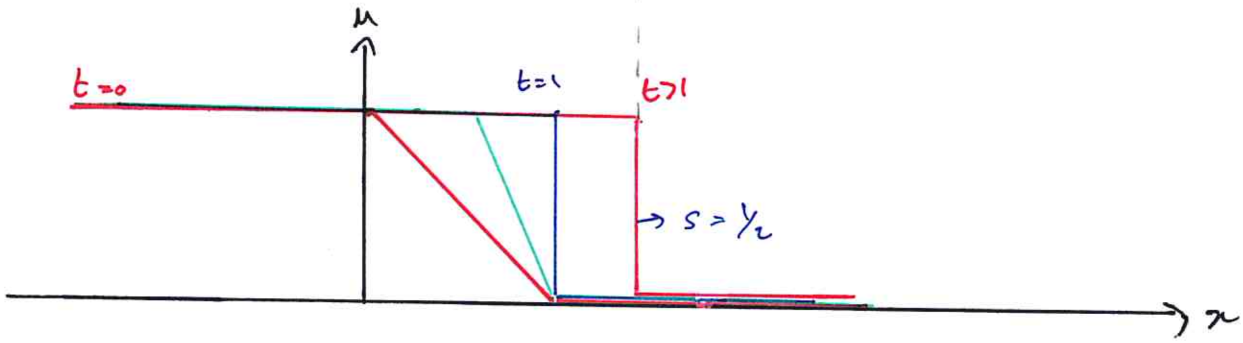
We search for a weak solution which is piecewise smooth for $t > 1$. At the point $(x,t) = (1,1)$ the solution is discontinuous. Hence, a shock curve passes through the point $(1,1)$. Let $t \mapsto x(t)$ be the equation of this shock curve. We determine this function by applying the Rankin-Hugoniot jump relation. Denoting $u_L(t)$ and $u_R(t)$ the left and right limits of the solution at the shock, one must have:

$$x'(t)\,(u_R(t) - u_L(t)) = f(u_R(t)) - f(u_L(t)) \quad , \forall t > 1$$

$$\Leftrightarrow \quad x'(t)\,(u_R(t) - u_L(t)) = \frac{u_R(t)^2}{2} - \frac{u_L(t)^2}{2} \quad , \forall t > 1$$

$$\Leftrightarrow \quad x'(t) = \frac{u_R(t) + u_L(t)}{2} \quad , \forall t > 1 \text{ where } u_R(t) \neq u_L(t).$$



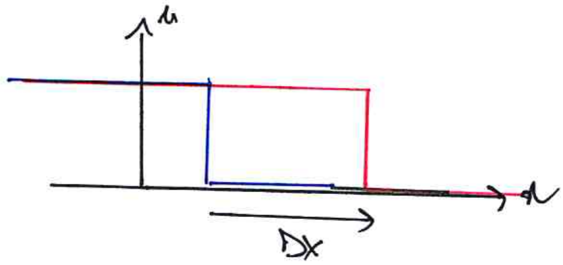


Extending naturally the solution constructed by the method of characteristics beyond $t = 1$, we look for a weak solution in the domain $\{t > 1\}$ such that $u_L(t) = 1$ and $u_R(t) = 0$.

Hence $x'(t) = \frac{1+0}{2} = \frac{1}{2} \Rightarrow \boxed{x(t) = \frac{1}{2}t + \frac{1}{2}}$ since at $t = 1$ the shock is at $x = 1$.

Hence, for $t > 1$ the solution is $u(x,t) = \begin{cases} 1 & \text{if } x < \frac{1}{2}t + \frac{1}{2} \\ 0 & \text{if } x > \frac{1}{2}t + \frac{1}{2} \end{cases}$

Remark: $x'(t) = s(t) = \frac{1}{2}$ is the propagation speed of the shock.



— $u(x, t_1)$
— $u(x, t_2)$

$$\Delta x = s\,(t_2 - t_1) = \frac{t_2 - t_1}{2}$$

Combining this solution (for $t > 1$) with the classical solution for $(t \leq 1)$:

For $t \leq 1$: $u(x,t) = \begin{cases} 1 & \text{if } x \leq t \\ \frac{1-x}{1-t} & \text{if } t \leq x \leq 1 \\ 0 & \text{if } x \geq 1 \end{cases}$

For $t > 1$: $u(x,t) = \begin{cases} 1 & \text{if } x < \frac{t}{2} + \frac{1}{2} \\ 0 & \text{if } x > \frac{t}{2} + \frac{1}{2}. \end{cases}$

We have defined almost everywhere an L^∞_{loc} function on $\mathbb{R} \times \mathbb{R}_+$.
This function is a weak solution because:

1) outside the shock curve it is a classical solution
2) On the shock curve, it satisfies the R-H jump relations.

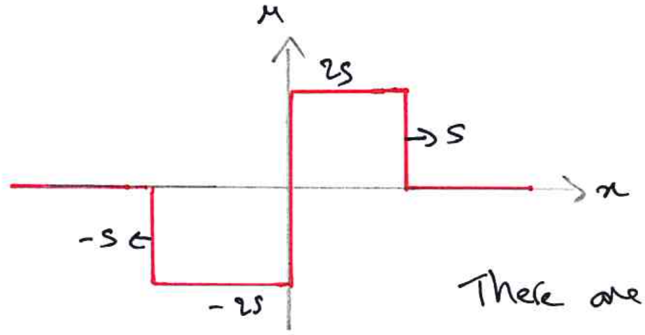
Remark: For the initial condition $u_0(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$, a weak solution is given by $u(x,t) = \begin{cases} 1 & \text{if } x < \frac{t}{2} \\ 0 & \text{if } x > t/2 \end{cases}$

Problem: there are two many weak solutions: no uniqueness!

Example: Burgers:

$$\begin{cases} \partial_t u + \partial_x \frac{u^2}{2} = 0, \\ u(x,0) = 0 \end{cases}$$

For every $s > 0$, the function $u(x,t) = \begin{cases} 0 & \text{if } x < -st \\ -2s & \text{if } -st < x < 0 \\ +2s & \text{if } 0 < x < st \\ 0 & \text{if } st < x \end{cases}$ is a weak solution.



There are infinitely many solutions!

Conclusion:

classical solutions, uniqueness but not existence (for $t \geq T^*$)
Weak solutions, existence but no uniqueness!

IV) Entropy weak solutions:

We search for a criterion which allows to pick up the "physical" solution among all the existing weak solutions.

The conservation laws of the form

$$\partial_t u + \partial_x f(u) = 0 \qquad (1)$$

are often approximations of laws of the following form

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_{xx} u \qquad (1-\varepsilon)$$

where the diffusion term has been neglected (ε = viscosity). One may consider that the "right" weak solutions of (1) are those which are limits of solutions u_ε of $(1-\varepsilon)$ in the limit $\varepsilon \to 0$.

One has to find an information on the solutions u_ε of $(1-\varepsilon)$ which "persists" when passing to the limit $u_\varepsilon \to u$.

1) A fundamental computation and definition of the entropy.

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a convex $\mathcal{C}^2$ function : $\eta'' \geq 0$.
Multiplying (1-ε) by $\eta'(u_\varepsilon)$:

$$\eta'(u_\varepsilon)\,\partial_t u_\varepsilon + \eta'(u_\varepsilon) f'(u_\varepsilon)\,\partial_x u_\varepsilon = \varepsilon\,\eta'(u_\varepsilon)\,\partial_{xx} u_\varepsilon .$$

Let ψ be defined by $\psi' = \eta' f'$ ie $\psi(u) = \int^u \eta'(s) f'(s)$.

Then : $\eta'(u_\varepsilon)\,\partial_t u_\varepsilon + \psi'(u_\varepsilon)\,\partial_x u_\varepsilon = \varepsilon\,\eta'(u_\varepsilon)\,\partial_{xx} u_\varepsilon$

$$\Longrightarrow \quad \partial_t \eta(u_\varepsilon) + \partial_x \psi(u_\varepsilon) = \varepsilon\,\partial_{xx}\eta(u_\varepsilon) \underbrace{- \varepsilon\,(\partial_x u_\varepsilon)^2\,\eta''(u_\varepsilon)}_{\leq 0}$$

$$\Longrightarrow \quad \partial_t \eta(u_\varepsilon) + \partial_x \psi(u_\varepsilon) \leq \varepsilon\,\partial_{xx}\eta(u_\varepsilon).$$

Formally, when passing to the limit $\varepsilon \to 0$:

$$\boxed{\partial_t \eta(u) + \partial_x \psi(u) \leq 0}$$

Multiplying this inequality by a positive test function ϕ and integrating on $\mathbb{R} \times [0,+\infty[$:

$$\int_0^{+\infty}\int_{\mathbb{R}} \left(\partial_t \eta(u)\,\phi + \partial_x \psi(u)\,\phi\right) dx\,dt \leq 0.$$

After integrating by parts:

$$\boxed{\int_0^{+\infty}\int_{\mathbb{R}} \left(\eta(u)\,\partial_t\phi + \psi(u)\,\partial_x\phi\right) dx\,dt + \int_{\mathbb{R}} \eta(u(x,0))\,\phi(x,0)\,dx \geq 0}$$

Conclusion:

If u is the limit of u_ε (the solution of the diffusive problem) when $\varepsilon \to 0$, then u should satisfy the above inequality for all convex function η and all positive test function ϕ.

Def: Consider a (SCL): $\partial_t u + \partial_x f(u) = 0$.
A $\mathcal{C}^1$ convex function η is an entropy for this (SCL) if there exists a function ψ called the entropy flux such that $\partial_t \eta(u) + \partial_x \psi(u) = 0$ for every classical solution u of the scalar conservation law.

Def: A weak solution is said to be an entropy weak solution if for every entropy/entropy flux pair, one has $\partial_t \eta(u) + \partial_x \psi(u) \leq 0$ in the weak sense, ie:

$$\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R} \times [0, +\infty[),$$

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(\eta(u) \partial_t \phi + \psi(u) \partial_x \phi \right) dx\, dt + \int_{\mathbb{R}} \eta(u(x,0)) \phi(x,0)\, dx \geq 0$$

Remarks:

1) The definition of an entropy is significant for systems of conservation laws. For a scalar conservation law, any $\mathcal{C}^1$ convex function is an entropy. It suffices to take ψ a primitive of $\eta' f'$.

2) Any classical solution is an entropy weak solution.

Theorem (Kruzhkov):

> Let $f \in \mathcal{C}^1(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$.
> The Cauchy problem $\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x,0) = u_0(x) \end{cases}$
> has a unique entropy weak solution $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$

2) Discontinuous entropy weak solution:

Proposition: Let u be a piecewise $\mathcal{C}^1$ solution of $\partial_t u + \partial_x f(u) = 0$.
Then u is an entropy weak solution if and only if at any point of a shock curve $t \mapsto x(t)$, one has:

$$\psi(u_R(t)) - \psi(u_L(t)) \leq x'(t) \left(\eta(u_R(t)) - \eta(u_L(t)) \right)$$

for all entropy / entropy flux pairs.

Remark: This is the analogue of the Rankine-Hugoniot jump-relation (applied to $\partial_t \eta(u) + \partial_x \psi(u) \leq 0$). This characterization is not easy to check for every pair (η, ψ)!

Proposition:
1) If f is strictly convex, then a shock between u_L and u_R is entropic iff $u_L > u_R$.
2) If f is strictly concave, the a shock between u_L and u_R is entropic iff $u_L < u_R$.

We know that $\begin{cases} \partial_t u + \partial_x \frac{u^2}{2} = 0 \\ u(x,0) = u_0(x) \end{cases}$ with $u_0(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$

admits a shock type weak solution which is $u(x,t) = \begin{cases} 1 & \text{if } x < x(t) \\ 0 & \text{if } x > x(t) \end{cases}$

where $x(t) = \frac{t}{2}$.

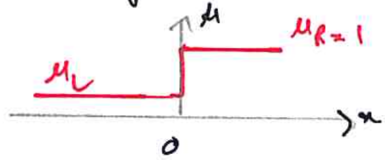
Question: Is this the entropy weak solution ?

Answer: Yes because its $f(u) = \frac{u^2}{2}$ is strictly convex and $u_L = 1 > u_R = 0$.

Another example: the formation of a traffic jam in the traffic flow model.

$\begin{cases} \partial_t u + \partial_x (u(1-u)) = 0 \\ u(x,0) = u_0(x) \end{cases}$ with $u_0(x) = \begin{cases} u_L & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$ where $0 < u_L < 1$.

Here, $V_m = 1$ and $u_m = 1$.



We expect a shock type solution with a discontinuity travelling back to the left. We look for an weak solution of the form:

$u(x,t) = \begin{cases} u_L & \text{if } x < x(t) \\ 1 & \text{if } x > x(t) \end{cases}$. The shock speed is given by the

RH jump relation: $x'(t) = \dfrac{f(u_R) - f(u_L)}{u_R - u_L} = \dfrac{\cancel{f(1)} - f(u_L)}{1 - u_L} = -\dfrac{u_L(1-u_L)}{1-u_L} = -u_L$

Hence $x'(t) \leq 0$ because $u_L \in]0,1[$. So the shock is indeed travelling to the left. At $t=0$, the discontinuity is at $x=0$. The eq. of the shock curve is $x(t) = -u_L t$.

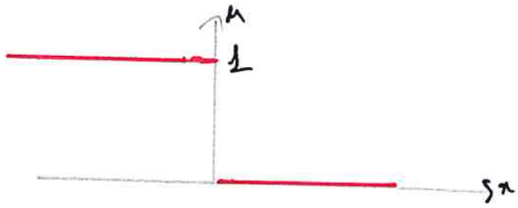
This is a weak solution. Is it the only entropy weak solution ?

Yes: $f(u) = u(1-u)$ is concave and $u_R = 1 > u_L$.

Another example: A red light turning green at $t=0$.

$$\begin{cases} \partial_t u + \partial_x (u(1-u)) = 0 \\ u(x,0) = u_0(x) \end{cases} \qquad \text{at } t=0: \ u_0(x) = \begin{cases} 1 & \text{if } x<0 \\ 0 & \text{if } x>0 \end{cases}$$

We look for a shock type solution: $u(x,t) = \begin{cases} 1 & \text{if } x < x(t) \\ 0 & \text{if } x > x(t) \end{cases}$



Shock speed: (RH) $x'(t) = \dfrac{f(u_R)-f(u_L)}{u_R-u_L} = \dfrac{0}{-1} = 0$.

At $t=0$, $x(0)=0 \Rightarrow x(t)=0 \quad \forall t>0$.

We obtain a standing discontinuity at $x=0$!

$\Rightarrow$ This solution is not what is intuitively expected. Indeed we rather expect that for $x>0$, the density of vehicles $u(x,t)$ increases along time while for $x<0$, the density of vehicles is expected to decrease with time. This is because this shock solution is not the entropy weak solution. Indeed $f(u) = u(1-u)$ is concave and $u_L > u_R$.

So the entropy weak solution is not a shock!

3) Rarefaction waves

Definition: Let $\partial_t u + \partial_x f(u) = 0$ be a (SCL).
A rarefaction wave is a solution of the form $u(x,t) = v\left(\frac{x}{t}\right)$ where $\xi \mapsto v(\xi)$ is a continuous and piecewise C^1 function.

Proposition: The rarefaction waves are of two types:
$u(x,t) = cst$ or $u(x,t) = v\left(\frac{x}{t}\right)$ where $v(\xi) = (f')^{-1}(\xi)$

Proof: Let $u(x,t) = v\left(\frac{x}{t}\right)$ be a solution. Cast this in $\partial_t u + \partial_x f(u) = 0$.

$\partial_t u = -\frac{x}{t^2} v'\left(\frac{x}{t}\right)$ and $\partial_x f(u) = f'\left(v\left(\frac{x}{t}\right)\right) \frac{1}{t} v'\left(\frac{x}{t}\right)$.

So : $-\frac{x}{t^2} v'\left(\frac{x}{t}\right) + f'\left(v\left(\frac{x}{t}\right)\right) \frac{1}{t} v'\left(\frac{x}{t}\right) = 0, \ \forall (x,t)$

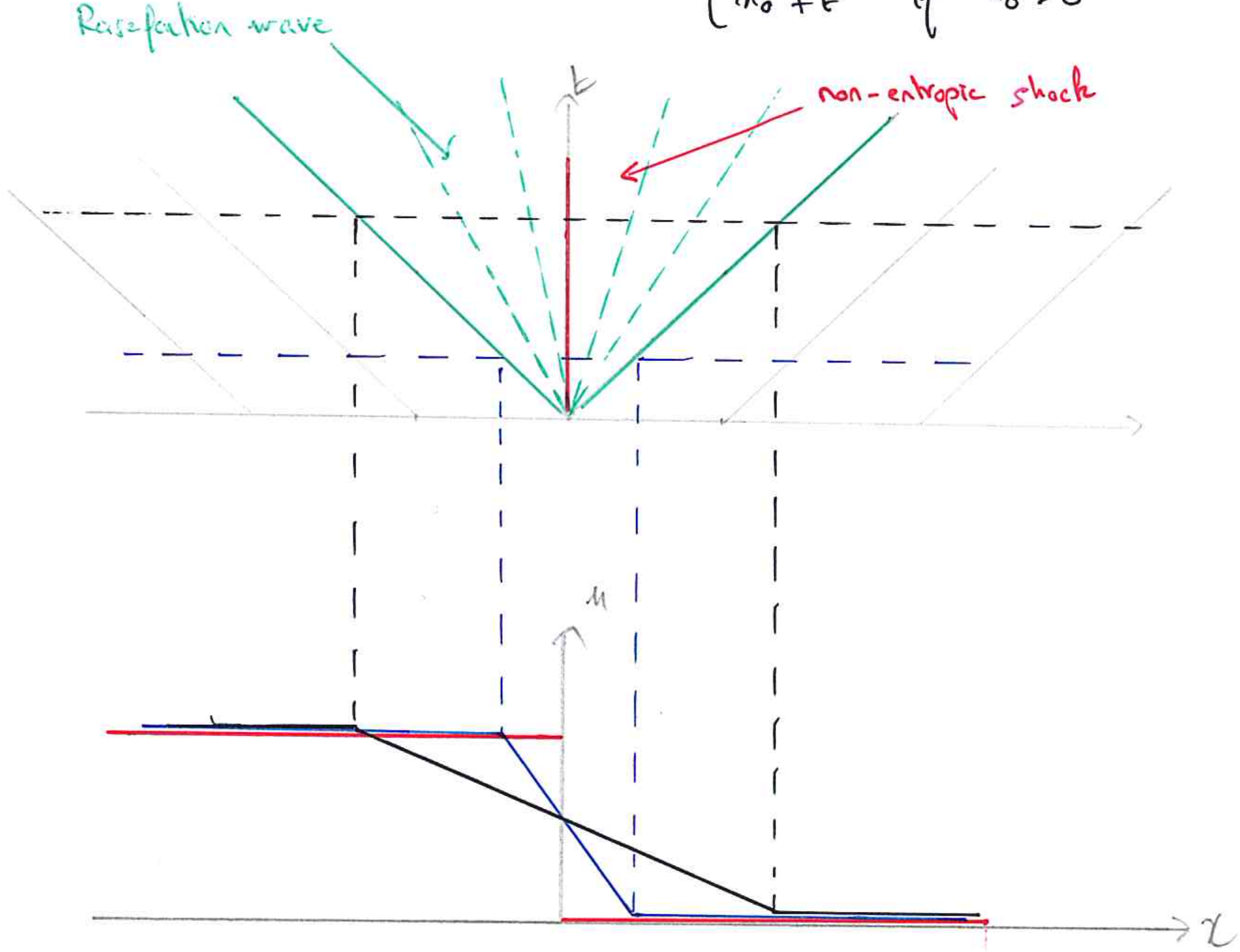
$\Rightarrow \quad \frac{1}{t}\left(-\frac{x}{t} + f'\left(v\left(\frac{x}{t}\right)\right)\right) v'\left(\frac{x}{t}\right) = 0$

Then either $v' = 0$ or $\forall \xi \ (f' \circ v)(\xi) = \xi$ ie $v(\xi) = (f')^{-1}(\xi)$ ∎

Example: Let us look for a rarefaction wave solution to the traffic flow model with the green light initial condition.

Eq of the characteristic curves: $f(u) = u(1-u)$, $a(u) = f'(u) = 1 - 2u$.

Hence : $x(t) = x_0 + (1 - 2u_0(x_0))t = \begin{cases} x_0 - t & \text{if } x_0 < 0 \\ x_0 + t & \text{if } x_0 > 0 \end{cases}$



For x s.t $x+t<0$ ie $x<-t$ we have $u(x,t)=1$.

For x s.t $x-t>0$ ie $x>t$ we have $u(x,t)=0$.

For $-t<x<t$ we search for a rarefaction wave solution:

$u(x,t)=v\left(\frac{x}{t}\right)$. We must have $(a\circ v)(\xi)=\xi$:

$$1-2v(\xi)=\xi \iff v(\xi)=\frac{1-\xi}{2}.$$

Hence $u(x,t)=v\left(\frac{x}{t}\right)=\dfrac{1-x/t}{2}$ for $-t<x<t$.

<u>Remark:</u> For $x=-t$, this formula gives $u=1$.
For $x=t$, this formula gives $u=0$.

Hence the solution is <u>continuous</u> :

$$u(x,t)=\begin{cases} 1 & \text{if } x\leq -t \\ \dfrac{1-\frac{x}{t}}{2} & \text{if } -t\leq x\leq t \\ 0 & \text{if } x\geq t \end{cases}$$

Despite u_0 is discontinuous, the solution is continuous for $t>0$. Hence, it is a classical solution. Hence it is the only entropy weak solution. This solution is more consistent with what is physically expected.

<u>Remarks:</u>

1) In the area of the (x,t) plan which is not covered by the characteristic curves we have constructed a rarefaction wave. In this area, we plot straight lines $\frac{x}{t}=$ cste in order to recall the fact that a rarefaction wave is constant along these lines.

2) In this example, for fixed t, the function $x \mapsto u(x,t)$ is affine in the area $-1 \leq \frac{x}{t} \leq 1$. This is not true in general. It is due to the fact that $v \mapsto a(v)$ is affine and hence so is $\xi \mapsto v(\xi)$. It would be false for $f(u) = u^4$ for instance.

V) The Riemann problem

Def: A Riemann problem is a Cauchy problem of the form:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(x,0) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases} \end{cases}$$

where u_L and u_R are two constants.

<u>Resolution:</u>

* If $\begin{cases} u_L > u_R & \text{and } f \text{ strictly convex} \\ u_L < u_R & \text{and } f \text{ strictly concave} \end{cases}$

then the unique entropy weak solution is a shock travelling at the speed $s = \dfrac{f(u_R) - f(u_L)}{u_R - u_L}$ ie $u(x,t) = \begin{cases} u_L & \text{if } x < st \\ u_R & \text{if } x > st \end{cases}$

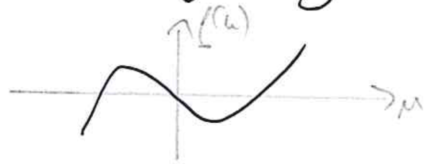
* If $\begin{cases} u_L < u_R & \text{and } f \text{ strictly convex} \\ u_L > u_R & \text{and } f \text{ strictly concave} \end{cases}$

then the unique entropy solution is a rarefaction wave:

$$u(x,t) = \begin{cases} u_L & \text{if } \frac{x}{t} \leq f'(u_L) \\ v\left(\frac{x}{t}\right) & \text{if } f'(u_L) \leq \frac{x}{t} \leq f'(u_R) \\ u_R & \text{if } \frac{x}{t} \geq f'(u_R) \end{cases}$$

where $v = (f')^{-1}$

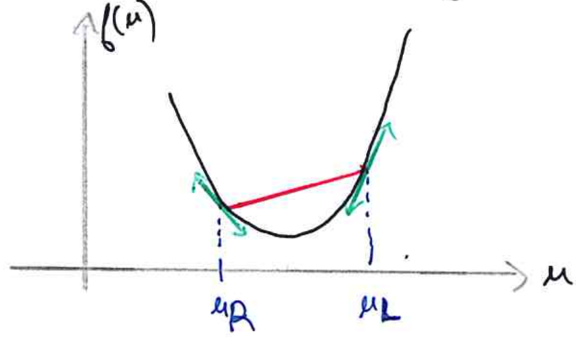
* If f has convexity changes : beyond the scope of this course.



Important remarks:

1) Entropic shock:

For instance : $u_L > u_R$ and f is strictly convex

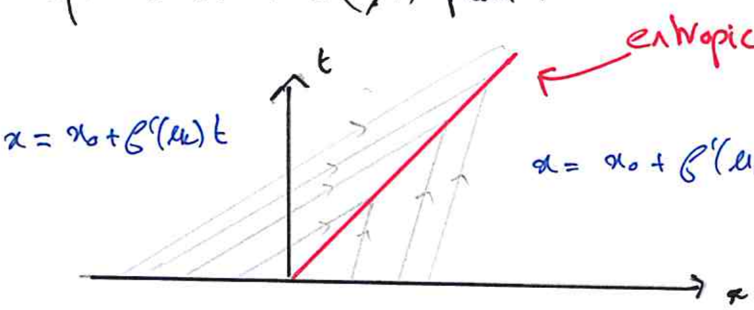


Convexity:

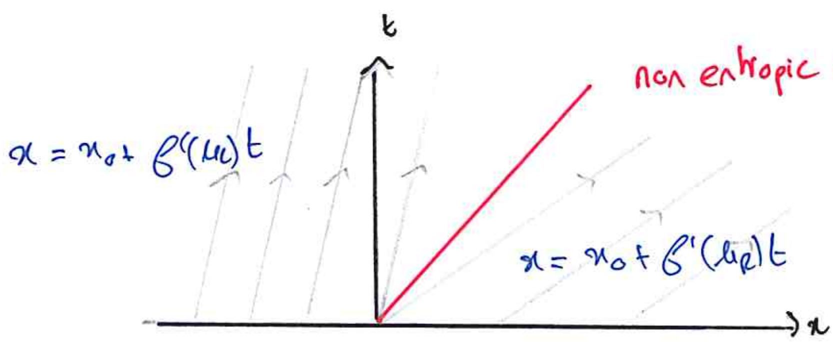
$$f'(u_R) \leq \frac{f(u_R)-f(u_L)}{u_R - u_L} \leq f'(u_L)$$

speed of the shock between u_L and u_R

Consequence in the (x,t) plan :



$(u_L > u_R)$



$(u_L < u_R)$

$\Rightarrow f'(u_L) < s < f'(u_R)$

For an entropic shock, the characteristics must enter into the shock.

2) Notations

In every case (shock or rarefaction wave), the solution of the Riemann problem can be written :

$$\boxed{u(x,t) = \mathcal{U}_R\left(\frac{x}{t}\,;\, u_L, u_R\right)}$$

where $\mathcal{U}_R$ is a function which only depends on f and which consists in two constant states u_L and u_R, separated by a wave (either shock wave or rarefaction wave) the speed of which is always bounded by $\displaystyle\max_{\substack{u_L \leq \xi \leq u_R \\ \text{or } u_R \leq \xi \leq u_L}} |f'(\xi)|$.

2) Naturally, the entropy weak solution of the following Riemann problem :

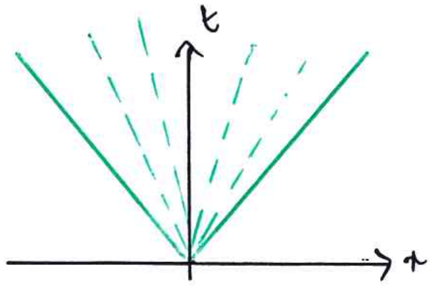
$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x, t_0) = \begin{cases} u_L & \text{if } x < x_0 \\ u_R & \text{if } x > x_0 \end{cases} \end{cases}$$

is given by $u(x,t) = \mathcal{U}_R\left(\dfrac{x - x_0}{t - t_0}\,;\, u_L, u_R\right)$.

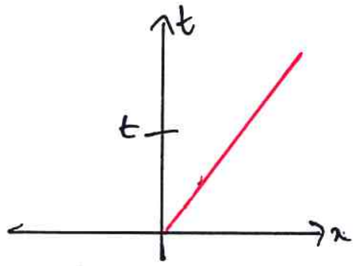
3) The function $h : \xi \mapsto f(\mathcal{U}_R(\xi\,;\, u_L, u_R))$ is always continuous at $\xi = 0$.

Indeed: there are 3 different senarios :

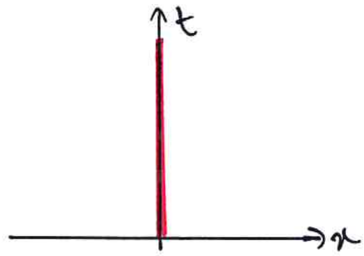
① $\mathcal{U}_R$ is a rarefaction wave



② $\mathcal{U}_R$ is a shock wave with $s \neq 0$



③ $\mathcal{U}_R$ is a shock wave with $s = 0$



Case ①: $\mathcal{U}_R$ is continuous and so is h.

Case ②: For $t>0$, since $s\neq 0$, the shock is no longer at $x=0$. So in a neighbourhood of $x=0$, the solution $\mathcal{U}_R$ is constant and therefore continuous.

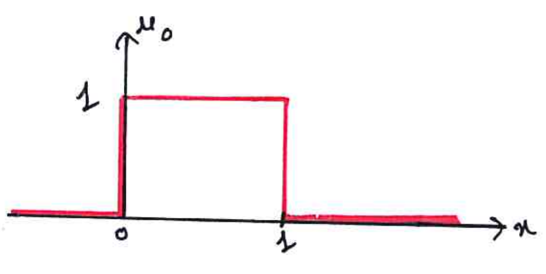
case ③: $s=0$. We have a standing shock at $x=0$. So $u(x,t)$ is discontinuous at $x=0$, $\forall t>0$. But the RH jump relation gives:

$$f(\mathcal{U}_R(0^+; u_L, u_R)) - f(\mathcal{U}_R(0^-; u_L, u_R))$$
$$= s\left(\mathcal{U}_R(0^+; u_L, u_R) - \mathcal{U}_R(0^-; u_L; u_R)\right)$$
$$= 0 \quad \text{since } s=0.$$

Exercise (very important):

Solve the following cauchy problem (ie find the entropy weak solution):

$$\begin{cases} \partial_t u + \partial_x(u(1-u)) = 0 \\ u(x,0) = u_0(x) \end{cases} \quad \text{where } u_0(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$$



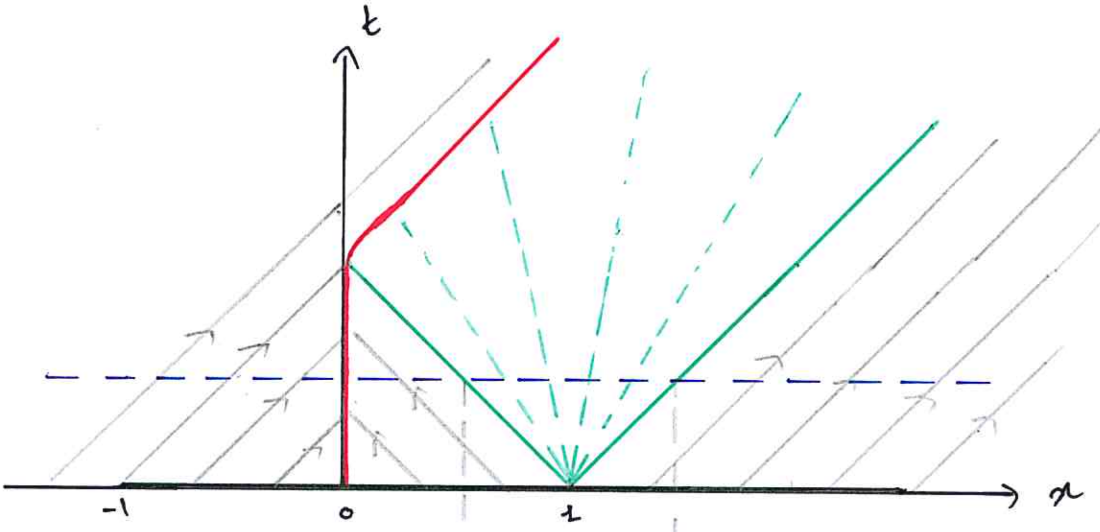
Resolution:

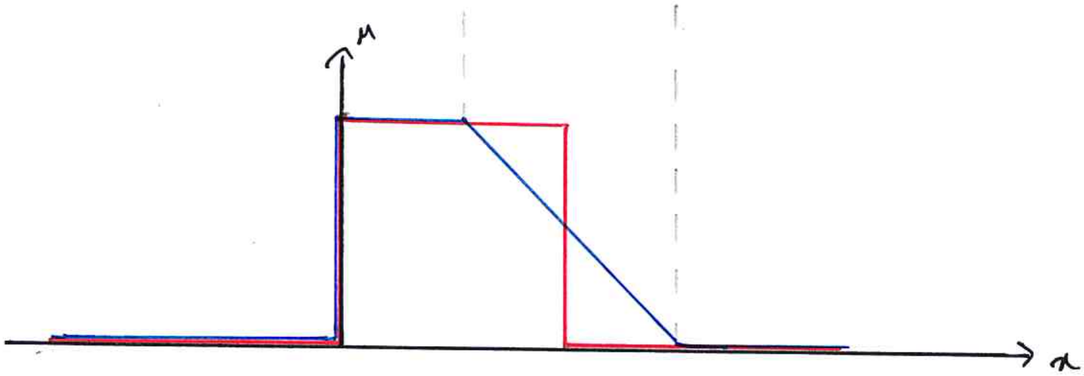
* $f: u \mapsto u(1-u)$ is concave $\Longrightarrow \begin{cases} \text{at } x=0: \text{ shock} \\ \text{at } x=1: \text{ rarefaction wave} \end{cases}$
* speed of the shock: $s = \dfrac{f(u_R)-f(u_L)}{u_R - u_L} = \dfrac{f(1)-f(0)}{1-0} = 0.$
* Speed of the left boundary of the rarefaction wave: $f'(1) = -1$.
  speed of the right boundary of the rarefaction wave: $f'(0) = 1$.

* Equation of characteristics :

$$x(t) = f'(u_0(x_0))t + x_0$$

$$= (1 - 2u_0(x_0))t + x_0 = \begin{cases} t + x_0 & \text{if } x_0 < 0 \\ -t + x_0 & \text{if } x_0 \in]0,1[\\ t + x_0 & \text{if } x_0 > 1 \end{cases}$$





Shock speed = 0 > speed of the left boundary of the rarefaction wave = -1.

So, there is a point (X_1, T_1) at which the rarefaction wave will touch the stationnary shock wave. This point (X_1, T_1) satisfies : $\begin{cases} X_1 = 0 \\ T_1 = 1 - X_1 \end{cases} \iff (X_1, T_1) = (0, 1)$

Hence, for $t \leq 1$, the solution is given by:

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \text{ and } \frac{x-1}{t} \leq f'(1) = -1 \\ v\left(\frac{x-1}{t}\right) & \text{if } -1 \leq \frac{x-1}{t} \leq f'(0) = 1 \\ 0 & \text{if } \frac{x-1}{t} \geq 1 \end{cases}$$

where the function v satisfies $f' \circ v(\xi) = \xi$ i.e.
$1 - 2v(\xi) = \xi \iff v(\xi) = \frac{1-\xi}{2}$. So for $-1 \leq \frac{x-1}{t} \leq 1$
we have $u(x,t) = \dfrac{1 - \frac{x-1}{t}}{2} = \frac{1}{2} - \frac{x-1}{2t}$.

Remark: Here for fixed $t \in]0,1[$, $x \mapsto u(x,t)$ is affine. This is because $\xi \mapsto f'(\xi)$ is linear so $\xi \mapsto v(\xi)$ is also linear. But it is not always the case.

what happens for $t \geq 1$?

A shock $t \mapsto x(t)$ starts at the point $(1,1)$ between the constant state $u_L = 0$ on the left and the time-dependent state given by the rarefaction wave on the right: $u_R(t) = \frac{1}{2} - \frac{x(t)-1}{2t}$.
the RH jump relation applied to this shock gives:

$$\begin{cases} x'(t) = \dfrac{f(u_R(t)) - \cancel{f(u_L)}}{u_R(t) - \cancel{u_L}} = \dfrac{f(u_R(t))}{u_R(t)} = \dfrac{u_R(t)(1-u_R(t))}{u_R(t)} = 1 - u_R(t) \\ \quad = 1 - \left(\frac{1}{2} - \frac{x(t)-1}{2t}\right) = \frac{1}{2}\left(1 + \frac{x(t)-1}{t}\right) \\ \text{and } x(1) = 0 \quad \text{since at } t = 1, \text{ the shock starts at } x = 0. \end{cases}$$

So we have to solve the ODE :

$$\begin{cases} x'(t) - \frac{1}{2t}x(t) = \frac{1}{2} - \frac{1}{2t} \\ x(1) = 0 \end{cases}$$

We find : $x(t) = (\sqrt{t} - 1)^2 = t - 2\sqrt{t} + 1$ for $t \geq 1$.

This is the eq. of a curved shock.

Remark : Speed of this curved shock : $x'(t) = 1 - \frac{1}{\sqrt{t}} < 1$.

Hence $x'(t) <$ speed of the right boundary of the rarefaction wave: The curved shock will not touch the right boundary of the rarefaction wave

Conclusion : The solution is given by :

For $t \leq 1$: $u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \text{ and } \frac{x-1}{t} \leq -1 \\ \frac{1}{2} - \frac{x-1}{2t} & \text{if } -1 \leq \frac{x-1}{t} \leq 1 \\ 0 & \text{if } 1 \leq \frac{x-1}{t} \end{cases}$

For $t > 1$: $u(x,t) = \begin{cases} 0 & \text{if } x < x(t) \\ \frac{1}{2} - \frac{x-1}{2t} & \text{if } x(t) < x \text{ and } \frac{x-1}{t} \leq 1 \\ 0 & \text{if } 1 \leq \frac{x-1}{t} \end{cases}$

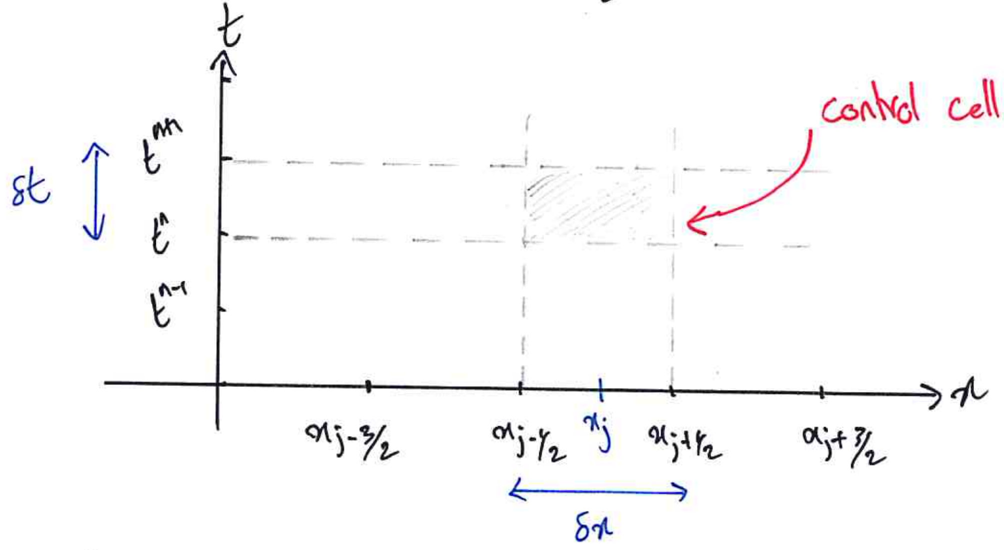
where $x(t) = (\sqrt{t} - 1)^2$.

VI) An introduction to the finite volume method :

1) Definitions :

We want to approximate the entropy weak solution of the Cauchy problem:
$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \quad , \quad \mathbb{R}\times\mathbb{R}^+ \\ u(x,0) = u_0(x) \quad , \quad x \in \mathbb{R} \end{cases}$$

Let be given a subdivision (or a mesh) of $\mathbb{R}\times\mathbb{R}^+$:
$$\mathbb{R}\times\mathbb{R}^+ = \bigcup_{\substack{j\in\mathbb{Z}\\ n\in\mathbb{N}}} [x_{j-1/2}, x_{j+1/2}]\times[t^n, t^{n+1}]$$
where for $j\in\mathbb{Z}$, $x_{j+1/2} = (j+\frac{1}{2})\delta x$ and for $n\in\mathbb{N}$, $t^n = n\delta t$ with,
$\delta x>0$: the space step, $\delta t>0$: the time step.
We define $x_j = j\delta x = \dfrac{x_{j+1/2} + x_{j-1/2}}{2}$.



Let us integrate the eq. on the control cell:
$$\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^{n+1})dx - \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^n)dx + \int_{t^n}^{t^{n+1}} f(x_{j+1/2},t)dt - \int_{t^n}^{t^{n+1}} f(x_{j-1/2},t)dt = 0.$$

In the finite volume method (FV), we denote u_j^n an approximation of $\frac{1}{\delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^n)\,dx$ the mean value of the solution over $[x_{j-1/2}, x_{j+1/2}]$ at t^n.

Remark: In the finite difference method (FD) u_j^n is an approximation of $u(x_j,t^n)$.

Defining a (FV) numerical method (or a numerical scheme) amounts to defining a rule to approximate the quantities $\frac{1}{\delta t}\int_{t^n}^{t^{n+1}} f(x_{j+1/2},t)\,dt$.

Def: A conservative scheme is a numerical scheme of the following form:

$$\frac{u_j^{n+1}-u_j^n}{\delta t} + \frac{F(u_j^n,u_{j+1}^n)-F(u_{j-1}^n,u_j^n)}{\delta x} = 0 \qquad (*)$$

where F is a Lipschitz continuous function of two variables called the numerical flux.

Def: We say that a conservative numerical scheme is consistent if the numerical flux F satisfies:

$$F(u,u) = f(u), \quad \forall u \in \mathbb{R}.$$

Remarks:

1) $F(u_j^n, u_{j+1}^n)$ is an approximation of $\frac{1}{\delta t}\int_{t^n}^{t^{n+1}} f(u(x_{j+1/2},t))\,dt$

2) In practice, when implementing the scheme, we use the following reformulation of $(*)$:

$$u_j^{n+1} = u_j^n - \frac{\delta t}{\delta x}\left(F(u_j^n,u_{j+1}^n) - F(u_{j-1}^n, u_j^n)\right)$$

The implementation on a computer looks like:

```
for n = 1 to N
    for j = 1 to J
        v_j = u_j - (δt/δx) (F(u_j, u_{j+1}) - F(u_{j-1}, u_j))
    End for
    for j = 1 to J
        u_j = v_j
    End for
End for.
```

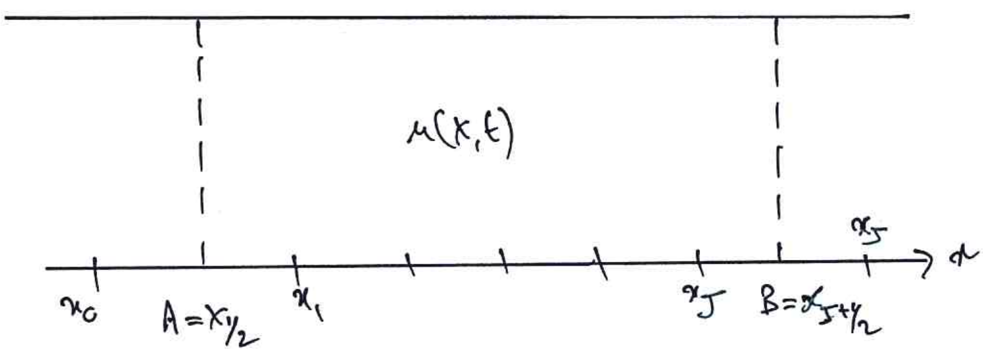
We can optimize this scheme by making a loop on the interfaces $x_{j+1/2}$ rather than on the cells x_j.

3) This is an explicit scheme. The implicit version of this scheme is:

$$u_j^{n+1} = u_j^n - \frac{\delta t}{\delta x}\left(F(u_j^{n+1}, u_{j+1}^{n+1}) - F(u_{j-1}^{n+1}, u_j^{n+1}) \right)$$

One has to solve a (possibly non linear) fixed-point problem in order to compute $(u_j^{n+1})_{j\in\mathbb{Z}}$ from $(u_j^n)_{j\in\mathbb{Z}}$.

4) Why is it called a conservative scheme ?



* At the continuous level (ie for the exact solution):

$$\partial_t u + \partial_x f(u) = 0$$

$$\Rightarrow \int_A^B \partial_t u \, dx + \int_A^B \partial_x f(u)\, dx = 0$$

$$\Rightarrow \underbrace{\frac{d}{dt}\left(\int_A^B u(x,t)\,dx\right)}_{\text{Variation of the total mass between A and B at time } t} = \underbrace{f(u(A,t))}_{\text{Inflow at A}} - \underbrace{f(u(B,t))}_{\text{outflow at B.}}$$

* At the discrete level: (ie for the approximate solution)

$$\sum_{j=1}^{J} \frac{u_j^{n+1} - u_j^n}{\Delta t}\,\delta x + \sum_{j=1}^{J} \frac{F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)}{\cancel{\delta x}}\,\cancel{\delta x} = 0$$

$$\Rightarrow \frac{1}{\delta t}\left(\underbrace{\sum_{j=1}^{J} u_j^{n+1}\,\Delta x}_{\approx \int_A^B u(x,t^{n+1})dx} - \sum_{j=1}^{J} u_j^n\,\delta x\right) = \underbrace{F(u_0^n, u_1^n)}_{\text{approximation of the inflow at A}} - \underbrace{F(u_J^n, u_{J+1}^n)}_{\text{approximation of the outflow at B.}}$$

It is called a conservative scheme because it reproduces the "mass conservation" property at the discrete level.

Examples of conservative schemes:

1) Burgers eq. $\partial_t u + \partial_x f(u) = 0$, $f(u) = \frac{u^2}{2}$.

We choose the numerical flux $F(u,v) = \frac{v^2}{2}$.

The conservative scheme reads:

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{\frac{(u_{j+1}^n)^2}{2} - \frac{(u_j^n)^2}{2}}{\delta x} = 0$$

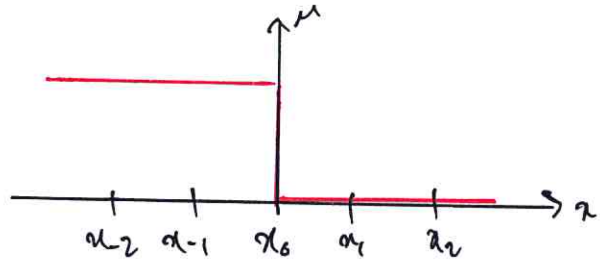
App. of : $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$

2) Non conservative Burgers equation: $\partial_t u + u\,\partial_x u = 0$.
A natural way to approximate this equation is,

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + u_j^n \, \frac{u_j^n - u_{j-1}^n}{\delta x} = 0 .$$

For the initial condition $u^0(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0 \end{cases}$ which we may discretize as follows:

$$u_j^0 = \begin{cases} 1 & \text{if } j < 0 \\ 0 & \text{if } j \geq 0 \end{cases}$$



We know that the exact solution is a shock propagating at speed $s = \frac{1}{2}$.
The numerical solution computed by this scheme is a stationary shock ($s=0$)! So the scheme does not compute the right entropy weak solution. It is because the exact solution is discontinuous and the scheme is <u>not a conservative scheme!</u>

__Theorem__ (Lax - Wendroff)

If the approximate solution of a conservative, consistent scheme converges when $\delta x, \delta t \to 0$ towards a function u, then u is a weak solution of the scalar conservation law.

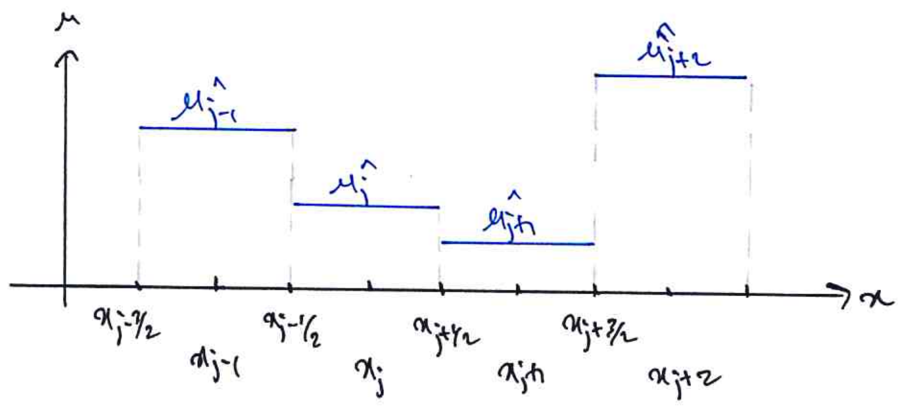
2) The Godunov scheme:

In this section, we assume that f is either linear, or strictly convex or strictly concave.

Assume that we know the values $(u_j^n)_{j\in\mathbb{Z}}$ at $t=t^n$.

__Notation:__ Denote $u_\delta(x)$ the function defined by

$$u_\delta(x) = u_j^n \quad \forall\, x \in [x_{j-1/2}, x_{j+1/2}]$$



The Godunov scheme is a two-step scheme:

__1st step:__ Compute the exact solution of the Cauchy problem:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & \text{on } \mathbb{R}\times[t^n, t^{n+1}] \\ u(x, t^n) = u_\delta(x), & x\in\mathbb{R} \end{cases}$$

Hence, we have to solve a family of Riemann problems at the interfaces $(x_{j+1/2})_{j\in\mathbb{Z}}$:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x, t^n) = \begin{cases} u_j^n & \text{if } x < x_{j+1/2}. \\ u_{j+1}^n & \text{if } x > x_{j+1/2}. \end{cases} \end{cases}$$

We know that the solution of these problems is a wave (either shock or rarefaction wave), propagating at a speed, the absolute value of which is less than $\max\limits_{\substack{\xi \text{ between} \\ u_j^n \text{ and } u_{j+1}^n}} |f'(\xi)|$.

Hence, choosing δt small enough such that :

$$\boxed{\delta t \left(\max_{j\in\mathbb{Z}} \max_{\substack{\xi \text{ between} \\ u_j^n \text{ and } u_{j+1}^n}} |f'(\xi)| \right) \leq \frac{\delta x}{2}}$$

the waves of the various Riemann problems won't interact.

<u>Remark:</u> This condition linking the space and time steps as well as the the maximum propagation speed is called a CFL condition (CFL for Courant - Friedrichs - Lewy).

Hence we have $u(x,t) = \mathcal{U}_R\left(\frac{x - x_{j+1/2}}{t - t^n} ; u_j^n, u_{j+1}^n\right)$ for all (x,t) such that $x \in \,]x_j, x_{j+1}[$ and $t \in \,]t^n, t^{n+1}]$. In particular for $t = t^{n+1}$: $u(x, t^{n+1}) = \mathcal{U}_R\left(\frac{x - x_{j+1/2}}{\delta t} ; u_j^n, u_{j+1}^n\right) \; \forall x \in \,]x_j, x_{j+1}[$.

2nd step: Define $u_j^{n+1} := \frac{1}{\delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^{n+1})\,dx$, in order to obtain a piecewise constant approximate solution at time $t = t^{n+1}$.

An other formulation of the Godunov scheme:

The function $u(x,t)$ constructed in the first step is an exact solution of $\partial_t u + \partial_x f(u) = 0$. Hence integrating the (SCL) on $]x_{j-1/2}, x_{j+1/2}[\times [t^n, t^{n+1}]$, we obtain:

$$\underbrace{\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^{n+1})\,dx}_{= \delta x\, u_j^{n+1}} - \underbrace{\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n)\,dx}_{= \delta x\, u_j^n} + \underbrace{\int_{t^n}^{t^{n+1}} f(u(x_{j+1/2}^-, t))\,dt}_{\delta t\, F_{j+1/2}^-} - \underbrace{\int_{t^n}^{t^{n+1}} f(u(x_{j-1/2}^+, t))\,dt}_{\delta t\, F_{j-1/2}^+} = 0$$

Computation of the numerical fluxes:

$$F_{j+1/2}^- = \frac{1}{\delta t} \int_{t^n}^{t^{n+1}} f(u(x_{j+1/2}^-))\,dt = \frac{1}{\delta t} \int_{t^n}^{t^{n+1}} f(\mathcal{U}_R(0^-; u_j^n, u_{j+1}^n))\,dt \quad \text{independent of } t$$

$$= f(\mathcal{U}_R(0^-; u_j^n, u_{j+1}^n))$$

$$F_{j-1/2}^+ = f(\mathcal{U}_R(0^+, u_{j-1}^n, u_j^n)).$$

Recalling that the function $\xi \mapsto f(\mathcal{U}_R(\xi; u_L, u_R))$ is continuous at $\xi = 0$ we obtain that $\forall j \in \mathbb{Z}$:

$$F_{j+1/2}^{\pm} = F(u_j^n, u_{j+1}^n) = f(\mathcal{U}_R(0; u_j^n, u_{j+1}^n)).$$

Hence the Godunov scheme can be written in the form of a conservative scheme where the numerical flux is

$$F(u,v) = f(\mathcal{U}_R(0;u,v)).$$

This flux is consistent: $F(u,u) = f(u)$ because the entropy weak solution of the Riemann problem:

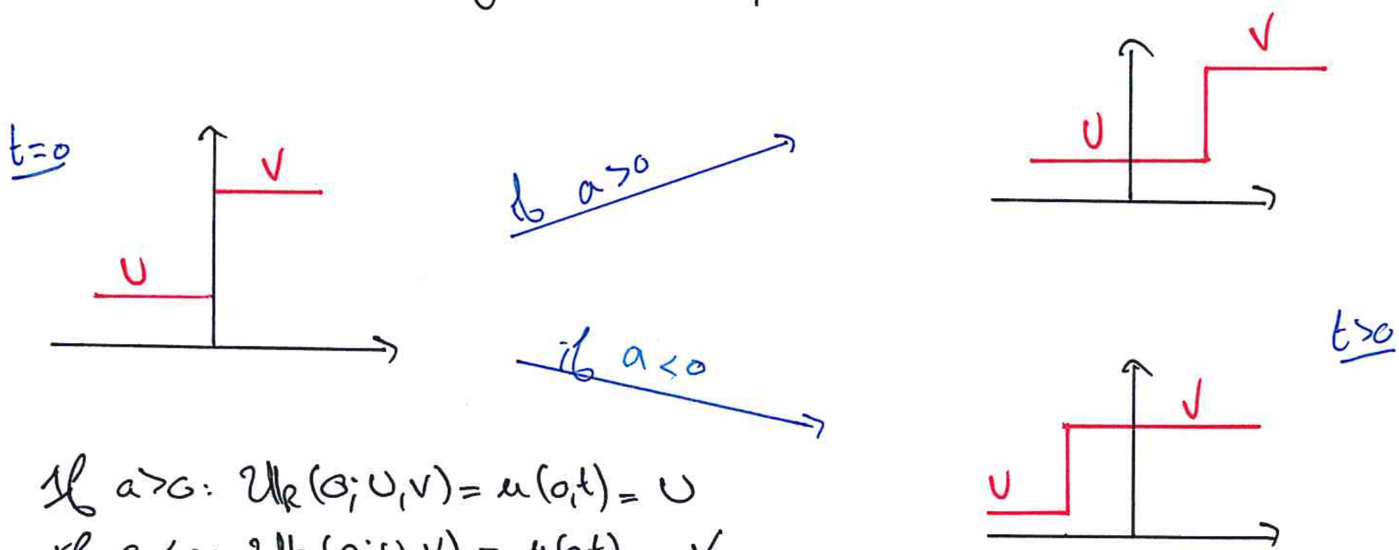
$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x,0) = \begin{cases} \bar{U} & \text{if } x<0 \\ \bar{U} & \text{if } x>0 \end{cases} \end{cases} \quad \text{is} \quad u(x,t) = \bar{U} \quad \forall (x,t).$$

<u>Godunov scheme for the transport equation:</u>

$$\partial_t u + a\partial_x u = 0, \quad a \neq 0.$$

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)}{\delta x} = 0$$

where $F(U,V) = f(\mathcal{U}_R(0;U,V))$.



If $a>0$: $\mathcal{U}_R(0;U,V) = u(0,t) = U$

If $a<0$: $\mathcal{U}_R(0;U,V) = u(0,t) = V$

<u>Hence:</u> If $a>0$: $F(U,V) = f(U) = aU \rightarrow \dfrac{U_j^{n+1} - u_j^n}{\delta t} + a\dfrac{u_j^n - u_{j-1}^n}{\delta x} = 0$

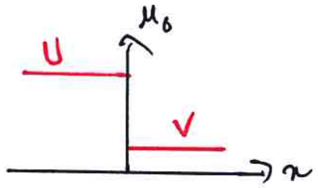
If $a<0$: $F(U,V) = f(V) = aV \rightarrow \dfrac{u_j^{n+1} - u_j^n}{\delta t} + a\dfrac{u_{j+1}^n - u_j^n}{\delta x} = 0$

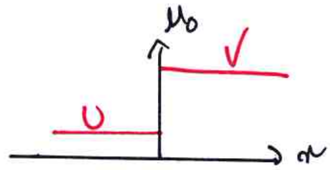
Godunov scheme for the Burgers equation:

$$\begin{cases} \partial_t u + \partial_x\left(\dfrac{u^2}{2}\right) = 0, \\ u(x,0) = u_0(x) = \begin{cases} U & \text{if } x<0, \\ V & \text{if } x>0. \end{cases} \end{cases}$$

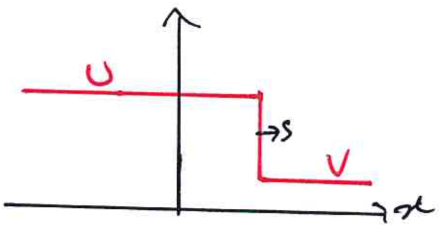
f is strictly convex.
We assume that $u_0 > 0$.

Initial condition: 2 cases:





If $U > V$, the solution of the Riemann problem is a schock propagating at the speed $s = \dfrac{U+V}{2} > 0$ because $U, V > 0$.



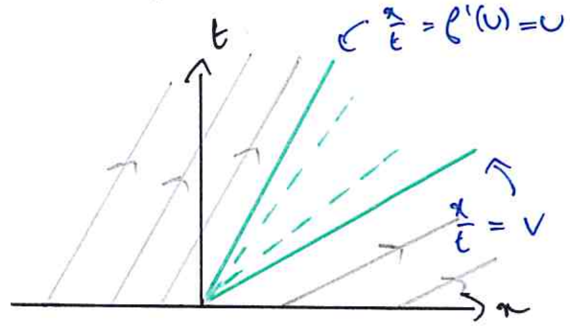
Hence: $\mathcal{U}_R(0; U,V) = U$ and
$F(U,V) = f(U)$

If $U < V$, the solution is a rarefaction wave. $a(u) = f'(u) = u$.

$$u(x,t) = \begin{cases} U & \text{if } \frac{x}{t} \le f'(U) = U \\ v\left(\frac{x}{t}\right) & \text{if } U \le \frac{x}{t} \le V \\ V & \text{if } V \le \frac{x}{t} \end{cases}$$

where $a \circ v(\xi) = \xi$ ie $v(\xi) = \xi$.

Since $U > 0$, all the rarefaction wave is in the quarter plan $\{x>0, t>0\}$:



Hence $\mathcal{U}_R(0; U,V) = U$ and
$F(U,V) = f(U)$.

If u_0 is not > 0: more cases must be studied.