A Robust Entropy-Satisfying Finite Volume Scheme for the Isentropic Baer-Nunziato Model

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Abstract

We construct an approximate Riemann solver for the isentropic Baer-Nunziato two-phase flow model, that is able to cope with arbitrarily small values of the statistical phase fractions. The solver relies on a relaxation approximation of the model for which the Riemann problem is exactly solved for subsonic relative speeds. In an original manner, the Riemann solutions to the linearly degenerate relaxation system are allowed to dissipate the total energy in the vanishing phase regimes, thereby enforcing the robustness and stability of the method in the limits of small phase fractions. The scheme is proved to satisfy a discrete entropy inequality and to preserve positive values of the statistical fractions and densities. The numerical simulations show a much higher precision and a more reduced computational cost (for comparable accuracy) than standard numerical schemes used in the nuclear industry. Finally, two test-cases assess the good behavior of the scheme when approximating vanishing phase solutions.

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1 Introduction

The modeling and numerical simulation of two-phase flows is a relevant approach for a detailed investigation of some patterns occurring in water-vapor flows such as those encountered in nuclear power plants. The targeted applications are the normal operating mode of pressurized water reactors as well as incidental configurations such as the Departure from Nucleate Boiling (DNB) [40], the Loss of Coolant Accident (LOCA) [41] or the re-flooding phase following a LOCA. In the normal operating mode, the flow in the primary circuit is quasi-monophasic as there is *a priori* no vapor in the fluid. In the incidental configurations however, the vapor statistical fraction may take values ranging from zero to nearly one if some areas of the fluid have reached the boiling point. The modeling as well as the numerical simulation of such phenomena remains challenging since both models that can handle phase transitions and robust numerical schemes are needed. The derived schemes are expected to ensure important stability properties such as the positivity of the densities and discrete entropy inequalities. In addition, as explicit schemes are needed for the simulation of these potentially highly unsteady phenomena, one major challenge is the control of the time step. In this context, the aim of this work is to design a robust and entropy-satisfying scheme for the numerical approximation of two-phase flows with vapor or liquid fractions arbitrarily close to zero.

The model concerned by this paper is the isentropic version of the two-fluid model introduced by Baer and Nunziato in [5], and studied in various papers [12, 22, 29] (see also [35] for a related

framework). The model consists in two sets of partial differential equations accounting for the evolution of mass, momentum and total energy for each phase, in addition to a transport equation for the phase fraction. The evolution equations of the two phases are coupled through first order non-conservative terms depending on the phase fraction gradient. A major feature of the Baer-Nunziato model is to assume two different velocities and two different pressures for the two phases. This approach is not genuinely usual in the nuclear industry where the commonly implemented methods assume the same pressure for the two phases at every time and everywhere in the flow. This latter assumption is justified by the very short time-scale associated with the relaxation of the phasic pressures towards an equilibrium. In the two-fluid two-pressure models (such as Baer-Nunziato's), zero-th order source terms may be added in order to account for this pressure relaxation phenomenon as well as friction terms for the relaxation of the phasic velocities towards an equilibrium. However, this work is mainly concerned with the convective effects and these relaxation source terms are not considered here (see [12] for some modeling choices of these terms and [27] for their numerical treatment).

Contrary to the single pressure models, the Baer-Nunziato model provides a pleasant property which is the weak hyperbolicity of its convective part. Indeed, unlike single pressure models, where the characteristic eigenvalues may be complex, the Baer-Nunziato model admits seven real eigenvalues and the associated right eigenvectors form a basis unless the relative velocity between the phases equals the speed of sound in the liquid (see [20]). A first consequence is that the initial value problem is not *a priori* ill-posed for this model as long as the system remains strictly hyperbolic, that is to say as long as no interaction occurs between the acoustic waves and the material wave transporting the phase fraction. Another important consequence is that the definition of the non-conservative term in the model is **not ambiguous**, still as long as the system remains strictly hyperbolic. Indeed, the non-conservative term in the Baer-Nunziato model is transported by the linearly degenerate phase fraction wave. Hence, as long as the system is hyperbolic, this non-conservative product is naturally defined by the Riemann invariants associated with this wave (see [31, 21]). One of these Riemann invariants expresses the conservation of the total mixture energy across the phase fraction wave in the domain of hyperbolicity of the model. When the strict hyperbolicity is not satisfied, one speaks of a resonance phenomenon and the definition of the non-conservative product is no longer straightforward. In these cases, one must define regularization models in order to prescribe the behavior of such resonant solutions. One usual model used in a related framework [28, 23] that encompasses the Baer-Nunziato model, assumes a monotonic evolution of the phase fraction within the resonant wave. Other regularizations processes have been introduced in the context of non-conservative interface coupling between hyperbolic systems [2, 9]. In [33], viscous regularization methods are used for related two-phase flow models and in the works of Yong [42], Kawashima-Yong [30], Chen *et al.* [11] and Hanouzet *et al.* [25], the authors investigate the stabilization effects of zero-th order source terms in the context of strictly hyperbolic balance equations. One common consequence of all these stabilization procedures is that they imply a dissipation of the system's energy either globally (for viscous and zero-th order stabilizations) or locally around the resonant wave. In this latter case, the regularization is associated with a so-called **kinetic relation** which is a generalized Rankine-Hugoniot type relation on the system's energy expressing its dissipation through the resonant linearly degenerate wave. For a general review on kinetic relations, we refer to [6].

In our context of interest, namely nuclear liquid-vapor flows, the resonance due to wave interaction between acoustic fields and the phase transport equation is unlikely to arise since it would imply sonic or supersonic values of the relative velocity between the phases. However, the aim of this work is to capitalize on the above considerations concerning the stabilization of resonant solutions in order to propose an original framework for the stable computation of vanishing phase solutions. The method is the following. Among all the energy dissipative mechanisms, we seek those that might ensure stable approximate solutions for arbitrarily small values of the phase fractions. For instance, an interesting question would be to know if in the Riemann problem, weakening the energy conservation across the phase fraction wave through a suitable kinetic relation, may ensure stable solutions in the regimes of small phase fractions. Unfortunately, despite some interesting works in that sense ([36, 18, 4]), calculating the exact solution of the Riemann problem for the Baer-Nunziato model with any initial data is so far out of reach. One main obstruction is that the characteristic eigenvalues of the system

are not naturally ordered, and no method has been found yet that could determine *a priori* their ordering, with respect to the initial data. In addition, the strong non-linearities of the pressure laws make even more difficult the derivation of an exact Riemann solver. Following the pioneering work of Harten, Lax and van Leer [26], other approaches consider approximate Riemann solvers [39, 3], and the method described in the present paper enters this category. Let us mention some other schemes grounded on operator splitting techniques [10, 15, 32, 34, 37, 38].

Actually, the method considered here relies on a *relaxation approximation* of the model, similar to that in Ambroso, Chalons, Coquel and Galié [1]. The idea consists in introducing a larger system, in which the pressure laws have been linearized, and which relaxes towards the actual system of Baer-Nunziato in the regime of a small relaxation parameter (for a general framework on relaxation schemes we refer to [13, 14, 7]). While in [1], the authors calculate an *approximate* solution of the relaxation system thanks to a prediction of the non-conservative product, one major contribution of our work is the **effective resolution** of the relaxation Riemann problem in the framework of subsonic wave ordering, by calculating **exact solutions**. In particular, we provide **explicit conditions on the initial data** that enable the *a priori* determination of the relative ordering of the waves. The proof relies on a fixed-point procedure that consists in iteratively considering the evolution equations of each phase, taken separately. An outstanding property is that this iterative procedure boils down to a fixed-point research on a monotonic **scalar function**. This property is crucial for the numerical computations using the relaxation Riemann solver since it avoids heavy computational costs as assessed by the numerical tests. In order to enforce the stability of the solutions when the initial phase fractions have arbitrarily small values, the solutions are allowed in these regimes to dissipate the total energy of the system across the phase fraction wave, through the definition of a suitable kinetic relation. Hence, despite the linear degeneracy of all the waves implied by the relaxation approximation procedure, some extra dissipation is introduced and one may speak of *dissipative relaxation*. This original idea has been introduced in order to control the time step in the vanishing phase regimes. Indeed, another way of stabilizing the numerical solutions in these regimes is to take larger values of the relaxation parameters, thus introducing more diffusion to the relaxation approximation. However, too large values of the relaxation parameters may involve too small time steps because of the CFL restrictions, since these parameters directly control the size of the wave fan in the approximate Riemann solver.

The resulting scheme is proved to preserve positive densties and to satisfy a discrete entropy-inequality under a sub-characteristic condition (Whitham's condition). To our knowledge, there exists no other scheme that is proved to satisfy these two properties. In addition, for the same level of refinement, the scheme is shown to be much more accurate than the Rusanov scheme, and for a given level of approximation error, the relaxation scheme is shown to perform much better in terms of computational cost than this classical scheme. Actually, comparing with Lax-Friedrichs type schemes is quite significant since for such stiff configurations as vanishing phase cases, these schemes are commonly used in the industrial context because of their known robustness [27]. Our relaxation scheme is first-order accurate and an interesting further work is the extension to higher orders (see [19, 39] for example of high order schemes). Nevertheless this work is focused on the design of stable computations for vanishing phase configurations. In these regimes indeed, even high order methods may develop instabilities.

The paper is organized as follows. Sections 2 and 3 are devoted to the presentation of the Baer-Nunziato model and its relaxation approximation. Some issues concerning the resonance, kinetic relations and vanishing phase regimes are also discussed. Section 4 is the core of the paper. It displays an existence theorem for the relaxation Riemann problem, for a generalized class of dissipative solutions. For the sake of understanding, the constructive proof is explained in detail. In Section 5, we use the relaxation exact Riemann solver to derive an approximate solver in the sense of Harten, Lax and van Leer [26] and thus, a Finite Volume scheme. Finally, Section 6 is devoted to the numerical tests. In addition to a convergence and CPU cost study, two test-cases assess that the scheme provides a robust numerical treatment of vanishing phase solutions.

2 The isentropic model of Baer-Nunziato

The isentropic Baer-Nunziato model is a two-phase flow model formulated in Eulerian coordinates where balance equations account for the evolution of mass and momentum of each phase. The velocities of each phase are denoted $u_i, i \in \{1, 2\}$, while the densities are denoted $\rho_i, i \in \{1, 2\}$. Each phase has a statistical phase fraction $\alpha_i, i \in \{1, 2\}$, with the saturation constraint $\alpha_1 + \alpha_2 = 1$. The model reads:

$$\partial_t \mathbb{U} + \partial_x \mathbf{f}(\mathbb{U}) + \mathbf{c}(\mathbb{U})\partial_x \mathbb{U} = 0, \quad x \in \mathbb{R}, t > 0, \tag{2.1}$$

with

$$\mathbb{U} = \begin{bmatrix} \alpha_1 \\ \alpha_1 \rho_1 \\ \alpha_1 \rho_1 u_1 \\ \alpha_2 \rho_2 \\ \alpha_2 \rho_2 u_2 \end{bmatrix}, \qquad \mathbf{f}(\mathbb{U}) = \begin{bmatrix} 0 \\ \alpha_1 \rho_1 u_1 \\ \alpha_1 \rho_1 u_1^2 + \alpha_1 p_1(\rho_1) \\ \alpha_2 \rho_2 u_2 \\ \alpha_2 \rho_2 u_2^2 + \alpha_2 p_2(\rho_2) \end{bmatrix}, \qquad \mathbf{c}(\mathbb{U})\partial_x \mathbb{U} = \begin{bmatrix} u_2 \\ 0 \\ -p_1(\rho_1) \\ 0 \\ +p_1(\rho_1) \end{bmatrix} \partial_x \alpha_1. \tag{2.2}$$

The state vector $\mathbb{U}$ is expected to belong to the natural physical space

$$\Omega = \left\{ \mathbb{U} \in \mathbb{R}^5, 0 < \alpha_1 < 1 \text{ and } \alpha_i \rho_i > 0 \text{ for } i \in \{1, 2\} \right\}. \tag{2.3}$$

We assume barotropic pressure laws for each phase $\rho_i \mapsto p_i(\rho_i)$, $i \in \{1, 2\}$ with smooth dependence on the density, and which satisfy the following natural assumptions for all $\rho_i > 0$:

$$p_i(\rho_i) > 0, \quad p_i'(\rho_i) > 0, \quad \lim_{\rho_i \to 0} p_i(\rho_i) = 0, \quad \lim_{\rho_i \to +\infty} p_i(\rho_i) = +\infty. \tag{2.4}$$

We define the mapping $\tau \mapsto \mathcal{P}_i(\tau) := p_i(\tau^{-1})$ which is the phasic pressure seen as a function of the specific volume $\tau = \rho^{-1}$. In the whole paper, this smooth function is assumed to be strictly convex:

$$\mathcal{P}_i''(\tau_i) > 0, \quad \text{for all } \tau_i > 0, i \in \{1, 2\}. \tag{2.5}$$

2.1 Main mathematical properties

The following proposition characterizes the fields of this system:

Proposition 2.1. *System (2.1) is weakly hyperbolic since it admits the following real eigenvalues*

$$\sigma_1(\mathbb{U}) = u_2, \quad \sigma_2(\mathbb{U}) = u_1 - c_1(\rho_1), \quad \sigma_3(\mathbb{U}) = u_1 + c_1(\rho_1), \quad \sigma_4(\mathbb{U}) = u_2 - c_2(\rho_2), \quad \sigma_5(\mathbb{U}) = u_2 + c_2(\rho_2), \tag{2.6}$$

where $c_i(\rho_i) = \sqrt{p_i'(\rho_i)}$ is the speed of sound for phase i. The corresponding right eigenvectors are linearly independent if, and only if,

$$\alpha_1 \neq 0, \quad \alpha_2 \neq 0, \quad |u_1 - u_2| \neq c_1(\rho_1). \tag{2.7}$$

*When (2.7) is not satisfied, the system is said to be **resonant**. The characteristic fields associated with σ_2, σ_3, σ_4 and σ_5 are genuinely non-linear, while the characteristic field associated with σ_1 is linearly degenerate.*

Remark 2.1. *Actually, following the definition of the admissible physical space (2.3), one never has $\alpha_1 = 0$ or $\alpha_2 = 0$. However, $\alpha_i = 0$ is to be understood in the sense $\alpha_i \to 0$ since the aim of this work is to construct a robust enough Riemann solver that could handle all the possible values of α_i, $i \in \{1, 2\}$, especially, arbitrarily small values.*

Proof. Denoting $\mathcal{U} = (\alpha_1, \rho_1, u_1, \rho_2, u_2)^T$, the smooth solutions of system (2.1) satisfy the equivalent following system

$$\partial_t \mathcal{U} + \mathbf{A}(\mathcal{U})\partial_x \mathcal{U} = 0,$$

4

where

$$\mathbf{A}(\mathcal{U}) = \begin{bmatrix} u_2 & 0 & 0 & 0 & 0 \\ \frac{\rho_1}{\alpha_1}(u_1 - u_2) & u_1 & \rho_1 & 0 & 0 \\ 0 & \frac{p_1'(\rho_1)}{\rho_1} & u_1 & 0 & 0 \\ 0 & 0 & 0 & u_2 & \rho_2 \\ \frac{p_1(\rho_1) - p_2(\rho_2)}{(1-\alpha_1)\rho_2} & 0 & 0 & \frac{p_2'(\rho_2)}{\rho_2} & u_2 \end{bmatrix}.$$

This matrix admits five real eigenvalues that are provided in (2.6). Denoting $M = \frac{u_1 - u_2}{c_1}$ the relative Mach number, the corresponding right eigenvectors can be chosen as

$$r_1(\mathcal{U}) = \left(\alpha_1 (1 - \alpha_1)(M^2 - 1), -(1 - \alpha_1)\rho_1 M^2, (1 - \alpha_1)c_1 M, \alpha_1 \frac{p_2(\rho_2) - p_1(\rho_1)}{c_2^2}(M^2 - 1), 0 \right)^T,$$

$$r_2(\mathcal{U}) = \frac{-2\sqrt{p_1'(\rho_1)}}{p_1''(\rho_1)\rho_1 + 2p_1'(\rho_1)} \left(0, \rho_1, -\sqrt{p_1'(\rho_1)}, 0, 0 \right)^T,$$

$$r_3(\mathcal{U}) = \frac{2\sqrt{p_1'(\rho_1)}}{p_1''(\rho_1)\rho_1 + 2p_1'(\rho_1)} \left(0, \rho_1, \sqrt{p_1'(\rho_1)}, 0, 0 \right)^T,$$

$$r_4(\mathcal{U}) = \frac{-2\sqrt{p_2'(\rho_2)}}{p_2''(\rho_2)\rho_2 + 2p_2'(\rho_2)} \left(0, 0, 0, \rho_2, -\sqrt{p_2'(\rho_2)} \right)^T,$$

$$r_5(\mathcal{U}) = \frac{2\sqrt{p_2'(\rho_2)}}{p_2''(\rho_2)\rho_2 + 2p_2'(\rho_2)} \left(0, 0, 0, \rho_2, \sqrt{p_2'(\rho_2)} \right)^T.$$

Hence, the system is hyperbolic, *i.e.* the five eigenvectors $r_k(\mathcal{U})$, k in $\{1, .., 5\}$ span $\mathbb{R}^5$ if and only if $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $(u_1 - u_2)^2 \neq c_1^2(\rho_1)$. Moreover, one can easily verify that

$$\nabla_{\mathcal{U}} \sigma_1(\mathcal{U}).r_1(\mathcal{U}) = 0,$$
$$\nabla_{\mathcal{U}} \sigma_k(\mathcal{U}).r_k(\mathcal{U}) = 1, \quad k = 2, 3, 4, 5$$

which proves that the first characteristic field is linearly degenerate while the four others are genuinely non-linear. □

The following proposition states the existence of two phasic energy equations satisfied by the smooth solutions of system (2.1).

Proposition 2.2. *Defining $E_i := E_i(u_i, \tau_i) = \frac{u_i^2}{2} + e_i(\tau_i)$, $i \in \{1, 2\}$ with $\tau \mapsto e_i(\tau)$ an antiderivative of $\tau \mapsto -\mathcal{P}_i(\tau)$, the smooth solutions of system (2.1) satisfy the following phasic energy equations:*

$$\partial_t(\alpha_i \rho_i E_i) + \partial_x (\alpha_i \rho_i E_i + \alpha_i p_i(\rho_i)) u_i - u_2 p_1(\rho_1)\partial_x \alpha_i = 0, \quad i \in \{1, 2\}. \tag{2.8}$$

Summing over $i = 1, 2$ yields the following additional conservation law, which expresses the total mixture energy conservation by the smooth solutions of system (2.1):

$$\partial_t (\alpha_1 \rho_1 E_1 + \alpha_2 \rho_2 E_2) + \partial_x ((\alpha_1 \rho_1 E_1 + \alpha_1 p_1(\rho_1)) u_1 + (\alpha_2 \rho_2 E_2 + \alpha_2 p_2(\rho_2)) u_2) = 0. \tag{2.9}$$

Proof. The proof is classical. It consists in multiplying the phasic momentum equations (third and fifth components of system (2.1)) respectively by u_i and re-arranging the equations using the definition of $e_i(\tau_i)$ and the mass conservation equations (second and fourth components of (2.1)). □

As regards the non-smooth weak solutions of (2.1), one has to add a so-called entropy criterion in order to select the relevant physical solutions. Thus, an entropy weak solution of (2.1) is a function $\mathbb{U}(x, t)$ that satisfies (2.1) in the sense of distributions as well as the following entropy inequality:

$$\partial_t (\alpha_1 \rho_1 E_1 + \alpha_2 \rho_2 E_2) + \partial_x ((\alpha_1 \rho_1 E_1 + \alpha_1 p_1(\rho_1)) u_1 + (\alpha_2 \rho_2 E_2 + \alpha_2 p_2(\rho_2)) u_2) \leq 0. \tag{2.10}$$

When the solution contains shock waves, inequality (2.10) is strict in order to account for the physical loss of energy due to viscous phenomena that are not modeled in system (2.1).

2.2 Riemann weak solutions in the case of strict hyperbolicity

The definition of weak solutions to system (2.1) is not straightforward because of the non-conservative product $p_1\partial_x\alpha_1$. Indeed, considering a discontinuity of the phase fraction α_1 across which the pressure p_1 may be discontinuous, it is clear that the product $p_1\partial_x\alpha_1$ is not *a priori* defined in the sense of distributions. However, since α_1 discontinuities coincide with a linearly degenerate field, there is no ambiguity in the definition of the product as long as the system is hyperbolic, *i.e.* as long as the three conditions (2.7) are met. In this case indeed, there are two equivalent ways for defining the product $p_1\partial_x\alpha_1$. Let us describe briefly these too equivalent alternatives in the context of a Riemann problem. System (2.1) is supplied with an initial condition of the type

$$\mathbb{U}(x, t=0) = \begin{cases} \mathbb{U}_L & \text{if} \quad x < 0, \\ \mathbb{U}_R & \text{if} \quad x > 0. \end{cases} \tag{2.11}$$

In most cases (hyperbolic cases, see thereafter), if the initial data is such that $\alpha_{1,L} \neq \alpha_{1,R}$, the solution of the Riemann problem is a self-similar function composed of intermediate states separated by waves associated with the acoustic fields (rarefaction or shock waves) or by a contact discontinuity associated with the eigenvalue u_2. Actually, the phase fractions α_i only jump through this contact discontinuity from $\alpha_{i,L}$ to $\alpha_{i,R}$. Away from this wave, the phase fractions are constant and the system behaves as two independent (isentropic) Euler systems associated with both phases since the term $\partial_x\alpha_i$ vanishes.
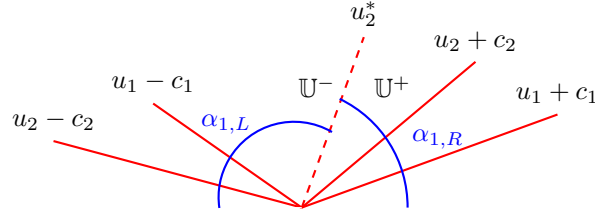
For the actual computation of the Riemann solution, one needs jump relations across each wave in order to link the different intermediate states. For the acoustic waves, the jump relations are the same as for the isentropic Euler equations associated with each phase. For the u_2-wave however, the two phases are coupled and there are two equivalent ways of defining the relation between two intermediate states $\mathbb{U}^-$ and $\mathbb{U}^+$ separated by such a discontinuity.

2.2.1 Definition through Riemann invariants

When the system is hyperbolic, there are four Riemann invariants associated with the linearly degenerate field $\sigma_1(\mathbb{U}) = u_2$, that is to say four independent functions $\Phi_k(\mathbb{U})$, $k = 1..4$ satisfying $\nabla_{\mathbb{U}}\Phi_k(\mathbb{U}).r_1(\mathbb{U}) = 0$, for all $k = 1..4$. In the hyperbolic case, two states $\mathbb{U}^-$ and $\mathbb{U}^+$ are separated by a u_2-contact discontinuity if and only if $\Phi_k(\mathbb{U}^-) = \Phi_k(\mathbb{U}^+)$ for all $k = 1..4$. For a proof of this result in the context of the full Baer-Nunziato model with energies, we refer to [21]. The derivative $\partial_x\alpha_1$ corresponds to a Dirac measure of the form $(\alpha_1^+ - \alpha_1^-)\delta_{x-u_2^*t}$, where u_2^* is the propagation speed of the α_1-discontinuity, and the non-conservative product $p_1\partial_x\alpha_1$ identifies with a measure

$$p_1\partial_x\alpha_1 := p_1^*(\alpha_1^+ - \alpha_1^-)\delta_{x-u_2^*t} \tag{2.12}$$

whose weight $p_1^*(\alpha_1^+ - \alpha_1^-)$ is obtained by applying Rankine-Hugoniot's jump relation equivalently to one of the momentum equations:

$$\begin{aligned} p_1^*(\alpha_1^+ - \alpha_1^-) &= -u_2^*[\alpha_1\rho_1 u_1]_{\frac{x}{t}=u_2^*} + [\alpha_1\rho_1 u_1^2 + \alpha_1 p_1]_{\frac{x}{t}=u_2^*} \\ &= - \left\{ -u_2^*[\alpha_2\rho_2 u_2]_{\frac{x}{t}=u_2^*} + [\alpha_2\rho_2 u_2^2 + \alpha_2 p_2]_{\frac{x}{t}=u_2^*} \right\}. \end{aligned} \tag{2.13}$$

Here, $[X]_{\frac{x}{t}=u_2^*} = X^+ - X^-$ denotes the difference between the values taken by the quantity X on the right and on the left of the u_2-contact discontinuity. As the eigenvalue u_2 is a Riemann invariant, the second line of (2.13) gives

$$p_1\partial_x\alpha_1 = p_1^*(\alpha_1^+ - \alpha_1^-)\delta_{x-u_2^*t} = -[\alpha_2 p_2]_{\frac{x}{t}=u_2^*}\,\delta_{x-u_2^*t}. \tag{2.14}$$

2.2.2 Definition through Rankine-Hugoniot jump relations

In the hyperbolic case, an equivalent way of defining the non-conservative product $p_1 \partial_x \alpha_1$ across a u_2-contact discontinuity is through Rankine-Hugoniot jump relations. A first relation is given by the continuity of the eigenvalue u_2 across this linearly degenerate wave, and we get two more independent jump relations by applying Rankine-Hugoniot's formula to the mass conservation equation of phase 1 and to the total momentum conservation:

$$[u_2]_{\frac{x}{t}=u_2^*} = 0, \tag{2.15}$$

$$-u_2^* [\alpha_1 \rho_1]_{\frac{x}{t}=u_2^*} + [\alpha_1 \rho_1 u_1]_{\frac{x}{t}=u_2^*} = 0, \tag{2.16}$$

$$-u_2^* [\alpha_1 \rho_1 u_1 + \alpha_2 \rho_2 u_2]_{\frac{x}{t}=u_2^*} + [\alpha_1 \rho_1 u_1^2 + \alpha_1 p_1 + \alpha_2 \rho_2 u_2^2 + \alpha_2 p_2]_{\frac{x}{t}=u_2^*} = 0. \tag{2.17}$$

Observe that the mass conservation equation for phase 2 gives no additional information since the corresponding jump relation is naturally satisfied because u_2 is constant through the discontinuity. Hence, there is one missing information in order to fully define the discontinuity and therefore the non-conservative product. If we assume the hyperbolicity of the system, this last jump relation is obtained by applying Rankine-Hugoniot's formula to the total energy conservation (2.9) which yields

$$-u_2^* [\alpha_1 \rho_1 E_1 + \alpha_2 \rho_2 E_2]_{\frac{x}{t}=u_2^*} + [(\alpha_1 \rho_1 E_1 + \alpha_1 p_1) u_1 + (\alpha_2 \rho_2 E_2 + \alpha_2 p_2) u_2]_{\frac{x}{t}=u_2^*} = 0. \tag{2.18}$$

Indeed, a theoretical result (see [31]) states that for a hyperbolic system, any additional conservation law satisfied by the smooth solutions of the system is also satisfied in the weak sense through contact discontinuities. Actually, a sufficient condition for the proof of this result is the existence of a right eigenvector basis. The non-conservative product is then obtained by applying Rankine-Hugoniot's relation to the momentum equations just as previously, which yields (2.14).

2.2.3 Local energy equations around the α_1-discontinuity

In the case of hyperbolicity, the phasic energy equations (2.8) are satisfied in the weak sense in the neighborhood of a u_2-contact discontinuity. For phase 2, this is a direct consequence of the fact that u_2 is constant across this wave according to (2.15). Indeed, one may write

$$-u_2^* [\alpha_2 \rho_2 E_2]_{\frac{x}{t}=u_2^*} + [(\alpha_2 \rho_2 E_2 + \alpha_2 p_2) u_2]_{\frac{x}{t}=u_2^*} - u_2^* [\alpha_2 p_2] = 0, \tag{2.19}$$

which by the definition (2.14) of the non-conservative product $p_1 \partial_x \alpha_1$, is the Rankine-Hugoniot relation of

$$\partial_t (\alpha_2 \rho_2 E_2) + \partial_x (\alpha_2 \rho_2 E_2 + \alpha_2 p_2(\rho_2)) u_2 - u_2 p_1(\rho_1) \partial_x \alpha_2 = 0. \tag{2.20}$$

As for phase 1, the local phasic energy equation is a consequence of the constancy (2.15) of u_2 **and** of the conservation of the total mixture energy (2.18). Thus in the hyperbolicity areas of the system, the weak solutions satisfy

$$-u_2^* [\alpha_1 \rho_1 E_1]_{\frac{x}{t}=u_2^*} + [(\alpha_1 \rho_1 E_1 + \alpha_1 p_1) u_1]_{\frac{x}{t}=u_2^*} - u_2^* [\alpha_2 p_2] = 0, \tag{2.21}$$

which is the weak formulation of the following PDE satisfied locally near the α_1-discontinuity:

$$\partial_t (\alpha_1 \rho_1 E_1) + \partial_x (\alpha_1 \rho_1 E_1 + \alpha_1 p_1(\rho_1)) u_1 - u_2 p_1(\rho_1) \partial_x \alpha_1 = 0. \tag{2.22}$$

Let us emphasize again, that this last property (2.21)-(2.22) is a consequence of the assumed hyperbolicity of the system since, this is the very property that allows the total energy conservation to be used in the weak sense across a u_2-contact discontinuity. If the system is not strictly hyperbolic, this property is not guaranteed as explained in the next section.

2.3 Riemann weak solutions in the resonant case

When (2.7) is not satisfied, for instance if at some point of the flow, the physical quantities satisfy $|u_1 - u_2| = c_1(\rho_1)$, the hyperbolicity is lost and the system is said to be resonant. In that case, the existence of four independent Riemann invariants is no longer guaranteed and the jump relation associated with the total mixture energy conservation is potentially lost. Indeed, the existing theoretical results stating that for a hyperbolic system, any additional conservation law satisfied by the smooth solutions is also satisfied in the weak sense through contact discontinuities, assume the existence of a right eigenvector basis. Therefore, in these resonant cases, if the total mixture energy can no more be conserved across the u_2-contact discontinuity, enforcing its dissipation appears to be necessary for stability reasons. To this end, one has to model regularization processes along the u_2-contact discontinuity by adding some information on the expected behavior of such resonant solutions. With any regularization process, is associated a so-called **kinetic relation**, which is an additional Rankine-Hugoniot type relation, that enables the actual computation of jumps across a resonant contact discontinuity, for instance when solving the Riemann problem. In our context, a kinetic relation takes the form

$$- u_2^* \left[\alpha_1 \rho_1 E_1 + \alpha_2 \rho_2 E_2 \right]_{\frac{x}{t} = u_2^*} + \left[\left(\alpha_1 \rho_1 E_1 + \alpha_1 p_1 \right) u_1 + \left(\alpha_2 \rho_2 E_2 + \alpha_2 p_2 \right) u_2 \right]_{\frac{x}{t} = u_2^*} = -\mathcal{Q}(u_2^*, \mathbb{U}_L, \mathbb{U}_R),$$
(2.23)

where $\mathcal{Q}(u_2^*, \mathbb{U}_L, \mathbb{U}_R)$ is a positive number measuring the amount of dissipated energy across the u_2-wave. In practice, the definition of such a kinetic relation has to obey several natural requirements. First of all, in the areas of the phase space Ω where the system is hyperbolic, $\mathcal{Q}(u_2^*, \mathbb{U}_L, \mathbb{U}_R)$ is expected to vanish in order to restore the total mixture energy conservation. On the other hand, in the resonance areas, $\mathcal{Q}(u_2^*, \mathbb{U}_L, \mathbb{U}_R)$ has to be designed so as to ensure the existence of stable solutions of the Riemann problem. Of course, several choices could be made for the regularization process, leading to several possible definitions of $\mathcal{Q}(u_2^*, \mathbb{U}_L, \mathbb{U}_R)$. Thus, the uniqueness of solutions can only be recovered for a *given* choice of the kinetic relation.

To exemplify the energy dissipation across a u_2-contact discontinuity in the resonant case, one may consider the case of a $\{u_1 - c_1\}$-shock superimposing with the u_2-contact discontinuity. If one considers the jump relation of the total energy through such a discontinuity, it reads

$$- u_2^* \left[\alpha_1 \rho_1 E_1 + \alpha_2 \rho_2 E_2 \right]_{\frac{x}{t} = u_2^*} + \left[\left(\alpha_1 \rho_1 E_1 + \alpha_1 p_1 \right) u_1 + \left(\alpha_2 \rho_2 E_2 + \alpha_2 p_2 \right) u_2 \right]_{\frac{x}{t} = u_2^*} < 0.$$
(2.24)

Actually, on the one hand, the energy equation for phase 2 (2.19)-(2.20) is still satisfied. On the other hand, the energy of phase 1 is strictly dissipated in this case due to the shock lying inside the discontinuity. The jump relation (2.24) is equivalent to

$$- u_2^* \left[\alpha_1 \rho_1 E_1 \right]_{\frac{x}{t} = u_2^*} + \left[\left(\alpha_1 \rho_1 E_1 + \alpha_1 p_1 \right) u_1 \right]_{\frac{x}{t} = u_2^*} - u_2^* \left[\alpha_2 p_2 \right] < 0,$$
(2.25)

which is the weak formulation of

$$\partial_t (\alpha_1 \rho_1 E_1) + \partial_x (\alpha_1 \rho_1 E_1 + \alpha_1 p_1(\rho_1)) u_1 - u_2 p_1(\rho_1) \partial_x \alpha_1 < 0.$$
(2.26)

Considering now the physically relevant case of nearly vanishing phases, where the phase fractions may be arbitrarily close (but not equal) to zero, the strict hyperbolicity of the system is not actually lost in the sense that the right eigenvector basis still exists as long as $\alpha_i \neq 0$, $i \in \{1, 2\}$. Nevertheless, the matrix composed of the right eigenvectors becomes singular as one of the phase fractions tends to zero. Obviously, this may cause instabilities of the model in these vanishing phase regimes. Yet, in many application contexts where one of the phases is naturally expected to disappear, one is interested in stable (approximate) computations of such stiff regimes. The purpose of this paper is to construct an approximate Riemann solver, for which the solutions are controlled in the regimes $\alpha_i \to 0$. This Riemann solver relies on a relaxation approximation which is presented in the next section.

Remark 2.2. *In the physical configurations aimed at in the present work (such as two-phase flows in nuclear reactors), resonant configurations corresponding to wave interaction between acoustic fields*

and the u_2-contact discontinuity are unlikely to occur. Indeed, for these applications, the flows have strongly subsonic relative velocities, i.e. a relative Mach number much smaller than one:

$$M = \frac{|u_1 - u_2|}{c_1} << 1. \tag{2.27}$$

Having set aside the sonic ($M = 1$) and supersonic ($M > 1$) flows, the main purpose of this work is to tackle the issue of vanishing phase instabilities.

3 A relaxation approximation

We now introduce a relaxation approximation of system (2.1). The new system possesses the same structure as the initial model (2.1) but the differential part is linearly degenerate while the non-linear part is postponed to the source term. The goal is to construct and analyze the linearly degenerate differential part. In general, the Riemann problem for such systems is simple to solve but here, we inherit the non-conservative and resonant structure from (2.1). As a consequence, we have to introduce a decrease of the associated mixture energy in some extreme cases by the use of a so-called kinetic relation in order to obtain admissible solutions.

Denoting $\mathbb{W} = (\alpha_1, \alpha_1\rho_1, \alpha_1\rho_1 u_1, \alpha_2\rho_2, \alpha_2\rho_2 u_2, \alpha_1\rho_1\mathcal{T}_1, \alpha_2\rho_2\mathcal{T}_2)^T$ the relaxation state vector, we propose the following relaxation approximation for system (2.1):

$$\partial_t \mathbb{W}^\varepsilon + \partial_x \mathbf{g}(\mathbb{W}^\varepsilon) + \mathbf{d}(\mathbb{W}^\varepsilon)\partial_x \mathbb{W}^\varepsilon = \frac{1}{\varepsilon}\mathcal{R}(\mathbb{W}^\varepsilon), \tag{3.1}$$

where

$$\mathbf{g}(\mathbb{W}) = \begin{bmatrix} 0 \\ \alpha_1\rho_1 u_1 \\ \alpha_1\rho_1 u_1^2 + \alpha_1\pi_1(\tau_1, \mathcal{T}_1) \\ \alpha_2\rho_2 u_2 \\ \alpha_2\rho_2 u_2^2 + \alpha_2\pi_2(\tau_2, \mathcal{T}_2) \\ \alpha_1\rho_1\mathcal{T}_1 u_1 \\ \alpha_2\rho_2\mathcal{T}_2 u_2 \end{bmatrix}, \quad \mathbf{d}(\mathbb{W})\partial_x \mathbb{W} = \begin{bmatrix} u_2 \\ 0 \\ -\pi_1(\tau_1, \mathcal{T}_1) \\ 0 \\ +\pi_1(\tau_1, \mathcal{T}_1) \\ 0 \\ 0 \end{bmatrix} \partial_x \alpha_1, \quad \mathcal{R}(\mathbb{W}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_1\rho_1(\tau_1 - \mathcal{T}_1) \\ \alpha_2\rho_2(\tau_2 - \mathcal{T}_2) \end{bmatrix}. \tag{3.2}$$

For each phase i in $\{1, 2\}$ the linearized pressure $\pi_i(\tau_i, \mathcal{T}_i)$ is a function defined as

$$\pi_i(\tau_i, \mathcal{T}_i) = \mathcal{P}_i(\mathcal{T}_i) + a_i^2(\mathcal{T}_i - \tau_i). \tag{3.3}$$

We can see that in the formal limit $\varepsilon \to 0$, the additional variable $\mathcal{T}_i$ tends towards the specific volume τ_i, and the linearized pressure π_i tends towards the original non-linear pressure p_i, thus recovering system (2.1) in the first five equations of (3.1). From this point and to ease the notation, we will omit the superscript $^\varepsilon$. In the sequel, the original system (2.1) will be referred to as the *equilibrium system* as opposed to the *relaxation system*. The constants a_i in (3.3) are two constant positive parameters that must be taken large enough so as to satisfy the so-called Whitham condition:

$$a_i^2 > \max_{\mathcal{T}_i \in K_i}\{-\mathcal{P}_i'(\mathcal{T}_i)\} \tag{3.4}$$

where K_i is a connected compact set of $\mathbb{R}$ that contains all the values of τ_i and $\mathcal{T}_i$ of the solution.

3.1 Main mathematical properties

Let us now focus on the convective part of system (3.1):

$$\partial_t \mathbb{W} + \partial_x \mathbf{g}(\mathbb{W}) + \mathbf{d}(\mathbb{W})\partial_x \mathbb{W} = 0. \tag{3.5}$$

The solutions are sought in the domain of positive densities ρ_i and positive $\mathcal{T}_i$:

$$\Omega^r = \left\{\mathbb{W} \in \mathbb{R}^7, 0 < \alpha_1 < 1, \ \alpha_i\rho_i > 0, \ \alpha_i\rho_i\mathcal{T}_i > 0, \ i \in \{1, 2\}\right\}. \tag{3.6}$$

Proposition 3.1. *System (3.5) is weakly hyperbolic since it admits the following real eigenvalues*

$$\sigma_1(\mathbb{W}) = u_1 - a_1\tau_1, \ \sigma_2(\mathbb{W}) = u_1, \ \sigma_3(\mathbb{W}) = u_1 + a_1\tau_1,$$
$$\sigma_4(\mathbb{W}) = u_2 - a_2\tau_2, \ \sigma_5(\mathbb{W}) = \sigma_6(\mathbb{W}) = u_2, \ \sigma_7(\mathbb{W}) = u_2 + a_2\tau_2. \tag{3.7}$$

The corresponding right eigenvectors are linearly independent if, and only if

$$\alpha_1 \neq 0, \quad \alpha_2 \neq 0, \quad |u_1 - u_2| \neq a_1\tau_1. \tag{3.8}$$

All the characteristic fields associated with these eigenvalues are linearly degenerate.

Proof. The proof is similar to that of Proposition 2.1. It is left to the reader. $\qquad\square$

Remark 3.1. *Here again, one never has $\alpha_1 = 0$ or $\alpha_2 = 0$. However, $\alpha_i = 0$ is to be understood in the sense $\alpha_i \to 0$.*

Unlike system (2.1), one remarkable property of the relaxation system (3.5) is the linear degeneracy of all the characteristic fields. This has the helpful consequence that jump relations can be easily derived through each wave. The relaxation approximation is therefore a pleasant way to get around the difficulties due to non-linearities (discrimination between shocks and rarefaction waves, jump relations for shocks...) which arise when solving the Riemann problem for (2.1).

In a similar way to that for the equilibrium system, we have balance equations on the phasic energies as well as a total mixture energy conservation equation satisfied by the smooth solutions of system (3.5):

Proposition 3.2. *The smooth solutions of system (3.5) satisfy the following phasic energy equations:*

$$\partial_t\left(\alpha_i\rho_i\mathcal{E}_i\right) + \partial_x\left(\alpha_i\rho_i\mathcal{E}_i + \alpha_i\pi_i\right)u_i - u_2\pi_1(\tau_1,\mathcal{T}_1)\partial_x\alpha_i = 0, \quad i \in \{1,2\}. \tag{3.9}$$

where the phasic energies are defined by

$$\mathcal{E}_i := \mathcal{E}_i(u_i,\tau_i,\mathcal{T}_i) = \frac{u_i^2}{2} + e_i(\mathcal{T}_i) + \frac{\pi_i^2(\tau_i,\mathcal{T}_i) - \mathcal{P}_i^2(\mathcal{T}_i)}{2a_i^2}, \qquad i \in \{1,2\}. \tag{3.10}$$

Summing over $i = 1,2$ yields the following additional conservation law, also satisfied by the smooth solutions of system (3.5):

$$\partial_t\left(\alpha_1\rho_1\mathcal{E}_1 + \alpha_2\rho_2\mathcal{E}_2\right) + \partial_x\left((\alpha_1\rho_1\mathcal{E}_1 + \alpha_1\pi_1)u_1 + (\alpha_2\rho_2\mathcal{E}_2 + \alpha_2\pi_2)u_2\right) = 0. \tag{3.11}$$

Proof. The proof is left to the reader. $\qquad\square$

3.2 Jump relations and kinetic relation for vanishing phases

Being given a pair of initial states $(\mathbb{W}_L, \mathbb{W}_R)$ in $\Omega^r \times \Omega^r$, we supply the homogeneous system (3.5) with the following initial condition

$$\mathbb{W}(x, t=0) = \begin{cases} \mathbb{W}_L & \text{if} \quad x < 0, \\ \mathbb{W}_R & \text{if} \quad x > 0. \end{cases} \tag{3.12}$$

The solution of the Riemann problem is a self-similar function composed of intermediate states separated by contact discontinuities associated with the eigenvalues of the system. The phase fractions α_i only jump through the u_2-contact discontinuity from $\alpha_{i,L}$ to $\alpha_{i,R}$. Away from this wave, the phase fractions are constant and the system behaves as two independent relaxation systems for the isentropic Euler equations of each phase.
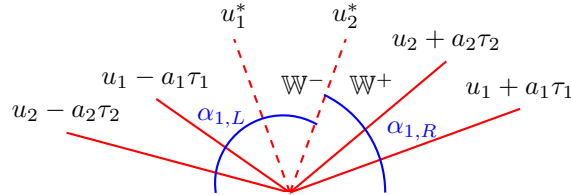
For all the discontinuities except the one associated with the eigenvalue u_2, the system is locally conservative (the product $\pi_1 \partial_x \alpha_i$ locally vanishes) and the jump conditions are simply obtained by the Rankine-Hugoniot relations applied to the conservative equations of the system.

On the contrary, for the u_2-wave, we have $\partial_x \alpha_i \neq 0$. In fact $\partial_x \alpha_i$ identifies with a Dirac measure and the pressure π_1 may be discontinuous across this wave. The product $\pi_1 \partial_x \alpha_i$ is determined by the Riemann invariants of this linearly degenerate field, or equivalently by the Rankine-Hugoniot jump relations. A first relation is given by the continuity of the eigenvalue u_2, and we get three more independent jump relations by applying Rankine-Hugoniot's formula to the conservative equations of phase 1 and to the total momentum conservation:

$$[u_2]_{\frac{x}{t}=u_2^*} = 0, \tag{3.13}$$

$$-u_2^* [\alpha_1 \rho_1]_{\frac{x}{t}=u_2^*} + [\alpha_1 \rho_1 u_1]_{\frac{x}{t}=u_2^*} = 0, \tag{3.14}$$

$$-u_2^* [\alpha_1 \rho_1 \mathcal{T}_1]_{\frac{x}{t}=u_2^*} + [\alpha_1 \rho_1 u_1 \mathcal{T}_1]_{\frac{x}{t}=u_2^*} = 0, \tag{3.15}$$

$$-u_2^* [\alpha_1 \rho_1 u_1 + \alpha_2 \rho_2 u_2]_{\frac{x}{t}=u_2^*} + [\alpha_1 \rho_1 u_1^2 + \alpha_1 \pi_1 + \alpha_2 \rho_2 u_2^2 + \alpha_2 \pi_2]_{\frac{x}{t}=u_2^*} = 0. \tag{3.16}$$

Finally, if the system is hyperbolic, a last jump relation (recall that the eigenvalue u_2 has multiplicity 2) is obtained by applying Rankine-Hugoniot's formula to the total energy preservation (3.11) which yields

$$- u_2^* [\alpha_1 \rho_1 \mathcal{E}_1 + \alpha_2 \rho_2 \mathcal{E}_2]_{\frac{x}{t}=u_2^*} + [(\alpha_1 \rho_1 \mathcal{E}_1 + \alpha_1 \pi_1) u_1 + (\alpha_2 \rho_2 \mathcal{E}_2 + \alpha_2 \pi_2) u_2]_{\frac{x}{t}=u_2^*} = 0. \tag{3.17}$$

In the sequel, when solving the Riemann problem for given initial data $(\mathbb{W}_L, \mathbb{W}_R)$ and a given pair of parameters (a_1, a_2), it appears that the exact solution may have non-positive densities if the ratio of initial phase fractions $\frac{\alpha_{1,L}}{\alpha_{1,R}}$ (or its inverse) is too large. One way to avoid this problem is to take larger values of the parameters (a_1, a_2) and solve the problem again. At the numerical level, taking larger values of these parameters introduces some more diffusion to the relaxation approximation, thus stabilizing the simulation. However, too large values of these parameters may involve too small time steps because of the CFL restrictions, since (a_1, a_2) directly control the size of the wave fan in the approximate Riemann solver. In this work, in order to control the time step, we propose to add the needed extra diffusion directly at the PDE level in the relaxation approximation, in the form of an energy dissipation. This is actually an original idea in the context of relaxation approximations, since usually, the first step of the method (treatment of the convective system (3.5)) is energy preserving because of the linear degeneracy of all the fields. Instead, the total energy conservation is replaced in the regimes of vanishing phases by a **kinetic relation** of the form

$$- u_2^* [\alpha_1 \rho_1 \mathcal{E}_1 + \alpha_2 \rho_2 \mathcal{E}_2]_{\frac{x}{t}=u_2^*} + [(\alpha_1 \rho_1 \mathcal{E}_1 + \alpha_1 \pi_1) u_1 + (\alpha_2 \rho_2 \mathcal{E}_2 + \alpha_2 \pi_2) u_2]_{\frac{x}{t}=u_2^*} = -\mathcal{Q}(u_2^*, \mathbb{W}_L, \mathbb{W}_R), \tag{3.18}$$

where $\mathcal{Q}(u_2^*, \mathbb{W}_L, \mathbb{W}_R)$ is a positive scalar function measuring the amount of dissipated energy across the u_2-wave. This kinetic relation allows to completely define the jump between two states $\mathbb{W}^-$ and $\mathbb{W}^+$ separated by a u_2-contact discontinuity. The non-conservative product $\pi_1 \partial_x \alpha_1$ then identifies with a Dirac measure

$$\pi_1^* \Delta \alpha_1 \delta_{x-u_2^* t}, \qquad \Delta \alpha_1 := \alpha_{1,R} - \alpha_{1,L}, \tag{3.19}$$

whose weight is obtained by applying Rankine-Hugoniot jump relation to the momentum equations:

$$\begin{aligned}
\pi_1^* \Delta \alpha_1 &= -u_2^* [\alpha_1 \rho_1 u_1]_{\frac{x}{t}=u_2^*} + [\alpha_1 \rho_1 u_1^2 + \alpha_1 \pi_1]_{\frac{x}{t}=u_2^*} \\
&= -\left\{ -u_2^* [\alpha_2 \rho_2 u_2]_{\frac{x}{t}=u_2^*} + [\alpha_2 \rho_2 u_2^2 + \alpha_2 \pi_2]_{\frac{x}{t}=u_2^*} \right\} = -[\alpha_2 \pi_2]_{\frac{x}{t}=u_2^*}.
\end{aligned} \tag{3.20}$$

4   Solving the Riemann problem for the relaxation system

The aim of this section is to solve the Riemann problem associated with the homogeneous part of the relaxation system. Being given a pair of initial states $(\mathbb{W}_L, \mathbb{W}_R)$ in $\Omega^r \times \Omega^r$, we seek solutions of the

following Cauchy problem:

$$\partial_t \mathbb{W} + \partial_x \mathbf{g}(\mathbb{W}) + \mathbf{d}(\mathbb{W})\partial_x \mathbb{W} = 0, \tag{4.1}$$

with the initial condition

$$\mathbb{W}(x, t = 0) = \begin{cases} \mathbb{W}_L & \text{if} & x < 0, \\ \mathbb{W}_R & \text{if} & x > 0. \end{cases} \tag{4.2}$$

4.1 Definition of the solutions to the Riemann problem

The solution is sought in the form of a *self-similar* function only depending on the variable $\xi = \frac{x}{t}$, that is to say $\mathbb{W}(x, t) = \mathbb{W}_r(x/t)$. As all the fields are linearly degenerate, if the solution remains in the domain of hyperbolicity, the function $\mathbb{W}_r(\xi)$ is a piecewise constant function, composed of (at most) seven constant states separated by (at most) six contact discontinuities, associated with the six eigenvalues $u_1 \pm a_1\tau_1$, $u_2 \pm a_2\tau_2$, u_1 and u_2 (see Lax's theory for Riemann problems [24]). More precisely, we give the following definition for the solutions of the Riemann problem (4.1)-(4.2).

Definition 4.1. *Let $(\mathbb{W}_L, \mathbb{W}_R)$ be two states in Ω^r. A solution to the Riemann problem (4.1)-(4.2) **with subsonic wave ordering** is a self-similar mapping $\mathbb{W}(x, t) = \mathbb{W}_r(x/t; \mathbb{W}_L, \mathbb{W}_R)$ where the function $\xi \mapsto \mathbb{W}_r(\xi; \mathbb{W}_L, \mathbb{W}_R)$ satisfies the following properties:*

(i) $\mathbb{W}_r(\xi; \mathbb{W}_L, \mathbb{W}_R)$ is a piecewise constant function, composed of (at most) seven intermediate states belonging to Ω^r, separated by (at most) six contact discontinuities associated with the eigenvalues $u_1 \pm a_1\tau_1$, $u_2 \pm a_2\tau_2$, u_1, u_2 and such that

$$\begin{aligned} \xi < \min_{i\in\{1,2\}} \{u_{i,L} - a_i\tau_{i,L}\} &\Longrightarrow \mathbb{W}_r(\xi; \mathbb{W}_L, \mathbb{W}_R) = \mathbb{W}_L, \\ \xi > \max_{i\in\{1,2\}} \{u_{i,R} + a_i\tau_{i,R}\} &\Longrightarrow \mathbb{W}_r(\xi; \mathbb{W}_L, \mathbb{W}_R) = \mathbb{W}_R. \end{aligned} \tag{4.3}$$

(ii) There exists a real number u_2^* (depending on $(\mathbb{W}_L, \mathbb{W}_R)$) with the following properties. Denoting $\Xi^- = \{\xi \in \mathbb{R} \text{ such that } \xi < u_2^*\}$ and $\Xi^+ = \{\xi \in \mathbb{R} \text{ such that } \xi > u_2^*\}$, $\alpha_1(\xi)$ equals $\alpha_{1,L}$ on Ξ^- and $\alpha_{1,R}$ on Ξ^+ and the function $\mathbb{W}_r(\xi; \mathbb{W}_L, \mathbb{W}_R)$ is a weak solution of

$$- \xi \mathbb{W}_r' + \mathbf{g}(\mathbb{W}_r)' = 0, \qquad \xi \in \Xi^- \cup \Xi^+. \tag{4.4}$$

(iii) There exists a real number π_1^* (depending on $(\mathbb{W}_L, \mathbb{W}_R)$) such that the following jump relations across the u_2^*-discontinuity hold

$$[u_2]_{\xi=u_2^*} = 0, \tag{4.5}$$

$$-u_2^* [\mathbb{W}_r]_{\xi=u_2^*} + [\mathbf{g}(\mathbb{W}_r)]_{\xi=u_2^*} + \mathbf{D}^*(\mathbb{W}_L, \mathbb{W}_R) = 0, \tag{4.6}$$

with $\mathbf{D}^*(\mathbb{W}_L, \mathbb{W}_R) = \Delta\alpha_1(u_2^*, 0, -\pi_1^*, 0, \pi_1^*, 0, 0)^T$ and $\Delta\alpha_1 = \alpha_{1,R} - \alpha_{1,L}$.

(iv) The energy jump of phase 1 across the u_2^*-discontinuity satisfies

$$- u_2^* [\alpha_1\rho_1\mathcal{E}_1]_{\xi=u_2^*} + [\alpha_1\rho_1\mathcal{E}_1 u_1 + \alpha_1\pi_1 u_1]_{\xi=u_2^*} - u_2^*\pi_1^*\Delta\alpha_1 \leq 0. \tag{4.7}$$

*If (4.7) is a strict inequality, the solution is said to be **energy-dissipative**. Otherwise the solution is **energy-preserving**.*

*(v) The solution has a **subsonic wave ordering** in the following sense:*

$$u_{1,L} - a_1\tau_{1,L} < u_2^* < u_{1,R} + a_1\tau_{1,R}. \tag{4.8}$$

Some comments on Definition 4.1:

1. In the above definition, u_2^* is the the propagation velocity of the phase fraction wave.

2. The non-conservative product $\pi_1 \partial_x \alpha_1$ identifies with the Dirac measure $\pi_1^* \Delta \alpha_1 \delta_{x-u_2^* t}$.

3. Item *(ii)* states that, away from the u_2-wave, the system behaves as two independent relaxation systems for the isentropic Euler equations.

4. Items *(iii)* and *(iv)* define the re-coupling relations between the two phases at the phase fraction wave u_2. In particular, summing the third and fifth components of (4.6) yields the conservation of the total momentum. Note that since u_2 remains constant across this wave (4.5), the fourth (mass conservation of phase 2), and seventh (convection of $\alpha_2 \rho_2 \mathcal{T}_2$) components of (4.6) are automatically satisfied.

5. Inequality (4.7) expresses that the phase 1 energy is either conserved (when (4.7) is an equality) or dissipated (when (4.7) is a strict inequality) through the u_2-contact discontinuity. As for the phase 2 energy, since u_2 is constant through this wave (4.5), and by the fifth component of (4.6), it is exactly conserved in the following sense:

$$-u_2^* \left[\alpha_2 \rho_2 \mathcal{E}_2\right]_{\xi=u_2^*} + \left[\alpha_2 \rho_2 \mathcal{E}_2 u_2 + \alpha_2 \pi_2 u_2\right]_{\xi=u_2^*} + u_2^* \pi_1^* \Delta \alpha_1 = 0. \tag{4.9}$$

Of course, speaking of the "conservation" of the phasic energies is an abuse of the usual terminology since neither of the two phasic energies has a conservative equation. Nevertheless, in the whole paper, we use this term when the phasic energy $\mathcal{E}_i$ satisfies its natural equation (3.9) in the weak sense across the u_2-contact discontinuity. Summing (4.7) and (4.9), we see that the total mixture energy is either conserved as in (3.17) or dissipated across this discontinuity. In that, the conservation (resp. dissipation) of the phase 1 energy is equivalent to the conservation (resp. dissipation) of the total mixture energy across the u_2-field. As this field is linearly degenerate, the total energy is actually expected to be preserved when the system is hyperbolic. In the sequel indeed, we will see that in *most cases*, preserving the total energy (3.17) through the u_2-contact discontinuity is possible for constructing admissible solutions of the Riemann problem. In this case, one may speak of *energy-preserving solutions*. However, it appears that when the ratio $\frac{\alpha_{1,L}}{\alpha_{1,R}}$ (or its inverse) is large, constructing solutions with positive densities while maintaining the exact energy conservation (3.17) across the u_2-contact discontinuity is impossible. It will be shown that in these vanishing phase regimes, solutions with positive densities cannot be obtained unless one authorizes some dissipation of the total energy through the u_2-contact, and one speaks of *energy-dissipative solutions* in that particular case.

6. For the applications aimed at by this work, such as nuclear flows, we are only interested in solutions which have a *subsonic wave ordering*, *i.e.* solutions for which the propagation velocity u_2^* of the phase fraction α_1 lies in-between the acoustic waves of phase 1 which is what is required in item *(v)*. Observe that this requirement prevents the loss of hyperbolicity due to wave interactions. In the sequel, these solutions are classified in three categories depending on the ordering between the u_1-contact discontinuity, and the u_2-contact discontinuity.

A solution with $u_2^* < u_1^*$.   A solution with $u_2^* = u_1^*$.   A solution with $u_2^* > u_1^*$.
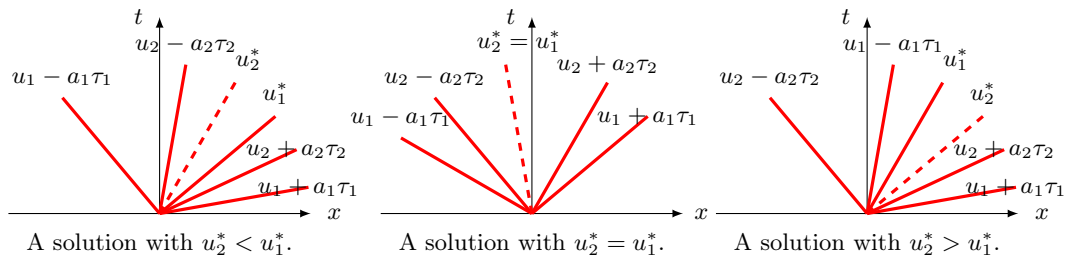
7. The considered solutions are allowed to have **phasic supersonic speeds** $|u_i| > a_i \tau_i$. Indeed, the subsonic property considered here is related to the **relative velocity** $u_1 - u_2$ with respect to the phase 1 speed of sound $a_1 \tau_1$.

4.2   The resolution strategy: a fixed-point procedure

The key challenge to solving the Riemann problem for the relaxation system consists in determining this non-conservative product $\pi_1^* \Delta \alpha_1 \delta_{x-u_2^* t}$ in the case $\alpha_{1,L} \neq \alpha_{1,R}$, *i.e.* in determining the values of u_2^* and π_1^* satisfying the constraints of Definition 4.1. Indeed, if $\alpha_{1,L} = \alpha_{1,R}$, the two phases are *decoupled* and the resolution is straightforward as stated in the following lemma.

Lemma 4.1. *Consider the Riemann problem (4.1)-(4.2) with $\alpha_{1,L} = \alpha_{1,R}$. Then, a self-similar solution $\xi \mapsto \mathbb{W}_r(\xi; \mathbb{W}_L, \mathbb{W}_R)$ is such that $\alpha_1(\xi) = cst = \alpha_{1,L} = \alpha_{1,R}$, so that the non-conservative product $\pi_1(\tau_1, \mathcal{T}_1) \partial_x \alpha_1$ vanishes. As a consequence, the evolutions of the two phases are completely decoupled and the intermediate states for each phase are given for $i \in \{1, 2\}$ by*

$$u_i^\sharp := \frac{1}{2}(u_{i,L} + u_{i,R}) - \frac{1}{2a_i}(\pi_{i,R} - \pi_{i,L}), \tag{4.10}$$

$$\pi_i^\sharp := \frac{1}{2}(\pi_{i,R} + \pi_{i,L}) - \frac{a_i}{2}(u_{i,R} - u_{i,L}), \tag{4.11}$$

$$\tau_{i,L}^\sharp := \tau_{i,L} + \frac{1}{a_i}(u_i^\sharp - u_{i,L}) = \tau_{i,L} + \frac{1}{2a_i}(u_{i,R} - u_{i,L}) - \frac{1}{2a_i^2}(\pi_{i,R} - \pi_{i,L}), \tag{4.12}$$

$$\tau_{i,R}^\sharp := \tau_{i,R} - \frac{1}{a_i}(u_i^\sharp - u_{i,R}) = \tau_{i,R} + \frac{1}{2a_i}(u_{i,R} - u_{i,L}) + \frac{1}{2a_i^2}(\pi_{i,R} - \pi_{i,L}). \tag{4.13}$$

Phase solutions in the case of constant initial phase fractions $\alpha_{i,L} = \alpha_{i,R}$.
In each case, $\mathcal{T}_{i,L}^\sharp = \mathcal{T}_{i,L}$ and $\mathcal{T}_{i,R}^\sharp = \mathcal{T}_{i,R}$.
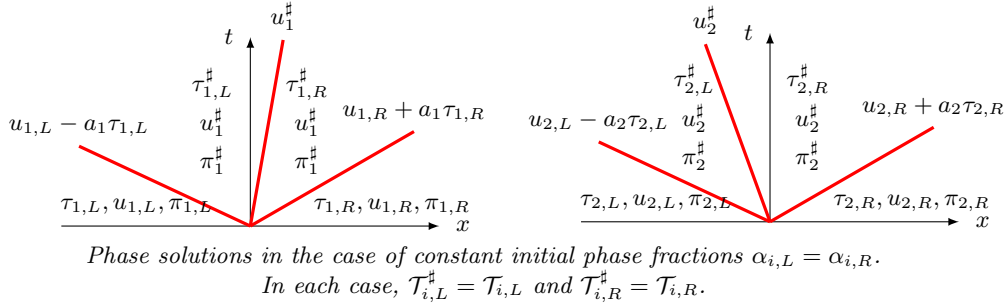
The value of u_2^* is given by $u_2^\sharp$ and the solution has a subsonic wave ordering if, and only if these quantities satisfy the following constraint:

$$u_1^\sharp - a_1 \tau_{1,L}^\sharp < u_2^\sharp < u_1^\sharp + a_1 \tau_{1,R}^\sharp \iff -a_1 \tau_{1,R}^\sharp < u_1^\sharp - u_2^\sharp < a_1 \tau_{1,L}^\sharp. \tag{4.14}$$

Observe that the quantities defined in (4.10) to (4.13) are independent of the phase fractions $\alpha_{1,L} = \alpha_{1,R}$. On the contrary, if $\alpha_{i,L} \neq \alpha_{i,R}$, the evolutions of both phases are affected by the u_2-wave (which has multiplicity 2) and the physical quantities of the two phases are **coupled** through this wave.

Phase solutions in the case of non-constant initial phase fractions $\alpha_{i,L} \neq \alpha_{i,R}$.
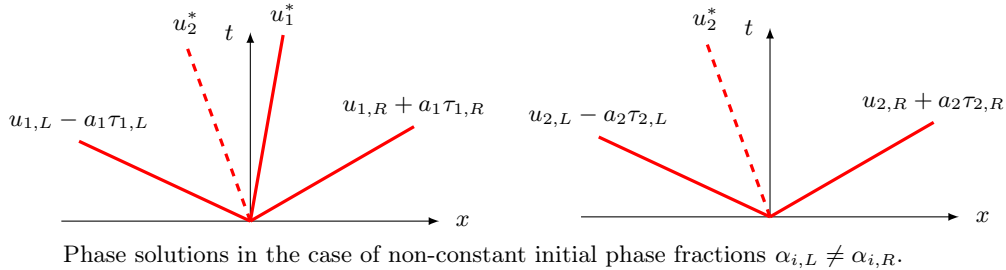
Starting from the known solution in the decoupled case $|\alpha_{1,L} - \alpha_{1,R}| = 0$, we seek to construct a *branch of solutions* with subsonic wave ordering, in the non-conservative cases $|\alpha_{1,L} - \alpha_{1,R}| \neq 0$. Actually the aim is to expose a subsonic type condition, similar to (4.14) which accounts for the subsonic ordering requirement

$$u_1^\sharp - a_1 \tau_{1,L}^\sharp < u_2^* < u_1^\sharp + a_1 \tau_{1,R}^\sharp, \tag{4.15}$$

and ensures the existence of such a solution. Of course, the main difficulty here is that the value of u_2^* is not *a priori* known with respect to the initial data unlike in the case $\alpha_{1,L} = \alpha_{1,R}$. However, the analysis carried out in this paper will expose a very simple generalization of condition (4.14) valid for the case $\alpha_{1,L} \neq \alpha_{1,R}$ and that can be explicitly tested with respect to the initial data just as (4.14).

For this purpose, we recall the following key remark, made for instance by Ambroso *et al.* [1] and which is the cornerstone of the whole resolution strategy.

Key remark: *Consider the case $\alpha_{1,L} \neq \alpha_{1,R}$. If one is able to **make a prediction** of the pressure π_1^* that defines the non-conservative product $\pi_1 \partial_x \alpha_i$ and therefore shift it to a **known right hand side of the system**, then one can see that the governing equations for phase 2 are completely independent of the phase 1 quantities, namely ρ_1, u_1 and $\mathcal{T}_1$.*

In [1], the authors calculate an *approximate solution* of the Riemann problem (4.1)-(4.2) by actually formulating a prediction $\pi_1^*(\mathbb{W}_L, \mathbb{W}_R)$ of the non-conservative product. Our approach is different. It consists in performing a fixed-point procedure in order to determine the "right" value of π_1^* and thus calculate an *exact solution* to the Riemann problem (4.1)-(4.2). Formally, the fixed-point procedure consists in iterating on the pair (u_2^*, π_1^*) by alternately considering each one of the two phases, as described hereunder.

First step: The pressure π_1^* defining the non-conservative product $\pi_1 \partial_x \alpha_1 = \pi_1^* \partial_x \alpha_1 = -\pi_1^* \partial_x \alpha_2$ is first assumed to be known. Hence, the governing equations for phase 2 are completely independent of the phase 1 quantities, and one determines u_2^* by solving the following Riemann problem for the phase 2 system:

$$\begin{cases} \partial_t \alpha_2 + u_2 \partial_x \alpha_2 = 0, \\ \partial_t(\alpha_2 \rho_2) + \partial_x(\alpha_2 \rho_2 u_2) = 0, \\ \partial_t(\alpha_2 \rho_2 u_2) + \partial_x(\alpha_2 \rho_2 u_2^2 + \alpha_2 \pi_2(\tau_2, \mathcal{T}_2)) - \pi_1^* \partial_x \alpha_2 = 0, \\ \partial_t(\alpha_2 \rho_2 \mathcal{T}_2) + \partial_x(\alpha_2 \rho_2 \mathcal{T}_2 u_2) = 0, \end{cases} \tag{4.16}$$

with the initial condition

$$\mathbb{W}_2(x, t = 0) = \begin{cases} \mathbb{W}_{2,L} & \text{if} & x < 0, \\ \mathbb{W}_{2,R} & \text{if} & x > 0, \end{cases} \tag{4.17}$$

where $\mathbb{W}_2 = (\alpha_2, \alpha_2 \rho_2, \alpha_2 \rho_2 u_2, \alpha_2 \rho_2 \mathcal{T}_2)^T$ denotes the state vector for phase 2 and $(\mathbb{W}_{2,L}, \mathbb{W}_{2,R})$ are the restriction of the complete initial data $(\mathbb{W}_L, \mathbb{W}_R)$ to the phase 2 variables. The sought solutions of (4.16)-(4.17) are asked to obey an additional equation on the energy which reads:

$$\partial_t(\alpha_2 \rho_2 \mathcal{E}_2) + \partial_x(\alpha_2 \rho_2 \mathcal{E}_2 + \alpha_2 \pi_2) u_2 - u_2 \pi_1^* \partial_x \alpha_2 = 0. \tag{4.18}$$

Solving this Riemann problem enables us to calculate the value u_2^* of the phase fraction propagation speed with respect to the assumed value of π_1^*, and thus to define a function

$$\mathscr{F}[\mathbb{W}_L, \mathbb{W}_R; a_2] : \begin{cases} \mathbb{R} & \longrightarrow & \mathbb{R} \\ \pi_1^* & \longmapsto & u_2^*. \end{cases} \tag{4.19}$$

Second step: The advection velocity u_2^* of the phase fraction α_1 is then assumed to be known. As a consequence, the evolution of the phase 1 quantities is independent of phase 2, and computing them amounts to solving the following phase 1 Riemann problem:

$$\begin{cases} \partial_t \alpha_1 + u_2^* \partial_x \alpha_1 = 0, \\ \partial_t(\alpha_1 \rho_1) + \partial_x(\alpha_1 \rho_1 u_1) = 0, \\ \partial_t(\alpha_1 \rho_1 u_1) + \partial_x(\alpha_1 \rho_1 u_1^2 + \alpha_1 \pi_1(\tau_1, \mathcal{T}_1)) - \pi_1 \partial_x \alpha_1 = 0, \\ \partial_t(\alpha_1 \rho_1 \mathcal{T}_1) + \partial_x(\alpha_1 \rho_1 \mathcal{T}_1 u_1) = 0, \end{cases} \tag{4.20}$$

with the initial condition

$$\mathbb{W}_1(x, t = 0) = \begin{cases} \mathbb{W}_{1,L} & \text{if} & x < 0, \\ \mathbb{W}_{1,R} & \text{if} & x > 0, \end{cases} \tag{4.21}$$

15

where $\mathbb{W}_1 = (\alpha_1, \alpha_1\rho_1, \alpha_1\rho_1 u_1, \alpha_1\rho_1\mathcal{T}_1)^T$ denotes the state vector for phase 1, and $(\mathbb{W}_{1,L}, \mathbb{W}_{1,R})$ are the restriction of the complete initial data $(\mathbb{W}_L, \mathbb{W}_R)$ to the phase 1 variables. Here, in accordance with (4.7), the sought solutions are asked to obey the following inequality on the phase 1 energy:

$$\partial_t (\alpha_1\rho_1\mathcal{E}_1) + \partial_x (\alpha_1\rho_1\mathcal{E}_1 + \alpha_1\pi_1) u_1 - u_2^*\pi_1\partial_x\alpha_1 \leq 0. \tag{4.22}$$

Once the Riemann problem for (4.20)-(4.21) is solved, applying Rankine-Hugoniot's jump relation to the momentum equation for the phase fraction wave of speed u_2^*, allows to compute the weight π_1^* of the non-conservative product $\pi_1\partial_x\alpha_1$. Hence, this second step enables to define a function

$$\mathscr{G}[\mathbb{W}_L, \mathbb{W}_R; a_1] : \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ u_2^* & \longmapsto & \pi_1^*. \end{array} \right. \tag{4.23}$$

Fixed-point: Performing an iterative procedure on these two steps actually boils down to the following fixed-point research.

Find u_2^* in $(u_1^\sharp - a_1\tau_{1,L}^\sharp, u_1^\sharp + a_1\tau_{1,R}^\sharp)\bigcap(u_2^\sharp - a_2\tau_{2,L}^\sharp, u_2^\sharp + a_2\tau_{2,R}^\sharp)$ such that

$$u_2^* = \left(\mathscr{F}[\mathbb{W}_L, \mathbb{W}_R; a_2] \circ \mathscr{G}[\mathbb{W}_L, \mathbb{W}_R; a_1]\right)(u_2^*). \tag{4.24}$$

The interval where u_2^* must be sought corresponds to the subsonic wave ordering condition (4.8) on the one hand, and to the positivity of the intermediate states of phase 2 on the other hand (see Proposition (4.4)).

Section 4.3 is devoted to presenting and commenting the main result of the paper, which is an existence theorem for the Riemann problem (4.1)-(4.2). In Section 4.4, the first step of the iterative process is performed and we give the explicit formula of function $\mathscr{F}[\mathbb{W}_L, \mathbb{W}_R; a_2]$ defined in (4.19). In Section 4.4, the second step of the iterative procedure is performed. We restrict the presentation to the research of solutions with the wave configurations $u_2^* < u_1^*$. The results for the other wave configurations $u_2^* = u_1^*$ and $u_2^* > u_1^*$ can be obtained through the same process, or can be inferred from the Galilean invariance of the equations. This second step allows us to define an explicit formula for function $\mathscr{G}[\mathbb{W}_L, \mathbb{W}_R; a_1]$ introduced in (4.23). Finally, in Section 4.6, we prove that, under some explicitly formulated assumptions on the initial data $(\mathbb{W}_L, \mathbb{W}_R)$ (see Theorem 4.2), there exists a unique energy-preserving solution to the fixed-point problem (4.24). This unique solution corresponds to the exact conservation of the total energy across the u_2-contact discontinuity. It is shown however that in some cases where the ratio $\frac{\alpha_{1,L}}{\alpha_{1,R}}$ is large, this solution may have non-positive densities. By defining an appropriate **kinetic relation** which amounts to relaxing the conservation of the total energy (through the phase 1 energy), we recover the existence of positive solutions.

4.3 An existence theorem for solutions with subsonic wave ordering

We may now state the existence theorem for the Riemann problem (4.1)-(4.2). We refer to equations (4.10) to (4.13) for the definition of the quantities $\sharp$ used in the theorem, and we define the following number which solely depends on the initial phase fractions:

$$\Lambda^\alpha := \frac{\alpha_{2,R} - \alpha_{2,L}}{\alpha_{2,R} + \alpha_{2,L}}. \tag{4.25}$$

Theorem 4.2. *Let be given a pair of admissible initial states $(\mathbb{W}_L, \mathbb{W}_R) \in \Omega^r \times \Omega^r$ and assume that the parameter a_i is such that $\tau_{i,L}^\sharp > 0$ and $\tau_{i,R}^\sharp > 0$ for i in $\{1, 2\}$. There exists solutions with subsonic wave ordering to the Riemann problem (4.1)-(4.2) in the sense of Definition 4.1 if the following condition holds:*

$$(\boldsymbol{A}) \qquad -a_1\tau_{1,R}^\sharp < \frac{u_1^\sharp - u_2^\sharp - \frac{1}{a_2}\Lambda^\alpha(\pi_1^\sharp - \pi_2^\sharp)}{1 + \frac{a_1}{a_2}|\Lambda^\alpha|} < a_1\tau_{1,L}^\sharp.$$

*Moreover, condition (**A**) can be decomposed into the three following conditions defining the wave configuration:*

(i) Either

$$(\mathbf{A1}) \qquad 0 < \frac{u_1^\sharp - u_2^\sharp - \frac{1}{a_2}\Lambda^\alpha(\pi_1^\sharp - \pi_2^\sharp)}{1 + \frac{a_1}{a_2}|\Lambda^\alpha|} < a_1\tau_{1,L}^\sharp,$$

and the solutions have the wave configuration $u_2^* < u_1^*$.

(ii) Or

$$(\mathbf{A2}) \qquad -a_1\tau_{1,R}^\sharp < \frac{u_1^\sharp - u_2^\sharp - \frac{1}{a_2}\Lambda^\alpha(\pi_1^\sharp - \pi_2^\sharp)}{1 + \frac{a_1}{a_2}|\Lambda^\alpha|} < 0,$$

and the solutions have the wave configuration $u_2^* > u_1^*$.

(iii) Or

$$(\mathbf{A3}) \qquad u_1^\sharp - u_2^\sharp - \frac{1}{a_2}\Lambda^\alpha(\pi_1^\sharp - \pi_2^\sharp) = 0,$$

and the solutions have the wave configuration $u_2^* = u_1^*$.

The proof of this theorem follows from the steps described in the three following Sections 4.4, 4.5 and 4.6. Before giving the details of these steps, let us first make some comments on this result.

Some comments on Theorem 4.2:

1. Assumption (**A**) (actually (**A1**), (**A2**) or (**A3**)) **can be explicitly tested** in terms of the initial data and the parameters a_i, $i \in \{1, 2\}$. Of course, there is no similar result concerning the Riemann problem for the isentropic equilibrium Baer-Nunziato system (2.1).

2. Assumption (**A**) reduces to (4.14) when $\alpha_{1,L} = \alpha_{1,R}$ since in this case $\Lambda^\alpha = 0$. In this sense, assumption (**A**) is a generalization of (4.14) for the non-conservative case $\alpha_{1,L} \neq \alpha_{1,R}$.

3. The quantities $a_1\tau_{1,L}^\sharp$ and $a_1\tau_{1,R}^\sharp$ can be seen as two sound propagation speeds, while the quantity $\frac{u_1^\sharp - u_2^\sharp - \frac{1}{a_2}\Lambda^\alpha(\pi_1^\sharp - \pi_2^\sharp)}{1 + \frac{a_1}{a_2}|\Lambda^\alpha|}$, which has the dimension of a velocity, measures the difference between the pressures and kinematic velocities of the two phases, in the initial data. Observe that if the initial data is close to the pressure and velocity equilibrium between the two phases, this quantity is expected to be small compared to $a_1\tau_{1,L}^\sharp$ and $a_1\tau_{1,R}^\sharp$.

4. One may formulate a more geometrical interpretation of Theorem 4.2. Assuming that there exists a solution with subsonic relative speeds when $|\alpha_{1,R} - \alpha_{1,L}| = 0$ (*i.e.* assuming (4.14)), the theorem shows that if $|\alpha_{1,R} - \alpha_{1,L}| \neq 0$ is sufficiently small, then the Riemann problem still admits energy-preserving subsonic solutions. Provided that one allows some energy-dissipation across the u_2-wave, this branch of solutions can be followed for Riemann problems in which $|\alpha_{1,R} - \alpha_{1,L}|$ increases (holding the other quantities in the initial left and right data fixed) until assumption (**A**) is violated, or until $|\alpha_{1,R} - \alpha_{1,L}| = 1$.

5. **Positivity of phase 1 densities.** If the ratio $\frac{\alpha_{1,L}}{\alpha_{1,R}}$ is in a neighborhood of 1, condition (**A**) is a *necessary and sufficient* condition for the existence of a **unique** energy-preserving solution. If $\frac{\alpha_{1,L}}{\alpha_{1,R}}$ is too large (under (**A1**)), or too small (under (**A2**)), depending on the wave ordering, ensuring positive densities for phase 1 may require a strict dissipation of the phase 1 energy:

$$\partial_t(\alpha_1\rho_1\mathcal{E}_1) + \partial_x(\alpha_1\rho_1\mathcal{E}_1 + \alpha_1\pi_1)u_1 - u_2^*\pi_1^*\partial_x\alpha_1 < 0. \tag{4.26}$$

This, combined with the energy conservation in phase 2 (4.18) (see Proposition 4.4), implies that the total mixture energy is dissipated. In Sections 4.5.2 and 4.6.2, we propose a *kinetic relation* for the determination of one solution, among all the admissible dissipative solutions given by the theorem.

6. **Positivity of phase 2 densities.** Assumption (**A**) allows to compute the value of the wave propagation velocity u_2^* (see Section 4.6). With this value, one has to *verify* that the following property, which is equivalent to the positivity of the phase 2 densities, is satisfied:

$$(\mathbf{B}) \qquad u_2^\sharp - a_2 \tau_{2,L}^\sharp < u_2^* < u_2^\sharp + a_2 \tau_{2,R}^\sharp. \tag{4.27}$$

In the numerical applications using this Riemann solver (see Sections 5 and 6), it will always be possible to ensure property (**B**) by taking a large enough value of the relaxation parameter a_2.

4.4 First step of the fixed-point procedure

In this first step, the pressure π_1^* defining the non-conservative product $\pi_1 \partial_x \alpha_1 = \pi_1^* \partial_x \alpha_1 = -\pi_1^* \partial_x \alpha_2$ is assumed to be known, and one solves the Riemann problem (4.16)-(4.17) in order to compute u_2^*, the propagation speed of the phase fraction α_2. Observe that the non-conservative product $\pi_1^* \partial_x \alpha_2$ is not ambiguous here since π_1^* is a **known constant**. The following proposition characterizes the convective behavior of system (4.16).

Proposition 4.3. *System (4.16) is a hyperbolic system with linearly degenerate fields associated with the eigenvalues $u_2 - a_2 \tau_2$, u_2 and $u_2 + a_2 \tau_2$. The eigenvalue u_2 has multiplicity 2.*

Proof. The proof is left to the reader. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We have the following well-posedness result for the governing equations of phase 2:

Proposition 4.4. *Assume that the parameter a_2 is such that $\tau_{2,L}^\sharp > 0$ and $\tau_{2,R}^\sharp > 0$. Then the Riemann problem (4.16)-(4.17) admits a unique solution whose intermediate states are defined by:*
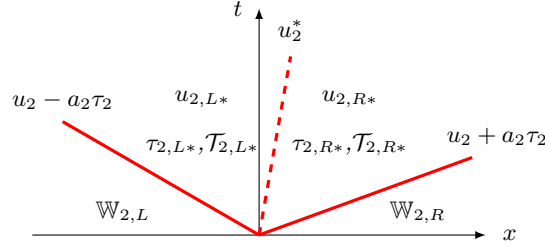
$$\tau_{2,L*} = \tau_{2,L}^\sharp + \frac{\Delta\alpha_1}{a_2^2}\frac{\pi_2^\sharp - \pi_1^*}{\alpha_{2,L} + \alpha_{2,R}}, \quad u_{2,L*} = u_2^* = u_2^\sharp + \frac{\Delta\alpha_1}{a_2}\frac{\pi_2^\sharp - \pi_1^*}{\alpha_{2,L} + \alpha_{2,R}}, \qquad \mathcal{T}_{2,L*} = \mathcal{T}_{2,L}, \quad (4.28)$$

$$\tau_{2,R*} = \tau_{2,R}^\sharp - \frac{\Delta\alpha_1}{a_2^2}\frac{\pi_2^\sharp - \pi_1^*}{\alpha_{2,L} + \alpha_{2,R}}, \quad u_{2,R*} = u_2^*, \qquad\qquad\qquad\qquad\quad \mathcal{T}_{2,R*} = \mathcal{T}_{2,R}. \quad (4.29)$$

The intermediate densities $\rho_{2,L*}$ and $\rho_{2,R*}$ are positive if and only if

$$u_2^\sharp - a_2 \tau_{2,L}^\sharp < u_2^* < u_2^\sharp + a_2 \tau_{2,R}^\sharp. \tag{4.30}$$

Moreover, this unique solution satisfies the following energy equation in the usual weak sense:

$$\partial_t \left(\alpha_2 \rho_2 \mathcal{E}_2\right) + \partial_x \left(\alpha_2 \rho_2 \mathcal{E}_2 + \alpha_2 \pi_2\right) u_2 - u_2 \pi_1^* \partial_x \alpha_2 = 0. \tag{4.31}$$

Proof. We only sketch the proof. The expressions of the intermediate states directly follow from classical manipulations of Rankine-Hugoniot's jump relations. The only non classical relation is the jump relation across the u_2-wave for the momentum equation, where the non-conservative product is taken into account:

$$\underbrace{-u_2^* \left[\alpha_2 \rho_2 u_2\right] + \left[\alpha_2 \rho_2 u_2^2\right]}_{=0} + \left[\alpha_2 \pi_2\right] = \pi_1^* \Delta\alpha_2 = -\pi_1^* \Delta\alpha_1.$$
$$\Longleftrightarrow \left[\alpha_2 \pi_2\right] = \pi_1^* \Delta\alpha_2 = -\pi_1^* \Delta\alpha_1. \tag{4.32}$$

18

The densities $\rho_{2,L*}$ and $\rho_{2,R*}$ are positive if and only if $u_2^* - a_2\tau_{2,L*} < u_2^* < u_2^* + a_2\tau_{2,R*}$. As the fields are linearly degenerate, the corresponding eigenvalues are Riemann invariants and we have $u_2^* - a_2\tau_{2,L*} = u_{2,L} - a_2\tau_{2,L} = u_2^\sharp - a_2\tau_{2,L}^\sharp$. In the same way, $u_2^* + a_2\tau_{2,R*} = u_{2,R} + a_2\tau_{2,R} = u_2^\sharp + a_2\tau_{2,R}^\sharp$. For the energy equation (4.31), the proof consists in verifying that the associated Rankine-Hugoniot jump relation is satisfied for all of the three waves $u_2 - a_2\tau_2$, u_2 and $u_2 + a_2\tau_2$. Across the two extreme waves where $\partial_x\alpha_2$ vanishes, the result directly follows from the decoupling of the two phases into two (relaxed) Euler systems for which the result is well-known (see for instance [13]). As for the u_2-wave, thanks to (4.32), the energy jump relation reads

$$\underbrace{-u_2^*[\alpha_2\rho_2\mathcal{E}_2]_{\xi=u_2^*} + [\alpha_2\rho_2\mathcal{E}_2 u_2]_{\xi=u_2^*}}_{=0} + [\alpha_2\pi_2 u_2]_{\xi=u_2^*} = u_2^*[\alpha_2\pi_2] = u_2^*\pi_1^*\Delta\alpha_2 = -u_2^*\pi_1^*\Delta\alpha_1.$$

$\square$

Remark 4.1. *The expression of u_2^* given in equation (4.28) defines the function $\mathscr{F}[\mathbb{W}_L, \mathbb{W}_R; a_2]$ introduced in (4.19), since u_2^* is expressed as a function of π_1^*. It clearly appears that if $\alpha_{1,L} = \alpha_{1,R}$, the non-conservative product vanishes and the resolution of the Riemann problem yields $u_2^* = u_2^\sharp$ as seen in Lemma 4.1.*

4.5 Second step of the fixed-point procedure

In this step, the velocity u_2^* of the wave supporting the α_1-discontinuity is assumed to be known, while the pressure π_1^* defining the non-conservative product $\pi_1\partial_x\alpha_1 = \pi_1^*\Delta\alpha_1\delta_{x-u_2^*t}$ is an unknown that must be calculated by solving the Riemann problem (4.20)-(4.21) and applying Rankine-Hugoniot's jump relation to the momentum equation.

4.5.1 A convenient change of variables

In order to solve the Riemann problem (4.20)-(4.21), it is judicious to rewrite the equations in the frame moving at the known constant speed u_2^*. For this purpose, we perform the following change of variables: $(x,t) \mapsto (y,t) = (x - u_2^*t, t)$. Any function $\mathbb{W}$ of the variables (x,t), is associated with a function $\mathcal{W}$ of the variables (y,t) such that $\mathcal{W}(y,t) = \mathbb{W}(x,t)$ *i.e.* $\mathbb{W}(x,t) = \mathcal{W}(x - u_2^*t, t)$. Denoting $w_1 = u_1 - u_2^*$ the fluid velocity of phase 1 in the frame of the u_2^*-wave, system (4.20) rewrites

$$\begin{cases} \partial_t\alpha_1 = 0, \\ \partial_t(\alpha_1\rho_1) + \partial_y(\alpha_1\rho_1 w_1) = 0, \\ \partial_t(\alpha_1\rho_1 w_1) + \partial_y(\alpha_1\rho_1 w_1^2 + \alpha_1\pi_1(\tau_1,\mathcal{T}_1)) - \pi_1\partial_y\alpha_1 = 0, \\ \partial_t(\alpha_1\rho_1\mathcal{T}_1) + \partial_y(\alpha_1\rho_1\mathcal{T}_1 w_1) = 0, \end{cases} \quad (4.33)$$

while the initial conditions (4.21) become, in the non-conservative variables:

$$\mathcal{W}_1(y, t=0) = \begin{cases} \mathcal{W}_{1,L} = (\alpha_{1,L}, \rho_{1,L}, w_{1,L} = u_{1,L} - u_2^*, \mathcal{T}_{1,L}), & y < 0, \\ \mathcal{W}_{1,R} = (\alpha_{1,R}, \rho_{1,R}, w_{1,R} = u_{1,R} - u_2^*, \mathcal{T}_{1,R}), & y > 0. \end{cases} \quad (4.34)$$

The solutions of (4.33)-(4.34) are asked to obey the following conservative energy inequality obtained by applying the same change of variables to the energy inequality (4.22):

$$\partial_t\left(\alpha_1\rho_1\overline{\mathcal{E}}_1\right) + \partial_y\left(\alpha_1\rho_1\overline{\mathcal{E}}_1 w_1 + \alpha_1\pi_1 w_1\right) \leq 0, \quad (4.35)$$

where $\overline{\mathcal{E}}_1 = \frac{w_1^2}{2} + e_1(\mathcal{T}_1) + \frac{1}{2a_1^2}(\pi_1^2(\tau_1,\mathcal{T}_1) - \mathcal{P}_1^2(\mathcal{T}_1))$. A solution which satisfies (4.35) with a strict inequality is an *energy-dissipative* solution. In the equality case, the solution is said to be *energy-preserving*. System (4.33) is nothing but the relaxation system introduced in [17] for the approximation of nozzle flows, and for which the associated Riemann problem has been fully resolved. Hence, we actually calculate a solution $\mathcal{W}(y,t)$ of the Riemann problem associated with system (4.33), and the

19

solution for the original Riemann problem (4.20)-(4.21) is obtained by $\mathbb{W}(x,t) = \mathcal{W}(x - u_2^* t, t)$, and by adding u_2^* to the velocities w_1. The following proposition characterizes the convective behavior of system (4.33).

Proposition 4.5. *System (4.33) admits four real eigenvalues that are $w_1 - a_1\tau_1$, w_1, $w_1 + a_1\tau_1$ and 0. All the fields are linearly degenerate and the system is hyperbolic if, and only if $|w_1| \neq a_1\tau_1$.*

Proof. The proof can be easily recovered following closely related steps developed in [7] for the Suliciu relaxation system. The details are left to the reader. □

The u_2^*-wave in (4.20) corresponds to the standing wave in (4.33) and the subsonic condition on the relative speed (4.8) now reads

$$w_{1,L} - a_{1,L}\tau_{1,L} < 0 < w_{1,R} + a_{1,R}\tau_{1,R}. \tag{4.36}$$

For the resolution, we only consider solutions with the subsonic wave ordering $u_2^* < u_1^*$ since the other possible wave orderings can be obtained by the Galilean invariance of the equations. This ordering for (4.20) corresponds to the wave ordering $w_1 - a_1\tau_1 < 0 < w_1 < w_1 + a_1\tau_1$ for (4.33), which in [17] is referred to as the $< 1, 2 >$ wave configuration:
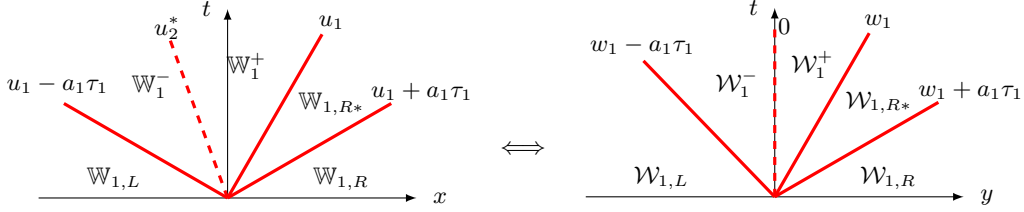
Following [17], we introduce the following notations, which are used for the definition of the solutions to the Riemann problem (4.33)-(4.34).

$$\nu = \frac{\alpha_{1,L}}{\alpha_{1,R}}, \qquad w^\sharp = u_1^\sharp - u_2^*, \qquad \mathcal{M}_L^* = \frac{w^\sharp}{a_1\tau_{1,L}^\sharp}, \qquad \omega = \frac{1 - \mathcal{M}_L^*}{1 + \mathcal{M}_L^*}. \tag{4.37}$$

where the $\sharp$quantites are defined in (4.10)-(4.11)-(4.12)-(4.13). We may now recall the main result (see Proposition (3.4) of [17]) which states that one can build a one-parameter family of solutions with the subsonic wave ordering $w_1 - a_1\tau_1 < 0 < w_1 < w_1 + a_1\tau_1$ for the Riemann problem (4.33)-(4.34) (*i.e.* the wave ordering $u_1 - a_1\tau_1 < u_2^* < u_1^* < u_1 + a_1\tau_1$ for the Riemann problem (4.20)-(4.21)), and the dissipation of energy across the standing wave is directly driven by the underlying parameter.

Proposition 4.6. *Assume that a_1 is such that $\tau_{1,L}^\sharp > 0$ and $\tau_{1,R}^\sharp > 0$. Then the Riemann problem (4.33)-(4.34) admits solutions with the subsonic wave ordering $w_1 - a_1\tau_1 < 0 < w_1 < w_1 + a_1\tau_1$, if and only if*

$$0 < \mathcal{M}_L^* < 1. \tag{4.38}$$

These solutions can be parametrized by $\mathcal{M} := \mathcal{M}^- = \frac{w_1^-}{a_1\tau_1^-}$, the Mach number of the state on the left of the standing wave, and the intermediate states are given by:

$$\tau_1^- = \tau_{1,L}^\sharp \frac{1 - \mathcal{M}_L^*}{1 - \mathcal{M}}, \qquad\qquad w_1^- = a_1\mathcal{M}\tau_1^-, \qquad\qquad \mathcal{T}_1^- = \mathcal{T}_{1,L}, \tag{4.39}$$

$$\tau_1^+ = \tau_{1,L}^\sharp \frac{1 + \mathcal{M}_L^*}{1 + \nu\mathcal{M}}, \qquad\qquad w_1^+ = \nu a_1\mathcal{M}\tau_1^+, \qquad\qquad \mathcal{T}_1^+ = \mathcal{T}_{1,L}, \tag{4.40}$$

$$\tau_{1,R*} = \tau_{1,R}^\sharp + \tau_{1,L}^\sharp \frac{\mathcal{M}_L^* - \nu\mathcal{M}}{1 + \nu\mathcal{M}}, \qquad w_{1,R*} = \nu a_1\mathcal{M}\tau_1^+, \qquad \mathcal{T}_{1,R*} = \mathcal{T}_{1,R}. \tag{4.41}$$

Besides, there exists a critical value $\nu^\sharp$ in $(1, +\infty]$ independent of $(\alpha_{1,L}, \alpha_{1,R})$ and possibly infinite such that the following alternative holds.

20

- If $\nu < \nu^\sharp$, $\mathcal{M}$ belongs to the interval $(0, \mathcal{M}_0(\nu, \omega)] \subsetneq (0, \min(1, 1/\nu))$ with

$$\mathcal{M}_0(\nu, \omega) = \frac{1}{2}\left(\frac{1+\omega^2}{1-\omega^2}\left(1+\frac{1}{\nu}\right) - \sqrt{\left(\frac{1+\omega^2}{1-\omega^2}\right)^2\left(1+\frac{1}{\nu}\right)^2 - \frac{4}{\nu}}\right). \qquad (4.42)$$

The value $\mathcal{M} = \mathcal{M}_0(\nu, \omega)$ gives the unique energy-preserving solution and for $0 < \mathcal{M} < \mathcal{M}_0(\nu, \omega)$, the solution is energy-dissipative.

- If $\nu \geq \nu^\sharp$, no energy-preserving solution has positive densities. The initial data is such that $0 < \frac{\mathcal{M}_L^*}{\nu} < \mathcal{M}_0(\nu, \omega) < \min(1, 1/\nu)$ where $\mathcal{M}_0(\nu, \omega)$ is given by (4.42). $\mathcal{M}$ must be strictly less than $\mathcal{M}_0(\nu, \omega)$, and by taking $\mathcal{M}$ close enough to $\frac{\mathcal{M}_L^*}{\nu}$ it is always possible to ensure that all the densities remain positive.

In both cases, the choice of the value of $\mathcal{M}$ determines the mass, momentum and energy jumps across the standing wave through

$$[\alpha_1 \rho_1 w_1]_{\frac{y}{t}=0} = 0, \qquad (4.43)$$

$$\left[\alpha_1 \rho_1 w_1^2 + \alpha_1 \pi_1\right]_{\frac{y}{t}=0} = \pi_1^\sharp \Delta\alpha_1 - a_1^2\left((\alpha_{1,R} + \alpha_{1,L})\mathcal{M}_L^* - 2\alpha_{1,L}\mathcal{M}\right)\tau_{1,L}^\sharp, \qquad (4.44)$$

$$\left[\alpha_1 \rho_1 \overline{\mathcal{E}}_1 w_1 + \alpha_1 \pi_1 w_1\right]_{\frac{y}{t}=0} = \frac{1}{2}(w^\sharp + a_1\tau_{1,L}^\sharp)^2\frac{\mathcal{Q}_0(\mathcal{M})}{(1+\nu\mathcal{M})(1-\mathcal{M})}\varphi(\mathcal{M}; \nu, \omega), \qquad (4.45)$$

where $\mathcal{Q}_0(\mathcal{M}) = \alpha_{1,L}\rho_1^- w_1^- = \alpha_{1,R}\rho_1^+ w_1^+ > 0$ is the constant mass flux across the standing wave and

$$\varphi(\mathcal{M}; \nu, \omega) = \omega^2(\nu\mathcal{M}+1)(\mathcal{M}+1) - (\nu\mathcal{M}-1)(\mathcal{M}-1). \qquad (4.46)$$

Remark 4.2. As $\mathcal{M}$ lies in the interval $(0, \min(1, 1/\nu))$, one has $(1+\nu\mathcal{M})(1-\mathcal{M}) > 0$. Hence, the sign of the energy-jump across the standing wave is determined by the sign of $\varphi(\mathcal{M}; \nu, \omega)$. If $\mathcal{M}$ is such that $\varphi(\mathcal{M}; \nu, \omega) = 0$, then the solution is energy-preserving. If $\mathcal{M}$ is such that $\varphi(\mathcal{M}; \nu, \omega) < 0$, the solution is energy-dissipative. The value $\mathcal{M}_0(\nu, \omega)$ given in (4.42) is the solution of $\varphi(\mathcal{M}; \nu, \omega) = 0$.

Proof. The proof is mainly given in [17] with slightly different notations. Indeed, it is easy to see that condition (3.20) in [17] is equivalent to "$0 < \mathcal{M}_L^\sharp < 1$" which here corresponds to $0 < \mathcal{M}_L^* < 1$. The proof consists in establishing that $0 < \mathcal{M}_L^* < 1$ is a necessary and sufficient condition for the existence of solutions $\mathcal{M}$ in the interval $(0, \min(1, 1/\nu))$ to the second order polynomial in-equation $\varphi(\mathcal{M}; \nu, \omega) \leq 0$, thus enforcing a non-positive energy jump across the standing wave according to (4.45). The value $\mathcal{M}_0(\nu, \omega)$ given in (4.42) is the unique solution of $\varphi(\mathcal{M}; \nu, \omega) = 0$ that lies in the interval $(0, \min(1, 1/\nu))$. As explained in [17], the existence of $\nu^\sharp$ is related to the expression of $\tau_{1,R*}$ in (4.41) which is the only intermediate specific volume that may be non-positive. It is possible to show that the function

$$\nu \mapsto \tau_{1,R*}(\nu) = \tau_{1,R}^\sharp - \tau_{1,L}^\sharp \frac{\nu\mathcal{M}_0(\nu, \omega) - \mathcal{M}_L^*}{1+\nu\mathcal{M}_0(\nu, \omega)}, \qquad (4.47)$$

is a non-increasing function that may become negative for large values of ν. Observe that for $\nu = 1$, we have $\nu\mathcal{M}_0(\nu, \omega) = \mathcal{M}_L^*$ which implies that the pathological values of ν are larger than one (*i.e.* $\nu^\sharp > 1$). In such pathological cases, in order to impose the positivity of $\tau_{1,R*}$ we must no longer exactly conserve the energy at the standing wave (by taking $\mathcal{M} = \mathcal{M}_0(\nu, \omega)$) but dissipate it by taking $\mathcal{M}$ smaller than $\mathcal{M}_0(\nu, \omega)$. Indeed, $\varphi(\mathcal{M}; \nu, \omega) \leq 0$ for all $\mathcal{M} \in (0, \mathcal{M}_0(\nu, \omega)]$. The expression of $\tau_{1,R*}$ clearly shows that if $\mathcal{M}$ is taken close enough to $\frac{\mathcal{M}_L^*}{\nu}$ (remember that $\nu > 1$), we have $\tau_{1,R*}$ close to $\tau_{1,R}^\sharp$ which is positive by hypothesis.

Let us however prove (4.44) which is not proved in [17]. As $\mathcal{T}_1$ is constant across the standing wave and is equal to $\mathcal{T}_{1,L}$ for both states $\mathcal{W}_1^-$ and $\mathcal{W}_1^+$, one may write

$$\left[\alpha_1 \rho_1 w_1^2 + \alpha_1 \pi_1\right]_{\frac{y}{t}=0} = (\alpha_1^+ \rho_1^+ w_1^{+2} - \alpha_1^- \rho_1^- w_1^{-2}) - a_1^2(\alpha_1^+ \tau_1^+ - \alpha_1^- \tau_1^-) + (\mathcal{P}_1(\mathcal{T}_{1,L}) + a_1^2\mathcal{T}_{1,L})\Delta\alpha_1,$$

21

Moreover, $\alpha_1^- = \alpha_{1,L}$ and $\alpha_1^+ = \alpha_{1,R}$. Thus

$$\left[\alpha_1\rho_1 w_1^2 + \alpha_1\pi_1\right]_{\frac{y}{t}=0} = a_1^2\left(\alpha_{1,R}\frac{w_1^{+2}}{a_1^2\tau_1^{+2}}\tau_1^+ - \alpha_{1,L}\frac{w_1^{-2}}{a_1^2\tau_1^{-2}}\tau_1^-\right) - a_1^2(\alpha_{1,R}\tau_1^+ - \alpha_{1,L}\tau_1^-)$$

$$+ (\mathcal{P}_1(\mathcal{T}_{1,L}) + a_1^2\mathcal{T}_{1,L})\Delta\alpha_1$$

$$= a_1^2\left(\alpha_{1,R}((\nu\mathcal{M})^2 - 1)\tau_1^+ - \alpha_{1,L}(\mathcal{M}^2 - 1)\tau_1^-\right) + (\mathcal{P}_1(\mathcal{T}_{1,L}) + a_1^2\mathcal{T}_{1,L})\Delta\alpha_1.$$

With the expressions of ν (4.37), τ_1^- (4.39), and τ_1^+ (4.40) we get

$$\left[\alpha_1\rho_1 w_1^2 + \alpha_1\pi_1\right]_{\frac{y}{t}=0} = -a_1^2\left((1 - (\nu\mathcal{M})^2)\frac{\tau_1^+}{\tau_{1,L}^\sharp} - \nu(1 - \mathcal{M}^2)\frac{\tau_1^-}{\tau_{1,L}^\sharp}\right)\alpha_{1,R}\tau_{1,L}^\sharp$$

$$+ (\mathcal{P}_1(\mathcal{T}_{1,L}) + a_1^2\mathcal{T}_{1,L})\Delta\alpha_1$$

$$= -a_1^2\left((1 - \nu\mathcal{M})(1 + \mathcal{M}_L^*) - \nu(1 + \mathcal{M})(1 - \mathcal{M}_L^*)\right)\alpha_{1,R}\tau_{1,L}^\sharp$$

$$+ (\mathcal{P}_1(\mathcal{T}_{1,L}) + a_1^2\mathcal{T}_{1,L})\Delta\alpha_1$$

$$= (\mathcal{P}_1(\mathcal{T}_{1,L}) + a_1^2\mathcal{T}_{1,L})\Delta\alpha_1 - a_1^2\left(1 - \nu + (1 + \nu)\mathcal{M}_L^* - 2\nu\mathcal{M}\right)\alpha_{1,R}\tau_{1,L}^\sharp$$

$$= (\mathcal{P}_1(\mathcal{T}_{1,L}) + a_1^2(\mathcal{T}_{1,L} - \tau_{1,L}^\sharp))\Delta\alpha_1 - a_1^2\left((1 + \nu)\mathcal{M}_L^* - 2\nu\mathcal{M}\right)\alpha_{1,R}\tau_{1,L}^\sharp,$$

with $\mathcal{P}_1(\mathcal{T}_{1,L}) + a_1^2(\mathcal{T}_{1,L} - \tau_{1,L}^\sharp) = \pi_1^\sharp$ which yields (4.44). $\qquad\square$

4.5.2 Kinetic relation

As seen in Proposition 4.6, the admissible solutions are parametrized by a real number $\mathcal{M}$ lying in the interval $(0, \mathcal{M}_0(\nu, \omega)]$. The aim of this section is to propose an "instruction manual" for the practical choice of the parameter $\mathcal{M}$ in this interval, with respect to the data of the problem $(\mathbb{W}_{1,L}, \mathbb{W}_{1,R}, u_2^*)$. The rule consists in enforcing a lower bound on $\tau_{1,R*}$, the only specific volume that may be non-positive:

$$\tau_{1,R*} \geq \mu\tau_{1,R}^\sharp, \tag{4.48}$$

where μ is a fixed real number in $(0, 1)$. If the energy-preserving value $\mathcal{M} = \mathcal{M}_0(\nu, \omega)$ is such that (4.48) is satisfied, then one chooses this value. Otherwise, one chooses the value of $\mathcal{M}$ which verifies $\tau_{1,R*} = \mu\tau_{1,R}^\sharp$. As the function $\mathcal{M} \mapsto \tau_{1,R*}(\mathcal{M}) = \tau_{1,R}^\sharp - \tau_{1,L}^\sharp \frac{\nu\mathcal{M} - \mathcal{M}_L^*}{1 + \nu\mathcal{M}}$, is a non-increasing function, this amounts to take $\mathcal{M} = \mathcal{M}(\nu, \mathcal{M}_L^*)$ where

$$\mathcal{M}(\nu, \mathcal{M}_L^*) := \min\left(\mathcal{M}_0(\nu, \omega), \mathcal{M}_\mu(\nu, \mathcal{M}_L^*)\right), \quad \omega = \frac{1 - \mathcal{M}_L^*}{1 + \mathcal{M}_L^*}, \tag{4.49}$$

with

$$\mathcal{M}_\mu(\nu, \mathcal{M}_L^*) := \frac{1}{\nu}\frac{\mathcal{M}_L^* + (1 - \mu)\frac{\tau_{1,R}^\sharp}{\tau_{1,L}^\sharp}}{1 - (1 - \mu)\frac{\tau_{1,R}^\sharp}{\tau_{1,L}^\sharp}}. \tag{4.50}$$

Note that if $\nu \leq 1$, then $\mathcal{M}_0(\nu, \omega) < \mathcal{M}_\mu(\nu, \mathcal{M}_L^*)$, which means that no dissipative correction is added since the lower-bound on $\tau_{1,R*}$ is already satisfied by the energy-preserving choice $\mathcal{M}_0(\nu, \omega)$. If $\nu > 1$, for μ close enough to one, one has $\mathcal{M}_\mu(\nu, \mathcal{M}_L^*)$ close to $\frac{\mathcal{M}_L^*}{\nu}$ and then $\mathcal{M}(\nu, \mathcal{M}_L^*) \in (0, \mathcal{M}_0(\nu, \omega)]$ which implies that the energy is now dissipated since $\varphi(\mathcal{M}; \nu, \omega) \leq 0$ for all $\mathcal{M} \in (0, \mathcal{M}_0(\nu, \omega)]$. **Formula (4.49) is referred to as the _kinetic relation_** since the choice of $\mathcal{M}$ prescribes an energy-dissipation rate through (4.45) and determines a unique solution to the Riemann-problem (4.33)-(4.34) (or equivalently to the Riemann problem (4.20)-(4.21)).

4.5.3   Jump relations in the original frame

Defining

$$\pi_1^* \Delta\alpha_1 := \pi_1^\sharp \Delta\alpha_1 - a_1^2 \left((\alpha_{1,R} + \alpha_{1,L})\mathcal{M}_L^* - 2\alpha_{1,L}\mathcal{M}\right) \tau_{1,L}^\sharp, \tag{4.51}$$

$$\mathcal{Q}(u_2^*, \mathbb{W}_L, \mathbb{W}_R) := -\frac{1}{2}(w^\sharp + a_1\tau_{1,L}^\sharp)^2 \frac{\mathcal{Q}_0(\mathcal{M})}{(1+\nu\mathcal{M})(1-\mathcal{M})} \varphi\left(\mathcal{M}; \nu, \omega\right), \tag{4.52}$$

easy manipulations using the relation $w_1 = u_1 - u_2^*$ show that the jump relations (4.43)-(4.44)-(4.45) are equivalent to the following jump relations in the original frame:

$$- u_2^* \left[\alpha_1\rho_1\right]_{\frac{x}{t}=u_2^*} + \left[\alpha_1\rho_1\right]_{\frac{x}{t}=u_2^*} = 0, \tag{4.53}$$

$$- u_2^* \left[\alpha_1\rho_1 u_1\right]_{\frac{x}{t}=u_2^*} + \left[\alpha_1\rho_1 u_1^2 + \alpha_1\pi_1\right]_{\frac{x}{t}=u_2^*} - \pi_1^* \Delta\alpha_1 = 0, \tag{4.54}$$

$$- u_2^* \left[\alpha_1\rho_1\mathcal{E}_1\right]_{\frac{x}{t}=u_2^*} + \left[\alpha_1\rho_1\mathcal{E}_1 u_1 + \alpha_1\pi_1 u_1\right]_{\frac{x}{t}=u_2^*} - u_2^*\pi_1^* \Delta\alpha_1 = -\mathcal{Q}(u_2^*, \mathbb{W}_L, \mathbb{W}_R) \le 0. \tag{4.55}$$

Finally, observe that expression (4.51) defines the function $\mathscr{G}[\mathbb{W}_L, \mathbb{W}_R; a_1]$ introduced in (4.23) since it expresses π_1^* as a function of u_2^* (through the relative Mach number $\mathcal{M}_L^*$).

4.6   Solution to the fixed-point problem and proof of Theorem 4.2

In this section, we prove that condition (**A1**) is a necessary and sufficient condition for the existence of solutions to the following fixed-point problem :

Find u_2^* in $(u_1^\sharp - a_1\tau_{1,L}^\sharp, u_1^\sharp + a_1\tau_{1,R}^\sharp) \bigcap (u_2^\sharp - a_2\tau_{2,L}^\sharp, u_2^\sharp + a_2\tau_{2,R}^\sharp)$ such that

$$u_2^* = \left(\mathscr{F}[\mathbb{W}_L, \mathbb{W}_R; a_2] \circ \mathscr{G}[\mathbb{W}_L, \mathbb{W}_R; a_1]\right)(u_2^*), \quad \text{with } u_2^* < u_1^*, \tag{4.56}$$

and therefore, for the existence of solutions to the Riemann problem (4.1)-(4.2) with the subsonic wave ordering $u_{1,L} - a_1\tau_{1,L} < u_2^* < u_1^* < u_{1,R} + a_1\tau_{1,R}$. Let us first introduce some non-dimensional numbers built on the quantities defined in (4.10)-(4.11)-(4.12)-(4.13) :

$$\mathcal{M}_L^\sharp := \frac{u_1^\sharp - u_2^\sharp}{a_1\tau_{1,L}^\sharp}, \qquad \mathcal{P}_L^\sharp := \frac{\pi_1^\sharp - \pi_2^\sharp}{a_1^2\tau_{1,L}^\sharp}. \tag{4.57}$$

Solving the fixed-point (4.56) amounts to *re-coupling* the two phases that have been decoupled for a separate resolution. We start by rewriting the expression of $\pi_1^* \Delta\alpha_1$ obtained for phase 2 in (4.28):

$$\begin{aligned} \pi_1^* \Delta\alpha_1 &= \Delta\alpha_1\pi_2^\sharp + a_2(\alpha_{2,L} + \alpha_{2,R})\left(u_2^\sharp - u_2^*\right) \\ &= \Delta\alpha_1\pi_2^\sharp + a_2(\alpha_{2,L} + \alpha_{2,R})\left(u_1^\sharp - u_2^* + u_2^\sharp - u_1^\sharp\right). \end{aligned} \tag{4.58}$$

Hence, solving the fixed-point problem (4.56) amounts to seeking u_2^* such that the two expressions of $\pi_1^* \Delta\alpha_1$ given in (4.51) and (4.58) are equal, *i.e.* such that

$$\begin{aligned} \pi_1^\sharp \Delta\alpha_1 - a_1^2\left((\alpha_{1,R} + \alpha_{1,L})\mathcal{M}_L^* - 2\alpha_{1,L}\mathcal{M}\right)\tau_{1,L}^\sharp = \pi_2^\sharp \Delta\alpha_1 + a_2(\alpha_{2,L} + \alpha_{2,R})a_1\tau_{1,L}^\sharp\mathcal{M}_L^* \\ - a_2(\alpha_{2,L} + \alpha_{2,R})(u_1^\sharp - u_2^\sharp). \end{aligned} \tag{4.59}$$

4.6.1   The energy-preserving case

We first look for solutions that exactly preserve the energy equality across the u_2^*-wave. Therefore, we take $\mathcal{M} := \mathcal{M}_0\left(\frac{\alpha_{1,L}}{\alpha_{1,R}}, \frac{1-\mathcal{M}_L^*}{1+\mathcal{M}_L^*}\right)$, where $\mathcal{M}_0(\nu, \omega)$ is defined in (4.42). Introducing the non-dimensional quantities $\mathcal{M}_L^\sharp$, $\mathcal{P}_L^\sharp$ and Λ^α, equation (4.59) re-writes as

$$\mathcal{M}_L^\sharp - \frac{a_1}{a_2}\Lambda^\alpha \mathcal{P}_L^\sharp = \mathcal{M}_L^* + \frac{a_1}{a_2}\frac{1}{\alpha_{2,L} + \alpha_{2,R}}\left((\alpha_{1,R} + \alpha_{1,L})\mathcal{M}_L^* - 2\alpha_{1,L}\mathcal{M}_0\left(\frac{\alpha_{1,L}}{\alpha_{1,R}}, \frac{1-\mathcal{M}_L^*}{1+\mathcal{M}_L^*}\right)\right). \tag{4.60}$$

23

Now, considering the change of variables $u_2^* \mapsto \mathcal{M}_L^* = \frac{u_1^\sharp - u_2^*}{a_1 \tau_{1,L}^\sharp}$, solving the fixed-point problem (4.56) is equivalent to finding $\mathcal{M}_L^*$ such that equation (4.60) holds. Observe that by Proposition 4.6, the solution has the subsonic wave ordering $u_1 - a_1\tau_1 < u_2 < u_1$ if and only if $\mathcal{M}_L^*$ belongs to $(0,1)$. Defining the function

$$
\Psi_0 \; : \; \begin{cases} (0,1) \longrightarrow \mathbb{R} \\ m \longmapsto m + \dfrac{a_1}{a_2} \dfrac{1}{\alpha_{2,L} + \alpha_{2,R}} \left((\alpha_{1,R} + \alpha_{1,L})m - 2\alpha_{1,L}\mathcal{M}_0 \left(\frac{\alpha_{1,L}}{\alpha_{1,R}}, \frac{1-m}{1+m} \right) \right), \end{cases} \tag{4.61}
$$

the following proposition proves that condition (**A1**) is equivalent to the existence of a unique solution $\mathcal{M}_L^*$ in $(0,1)$ to our fixed-point problem:

Proposition 4.7. *Function $m \mapsto \Psi_0(m)$ is a differentiable increasing function on the interval $(0,1)$, whose limit values are*

$$
\lim_{m \to 0} \Psi_0(m) = 0, \qquad \lim_{m \to 1} \Psi_0(m) = 1 + \frac{a_1}{a_2}|\Lambda^\alpha|. \tag{4.62}
$$

*Hence, if the following condition, which is equivalent to (**A1**) holds,*

$$
0 < \mathcal{M}_L^\sharp - \frac{a_1}{a_2}\Lambda^\alpha \mathcal{P}_L^\sharp < 1 + \frac{a_1}{a_2}|\Lambda^\alpha|, \tag{4.63}
$$

then there exists a unique $\mathcal{M}_L^*$ in $(0,1)$ such that

$$
\Psi_0(\mathcal{M}_L^*) = \mathcal{M}_L^\sharp - \frac{a_1}{a_2}\Lambda^\alpha \mathcal{P}_L^\sharp. \tag{4.64}
$$

Proof. The function Ψ_0 is clearly differentiable on the interval $(0,1)$. Differentiating *w.r.t* m, one gets

$$
\Psi_0'(m) = 1 + \frac{a_1}{a_2}\frac{\alpha_{1,R} + \alpha_{1,L}}{\alpha_{2,L} + \alpha_{2,R}} - \frac{a_1}{a_2}\frac{2\alpha_{1,L}}{\alpha_{2,L} + \alpha_{2,R}}\frac{d}{dm}\left\{ \mathcal{M}_0 \left(\frac{\alpha_{1,L}}{\alpha_{1,R}}, \frac{1-m}{1+m} \right) \right\}
$$

$$
= 1 + \frac{a_1}{a_2}\frac{\alpha_{1,R} + \alpha_{1,L}}{\alpha_{2,L} + \alpha_{2,R}} - \frac{a_1}{a_2}\frac{2\alpha_{1,L}}{\alpha_{2,L} + \alpha_{2,R}}\frac{\partial \mathcal{M}_0}{\partial \omega}\left(\frac{\alpha_{1,L}}{\alpha_{1,R}}, \frac{1-m}{1+m} \right) \cdot \frac{d\omega}{dm},
$$

where $\omega = \frac{1-m}{1+m}$. We have $\frac{d\omega}{dm} = -\frac{2}{(1+m)^2}$, hence

$$
\Psi_0'(m) = 1 + \frac{a_1}{a_2}\frac{1}{\alpha_{2,L} + \alpha_{2,R}}\left\{ \alpha_{1,R} + \alpha_{1,L} + 2\alpha_{1,L}\frac{\partial \mathcal{M}_0}{\partial \omega}\left(\frac{\alpha_{1,L}}{\alpha_{1,R}}, \frac{1-m}{1+m} \right) \cdot \frac{2}{(1+m)^2} \right\}.
$$

Therefore, again denoting $\nu = \frac{\alpha_{1,L}}{\alpha_{1,R}}$ and $\omega = \frac{1-m}{1+m}$, it is sufficient for the derivative $\Psi_0'(m)$ to be positive, that

$$
1 + \nu + \nu(1 + \omega)^2\frac{\partial \mathcal{M}_0}{\partial \omega}(\nu, \omega) \geq 0, \qquad \text{for all } \omega \text{ in } (0,1),
$$

$$
\iff \quad 1 + (1 + \omega)^2\frac{\partial}{\partial \omega}\left\{ \frac{\nu}{1 + \nu}\mathcal{M}_0(\nu, \omega) \right\} \geq 0, \quad \text{for all } \omega \text{ in } (0,1). \tag{4.65}
$$

With the expression of $\mathcal{M}_0(\nu, \omega)$ in (4.42), one gets

$$
\frac{\nu}{1 + \nu}\mathcal{M}_0(\nu, \omega) = \frac{1}{2}\left(\frac{1 + \omega^2}{1 - \omega^2} - \sqrt{\left(\frac{1 + \omega^2}{1 - \omega^2} \right)^2 - \frac{4\nu}{(1 + \nu)^2}} \right).
$$

Differentiating this with respect to ω yields

$$
\frac{\partial}{\partial \omega}\left\{ \frac{\nu}{1 + \nu}\mathcal{M}_0(\nu, \omega) \right\} = \frac{1}{2}\left(\frac{4\omega}{(1 - \omega^2)^2} - \frac{4\omega}{(1 - \omega^2)^2}\frac{1 + \omega^2}{1 - \omega^2}\left(\left(\frac{1 + \omega^2}{1 - \omega^2} \right)^2 - \frac{4\nu}{(1 + \nu)^2} \right)^{-1/2} \right)
$$

$$
= \frac{2\omega}{(1 - \omega^2)^2}\left(1 - \left(1 - \frac{4\nu}{(1 + \nu)^2}\left(\frac{1 - \omega^2}{1 + \omega^2} \right)^2 \right)^{-1/2} \right). \tag{4.66}
$$

24

Casting this in (4.65), the sufficient condition for the function Ψ_0 to be increasing becomes

$$1 + 2\omega \frac{(1+\omega)^2}{(1-\omega^2)^2} \left(1 - \left(1 - \frac{4\nu}{(1+\nu)^2}\left(\frac{1-\omega^2}{1+\omega^2}\right)^2\right)^{-1/2}\right) \geq 0, \quad \text{for all } \omega \text{ in } (0,1). \tag{4.67}$$

Now, isolating the terms in ν and those in ω, (4.67) is equivalent to

$$\frac{4\nu}{(1+\nu)^2} \leq \left(\frac{1+\omega^2}{1-\omega^2}\right)^2 \left(1 - \frac{1}{\left(1 + \frac{(1-\omega^2)^2}{2\omega(1+\omega)^2}\right)^2}\right), \quad \text{for all } \omega \text{ in } (0,1). \tag{4.68}$$

A calculation shows that the right-hand side term of (4.68) is independent of ω and equals 1. Hence, a sufficient condition for the function Ψ_0 to be strictly increasing is

$$\frac{4\nu}{(1+\nu)^2} \leq 1, \tag{4.69}$$

which is true for any ν in $\mathbb{R}^+$. As for the limit values of Ψ_0, observe that the function $\mathcal{M}_0(\nu, \omega)$ is such that $\lim_{\omega \to 0} \mathcal{M}_0(\nu, \omega) = \min\left(1, \frac{1}{\nu}\right)$ and $\lim_{\omega \to 1} \mathcal{M}_0(\nu, \omega) = 0$, hence the limits (4.62) as m tends to 0 and 1. Finally, Proposition 4.7 follows from the intermediate value theorem. $\square$

Thus, provided positive values of the densities, Proposition 4.7 proves that (**A1**) is a **necessary and sufficient** condition for the **existence and uniqueness** of an energy-preserving solution. If the phase 1 densities are not positive, one must authorize some energy dissipation as detailed hereunder.

4.6.2 The energy-dissipating case

According to the proof of Proposition 4.6, it may occur when the ratio $\nu = \frac{\alpha_{1,L}}{\alpha_{1,R}}$ is large, that the solution $\mathcal{M}_L^*$ of the fixed-point problem (4.64) is such that the specific volume $\tau_{1,R*}$ (see (4.41) with $\mathcal{M} = \mathcal{M}_0\left(\frac{\alpha_{1,L}}{\alpha_{1,R}}, \frac{1-\mathcal{M}_L^*}{1-\mathcal{M}_L^*}\right)$) is non-positive. In such pathological case, the unique energy-preserving solution of the previous section is not admissible and one has to authorize energy dissipation introducing some *kinetic relation*. As explained in Section 4.5.2, this kinetic relation is obtained by prescribing a lower-bound on $\tau_{1,R*}$, namely $\tau_{1,R*} \geq \mu \tau_{1,R}^\sharp$ where $\mu \in (0,1)$, through the definition of a new value of the Mach number $\mathcal{M} = \mathcal{M}(\nu, \mathcal{M}_L^*)$ given by (4.49). The fixed-point research must now be performed with the new function

$$\Psi : \begin{cases} (0,1) \longrightarrow \mathbb{R} \\ m \longmapsto m + \dfrac{a_1}{a_2} \dfrac{1}{\alpha_{2,L} + \alpha_{2,R}} \left((\alpha_{1,R} + \alpha_{1,L})m - 2\alpha_{1,L}\mathcal{M}\left(\frac{\alpha_{1,L}}{\alpha_{1,R}}, m\right)\right). \end{cases} \tag{4.70}$$

Observe that if m is such that $\mathcal{M}_0\left(\nu, \frac{1-m}{1+m}\right) \leq \mathcal{M}_\mu(\nu, m)$ then $\Psi(m) = \Psi_0(m)$. In particular, if $\alpha_{1,L} \leq \alpha_{1,R}$, then Ψ identifies with Ψ_0 on the whole interval $(0,1)$. We have the following proposition which shows that, provided an appropriate choice of the parameter $\mu \in (0,1)$, there still exists a unique solution $\mathcal{M}_L^* \in (0,1)$ under condition (**A1**).

Proposition 4.8. *If the parameter $\mu \in (0,1)$ is close enough to one, the function $m \mapsto \Psi(m)$ is a Lipschitz-continuous increasing function on the interval $(0,1)$, whose limit values are*

$$\lim_{m \to 0} \Psi(m) = 0, \qquad \lim_{m \to 1} \Psi(m) = 1 + \frac{a_1}{a_2}|\Lambda^\alpha|. \tag{4.71}$$

*Hence, if condition (**A1**) holds, then there exists a unique $\mathcal{M}_L^*$ in $(0,1)$ such that*

$$\Psi(\mathcal{M}_L^*) = \mathcal{M}_L^\sharp - \frac{a_1}{a_2}\Lambda^\alpha \mathcal{P}_L^\sharp. \tag{4.72}$$

Proof. If $\alpha_{1,L} \leq \alpha_{1,R}$ then $\Psi \equiv \Psi_0$ and the result follows from the energy-preserving case. Let us turn to the case $\alpha_{1,L} > \alpha_{1,R}$. As the minimum of two differentiable functions, $\mathcal{M}(\nu,m)$ is Lipschitz-continuous and so is Ψ. Actually, Ψ is almost everywhere differentiable on $(0,1)$. For the limit values of Ψ, we know from the energy-preserving case that $\lim\limits_{m \to 0} \mathcal{M}_0\left(\nu, \frac{1-m}{1+m}\right) = 0$, $\lim\limits_{m \to 1} \mathcal{M}_0\left(\nu, \frac{1-m}{1+m}\right) = \min(1, 1/\nu)$, and if μ is close enough to one, we have

$$\mathcal{M}_\mu(\nu,0) = \frac{1}{\nu}\frac{(1-\mu)\frac{\tau_{1,R}^\sharp}{\tau_{1,L}^\sharp}}{1-(1-\mu)\frac{\tau_{1,R}^\sharp}{\tau_{1,L}^\sharp}} > 0, \qquad \mathcal{M}_\mu(\nu,1) = \frac{1}{\nu}\frac{1+(1-\mu)\frac{\tau_{1,R}^\sharp}{\tau_{1,L}^\sharp}}{1-(1-\mu)\frac{\tau_{1,R}^\sharp}{\tau_{1,L}^\sharp}} > \min\left(1,\frac{1}{\nu}\right). \qquad (4.73)$$

Hence, Ψ and Ψ_0 share the same limit values at 0 and 1. As for the monotony of function Ψ, we may write that for almost every m in $(0,1)$:

$$\Psi'(m) \geq \min\left(\Psi_0'(m), 1 + \frac{a_1}{a_2}\frac{1}{\alpha_{2,L}+\alpha_{2,R}}\left(\alpha_{1,R}+\alpha_{1,L}-2\frac{\alpha_{1,R}}{1-(1-\mu)\frac{\tau_{1,R}^\sharp}{\tau_{1,L}^\sharp}}\right)\right). \qquad (4.74)$$

For $\mu = 1$, this expression gives $\Psi'(m) \geq \min\left(\Psi_0'(m), 1 + \frac{a_1}{a_2}\frac{|\alpha_{1,R}-\alpha_{1,L}|}{\alpha_{2,L}+\alpha_{2,R}}\right)$ since $\alpha_{1,L} > \alpha_{1,R}$. As $\Psi_0'(m) > 0$ by the study of the energy-preserving case, this proves that Ψ is strictly increasing if μ is close enough to one, which concludes the proof. $\square$

4.6.3 Proof of Theorem 4.2 :

We may now complete the proof of Theorem 4.2. If the ratio $\frac{\alpha_{1,L}}{\alpha_{1,R}}$ is close to one, Proposition 4.6 concerning phase 1, asserts that no energy dissipation is needed for ensuring the positivity of the densities. Hence, for $\frac{\alpha_{1,L}}{\alpha_{1,R}}$ in a neighborhood of 1, by Proposition 4.7, condition (**A1**) is a necessary and sufficient condition for the existence of a unique solution to the fixed-point problem (4.60), *i.e.* for the existence and uniqueness of an energy-preserving solution to the Riemann problem (4.1)-(4.2) with the subsonic wave ordering $u_2^* < u_1^*$. For large values of the ratio $\frac{\alpha_{1,L}}{\alpha_{1,R}}$, Proposition 4.6 shows that ensuring positive densities for phase 1 may require strict energy dissipation across the u_2-wave. In this case, still assuming condition (**A1**), Proposition 4.8 proves that using the kinetic relation (4.49) defining $\mathcal{M}$ with respect to the pair $\left(\frac{\alpha_{1,L}}{\alpha_{1,R}}, \mathcal{M}_L^*\right)$, it is always possible to ensure the existence of a unique solution with positive densities for phase 1 by dissipating the total energy. Note that uniqueness is achieved once the kinetic relation is prescribed. If another choice had been made for the kinetic relation (4.49), another dissipative solution could be found.

Finally, thanks to the Galilean invariance of system (4.1), one can prove that the symmetric wave-configuration $u_2^* > u_1^*$ is implied by (**A2**) by exchanging the subscripts L and R and changing the velocities to their opposite values. As for condition (**A3**), it can be obtained by passing to the limit in (**A1**). The corresponding $\mathcal{M}_L^*$ is equal to zero, and we obtain the $u_2^* = u_1^*$ configuration. This observation concludes the proof of Theorem 4.2.

4.7 Expression of the Riemann solution

In this section, we construct the solution $\mathbb{W}(x,t;\mathbb{W}_L,\mathbb{W}_R) = \mathbb{W}_r(x/t;\mathbb{W}_L,\mathbb{W}_R)$ for a given pair of initial conditions $(\mathbb{W}_L,\mathbb{W}_R)$ in Ω^r and two parameters a_1 and a_2 meeting the conditions of Theorem 4.2. We distinguish the three different cases corresponding respectively to (**A1**), (**A2**) and (**A3**).

- If (**A1**) holds, the phasic solutions have the following form:
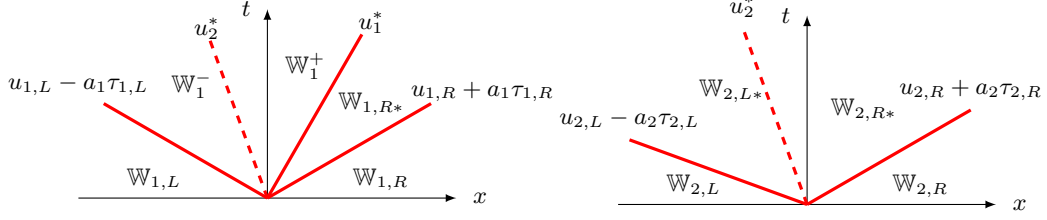
The values u_2^* and π_1^* are calculated as follows. First use an iterative method (Newton's method or a dichotomy algorithm for instance) to compute $\mathcal{M}_L^*$ such that

$$\Psi(\mathcal{M}_L^*) = \mathcal{M}_L^\sharp - \frac{a_1}{a_2} \Lambda^\alpha \mathcal{P}_L^\sharp. \tag{4.75}$$

According to Section 4.6, $\mathcal{M}_L^*$ always exists under $(\mathbf{A1})$ and is unique if μ is close enough to one. We then obtain u_2^* by $u_2^* = u_1^\sharp - a_1 \tau_{1,L}^\sharp \mathcal{M}_L^*$, while π_1^* is obtained through (4.58). Then the intermediate states for phase 2 are given by equations (4.28) and (4.29) in Proposition 4.4. Once prescribed the value $\mathcal{M} := \mathcal{M}(\nu, \mathcal{M}_L^*)$ according to (4.49), the intermediate states for phase 1 are given in equations (4.39) to (4.41) of Proposition 4.6, except for the velocities to which one must add u_2^*: $u_1^- = w_1^- + u_2^*$, $u_1^+ = w_1^+ + u_2^*$, $u_{1,R*} = w_{1,R*} + u_2^*$.

- If $(\mathbf{A2})$ holds, we exploit the Galilean invariance of the equations. The solution is obtained by the transformation

$$\mathbb{W}_r(\xi; \mathbb{W}_L, \mathbb{W}_R) := \mathcal{V}\mathbb{W}_r(-\xi; \mathcal{V}\mathbb{W}_R, \mathcal{V}\mathbb{W}_L), \tag{4.76}$$

where the operator $\mathcal{V}$ changes the velocities into their opposite values:

$$\mathcal{V} : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (x_1, x_2, -x_3, x_4, -x_5, x_6, x_7). \tag{4.77}$$

Of course, the function $\mathbb{W}_r(-\xi; \mathcal{V}\mathbb{W}_R, \mathcal{V}\mathbb{W}_L)$ is computed through the first case, since for these new initial data $(\mathcal{V}\mathbb{W}_R, \mathcal{V}\mathbb{W}_L)$, it is condition $(\mathbf{A1})$ that holds.

- If $(\mathbf{A3})$ holds, u_2^* is equal to $u_1^\sharp$ (*i.e.* $\mathcal{M}_L^* = 0$). The intermediate states for phase 2 are obtained through equations (4.28) and (4.29) in Proposition 4.4, and the intermediate states for phase 1 are computed by passing to the limit as $\mathcal{M}_L^*$ goes to zero (*i.e.* $\omega \to 1$). We obtain the intermediate states in equations (4.39) to (4.41), for $\mathcal{M} = 0$.

5 A positive and entropy-satisfying Finite Volume scheme

In this section, the exact Riemann solver $\mathbb{W}_r(\xi; \mathbb{W}_L, \mathbb{W}_R)$ build in Section 4 for the relaxation system is used to derive an approximate Riemann solver of Harten, Lax and van Leer [26] for the simulation of system (2.1). This approximate Riemann solver is proved to satisfy important stability properties such as the preservation of the densities positivity, and a discrete entropy inequality which is a discrete counterpart of the energy inequality (2.10) satisfied by the exact weak solutions of the model.

5.1 The HLL approximate Riemann solver

In order to define the approximate Riemann solver, let us introduce the mappings

$$\mathscr{M} : \begin{cases} \mathbb{R}^5 & \longrightarrow & \mathbb{R}^7 \\ (x_k)_{k=1,..,5} & \longmapsto & (x_1, x_2, x_3, x_4, x_5, x_1, 1 - x_1). \end{cases} \tag{5.1}$$

$$\mathscr{P} : \begin{cases} \mathbb{R}^7 & \longrightarrow & \mathbb{R}^5 \\ (x_k)_{k=1,..,7} & \longmapsto & (x_1, x_2, x_3, x_4, x_5). \end{cases} \tag{5.2}$$

An approximate solution of the Riemann problem (2.1)-(2.11) is obtained by

$$\mathbb{U}_{app}(\xi; \mathbb{U}_L, \mathbb{U}_R) := \mathscr{P}\mathbb{W}_r(\xi; \mathscr{M}(\mathbb{U}_L), \mathscr{M}(\mathbb{U}_R)). \tag{5.3}$$

Or course, $\mathbb{W}_r(\xi; \mathscr{M}(\mathbb{U}_L), \mathscr{M}(\mathbb{U}_R))$ depends on the parameters (a_1, a_2) which must be chosen so as to satisfy some stability properties (see Section 5.5 below). With this approximate Riemann solver, are associated the following two interface numerical fluxes:

$$\mathbf{F}^-(\mathbb{U}_L, \mathbb{U}_R) := \mathbf{f}(\mathbb{U}_L) - \int_{-\infty}^0 \{\mathbb{U}_{app}(\xi; \mathbb{U}_L, \mathbb{U}_R) - \mathbb{U}_L\} \, d\xi, \tag{5.4}$$

$$\mathbf{F}^+(\mathbb{U}_L, \mathbb{U}_R) := \mathbf{f}(\mathbb{U}_R) + \int_0^{+\infty} \{\mathbb{U}_{app}(\xi; \mathbb{U}_L, \mathbb{U}_R) - \mathbb{U}_R\} \, d\xi. \tag{5.5}$$

The integrals in (5.4) and (5.5) are well-defined thanks to (4.3) and to the property $\mathscr{P} \circ \mathscr{M} = Id_{\mathbb{R}^5}$. Observe that $\mathbf{F}^-(\mathbb{U}_L, \mathbb{U}_R) \neq \mathbf{F}^+(\mathbb{U}_L, \mathbb{U}_R)$ unless $\alpha_{1,L} = \alpha_{1,R}$ because of the non-conservative part of system (4.1). In the sequel $\mathbf{F}_k^\pm$ denotes the kth component of vector $\mathbf{F}^\pm$.

5.2 The Finite Volume relaxation scheme

We now derive a Finite Volume scheme for the approximation of the entropy weak solutions of a Cauchy problem associated with system (2.1):

$$\begin{cases} \partial_t \mathbb{U} + \partial_x \mathbf{f}(\mathbb{U}) + \mathbf{c}(\mathbb{U})\partial_x \mathbb{U} = 0, & x \in \mathbb{R}, t > 0, \\ \mathbb{U}(x, 0) = \mathbb{U}_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.6}$$

For simplicity in the notations, we assume constant positive time and space steps Δt and Δx, and we define $\lambda = \frac{\Delta t}{\Delta x}$. The space is partitioned into cells $\mathbb{R} = \bigcup_{j \in \mathbb{Z}} C_j$ where $C_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[$ with $x_{j+\frac{1}{2}} = (j + \frac{1}{2})\Delta x$ for all j in $\mathbb{Z}$. The centers of the cells are denoted $x_j = j\Delta x$ for all j in $\mathbb{Z}$. We also introduce the discrete intermediate times $t^n = n\Delta t$, $n \in \mathbb{N}$. The approximate solution at time t^n, $x \in \mathbb{R} \mapsto \mathbb{U}_\lambda(x, t^n) \in \Omega$ is a piecewise constant function whose value on each cell C_j is a constant value denoted by $\mathbb{U}_j^n$. We assume that Δt and Δx satisfy the CFL condition

$$\frac{\Delta t}{\Delta x} \max_{i \in \{1,2\}, j \in \mathbb{Z}} \max \left\{ |(u_i - a_i\tau_i)_j^n|, |(u_i + a_i\tau_i)_{j+1}^n| \right\} < \frac{1}{2}. \tag{5.7}$$

The Finite Volume relaxation scheme reads

$$\mathbb{U}_j^{n+1} = \mathbb{U}_j^n - \frac{\Delta t}{\Delta x} \left(\mathbf{F}^-(\mathbb{U}_j^n, \mathbb{U}_{j+1}^n) - \mathbf{F}^+(\mathbb{U}_{j-1}^n, \mathbb{U}_j^n) \right). \tag{5.8}$$

We do not precise at this point the computation of the constant a_i. This discussion is postponed to Section 5.5.

Remark 5.1. *The numerical fluxes formula that are used in practice in the implementation are*

$$\mathbf{F}^-(\mathbb{U}_L, \mathbb{U}_R) = \mathscr{P}\boldsymbol{g}\left(\mathbb{W}_r\left(0; \mathscr{M}(\mathbb{U}_L), \mathscr{M}(\mathbb{U}_R)\right)\right) + \mathscr{P}\mathbf{D}^*\left(\mathscr{M}(\mathbb{U}_L), \mathscr{M}(\mathbb{U}_R)\right)\mathbf{1}_{\{u_2^* < 0\}},$$

$$\mathbf{F}^+(\mathbb{U}_L, \mathbb{U}_R) = \mathscr{P}\boldsymbol{g}\left(\mathbb{W}_r\left(0; \mathscr{M}(\mathbb{U}_L), \mathscr{M}(\mathbb{U}_R)\right)\right) - \mathscr{P}\mathbf{D}^*\left(\mathscr{M}(\mathbb{U}_L), \mathscr{M}(\mathbb{U}_R)\right)\mathbf{1}_{\{u_2^* > 0\}},$$

where $\mathbf{D}^*(\mathbb{W}_L, \mathbb{W}_R) := (\alpha_{1,R} - \alpha_{1,L})(u_2^*(\mathbb{W}_L, \mathbb{W}_R), 0, -\pi_1^*(\mathbb{W}_L, \mathbb{W}_R), 0, \pi_1^*(\mathbb{W}_L, \mathbb{W}_R), 0, 0)^T$.

5.3 Basic properties of the scheme

The following theorem states some important properties of the relaxation scheme.

Theorem 5.1. *The approximate Riemann solver (5.3) has the following properties:*

- **Consistency.** *The numerical fluxes satisfy* $\mathbf{F}^-(\mathbb{U}, \mathbb{U}) = \mathbf{F}^+(\mathbb{U}, \mathbb{U}) = \mathbf{f}(\mathbb{U})$.
- **Positivity.** *Under the CFL condition (5.7), if the initial condition* $x \mapsto \mathbb{U}_0(x)$ *is in* Ω, *then the values* $(\mathbb{U}_j^n)_{j \in \mathbb{Z}, n \in \mathbb{N}}$ *computed by the scheme are such that,*

$$0 < \alpha_{i,j}^n < 1, \quad and \quad (\alpha_i\rho_i)_j^n > 0, \qquad for all i in \{1, 2\}, j in \mathbb{Z} and n in \mathbb{N}. \tag{5.9}$$

- **Mass conservation.** The fluxes (5.4) and (5.5) satisfy $\mathbf{F}_k^-(\mathbb{U}_L, \mathbb{U}_R) = \mathbf{F}_k^+$ for k in $\{2, 4\}$.
- **Total momentum conservation.** The fluxes (5.4) and (5.5) satisfy
  $$\mathbf{F}_3^-(\mathbb{U}_L, \mathbb{U}_R) + \mathbf{F}_5^-(\mathbb{U}_L, \mathbb{U}_R) = \mathbf{F}_3^+(\mathbb{U}_L, \mathbb{U}_R) + \mathbf{F}_5^+(\mathbb{U}_L, \mathbb{U}_R).$$

Proof. Following [26] or [7], the approximate Riemann solver is a Godunov type scheme where $\mathbb{U}_j^{n+1}$ is the cell-average over C_j of the function $\mathbf{1}_{x < x_j} \mathbb{U}_{app}(x/\Delta t; \mathbb{U}_{j-1}^n, \mathbb{U}_j^n) + \mathbf{1}_{x \geq x_j} \mathbb{U}_{app}(x/\Delta t; \mathbb{U}_j^n, \mathbb{U}_{j+1}^n)$. Hence, the positivity property is a direct consequence of Theorem 4.2 which ensures the positivity of the phase fractions (which are simply transported) and of the densities in the relaxation Riemann solver. For this purpose, entropy dissipation across the u_2-contact discontinuity may be necessary for enforcing this property when the ratio $\frac{\alpha_{1,L}}{\alpha_{1,R}}$ (or its inverse) is large. The proof of the other properties involves no particular difficulties. It consists in simple verifications using the expressions of the intermediate states. $\square$

5.4 Non-linear stability

Let us now turn to the discrete entropy inequality property. In order to ease the notations, we denote by η and $\mathcal{F}_\eta$ the energy-energy flux pair of the Baer-Nunziato model: $\eta(\mathbb{U}) := \sum_{i=1}^2 \alpha_i \rho_i E_i$ and $\mathcal{F}_\eta(\mathbb{U}) := \sum_{i=1}^2 \alpha_i (\rho_i E_i + p_i(\rho_i)) u_i$. The entropy weak solutions of (2.1) satisfy the following energy inequality in the weak sense:

$$\partial_t \eta(\mathbb{U}) + \partial_x \mathcal{F}_\eta(\mathbb{U}) \leq 0. \tag{5.10}$$

We also recall the following property which is proved in [16, 34]:

Proposition 5.2. *The mapping* $\eta : \left\{ \begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R} \\ \mathbb{U} & \longmapsto & \eta(\mathbb{U}) \end{array} \right.$ *is non-strictly convex.*

In the same way, we introduce the energy-energy flux pair for the relaxation system: $\eta^r(\mathbb{W}) := \sum_{i=1}^2 \alpha_i \rho_i \mathcal{E}_i$ and $\mathcal{F}_\eta^r(\mathbb{W}) := \sum_{i=1}^2 \alpha_i (\rho_i \mathcal{E}_i + \pi_i(\tau_i, \mathcal{T}_i)) u_i$. The weak solutions of (4.1)-(4.2) in the sense of Definition 4.1 satisfy the inequality

$$\partial_t \eta^r(\mathbb{W}) + \partial_x \mathcal{F}_\eta^r(\mathbb{W}) = -\mathcal{Q}(u_2^*, \mathbb{W}_L, \mathbb{W}_R) \delta_{x - u_2^* t}, \tag{5.11}$$

where $\mathcal{Q}(u_2^*, \mathbb{W}_L, \mathbb{W}_R)$ is a non-negative number defined in (4.52). In the sequel, we prove that if a so-called sub-characteristic condition (also known as Whitham's condition, see [7]) is verified by the parameters a_1 and a_2, then the relaxation Riemann solver satisfies a discrete entropy inequality, as defined in Theorem 5.4.

Definition 5.1. *Consider* $(\mathbb{U}_L, \mathbb{U}_R) \in \Omega \times \Omega$ *and let* $(\mathbb{W}_L, \mathbb{W}_R) = (\mathcal{M}(\mathbb{U}_L), \mathcal{M}(\mathbb{U}_R)) \in \Omega^r \times \Omega^r$ *be the corresponding relaxation initial data. Let* Δx *and* Δt *be two space and time steps satisfying the CFL condition (5.7). Denoting* $\tau_i(\xi) = \rho_i^{-1}(\xi)$ *the specific volumes in the solution* $\mathbb{W}_r(\xi; \mathbb{W}_L, \mathbb{W}_R)$ *of the Riemann problem (4.1)-(4.2), the parameters* (a_1, a_2) *are said to satisfy* **Whitham's condition for** $(\mathbb{U}_L, \mathbb{U}_R)$ *if*

$$\text{for } i \text{ in } \{1, 2\}, \quad a_i^2 > -\mathcal{P}_i'(\tau_i(\xi)), \text{ for almost every } \xi \text{ in } \left[-\frac{\Delta x}{2\Delta t}, \frac{\Delta x}{2\Delta t} \right]. \tag{5.12}$$

Lemma 5.3. *Take the same notations as in Definition 5.1 and denote* $\mathbb{U}_{app}(\xi)$ *and* $\mathbb{W}_r(\xi)$ *so as to ease the notation. If* (a_1, a_2) *satisfy Whitham's condition for* $(\mathbb{U}_L, \mathbb{U}_R)$, *then the relaxation approximate Riemann solver satisfies an interface entropy inequality (see [26, 7]) in the sense that*

$$\eta(\langle \mathbb{U} \rangle^L) - \eta(\mathbb{U}_L) + \frac{2\Delta t}{\Delta x} \left(\mathcal{F}_\eta^r \left(\mathbb{W}_r(0^+) \right) - \mathcal{F}_\eta(\mathbb{U}_L) \right) \leq 0, \tag{5.13}$$

$$\eta(\langle \mathbb{U} \rangle^R) - \eta(\mathbb{U}_R) + \frac{2\Delta t}{\Delta x} \left(\mathcal{F}_\eta(\mathbb{U}_R) - \mathcal{F}_\eta^r \left(\mathbb{W}_r(0^+) \right) \right) \leq 0, \tag{5.14}$$

where

$$\langle \mathbb{U} \rangle^L = \frac{2}{\Delta x} \int_{-\frac{\Delta x}{2}}^{0} \mathbb{U}_{app}(x/\Delta t)dx = \frac{2\Delta t}{\Delta x} \int_{-\frac{\Delta x}{2\Delta t}}^{0} \mathbb{U}_{app}(\xi)d\xi, \tag{5.15}$$

$$\langle \mathbb{U} \rangle^R = \frac{2}{\Delta x} \int_{0}^{\frac{\Delta x}{2}} \mathbb{U}_{app}(x/\Delta t)dx = \frac{2\Delta t}{\Delta x} \int_{0}^{\frac{\Delta x}{2\Delta t}} \mathbb{U}_{app}(\xi)d\xi. \tag{5.16}$$

Proof. We only prove inequality (5.13) (the proof of (5.14) is similar). By Jensen's inequality, the convexity of the map $\mathbb{U} \mapsto \eta(\mathbb{U})$ (see Proposition 5.2) implies that it is sufficient to prove

$$\frac{2\Delta t}{\Delta x} \int_{-\frac{\Delta x}{2\Delta t}}^{0} \eta\left(\mathbb{U}_{app}(\xi)\right)d\xi - \eta(\mathbb{U}_L) + \frac{2\Delta t}{\Delta x}\left(\mathcal{F}_\eta^r\left(\mathbb{W}_r(0^+)\right) - \mathcal{F}_\eta(\mathbb{U}_L)\right) \le 0, \tag{5.17}$$

under Whitham's condition (5.12). The solution of the Riemann problem (4.1)-(4.2) satisfies (5.11) in the weak sense, where $\mathcal{Q}(u_2^*, \mathbb{W}_L, \mathbb{W}_R)\delta_{x-u_2^*t}$ is a positive measure. Integrating this equation over $]-\frac{\Delta x}{2}, 0[\times]0, \Delta t[$, and dividing by $\frac{\Delta x}{2}$, we get

$$\frac{2\Delta t}{\Delta x} \int_{-\frac{\Delta x}{2\Delta t}}^{0} \eta^r(\mathbb{W}_r(\xi))d\xi - \eta^r(\mathbb{W}_L) + \frac{2\Delta t}{\Delta x}\left(\mathcal{F}_\eta^r(\mathbb{W}_r(0^-)) - \mathcal{F}_\eta^r(\mathbb{W}_L)\right) \le 0. \tag{5.18}$$

Now, as $(\mathbb{W}_L, \mathbb{W}_R) = (\mathscr{M}(\mathbb{U}_L), \mathscr{M}(\mathbb{U}_R))$ are at equilibrium, we have $\eta^r(\mathbb{W}_L) = \eta(\mathbb{U}_L)$ and $\mathcal{F}_\eta^r(\mathbb{W}_L) = \mathcal{F}_\eta(\mathbb{U}_L)$. Moreover, the Riemann solution is constructed such that $\mathcal{F}_\eta^r(\mathbb{W}_r(0^+)) - \mathcal{F}_\eta^r(\mathbb{W}_r(0^-)) \le 0$ (indeed, we have $\mathcal{F}_\eta^r(\mathbb{W}_r(0^+)) - \mathcal{F}_\eta^r(\mathbb{W}_r(0^-)) = 0$ unless $u_2^* = 0$ in which case $\mathcal{F}_\eta^r(\mathbb{W}_r(0^+)) - \mathcal{F}_\eta^r(\mathbb{W}_r(0^-)) = -\mathcal{Q}(u_2^*, \mathbb{W}_L, \mathbb{W}_R) \le 0$). Replacing in (5.18) this yields

$$-\eta(\mathbb{U}_L) + \frac{2\Delta t}{\Delta x}\left(\mathcal{F}_\eta^r\left(\mathbb{W}_r(0^+)\right) - \mathcal{F}_\eta(\mathbb{U}_L)\right) \le -\frac{2\Delta t}{\Delta x}\int_{-\frac{\Delta x}{2\Delta t}}^{0} \eta^r\left(\mathbb{W}_r(\xi)\right)d\xi. \tag{5.19}$$

Hence, a sufficient condition for (5.17) (and thus for (5.13)) to hold true is

$$\frac{2\Delta t}{\Delta x} \int_{-\frac{\Delta x}{2\Delta t}}^{0} \left\{\eta\left(\mathbb{U}_{app}(\xi)\right) - \eta^r\left(\mathbb{W}_r(\xi)\right)\right\}d\xi \le 0. \tag{5.20}$$

Now, for almost every ξ in $\left[-\frac{\Delta x}{2\Delta t}, 0\right]$, we have

$$\eta\left(\mathbb{U}_{app}(\xi)\right) - \eta^r\left(\mathbb{W}_r(\xi)\right) = \sum_{i=1}^{2}(\alpha_i\rho_i)(\xi)\left(e(\tau_i(\xi)) - e_i(\mathcal{T}_i(\xi)) - \frac{1}{2a_i^2}\left(\pi_i^2(\tau_i(\xi), \mathcal{T}_i(\xi)) - \mathcal{P}_i^2(\mathcal{T}_i(\xi))\right)\right). \tag{5.21}$$

Omitting the dependence on ξ, we have for $i = 1, 2$:

$$\begin{aligned}
\pi_i^2(\tau_i, \mathcal{T}_i) - \mathcal{P}_i^2(\mathcal{T}_i) &= \left(\pi_i(\tau_i, \mathcal{T}_i) - \mathcal{P}_i(\mathcal{T}_i)\right)\left(\pi_i(\tau_i, \mathcal{T}_i) + \mathcal{P}_i(\mathcal{T}_i)\right) \\
&= a_i^2(\mathcal{T}_i - \tau_i)\left(2\mathcal{P}_i(\mathcal{T}_i) + a_i^2(\mathcal{T}_i - \tau_i)\right) \\
&= -2a_i^2 e_i'(\mathcal{T}_i)(\mathcal{T}_i - \tau_i) + a_i^4(\mathcal{T}_i - \tau_i)^2,
\end{aligned}$$

since $e_i' = -\mathcal{P}_i$. Hence,

$$e_i(\tau_i) - e_i(\mathcal{T}_i) - \frac{1}{2a_i^2}\left(\pi_i^2(\tau_i, \mathcal{T}_i) - \mathcal{P}_i^2(\mathcal{T}_i)\right) = e_i(\tau_i) - e_i(\mathcal{T}_i) - e_i'(\mathcal{T}_i)(\tau_i - \mathcal{T}_i) - \frac{a_i^2}{2}(\mathcal{T}_i - \tau_i)^2. \tag{5.22}$$

A Taylor expansion with integral remainder gives

$$e_i(\tau_i) - e_i(\mathcal{T}_i) - e_i'(\mathcal{T}_i)(\tau_i - \mathcal{T}_i) = (\mathcal{T}_i - \tau_i)^2 \int_0^1 e_i''(s\tau_i + (1-s)\mathcal{T}_i)(1-s)ds. \tag{5.23}$$

Then, replacing in (5.22) and observing that $e_i'' = -\mathcal{P}_i'$ we get a sufficient condition for (5.17) (and thus for (5.13)):

$$2 \int_0^1 -\mathcal{P}_i'(s\tau_i(\xi) + (1-s)\mathcal{T}_i(\xi))(1-s)ds - a_i^2 \leq 0 \quad \text{for } a.e. \ \xi \text{ in } \left[-\frac{\Delta x}{2\Delta t}, 0\right]. \tag{5.24}$$

Noticing that in the solution $\mathcal{T}_i(\xi) = \tau_{i,L}$ or $\tau_{i,R}$ and using the strict convexity of $\tau \mapsto \mathcal{P}_i(\tau)$, we get for $a.e.$ ξ in $\left[-\frac{\Delta x}{2\Delta t}, 0\right]$:

$$
\begin{aligned}
2 \int_0^1 -\mathcal{P}_i'(s\tau_i(\xi) + (1-s)\mathcal{T}_i(\xi))(1-s)ds &\leq \max_{s \in [0,1]} \left\{ -\mathcal{P}_i'(s\tau_i(\xi) + (1-s)\mathcal{T}_i(\xi)) \right\} 2 \int_0^1 (1-s)ds \\
&\leq \operatorname*{ess\,sup}_{\xi \in [-\frac{\Delta x}{2\Delta t}, \frac{\Delta x}{2\Delta t}]} \left\{ -\mathcal{P}_i'(\tau_i(\xi)) \right\} \\
&< a_i^2
\end{aligned}
\tag{5.25}
$$

by Whitham's condition. This concludes the proof of inequality (5.13) under Whitham's condition. $\square$

We may now prove the following theorem which states that under the CFL condition (5.7), Whitham's condition (5.12) guarantees a discrete entropy inequality for the relaxation finite volume scheme. This is a classical corollary of Lemma 5.3. We refer to [7] for the proof.

Theorem 5.4. *Assume the CFL condition (5.7) and suppose that for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, the pair $(a_1, a_2)_{j+\frac{1}{2}}$ satisfies Whitham's condition for $(\mathbb{U}_j^n, \mathbb{U}_{j+1}^n)$. Then the relaxation scheme satisfies the following discrete entropy inequality:*

$$\eta(\mathbb{U}_j^{n+1}) - \eta(\mathbb{U}_j^n) + \frac{\Delta t}{\Delta x}(H(\mathbb{U}_j^n, \mathbb{U}_{j+1}^n) - H(\mathbb{U}_{j-1}^n, \mathbb{U}_j^n)) \leq 0, \tag{5.26}$$

where the numerical entropy flux is given by $H(\mathbb{U}_L, \mathbb{U}_R) = \mathcal{F}_\eta^r(\mathbb{W}_r(0^+; \mathcal{M}(\mathbb{U}_L), \mathcal{M}(\mathbb{U}_R)))$.

This can be seen as a stability condition because if one considers the discrete L^1-norm of the total mixture energy at time t^n: $\sum_{j \in \mathbb{Z}} \eta(\mathbb{U}_j^n)\Delta x$, then summing inequality (5.26) over the cells yields

$$\sum_{j \in \mathbb{Z}} \eta(\mathbb{U}_j^{n+1})\Delta x \leq \sum_{j \in \mathbb{Z}} \eta(\mathbb{U}_j^n)\Delta x, \quad \text{for all } n \text{ in } \mathbb{N}, \tag{5.27}$$

which means that the total mixture energy is decreasing in time.

5.5 Practical choice of the pair (a_1, a_2)

The pair of parameters (a_1, a_2), which is computed at each interface $x_{j+\frac{1}{2}}$ must be chosen large enough so as to satisfy several requirements:

- In order to ensure the stability of the relaxation approximation, a_i must satisfy Whitham's condition (5.12). For simplicity however, we do not impose Whitham's condition everywhere in the solution of the Riemann problem (2.1)-(2.11), but only for the left and right initial data at each interface:

$$\text{for } i \text{ in } \{1,2\}, \quad a_i^2 > \max(-\mathcal{P}_i'(\tau_{i,L}), -\mathcal{P}_i'(\tau_{i,R})). \tag{5.28}$$

  In practice, no instabilities were observed during the numerical simulations due to this simpler Whitham-like condition.

- The specific volumes $\tau_{i,L}^\sharp$ and $\tau_{i,R}^\sharp$ must be positive. As the initial conditions of the local Riemann problems are such that $(\mathbb{W}_L, \mathbb{W}_R) = (\mathcal{M}(\mathbb{U}_L), \mathcal{M}(\mathbb{U}_R))$, we have $\mathcal{T}_{i,L} = \tau_{i,L}$ and $\mathcal{T}_{i,R} = \tau_{i,R}$. Thus

$$
\begin{aligned}
\tau_{i,L}^\sharp &= \tau_{i,L} + \frac{1}{2a_i}(u_{i,R} - u_{i,L}) - \frac{1}{2a_i^2}\left(\mathcal{P}_i(\tau_{i,R}) - \mathcal{P}_i(\tau_{i,L})\right), \tag{5.29} \\
\tau_{i,R}^\sharp &= \tau_{i,R} + \frac{1}{2a_i}(u_{i,R} - u_{i,L}) + \frac{1}{2a_i^2}\left(\mathcal{P}_i(\tau_{i,R}) - \mathcal{P}_i(\tau_{i,L})\right). \tag{5.30}
\end{aligned}
$$

31

Equations (5.29) and (5.30) are two second order polynomials in a_i^{-1}, and by taking a_i large enough, one can guarantee that $\tau_{i,L}^\sharp > 0$ and $\tau_{i,R}^\sharp > 0$, since the initial specific volumes $\tau_{i,L}$ and $\tau_{i,R}$ are positive.

- Finally, (a_1, a_2) must be chosen so as to meet the condition (**A**) of Theorem 4.2 as well as the positivity condition of the phase 2 densities (**B**).

Thereafter, we propose an iterative algorithm for the computation of the parameters (a_1, a_2) at each interface. $\texttt{Fixedpoint}(a_1, a_2)$ is a subroutine that computes a numerical approximation of the solution u_2^* of the fixed-point problem (4.24), using some numerical method such as Newton's method or a dichotomy algorithm. The notation $\texttt{not}(\mathbf{P})$ is the negation of the logical statement $\mathbf{P}$.

Algorithm for the choice of (a_1, a_2)**:**

```
Choose κ a (small) parameter in the interval (0,1).
```
For all interface $x_{j+\frac12}$, $j \in \mathbb{Z}$, calculate $a_{1,j+\frac12}$ and $a_{2,j+\frac12}$ as follows

- For i in $\{1,2\}$ initialize $a_{i,j+\frac12}$:
$$a_{i,j+\frac12}^2 := (1+\kappa)\max\left(-\mathcal{P}_i'(\tau_{i,j}^n), -\mathcal{P}_i'(\tau_{i,j+1}^n)\right).$$

- For i in $\{1,2\}$, do $\{$ $a_{i,j+\frac12} := (1+\kappa)a_{i,j+\frac12}$

         compute $u_i^\sharp$, $\pi_i^\sharp$, $\tau_{i,L}^\sharp$ and $\tau_{i,R}^\sharp$

         $\}$ while $\left(\tau_{i,L}^\sharp \le 0 \text{ or } \tau_{i,R}^\sharp \le 0\right)$

- do $\{$ $a_{2,j+\frac12} := (1+\kappa)a_{2,j+\frac12}$

     compute $u_2^\sharp$, $\pi_2^\sharp$, $\tau_{2,L}^\sharp$ and $\tau_{2,R}^\sharp$

     do $\{$ $a_{1,j+\frac12} := (1+\kappa)a_{1,j+\frac12}$

         compute $u_1^\sharp$, $\pi_1^\sharp$, $\tau_{1,L}^\sharp$ and $\tau_{1,R}^\sharp$

     $\}$ while $\left(\texttt{not}(\mathtt{A})\right)$

     compute ν, $\mathcal{M}_L^\sharp$ and $\mathcal{P}_L^\sharp$,

     compute $u_2^* = \texttt{Fixedpoint}(a_{1,j+\frac12}, a_{2,j+\frac12})$

$\}$ while $\left(\texttt{not}(\mathtt{B})\right)$

It is possible to prove that this algorithm always converges in the sense that there is non infinite looping due to the while-conditions. Moreover, this algorithm provides reasonable values of a_1 and a_2, since in all the numerical simulations, the time step obtained through the CFL condition (5.7) remains reasonably large and does not go to zero. In fact, the obtained values of a_1 and a_2 are quite satisfying since the relaxation scheme compares very favorably with Rusanov's scheme, in terms of CPU-time performances (see Section 6).

6   Numerical tests

In this section, we present Riemann-type test-cases on which the performances of the relaxation scheme are tested. The phasic equations of state are given by the following ideal gas pressure laws:

$$\begin{aligned} p_1(\rho_1) &= \kappa_1\rho_1^{\gamma_1}, \quad \text{with } \kappa_1 = 1 \text{ and } \gamma_1 = 3,\\ p_2(\rho_2) &= \kappa_2\rho_2^{\gamma_2}, \quad \text{with } \kappa_2 = 1 \text{ and } \gamma_2 = 1.5. \end{aligned} \tag{6.1}$$

In the sequel, $\mathcal{U} = (\alpha_1, \rho_1, u_1, \rho_2, u_2)^T$ denotes the state vector in non-conservative variables.

6.1 Test-case 1: a complete Riemann problem

We consider the following initial data,

$$\mathcal{U}_L = (0.1, 0.85, 0.4609513139, 0.96, 0.0839315299) \qquad \text{if } x < 0,$$
$$\mathcal{U}_R = (0.6, 1.2520240113, 0.7170741165, 0.2505659851, -0.3764790609) \qquad \text{if } x > 0,$$

for which the exact solution is composed, from left to right, of a $\{u_1 - c_1\}$-shock wave, followed by a $\{u_2 - c_2\}$-rarefaction wave, followed by a u_2-contact discontinuity, followed by a $\{u_2 + c_2\}$-shock and finally followed by a $\{u_1 + c_1\}$-rarefaction wave (see Figure 6.1). The intermediate states are given by:

$$\mathcal{U}_1 = (0.1, 1., 0.2, 0.96, 0.0839315299),$$
$$\mathcal{U}_2 = (0.1, 1., 0.2, 0.8, 0.3),$$
$$\mathcal{U}_3 = (0.6, 1.0016192090, 0.2833602765, 0.5011319701, 0.3),$$
$$\mathcal{U}_4 = (0.6, 1.0016192090, 0.2833602765, 0.2505659851, -0.3764790609).$$

At each interface $x_{j+\frac{1}{2}}$, Newton's method is used in order to compute the solution $\mathcal{M}_L^*$ of (4.75). The iterative procedure is stopped when the error is less than 10^{-12}.

In Figure 6.1, the approximate solution computed with the relaxation scheme is compared with both the exact solution and the approximate solution obtained with Rusanov's scheme (a Lax-Friedrichs type scheme see [21]). The results show that unlike Rusanov's scheme, the relaxation method correctly captures the intermediate states even for this rather coarse mesh of 100 cells. This coarse mesh is a typical example of an industrial mesh, reduced to one direction, since 100 cells in 1D correspond to a 10^6-cell mesh in 3D. It appears that the contact discontinuity is captured more sharply by the relaxation method than by Rusanov's scheme for which the numerical diffusion is larger. We can also see that for the phase 1 variables, there are no oscillations as one can see for Rusanov's scheme: the curves are monotone between the intermediate states. As for phase 2, the intermediate states are captured by the relaxation method while with Rusanov's scheme, this weak level of refinement is clearly not enough to capture any intermediate state. These observations assess that, for the same level of refinement, the relaxation method is more accurate than Rusanov's scheme.

A mesh refinement process has also been implemented in order to check numerically the convergence of the method, as well as it's performances in terms of CPU-time cost. For this purpose, we compute the discrete L^1-error between the approximate solution and the exact one at the final time $T = N\Delta t$, normalized by the discrete L^1-norm of the exact solution:

$$E(\Delta x) = \frac{\sum_{\text{cells}j} |\varphi_j^N - \varphi_{ex}(x_j, T)| \Delta x}{\sum_{\text{cells}j} |\varphi_{ex}(x_j, T)| \Delta x}, \tag{6.2}$$

where φ is any of the **conservative** variables $(\alpha_1, \alpha_1\rho_1, \alpha_1\rho_1 u_1, \alpha_2\rho_2, \alpha_2\rho_2 u_2)$. The calculations have been implemented on several meshes composed of 100×2^n cells with $n = 0, 1, .., 10$ (knowing that the domain size is $L = 1$). In Figure 6.1, the error $E(\Delta x)$ at the final time $T = 0.14$, is plotted against Δx in a $log - log$ scale. Only the error on the phase fraction α_1 converges towards zero with the expected order of $\Delta x^{1/2}$, while the other variables seem to converge with a higher rate. However, $\Delta x^{1/2}$ is only an asymptotic order of convergence, and in this particular case, one would have to implement the calculation on much more refined meshes in order to reach the theoretically expected order of $\Delta x^{1/2}$.

Figure 6.1 also displays the error on the conservative variables with respect to the CPU-time of the calculation expressed in seconds. Each point of the plot corresponds to one single calculation for a given mesh size (going from 400 to 102400 cells for the relaxation scheme and from 800 to 102400 cells for Rusanov's scheme). One can see that, for all the variables except $\alpha_1\rho_1 u_1$, if one prescribes a given level of the error, the computational cost of Rusanov's scheme is significantly higher than that of the relaxation method. For instance, for the same error on the phase fraction α_1, the gain in computational cost is more than 13 when using the relaxation method rather than Rusanov's scheme. For the variable $\alpha_1\rho_1 u_1$, even if the two methods seem to provide similar results, the relaxation method seems to give slightly better results for mesh sizes beyond 10000 cells.
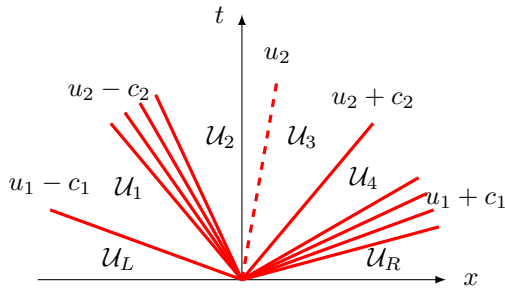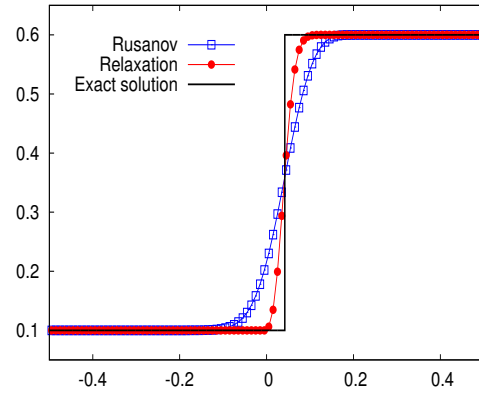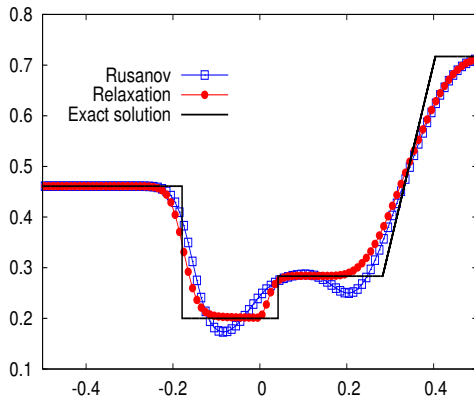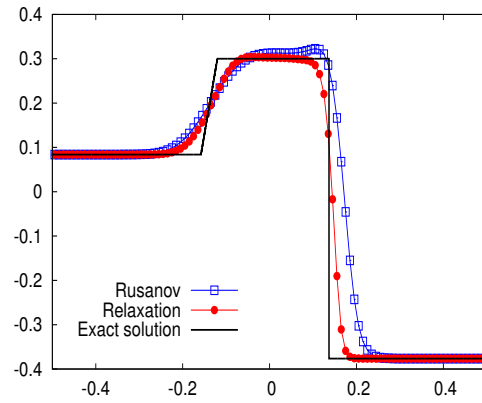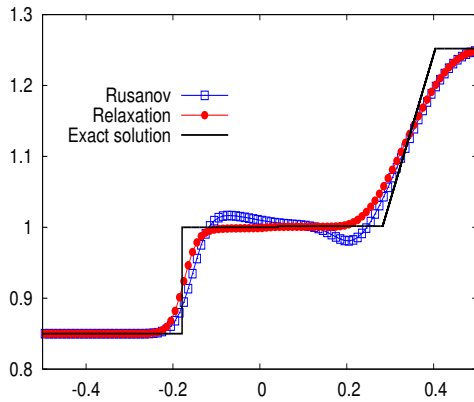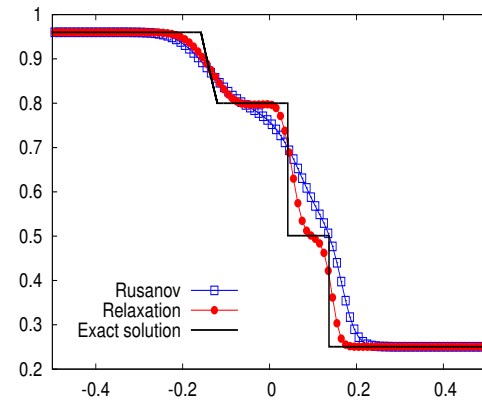
Figure 1: Test-case 1: Structure of the solution and space variations of the physical variables at the final time $T = 0.14$. Mesh size: 100 cells.

6.2 Test-case 2: a vanishing phase case

We now consider a Riemann problem in which one of the two phases vanishes in one of the initial states, which means that the corresponding phase fraction α_1 or α_2 is equal to zero. For this kind of Riemann problem, the u_2-contact separates a mixture region where the two phases coexist from a
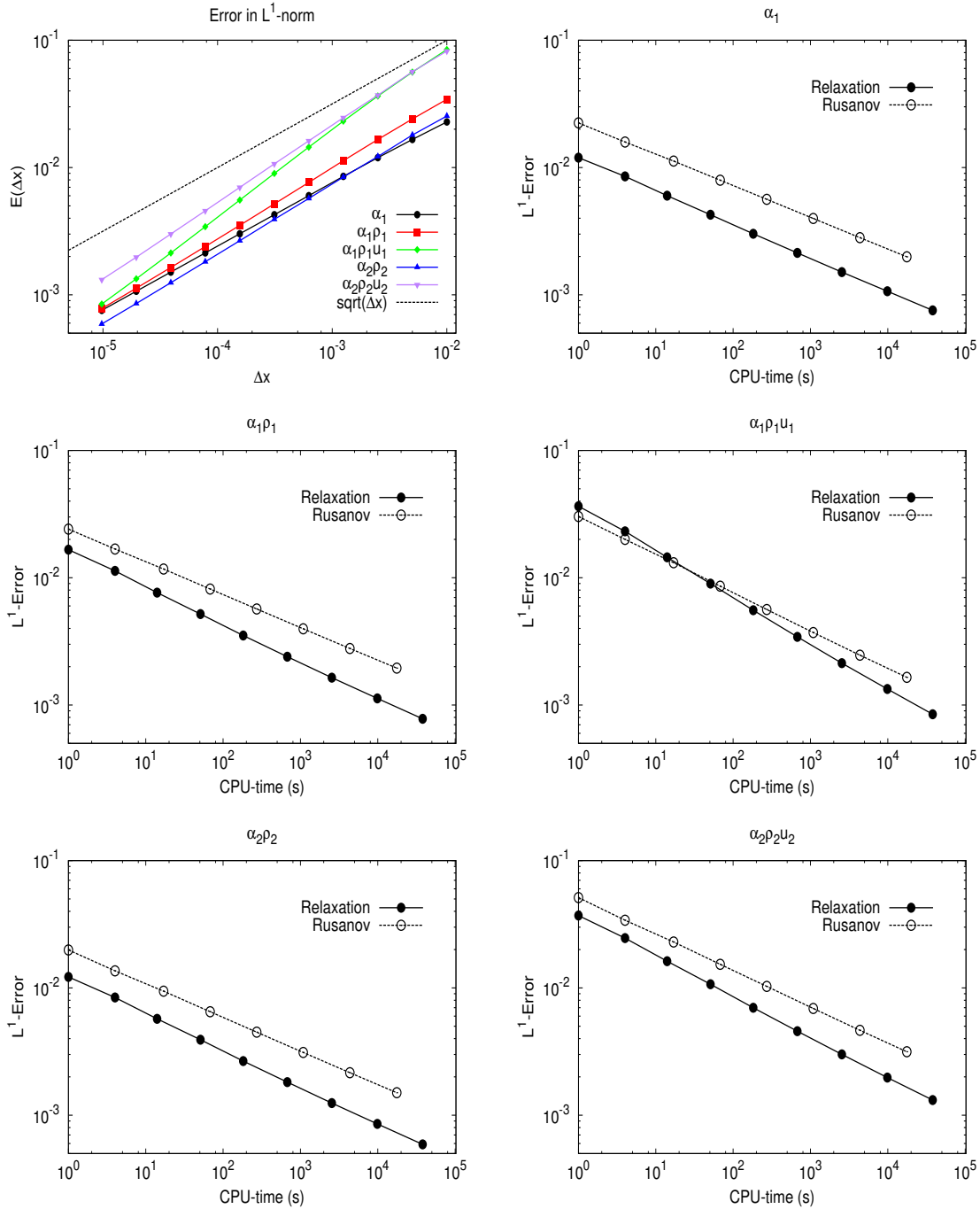
Figure 2: Test-case 1: L^1-Error with respect to Δx and L^1-Error with respect to computational cost (in seconds), for the conservative variables.

single phase region with the remaining phase. Assuming for instance that $\alpha_{1,L} = 1$ and $0 < \alpha_{1,R} < 1$, the right state is a mixture of both phases while the left initial state is composed solely of phase 1. This type of vanishing-phase Riemann solution is introduced in [36] in the more general context of the complete Baer-Nunziato system with energy equations. The construction of an exact solution with

vanishing phase fraction is not classical and we choose to follow the natural approach given in [36]. For more details, see appendix A.1. Consider the intermediate states

$$\mathcal{U}_L = (1.0, 1.8, 0.747051068928543, 3.979765198025580, 0.6),$$
$$\mathcal{U}_1 = (1.0, 2.0, 0.4, 3.979765198025580, 0.6),$$
$$\mathcal{U}_2 = (0.4, 1.982040094756841, 0.095469338564172, 3.979765198025580, 0.6),$$
$$\mathcal{U}_3 = (0.4, 1.9820400948, 0.0954693386, 5.1736947574, 1.0690676047),$$
$$\mathcal{U}_R = (0.4, 2.081142099494683, 0.267119045902047, 5.173694757433254, 1.069067604724276).$$

The solution is composed of a $\{u_1 - c_1\}$-shock wave from $\mathcal{U}_L$ to $\mathcal{U}_1$ in the left-hand side (LHS) region where only phase 1 is present. This region is separated by a u_2-contact discontinuity from the right-hand side (RHS) region where the two phases are mixed. In this RHS region, the solution is composed of a $\{u_2 + c_2\}$-rarefaction wave connecting $\mathcal{U}_2$ to $\mathcal{U}_3$ followed by a $\{u_1 + c_1\}$-rarefaction wave from $\mathcal{U}_3$ to $\mathcal{U}_R$ (see Figure 6.1).

In practice, the numerical method requires values of $\alpha_{1,L}$ and $\alpha_{1,R}$ that lie strictly in the interval $(0, 1)$. Therefore, in the numerical implementation, we take $\alpha_{1,L} = 1 - 10^{-9}$. The aim here is to give a qualitative comparison between the numerical approximation and the exact solution. Moreover, there is theoretically no need to specify left initial values for the phase 2 quantities since this phase is not present in the LHS region. For the sake of the numerical simulations however, one must provide such values. We choose to set $\rho_{2,L}$ and $u_{2,L}$ to the values on the right of the u_2-contact discontinuity. For the relaxation scheme, this choice enables to avoid oscillations of phase 2 quantities in the region where phase 2 in not present. However, some tests have been conducted that assess that taking other values of $(\rho_{2,L}, u_{2,L})$ has little impact on the phase 1 quantities as well as on the phase 2 quantities where this phase is present.

At each interface $x_{j+\frac{1}{2}}$, a dichotomy algorithm is used in order to compute the solution $\mathcal{M}_L^*$ of (4.75). Indeed for such a vanishing phase test-case, Newton's method fails to converge. The dichotomy algorithm is stopped when the error is less than 10^{-12}. As for the first test-case, we can see that for the same level of refinement, the relaxation method is more accurate than Rusanov's scheme, which can be seen especially for phase 1. As regards the region where phase 2 does not exist, we can see that the relaxation method is much more stable than Rusanov's scheme. Indeed, the relaxation scheme behaves better than Rusanov's scheme when it comes to divisions by small values of α_2, since the solution approximated by Rusanov's scheme develops important oscillations.

6.3 Test-case 3: Coupling between two pure phases

The last test-case considers the coupling between two pure phases. A left region, where only phase 1 exists ($\alpha_{1,L} = 1$), is separated by a u_2-contact discontinuity from a right region, where only phase 2 is present ($\alpha_{1,R} = 0$). The intermediate states are given by

$$\mathcal{U}_L = (1.0, 0.861773876012754, 3.552800564555003, 4.641588833612778, 1.0),$$
$$\mathcal{U}_1 = (1.0, 2.154434690031884, 1., 4.641588833612778, 1.),$$
$$\mathcal{U}_2 = (0., 2.154434690031884, 1., 4.641588833612778, 1.),$$
$$\mathcal{U}_R = (0., 2.154434690031884, 1.0, 6.962383250419167, 1.767119653712349).$$

$$(6.3)$$

The exact solution is composed of a $\{u_1 - c_1\}$-shock wave from $\mathcal{U}_L$ to $\mathcal{U}_1$ in the LHS region where only phase 1 is present. $\mathcal{U}_1$ is connected to $\mathcal{U}_2$ by a u_2-contact discontinuity separating the two pure phase regions. In the RHS region, where only phase 2 exists, $\mathcal{U}_2$ is connected to $\mathcal{U}_R$ by a $\{u_2 + c_2\}$-rarefaction wave. For more details on how the exact solution is constructed see appendix A.2.

In the numerical implementation, we set $\alpha_{1,L} = 1 - 10^{-9}$ and $\alpha_{1,R} = 10^{-9}$. A dichotomy procedure is used in order to compute the solution $\mathcal{M}_L^*$ of (4.75) at each interface $x_{j+\frac{1}{2}}$. The dichotomy algorithm is stopped when the error is less than 10^{-12}.

One can see that, in the LHS region, the quantities of the only present phase 1 are correctly approximated while the quantities of the vanishing phase 2 remain stable despite the division by small
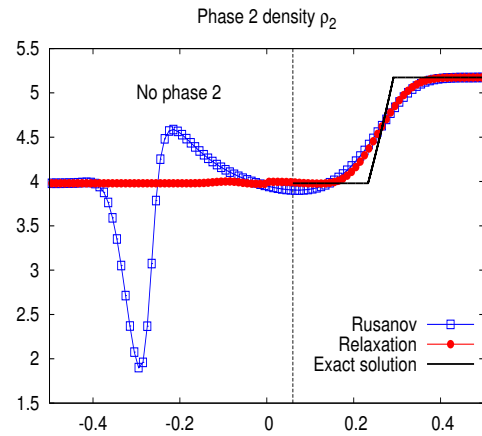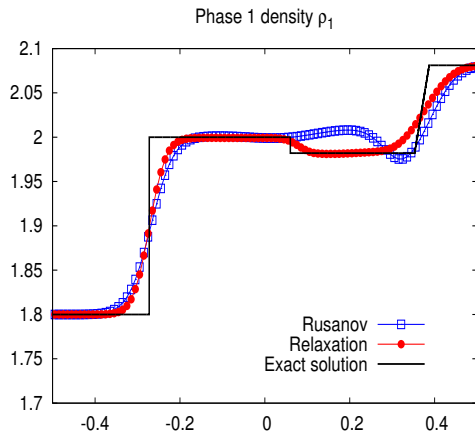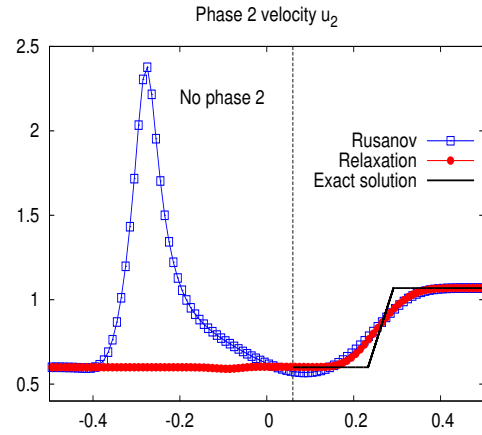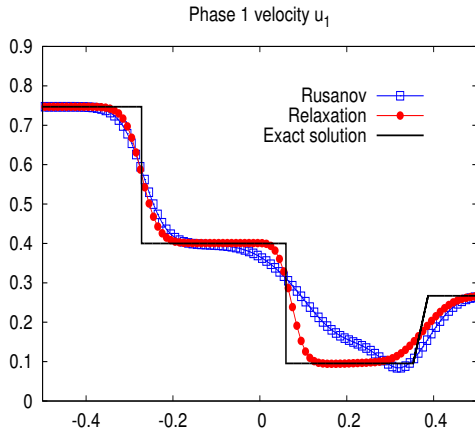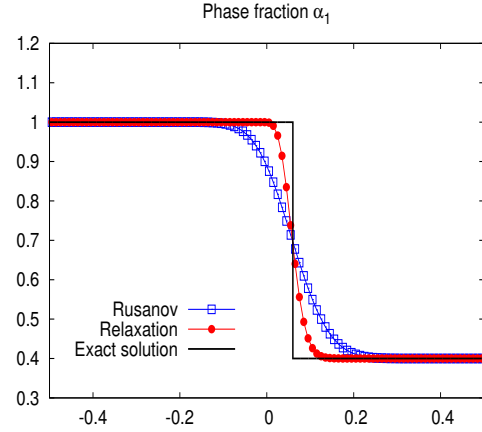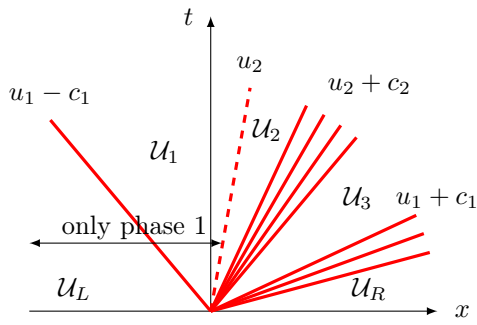
Figure 3: Test-case 2: Structure of the solution and space variations of the physical variables at the final time $T = 0.1$. Mesh size: 100 cells.

values of α_2. The same observation can be made for the RHS region. On the contrary, with Rusanov's scheme, strong oscillations pop up in the regions where a phase vanishes. Observe also that unlike Rusanov's scheme, the relaxation scheme does not fail to correctly approximate the evolution of the phase fraction α_1.
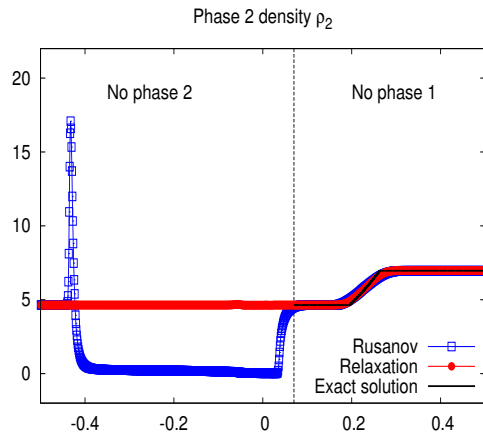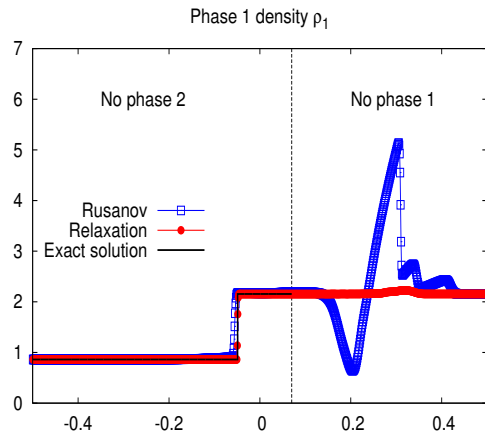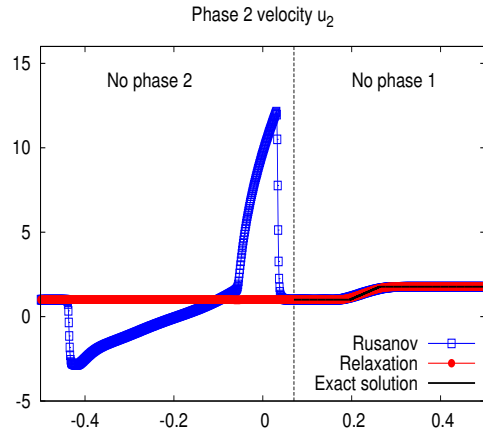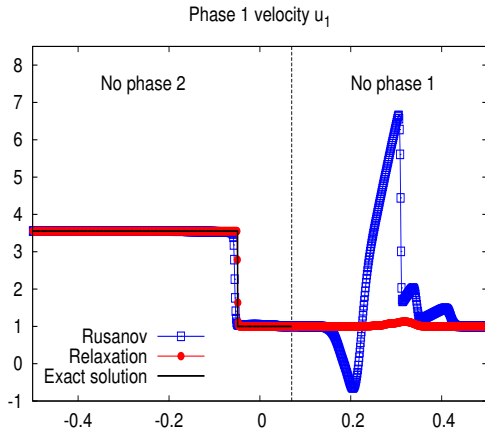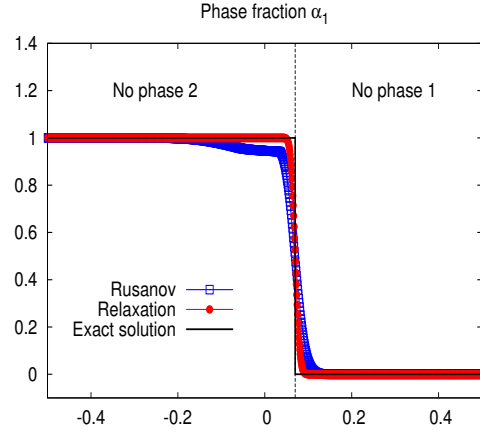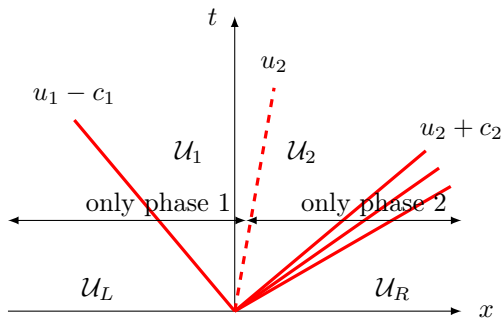
Figure 4: Test-case 3: Structure of the solution and space variations of the physical variables at the final time $T = 0.07$. Mesh size: 1000 cells.

7 Conclusion and further works

The work performed in this paper provides some interesting teachings. First of all, the relaxation approximation introduced in (3.1) seems to be a reasonable choice of continuous approximation for the isentropic Baer-Nunziato model. Indeed, thanks to the linear degeneracy of all the fields, one has

been able to exactly solve the associated relaxation Riemann problem in the framework of solutions with subsonic wave ordering. More surprising is the fact that we have been able to provide explicit necessary and sufficient conditions on the initial left and right data for the existence of such solutions with subsonic wave ordering.

One natural question is the possibility of extending this result to all the wave-orderings that may be considered for the solution, namely, supersonic and sonic wave orderings. The eventual aim would be to construct a partition of the whole space of initial conditions $\Omega^r \times \Omega^r$, where each element of the partition would be characterized by explicit conditions similar to ($\mathbf{A}$) that indicate the wave-ordering of the solution for a given pair $(\mathbb{W}_L, \mathbb{W}_R)$. Unfortunately, performing the study of all the wave-orderings is not a trivial task because it may involve δ-shock type solutions due to the linear resonance phenomena (see [8]) when a (linearized) acoustic field interacts with the transport wave u_2. As a matter of fact, Dirac solutions have already been observed for the relaxation system (4.33) in the context of Euler's equations in nozzle (see [17]). For this simpler system, which is a major building block in the resolution of the relaxation Riemann problem (4.1)-(4.2) for the Baer-Nunziato model, the Riemann problem has been completely solved for any initial data, and δ-shock solutions have been introduced for sonic flows.

As for Euler's equations in a nozzle, the relaxation approximation introduced here provides a simple Riemann solver of Harten, Lax and van Leer, which enables a stable treatment of vanishing phase cases. Indeed, thanks to the introduction of a generalized class of solutions for which energy dissipation through the linearly degenerate phase fraction wave has been allowed, Riemann problems with arbitrarily small initial phase fractions may be computed by the solver. Introducing energy dissipation in the first step of the relaxation approximation is an original idea since all the fields of the relaxation system are linearly degenerate. Another more natural choice would be to increase the relaxation parameters (a_1, a_2) thus introducing some dissipation in the method but this, contrary to our approach, would restrict the time-step to potentially small values in the regimes of vanishing phases. In addition, the relaxation Riemann solver is proved to preserve positive densities and to satisfy a discrete entropy inequality which guarantees the stability of the derived scheme. To our knowledge, there exists no other scheme that is proved to satisfy these two properties. The numerical tests prove the good behavior of the scheme in vanishing phase cases. Moreover, for a given error on the solution, it is shown that the relaxation scheme compares much favorably with Rusanov's scheme in terms of computational cost.

An important sequel of this work is the extension of the method to the full Baer-Nunziato model with energies thanks to the energy/entropy duality. Indeed, thanks to the second principle of thermodynamics which connects the phasic energies and the transported phasic entropies, one is able to extend this Riemann solver to the full model with energies through minor adaptations (see [13, 7] for a related framework). This is explained in [34] and is the purpose of a forthcoming article, together with the multi-dimension extension of the method.

A   Appendices

A.1   Vanishing-phase solution of test-case 2

The vanishing phase solution of test-case 2 is constructed as follows. First of all, two states $\mathcal{U}^-$ and $\mathcal{U}^+$ are constructed so as to be connected by a u_2-contact discontinuity with $\alpha_1^- = \alpha_{1,L} = 1$ and $\alpha_1^+ = \alpha_{1,R} = 0.4$ ($\mathcal{U}^-$ $resp.$ $\mathcal{U}^+$ is denoted $\mathcal{U}_1$ $resp.$ $\mathcal{U}_2$ in (6.2)). For this purpose, the jump relations (2.15)-(2.16)-(2.17)-(2.18) associated with the contact discontinuity reduce to

$$u_2^- = u_2^+,$$
$$\rho_1^-(u_1^- - u_2^-) = \alpha_1^+ \rho_1^+(u_1^+ - u_2^+),$$
$$\rho_1^-(u_1^- - u_2^-)u_1^- + p_1(\rho_1^-) = \alpha_1^+ \rho_1^+(u_1^+ - u_2^+)u_1^+ + \alpha_1^+ p_1(\rho_1^+) + \alpha_2^+ p_2(\rho_2^+),$$
$$\rho_1^-(u_1^- - u_2^-)E_1^- + p_1(\rho_1^-)u_1^- = \alpha_1^+ \rho_1^+(u_1^+ - u_2^+)E_1^+ + \alpha_1^+ p_1(\rho_1^+)u_1^+ + \alpha_2^+ p_2(\rho_2^+)u_2^+,$$

since $\alpha_1^- = 1$ and $\alpha_2^- = 0$. Thanks to these jump relations, given the values $\rho_1^- = 2.0$, $u_1^- = 0.4$ and $u_2^- = 0.6$ (no value of ρ_2^- is needed), we compute the values of ρ_1^+, u_1^+, ρ_2^+ and u_2^+ which are given in the intermediate state $\mathcal{U}_2$ in (6.2). The value of ρ_2^- is then imposed to be equal to ρ_2^+. Then the state $\mathcal{U}^- = \mathcal{U}_1$ is connected on its left with the state $\mathcal{U}_L$ through a $\{u_1 - c_1\}$-shock. The state $\mathcal{U}_2$ is connected to $\mathcal{U}_3$ through a $\{u_2 + c_2\}$-rarefaction wave, and finally $\mathcal{U}_3$ is connected to $\mathcal{U}_R$ through a $\{u_1 + c_1\}$-rarefaction wave.

A.2 Coupling solution of test-case 3

The same procedure is implemented for the construction of the exact solution of test-case 3, which corresponds to a coupling between a pure phase 1 ($\alpha_1^- = \alpha_{1,L} = 1$) on the right and a pure phase 2 ($\alpha_1^+ = \alpha_{1,R} = 0$) on the left. This time, the jump relations of the u_2-contact discontinuity reduce to

$$
\begin{aligned}
&u_2^- = u_2^+, \\
&\rho_1^- (u_1^- - u_2^-) = 0, \\
&p_1(\rho_1^-) = p_2(\rho_2^+), \\
&p_1(\rho_1^-)u_1^- = p_2(\rho_2^+)u_2^+.
\end{aligned}
$$

A solution is given by $u_2^- = u_2^+ = u_1^- = 1.0$ and $\rho_1^- = p_1^{-1}(p)$, $\rho_2^+ = p_2^{-1}(p)$ where $p = p_1^- = p_2^+ = 10$ is the common pressure. The values of ρ_1^+ and u_1^+, which are of no importance since phase 1 in not present on the right of the contact discontinuity, are taken equal to ρ_1^- and u_1^-. Similarly, we set $\rho_2^- := \rho_2^+$. This concludes the construction of the intermediate states $\mathcal{U}^- = \mathcal{U}_1$ and $\mathcal{U}^+ = \mathcal{U}_2$. Then the state $\mathcal{U}_1$ is connected on its left with the state $\mathcal{U}_L$ through a $\{u_1 - c_1\}$-shock and the state $\mathcal{U}_2$ is connected to $\mathcal{U}_R$ through a $\{u_2 + c_2\}$-rarefaction wave.

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