An easy control of the artificial numerical viscosity to get discrete entropy inequalities when approximating hyperbolic systems of conservation laws

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Abstract When considering the numerical approximation of weak solutions of systems of conservation laws, the derivation of discrete entropy inequalities are, in general, very difficult to obtain. In the present work, we present a suitable control of the numerical artificial viscosity in order to recover the expected discrete entropy inequalities. Moreover, the artificial viscosity control turns out to be very easy and the resulting numerical implementation is very convenient.

1 Main motivations

The present work is devoted to the numerical approximation of the weak solutions of hyperbolic systems made of $N \geq 1$ conservation laws in the form

$$\partial_t w + \partial_x f(w) = 0, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1}$$

where $w \in \mathbb{R}^N$ stands for the unknown vector and $f : \mathbb{R}^N \to \mathbb{R}^N$ denotes the flux function. According to some physical restriction, the solution may be imposed to belong to an invariant admissible domain $\Omega \subset \mathbb{R}^N$, so that we have $w \in \Omega$. The model is complemented with an initial data $w(x, t = 0) = w_0(x)$ where $w_0 : \mathbb{R} \to \mathbb{R}^N$ is given.

Since the system (1) is assumed to be hyperbolic, in a finite time the solution may contain discontinuities. Such discontinuities are governed by the well-known

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Rankine-Hugoniot relations (for instance, see [12]). Unfortunately, by only adopting theses relations, the uniqueness of the solution is lost. In order to rule out nonphysical discontinuities, the system is endowed with entropy inequalities given by

$$\partial_t \eta(w) + \partial_x G(w) \leq 0, \tag{2}$$

where $\eta : \Omega \to \mathbb{R}$ is a convex function, called the entropy function, and $G : \Omega \to \mathbb{R}$ is the entropy flux function such that ${}^t\nabla_w f(w) = {}^t\nabla_w \eta(w) \nabla_w G(w)$. A solution of (1) is said entropy satisfying if the entropy inequalities (2) are verified for all entropy pairs (η, G).

Now, let us consider the numerical approximation of w. To address such an issue, we first introduce a discretization of the space by adopting a uniform mesh made of cells $(x_{i-1/2}, x_{i+1/2})$ of constant size $\Delta x > 0$ so that $x_{i+1/2} = x_{i-1/2} + \Delta x$ for all i in $\mathbb{Z}$. We denote $x_i = (x_{i-1/2} + x_{i+1/2})/2$ for i in $\mathbb{Z}$. Concerning the time discretization, we fix $t^{n+1} = t^n + \Delta t$ with $\Delta t > 0$ the time increment restricted according to a suitable CFL condition (for instance, see [12]).

At time t^n, we consider an approximation of the solution given by a piecewise constant function as follows:

$$w_\Delta(x, t^n) = w_i^n \quad \text{if } x \in (x_{i-1/2}, x_{i+1/2}). \tag{3}$$

In order to evolve with respect to time this approximation, the updated states $(w_i^{n+1})_{i\in\mathbb{Z}}$ are evaluated by the following numerical scheme:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(f_\Delta(w_i^n, w_{i+1}^n) - f_\Delta(w_{i-1}^n, w_i^n)\right), \tag{4}$$

where $f_\Delta(w_L, w_R)$ is the numerical flux function. From now on, we underline that this numerical flux function must satisfy $f_\Delta(w, w) = f(w)$ for consistency reasons.

During the fifty last years, numerous works were devoted to propose relevant formulas to define $f_\Delta(w_L, w_R)$. Of course, it is not possible to give an exhaustive list of the numerical methods available to approximate the weak solutions of (1), but the reader is referred to [12, 3] for an overview of some well-known numerical techniques.

For stability reasons, the scheme (4) is complemented with a CFL condition (for instance, see [12, 3]) to restrict the time step by $\frac{\Delta t}{\Delta x}\Lambda \leq \frac{1}{2}$, where $\Lambda \geq 0$ is determined according to the numerical flux function definition.

At this level, according to the celebrated Lax-Wendroff Theorem [7], we may expect a convergence of $w_\Delta(x, t^n)$ to a weak solution $w(x, t)$ of system (1) as Δt and Δx tend to zero. In order to avoid some possible nonphysical solutions, the scheme must be enriched with discrete entropy inequalities. As a consequence, the updated states $(w_i^{n+1})_{i\in\mathbb{Z}}$, given by (4), must satisfy the following estimation:

$$\eta(w_i^{n+1}) - \eta(w_i^n) + \frac{\Delta t}{\Delta x}\left(G_\Delta(w_i^n, w_{i+1}^n) - G_\Delta(w_{i-1}^n, w_i^n)\right) \leq 0, \tag{5}$$

where $G_\Delta(w_L, w_R)$ denotes the numerical entropy flux function. From now on, we underline that this numerical flux function must satisfy $G_\Delta(w, w) = G(w)$ for all w in Ω.

If the discrete entropy inequalities (5) are satisfied by all entropy pairs (η, G), the scheme is said entropy stable or entropy preserving. The establishment of (5) may be very difficult to obtain as soon as $N > 1$. The Godunov scheme [4] or the HLL scheme [5] are known to be entropy preserving. Next, considering the isentropic gas dynamics or the Euler model, some schemes such as the HLLC scheme [13] or the Suliciu relaxation scheme [1] are also proved to be entropy preserving. But, in general, the proof of (5) is not reachable or, eventually, is violated. For instance, the Roe scheme [8] or the VF-Roe scheme [2] are known to be entropy violating and suitable entropy fixes must be adopted (for instance, see [6]).

In this paper, we introduce a simple extension of the artificial viscosity technique [9, 11] in order to easily recover the required entropy stability given by (5). In the next section, we propose a reformulation of the artificial viscosity within the framework of Godunov-type schemes. In Section 3, we give a direct control of the artificial viscosity to get (5) for a given entropy pair. The last section is devoted to illustrate the relevance of the suggested numerical procedure.

2 Godunov-type scheme with artificial viscosity

In order to enforce the required entropy stability, we introduce artificial viscosity into the scheme (4). As a consequence, the improved numerical method of interest now reads

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(f_\Delta(w_i^n, w_{i+1}^n) - f_\Delta(w_{i-1}^n, w_i^n)\right) + \frac{\gamma}{2}\frac{\Delta t}{\Delta x}\left(w_{i+1}^n - 2w_i^n + w_{i-1}^n\right), \quad (6)$$

where $\gamma \geq 0$ is a parameter to govern the artificial viscosity.

Now, we propose to reformulate (6) as a Godunov-type scheme. For this purpose, we introduce the following approximate Riemann solver:

$$w_{\mathcal{R}}\left(\frac{x}{t}; w_L, w_R\right) = \begin{cases} w_L & \text{if} \quad x < -(\Lambda+\gamma)t, \\ \bar{w}_L & \text{if} \quad -(\Lambda+\gamma)t < x < -\Lambda t, \\ w_L^\star & \text{if} \quad -\Lambda t < x < 0, \\ w_R^\star & \text{if} \quad 0 < x < \Lambda t, \\ \bar{w}_R & \text{if} \quad \Lambda t < x < (\Lambda+\gamma)t, \\ w_R & \text{if} \quad x > (\Lambda+\gamma)t, \end{cases} \quad (7)$$

where we have set

$$\bar{w}_L = w_L + \frac{1}{2}(w_R - w_L) = \frac{1}{2}(w_L + w_R), \quad \bar{w}_R = w_R - \frac{1}{2}(w_R - w_L) = \frac{1}{2}(w_L + w_R),$$
$$w_L^\star = w_L - \frac{1}{\Lambda}\left(f_\Delta(w_L, w_R) - f(w_L)\right), \quad w_R^\star = w_R + \frac{1}{\Lambda}\left(f_\Delta(w_L, w_R) - f(w_R)\right). \tag{8}$$

To simplify the notations, we set $\bar{w} = \bar{w}_L = \bar{w}_R$. Next, equipped with this approximate Riemann solver, we evolve the approximate solution (3) in times as follows:

$$w_\Delta(x, t^n + t) = w_\mathcal{R}\left(\frac{x - x_{i+1/2}}{t}; w_i^n, w_{i+1}^n\right) \qquad \text{for } (x,t) \in [x_i, x_{i+1}) \times (0, \Delta t),$$

where Δt is restricted by

$$\frac{\Delta t}{\Delta x}(\Lambda + \gamma) \leq \frac{1}{2}. \tag{9}$$

Such a time evolution coincides to a juxtaposition of approximate Riemann solvers $w_\mathcal{R}$ stated at each interface $x_{i+1/2}$. Because of the CFL condition (9), two successive approximate Riemann solvers never interact. After a straightforward computation, we notice that the updated states w_i^{n+1}, given by (6), reformulate as follows:

$$\begin{aligned} w_i^{n+1} &= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} w_\Delta(x, t^n + \Delta t)dx, \\ &= \frac{1}{\Delta x}\int_0^{\Delta x/2} w_\mathcal{R}\left(\frac{x}{\Delta t}; w_{i-1}^n, w_i^n\right)dx + \frac{1}{\Delta x}\int_{-\Delta x/2}^{0} w_\mathcal{R}\left(\frac{x}{\Delta t}; w_i^n, w_{i+1}^n\right)dx, \end{aligned} \tag{10}$$

Because of the Godunov-type reformulation, we are now able to control the artificial viscosity, parameterized by γ, in order to obtain the expected discrete entropy inequality (5).

3 Discrete entropy inequality

To establish the required entropy stability (5), we first recall a result stated by Harten, Lax and van Leer in [5]. In fact, this statement gives a local sufficient condition, interface per interface, that implies (5). The reader is referred to [5] for a proof.

Lemma 1 (Harten, Lax and van Leer [5])

Let w_L and w_R be given in Ω. Assume that the approximate Riemann solver satisfies

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \eta\left(w_\mathcal{R}\left(\frac{x}{\Delta t}; w_L, w_R\right)\right)dx \leq \frac{1}{2}\left(\eta(w_L) + \eta(w_R)\right) - \frac{\Delta t}{\Delta x}\left(G(w_R) - G(w_L)\right), \tag{11}$$

for a given entropy pair (η, G). Then, the Godunov-type scheme (10) satisfies a discrete entropy inequality (5) where the numerical entropy flux function reads as *follows:*

$$G_\Delta(w_L, w_R) = G(w_R) - \frac{\Delta x}{2\Delta t}\eta(w_R) + \frac{1}{\Delta x}\int_0^{\Delta x/2} \eta\left(w_{\mathcal{R}}\left(\frac{x}{\Delta t}; w_L, w_R\right)\right) dx.$$

From now on, let us underline that, as soon as the approximate Riemann solver is given by (7), the numerical entropy flux function rewrites

$$G_\Delta(w_L, w_R) = G(w_R) + \Lambda\eta(w_R^\star) + (\Lambda + \gamma)(\eta(\bar{w}) - \eta(w_R)).$$

Now, arguing Lemma 1, it is sufficient to satisfy (11) to obtain (5). After a straightforward computation, the inequality (11) reads as follows:

$$\mathcal{E}_0 + \gamma\mathcal{D} \leq 0, \tag{12}$$

where we have set

$$\mathcal{E}_0 = \Lambda\left(\eta(w_L^\star) + \eta(w_R^\star) - \eta(w_L) - \eta(w_R)\right) + \left(G(w_R) - G(w_L)\right), \tag{13}$$

$$\mathcal{D} = 2\eta(\bar{w}) - \eta(w_L) - \eta(w_R). \tag{14}$$

Let us emphasize that $\mathcal{E}_0$ coincides with the entropy dissipation rate for the initial scheme (4) when the artificial viscosity γ vanishes. Of course $\mathcal{E}_0$ turns out to be negative as long as the scheme (4) is entropy preserving. But $\mathcal{E}_0$ may become positive for an entropy violating scheme. Next, since η is a convex function, we immediately get $\mathcal{D} \leq 0$ with equality to zero if and only if $w_L = w_R$.

Moreover, we remark that neither $\mathcal{E}_0$ nor $\mathcal{D}$ depend on γ. As a consequence, as soon as $\mathcal{D} \neq 0$, it is possible to fix $\gamma \geq 0$ large enough such that the local entropy inequality (12) holds.

Lemma 2 *Let w_L and w_R be given in Ω. Then there exists $\gamma \geq 0$ such that (12) holds. Moreover, assume that the matrix $\nabla_w^2\eta(w)$ is positive definite then γ is bounded.*

Proof The existence of $\gamma \geq 0$ is obvious. Now, we establish that γ is bounded in a neighborhood, denoted $\mathcal{V}$, of $w_L = w_R$. Indeed, since $\mathcal{D}$ vanishes only into $\mathcal{V}$, γ is immediately bounded far away from $\mathcal{V}$. First, we give the expansion with respect of $\bar{w}$ of $\mathcal{D}$, given by (14), in $\mathcal{V}$. A direct evaluation gives $\mathcal{D} = \mathcal{O}(\| w_R - w_L \|^2)$ with optimal order since $\nabla_w^2\eta(\bar{w})$ is positive definite. Because of the convexity of η, never this quantity vanishes in $\mathcal{V}$ as long as $w_L \neq w_R$. Concerning the expansion of $\mathcal{E}_0$, we first introduce

$$w^{HLL} = \frac{1}{2}(w_L + w_R) - \frac{1}{2\Lambda}(f(w_R) - f(w_L)).$$

Following [5], we have $2\eta(w^{HLL}) \leq \eta(w_L) + \eta(w_R) - G(w_R) + G(w_L)$, with equality if and only if $w_L = w_R$. Then we obtain $\mathcal{E}_0/\Lambda \leq \eta(w_L^\star) + \eta(w_R^\star) - 2\eta(w^{HLL})$. Next, we notice that $w_L^\star + w_R^\star = 2w^{HLL}$, then we have $\eta(w_L^\star) + \eta(w_R^\star) - 2\eta(w^{HLL}) = \mathcal{O}(\| w_R - w_L \|^2)$, with equality to zero if and only if $w_L = w_R$. As a consequence, we

Table 1 Evaluation of L^1-norm of the positive part of the entropy budget.

Cells	φ_1	φ_2	φ_3
100	1.05291E-6	4.40092E-7	1.48806E-6
800	5.53458E-8	2.30603E-8	7.90713E-8
1600	3.42990E-8	1.42867E-8	4.90515E-8
12800	8.27230E-9	3.44473E-8	1.18418E-8
102400	1.23514E-9	5.14317E-9	1.76836E-9

have $\mathcal{E}_0 = \mathcal{O}(\| w_L - w_L \|^2)$ once again with optimal order since $\nabla_w^2 \eta(w^{HLL})$ is positive definite. It results $\gamma = \mathcal{O}(1)$ and the proof is achieved. □

4 Numerical results

To illustrate the procedure we adopt the Euler model, defined by $w = {}^t(\rho, \rho u, E)$ and $f(w) = {}^t(\rho u, \rho u^2 + p, (E + p)u)$, where p is defined by the usual perfect gas law [12] with adiabatic coefficient fixed equal to 1.4. Here, the entropy function is given by $\eta(w) = \rho\varphi(p/\rho^{1.4})$, where φ is a smooth function which satisfies the restrictions determined in [10].

In Fig 1, we display the numerical results obtained when simulating a Riemann problem with initial data given by $w_0(x) = {}^t(1, 0, 2.5)$ if $x < 0$ and $w_0(x) = {}^t(0.125, 0, 0.025)$ if $x > 0$. The adopted initial scheme is given by the VF-Roe method [2], known to be entropy violating. Then, we notice a nonphysical shock wave at the sonic point within the rarefaction. Next, we introduce the numerical artificial viscosity governed by γ fixed according to (12). We remark that the entropy violating shock wave no longer remains. In addition, Fig 1 displays the values of γ versus time. As expected, γ is bounded.

To conclude this numerical experiment, let us underline that the control of the artificial viscosity is performed according to a single entropy pair. Here, we have adopted the entropy defined with $\varphi_1(\theta) = \ln(\theta)$. As a consequence, a natural question arising concerns the behavior of the other entropies. In Tab 1, we present the value of L^1-norm of the positive part of the entropy budget, defined as the left hand side of inequality (5), obtained for two other entropies defined by $\varphi_2(\theta) = -\theta^{1/2.4}$ and $\varphi_3(\theta) = \theta^{-2/1.4}$ when considering the original VF-Roe scheme. Since we expect negative entropy dissipation rate, the value of the positive part must be zero. With the original VF-Roe scheme, we get positive values of the entropy dissipation rate (see Tab 1) while we obtain values less than 10^{-16} with the derived single entropy preserving scheme.

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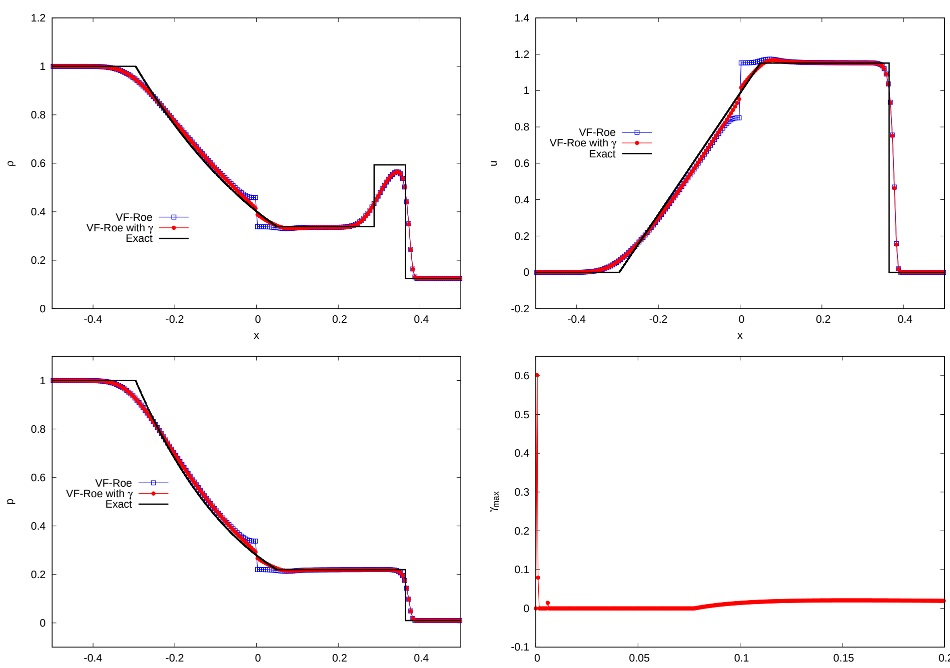

Fig. 1 Riemann solution for the density ρ, the velicity u, the pressure $p$and volution of γ.

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