TWO PROPERTIES OF TWO-VELOCITY TWO-PRESSURE MODELS FOR TWO-PHASE FLOWS

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Abstract. We study a class of models of compressible two-phase flows. This class, which includes the Baer–Nunziato model, is based on the assumption that each phase is described by its own pressure, velocity and temperature and on the use of void fractions obtained from averaging process. These models are nonconservative and non-strictly hyperbolic. We prove that the mixture entropy is non-strictly convex and that the system admits a symmetric form.

Key-words. Two-phase flows, entropy, symmetrizable system.
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1. Introduction

The modeling of compressible two-phase flows is a challenging task in Thermohydraulics. It is a crucial issue for many applications, for instance for water flows in some components of nuclear power plants such as the pressurized water reactors or the steam generators, especially in some specific situations, as the departure from nucleate boiling or the loss of coolant accident. When dealing with highly heterogeneous and disturbed flows, it is now commonly accepted that averaged models have to be considered. However, there is no consensus on the "good" model to use, especially when focusing on the two-fluid approach, where it is assumed that state variables within each phase (namely pressure, velocity and temperature) should not be confused. When restricting to the latter two-fluid framework, one can distinguish the Baer–Nunziato model [1] among the huge literature on the modeling of two-phase flows, both from the mathematical and physical point (see for instance [2, 5, 7, 8, 15] and references therein). Indeed, this system almost has the expected structure: if we only consider its convective part (i.e. the first order differential terms), its eigenvalues are always real and the associated eigenvectors are linearly independent except for some sonic cases (this is the resonance phenomenon). Even more, according to some choices of closure laws, it is possible to provide a wave-by-wave study, in spite of its nonconservative structure, and obtain some positive results on the solution of the associated Riemann problem [5, 3, 6].

In this work, we propose to investigate two properties which are crucial in the theory of nonlinear hyperbolic partial differential equations: the convexity of the entropy and the existence of a symmetric form. While such properties are very well understood for systems of conservation laws since Godunov [10] and Mock [14], it remains an open question for nonconservative and non-strictly hyperbolic systems, such as the Baer–Nunziato model. Equipped with these properties, it is possible to pursue our study of the Baer–Nunziato models towards the Cauchy problem, which will be the subject of forthcoming works.

Actually we will discuss these two properties, not restricting to the exact Baer-Nunziato model, and we will consider a larger framework of two-phase flow models,

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introduced in [6], that contains the former model [1].

2. The two-velocity two-pressure models

We focus on the first-order part of models for two-phase compressible flows [6] which can be written under the form

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) + \mathbf{c}(\mathbf{u})\partial_x \alpha_1 = 0 \tag{2.1}$$

where

$$\begin{aligned}
\mathbf{u}^T &= (\alpha, \alpha\mathbf{u}_1^T, (1-\alpha)\mathbf{u}_2^T), & \mathbf{f}(\mathbf{u})^T &= (0, \alpha\mathbf{f}_1(\mathbf{u}_1)^T, (1-\alpha)\mathbf{f}_2(\mathbf{u}_2)^T),\\
\mathbf{u}_k^T &= (\rho_k, \rho_k u_k, \rho_k E_k), & \mathbf{f}_k(\mathbf{u}_k)^T &= (\rho_k u_k, \rho_k u_k^2/2 + p_k, u_k(\rho_k E_k + p_k)),\\
\alpha &= \alpha_1 = 1 - \alpha_2, & \mathbf{c}(\mathbf{u})^T &= (v_i, 0, -p_i, -v_i p_i, 0, p_i, v_i p_i),
\end{aligned}$$

with $k = 1, 2$. The notations are classical: α_k is the void fraction, ρ_k the density, u_k the velocity, p_k the pressure and E_k the specific total energy of the phase k, with $k = 1, 2$. Besides, v_i and p_i, usually called the interfacial velocity and the interfacial pressure, are given functions of $\mathbf{u}$. The total energies are defined by

$$E_k = \varepsilon_k + \frac{u_k^2}{2},$$

where ε_k denotes the specific internal energy of the phase k. We assume in the sequel that each phase admits an entropy s_k, complying with

$$T_k ds_k = d\varepsilon_k + p_k d\tau_k, \tag{2.2}$$

noting T_k the temperature and $\tau_k = 1/\rho_k$ the specific volume of the phase k. The knowledge of these entropies enables us to deduce the temperature and the pressure of each phase:

$$\frac{\partial s_k}{\partial \tau_k}(\tau_k, \varepsilon_k) = \frac{p_k}{T_k}(\tau_k, \varepsilon_k) \quad \text{and} \quad \frac{\partial s_k}{\partial \varepsilon_k}(\tau_k, \varepsilon_k) = \frac{1}{T_k}(\tau_k, \varepsilon_k).$$

Besides, we assume that *each entropy s_k is a strictly concave function of τ_k and ε_k*. Moreover, one can define the sound speeds c_k by

$$\rho_k (c_k)^2 = \left(\frac{p_k}{\rho_k} - \rho_k(\partial_{\rho_k}\varepsilon_k)(\rho_k, p_k)\right)\big((\partial_{p_k}\varepsilon_k)(\rho_k, p_k)\big)^{-1}.$$

Let us recall the hyperbolicity property of system (2.1) for solutions in the set of admissible states

$$\Omega = \{\mathbf{u} \in \mathbb{R}^7; \alpha \in (0,1), \rho_k > 0, \varepsilon_k > 0, k = 1, 2\}. \tag{2.3}$$

Proposition 2.1. *System (2.1) admits seven real eigenvalues on Ω: v_i, u_k and $u_k \pm c_k$, $k = 1, 2$. The corresponding eigenvectors form a basis of $\mathbb{R}^7$ as soon as*

$$(u_k - v_i)^2 \neq (c_k)^2, \quad k = 1, 2. \tag{2.4}$$

This is called the non resonance condition.

3. Non strict convexity of the mixture entropy

Let us introduce the entropy pair (S_k, F_k) defined by

$$S_k(\mathbf{u}_k) = -\rho_k s_k \quad \text{and} \quad F_k'(\mathbf{u}_k) = (S_k'(\mathbf{u}_k))^T \mathbf{f}_k'(\mathbf{u}_k), \tag{3.1}$$

associated with the Euler systems $\partial_t \mathbf{u}_k + \partial_x \mathbf{f}_k(\mathbf{u}_k) = 0$.

It is classical to define the entropy of the two-phase model (2.1) by

DEFINITION 3.1. *The* mixture entropy *for system (2.1) is*

$$S(\mathbf{u}) = \alpha S_1(\mathbf{u}_1) + (1-\alpha) S_2(\mathbf{u}_2), \tag{3.2}$$

and the associated mixture entropy flux

$$F(\mathbf{u}) = \alpha F_1(\mathbf{u}_1) + (1-\alpha) F_2(\mathbf{u}_2). \tag{3.3}$$

These definitions are classical and may lead to a *conservative* entropy inequality under some assumptions on p_i, see [3, 6].

We state now the crucial property of the mixture entropy:

THEOREM 3.2. *The mixture entropy S is a non strictly convex function of $\mathbf{u}$.*

Proof. With a slight abuse of notation, let us rewrite the mixture entropy S as a function of $(\alpha, \alpha\mathbf{u}_1, (1-\alpha)\mathbf{u}_2)$:

$$S(\alpha, \alpha\mathbf{u}_1, (1-\alpha)\mathbf{u}_2) = \alpha S_1\Big(\frac{\alpha\mathbf{u}_1}{\alpha}\Big) + (1-\alpha) S_2\Big(\frac{(1-\alpha)\mathbf{u}_2}{1-\alpha}\Big).$$

Then, the Hessian matrix of S with respect to $(\alpha, \alpha u_1, (1-\alpha)u_2)$ has the form

$$S''(\mathbf{u}) = \begin{pmatrix} A & B^T & C^T \\ B & \frac{1}{\alpha} S_1''(\mathbf{u}_1) & 0 \\ C & 0 & \frac{1}{1-\alpha} S_2''(\mathbf{u}_2) \end{pmatrix}$$

with

$$A = \frac{1}{\alpha} \mathbf{u}_1^T S_1''(\mathbf{u}_1)\mathbf{u}_1 + \frac{1}{1-\alpha} \mathbf{u}_2^T S_2''(\mathbf{u}_2)\mathbf{u}_2,$$
$$B = -\frac{1}{\alpha} S_1''(\mathbf{u}_1)\mathbf{u}_1 \quad \text{and} \quad C = \frac{1}{1-\alpha} S_2''(\mathbf{u}_2)\mathbf{u}_2.$$

Let (a, b^T, c^T) be a non-null vector of $\mathbb{R}^7$. Let us check that the Hessian S'' is positive as soon as S_1'' and S_2'' are positive. We have

$$(a, b^T, c^T)\ S''(\mathbf{u}) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a^2 A + aB^T b + aC^T c$$
$$+ ab^T B + \frac{1}{\alpha} b^T S_1''(\mathbf{u}_1) b + ac^T C + \frac{1}{1-\alpha} c^T S_2''(\mathbf{u}_2) c.$$

Using the definitions of A, B and C we obtain

$$(a, b^T, c^T)\ S''(\mathbf{u}) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{\alpha}(b - a\mathbf{u}_1)^T S_1''(\mathbf{u}_1)(b - a\mathbf{u}_1)$$
$$+ \frac{1}{1-\alpha}(c + a\mathbf{u}_2)^T S_2''(\mathbf{u}_2)(c + a\mathbf{u}_2).$$

This right-hand side is clearly nonnegative since S_1 and S_2 are strictly convex, which concludes the proof. ▯

Let us mention that the case of the degeneracy of $S''(\mathbf{u})$ corresponds to

$$(a, b^T, c^T)\ S''(\mathbf{u}) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Longleftrightarrow (a, b, c) = a(1, \mathbf{u}_1, -\mathbf{u}_2). \tag{3.4}$$

As a consequence, for any state $\mathbf{u} = (\alpha, \alpha\mathbf{u}_1, (1-\alpha)\mathbf{u}_2) \in \Omega$, the degeneracy manifold of $S''(\mathbf{u})$ is

$$\mathscr{D}(\mathbf{u}) = \{\mathbf{v} \in \Omega; \mathbf{v} = (\beta, \beta\mathbf{u}_1, (1-\beta)\mathbf{u}_2)), \beta \in (0,1) \setminus \{\alpha\}\},$$

since a vector $\mathbf{v} - \mathbf{u}$ may be written as $a(1, \mathbf{u}_1, -\mathbf{u}_2)$, $a \in \mathbb{R}$, if and only if $\mathbf{v} \in \mathscr{D}(\mathbf{u})$. In other words, $S(\mathbf{v}) = S(\mathbf{u}) + S'(\mathbf{u})^T(\mathbf{v} - \mathbf{u})$ for all $\mathbf{v} \in \mathscr{D}(\mathbf{u})$, i.e. a variation of the void fraction α gives rise to a linear modification of the entropy.

4. The system is symmetrizable out of resonance

Definition 4.1. *The system (2.1) is said to be* symmetrizable *if there exists a $\mathcal{C}^1$-diffeomorphism from $\mathbb{R}^7$ to $\mathbb{R}^7$ $\varphi\colon \mathbf{u} \mapsto \mathbf{y}$, a symmetric positive definite matrix $P(\mathbf{y}) \in \mathbb{R}^{7\times 7}$ and a symmetric matrix $Q(\mathbf{y}) \in \mathbb{R}^{7\times 7}$ such that the smooth solutions of system (2.1) satisfy*

$$P(\mathbf{y})\partial_t\mathbf{y} + Q(\mathbf{y})\partial_x\mathbf{y} = 0. \tag{4.1}$$

Theorem 4.2. *System (2.1) is symmetrizable if and only if the non resonance condition (2.4) holds.*

Proof. Let us define $\mathbf{y} = \varphi(\mathbf{u}) := (\alpha_2, u_2, p_2, s_2, u_1, p_1, s_1)^t$ and introduce the partial masses $m_k = \alpha_k\rho_k$. One may check by classical manipulations that the smooth solutions of system (2.1) satisfy

$$\partial_t\mathbf{y} + M(\mathbf{y})\partial_x\mathbf{y} = 0$$

where

$$M = \begin{pmatrix} v_i & 0 & 0 \\ M_{2\alpha} & M_2 & 0 \\ M_{1\alpha} & 0 & M_1 \end{pmatrix},\ M_k = \begin{pmatrix} u_k & \tau_k & 0 \\ \rho_k(c_k)^2 & u_k & 0 \\ 0 & 0 & u_k \end{pmatrix}$$

and

$$\begin{aligned} M_{k\alpha}^T &= \left((-1)^k\frac{p_k - p_i}{m_k}, M_{k\alpha}^{(2)}, M_{k\alpha}^{(3)}\right), \\ M_{k\alpha}^{(2)} &= (-1)^k\frac{u_k - v_i}{\alpha_k}\left(\rho_k(c_k)^2 + \frac{p_i - p_k}{\rho_k}(\partial_{p_k}\varepsilon_k)^{-1}\right), \\ M_{k\alpha}^{(3)} &= (-1)^k(u_k - v_i)\frac{p_i - p_k}{m_k T_k}. \end{aligned}$$

The proof we provide here is constructive for the sake of understanding. We seek for a matrix of symmetrization $P(\mathbf{y})$ of the form

$$P = \begin{pmatrix} P_\alpha & P_{2\alpha}^T & P_{1\alpha}^T \\ P_{2\alpha} & P_2 & 0 \\ P_{1\alpha} & 0 & P_1 \end{pmatrix},\ P_k = \begin{pmatrix} (\rho_k c_k)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the associated symmetric convection matrix is $Q(\mathbf{y}) = P(\mathbf{y})M(\mathbf{y})$. We have to find P such that it is positive definite and that Q is symmetric. Let us first focus on this latter condition. We have

$$Q = \begin{pmatrix} P_\alpha v_i + P_{2\alpha}^T M_{2\alpha} + P_{1\alpha}^T M_{1\alpha} & P_{2\alpha}^T M_2 & P_{1\alpha}^T M_1 \\ v_i P_{2\alpha} + P_2 M_{2\alpha} & P_2 M_2 & 0 \\ v_i P_{1\alpha} + P_1 M_{1\alpha} & 0 & P_1 M_1 \end{pmatrix}.$$

This matrix is symmetric if and only if we have for $k = 1, 2$

$$(M_k^T - v_i \mathbb{I}) P_{k\alpha} = P_k M_{k\alpha}, \tag{4.2}$$

where $\mathbb{I}$ is the 3×3 identity matrix. Assume now that

$$\delta_k := (u_k - v_i)^2 - (c_k)^2 \neq 0,$$

i.e. inequality (2.4) holds. As a consequence, the first two equations of system (4.2) can be solved:

$$\begin{pmatrix} P_{k\alpha}^{(1)} \\ P_{k\alpha}^{(2)} \end{pmatrix} = \frac{1}{\delta_k} \begin{pmatrix} u_k - v_i & -\rho_k (c_k)^2 \\ -\tau_k & u_k - v_i \end{pmatrix} \begin{pmatrix} (-1)^k \rho_k (c_k)^2 \frac{p_k - p_i}{\alpha_k} \\ M_{k\alpha}^{(2)} \end{pmatrix}. \tag{4.3}$$

It remains to solve the third equation of system (4.2), which writes

$$(u_k - V_I) P_{k\alpha}^{(3)} = (-1)^k (u_k - v_i) \frac{p_i - p_k}{m_k T_k}. \tag{4.4}$$

Clearly, this equation admits a unique solution if $u_k \neq v_i$ and an infinity of solutions if $u_k = v_i$. Therefore, if $P_{k\alpha}$ is defined by (4.3) and (4.4), the matrices P and Q are symmetric.

Let us now check that P is a positive definite matrix, that is to say for all non-null vector (a, b_2^T, b_1^T) of $\mathbb{R}^7$, we have

$$(a, b_2^T, b_1^T)\ P \begin{pmatrix} a \\ b_2 \\ b_1 \end{pmatrix} = a^2 P_\alpha + 2a(P_{2\alpha}^T b_2 + P_{1\alpha}^T b_1) + b_2^T P_2 b_2 + b_1^T P_1 b_1 > 0.$$

This is a polynomial of degree 2 in a and its discriminant is

$$\begin{aligned} \Delta &= 4\Big[|P_{2\alpha}^T b_2 + P_{1\alpha}^T b_1|^2 - P_\alpha (b_2^T P_2 b_2 + b_1^T P_1 b_1)\Big] \\ &= 4\Big[\big|(P_2^{-1/2} P_{2\alpha})^T \bar{b}_2 + (P_1^{-1/2} P_{1\alpha})^T \bar{b}_1\big|^2 - P_\alpha (|\bar{b}_2|^2 + |\bar{b}_1|^2)\Big] \\ &= 4\Big[\big(|P_2^{-1/2} P_{2\alpha}|^2 + |P_1^{-1/2} P_{1\alpha}|^2 - P_\alpha\big)(|\bar{b}_2|^2 + |\bar{b}_1|^2) \\ &\qquad - \big|(P_2^{-1/2} P_{2\alpha})^T \bar{b}_1 - (P_1^{-1/2} P_{1\alpha})^T \bar{b}_2\big|^2\Big] \end{aligned}$$

where $\bar{b}_k = P_k^{1/2} b_k$ (as usual, $P_k^{1/2}$ is the symmetric positive definite matrix such that $P_k^{1/2} P_k^{1/2} = P_k$ and $P_k^{-1/2}$ is its inverse). The discriminant Δ is positive if

$$P_\alpha > |P_2^{-1/2} P_{2\alpha}|^2 + |P_1^{-1/2} P_{1\alpha}|^2, \tag{4.5}$$

which is realizable under the condition of non resonance (2.4). □

5. Consequences and further works

First of all, it is worth noting that Theorems 3.2 and 4.2 have been obtained for system (2.1) without any assumption on the definitions of v_i and p_i. Moreover, these results have been obtained for any admissible equations of state within each phase. It is then straightforward to extend these results to similar models (such as the barotropic Baer-Nunziato model, or the extended model introduced in [4]) and it also seems to be a reasonable assumption for the model discussed in [7]. Moreover, we have restricted ourselves to the one-dimensional case, but since this model is invariant under frame rotation (under some natural conditions on v_i and p_i), these properties are still verified in the multidimensional setting.

A first consequence of the existence of the symmetric form (4.1) is that, far from resonance, there exists a local-in-time smooth solution to the Cauchy problem. This is a direct application of Kato's theorem [13]. Of course, the blow-up in finite time still holds, but with the additional restriction due to the non resonance condition (2.4). Let us mention that the resonance phenomenon prevents us from proving a well-posedness result in a weaker setting (as entropy weak solutions with small total variation), since the Riemann problem is known to admit up to three solutions [12, 9].

Nonetheless, we must recall here that the full version of two-phase models also includes source terms which govern the trend of a flow to converge towards equilibrium: equality of pressures, velocities, temperatures and chemical potential. These respective equilibria correspond to mechanic, kinematic, thermal and thermodynamical equilibrium, and thus they tend to remove the occurence of the resonance phenomenon, since condition (2.4) is expected to be satisfied when an equilibrium is not far from being reached. Such source terms are entropy-dissipative, that is to say that their contribution to the mixture entropy balance law is non-positive (see for instance [4]). As a consequence, one may wonder if they may help to obtain a global-in-time solution to the Cauchy problem, following [11] and [16]. Even if these two papers are dedicated to systems of conservation laws, the analysis relies on the use of the entropy (and equivalently in the conservation case, upon the symmetric form of the equations). This work is under investigation.

Eventually, these properties may also be used for computational purposes.

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