LOW MACH NUMBER LIMIT OF A PRESSURE CORRECTION MAC SCHEME FOR COMPRESSIBLE BAROTROPIC FLOWS

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ABSTRACT. We study the incompressible limit of a pressure correction MAC scheme [3] for the unstationary compressible barotropic Navier-Stokes equations. Provided the initial data are well-prepared, the solution of the numerical scheme converges, as the Mach number tends to zero, towards the solution of the classical pressure correction *inf-sup* stable MAC scheme for the incompressible Navier-Stokes equations.

1. INTRODUCTION

Let Ω be parallelepiped of $\mathbb{R}^d$, with $d \in \{2,3\}$ and $T > 0$. The unsteady barotropic compressible Navier-Stokes equations, parametrized by the Mach number ε, read for $(\boldsymbol{x}, t) \in \Omega \times (0,T)$:

$$\partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon\, \boldsymbol{u}^\varepsilon) = 0, \tag{1a}$$

$$\partial_t(\rho^\varepsilon\, \boldsymbol{u}^\varepsilon) + \mathbf{div}(\rho^\varepsilon\, \boldsymbol{u}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) - \mathbf{div}(\boldsymbol{\tau}(\boldsymbol{u}^\varepsilon)) + \frac{1}{\varepsilon^2}\, \boldsymbol{\nabla} \wp(\rho^\varepsilon) = 0, \tag{1b}$$

$$\boldsymbol{u}^\varepsilon|_{\partial\Omega} = 0, \qquad \rho^\varepsilon|_{t=0} = \rho_0^\varepsilon, \qquad \boldsymbol{u}^\varepsilon|_{t=0} = \boldsymbol{u}_0^\varepsilon, \tag{1c}$$

where $\rho^\varepsilon > 0$ and $\boldsymbol{u}^\varepsilon = (u_1^\varepsilon, .., u_d^\varepsilon)^T$ are the density and velocity of the fluid. The pressure satisfies the ideal gas law $\wp(\rho^\varepsilon) = (\rho^\varepsilon)^\gamma$, with $\gamma \geq 1$, and

$$\mathbf{div}(\boldsymbol{\tau}(\boldsymbol{u})) = \mu \Delta \boldsymbol{u} + (\mu + \lambda) \boldsymbol{\nabla}(\operatorname{div} \boldsymbol{u}),$$

where the real numbers μ and λ satisfy $\mu > 0$ and $\mu + \lambda > 0$. The smooth solutions of (1) are known to satisfy a kinetic energy balance and a renormalization identity. In addition, under assumption on the initial data, it may be inferred from these estimates that the density ρ^ε tends to a constant $\bar\rho$, and the velocity tends, in a sense to be defined, to a solution $\bar u$ of the incompressible Navier-Stokes equations [4]:

$$\operatorname{div} \bar{\boldsymbol{u}} = 0, \tag{2a}$$

$$\bar\rho \partial_t \bar{\boldsymbol{u}} + \bar\rho\, \mathbf{div}(\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}) - \mu \boldsymbol{\Delta} \bar{\boldsymbol{u}} + \boldsymbol{\nabla} \pi = 0, \tag{2b}$$

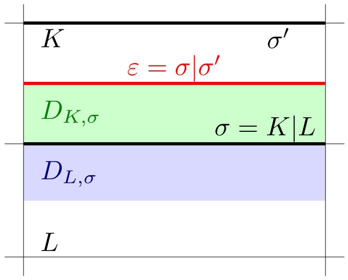
where π is the formal limit of $(\wp(\rho^\varepsilon) - \wp(\bar\rho))/\varepsilon^2$.

In this paper, we reproduce this theory for a pressure correction scheme, based on the Marker-And-Cell (MAC) space discretization: we first derive discrete analogues of the kinetic energy and renormalization identities, then establish from these relations that approximate solutions of (1) converge, as $\varepsilon \to 0$, towards the solution of the classical projection scheme for the incompressible Navier-Stokes equations (2).

For this asymptotic analysis, we assume that the initial data is "well prepared": $\rho_0^\varepsilon > 0$, $\rho_0^\varepsilon \in \mathrm{L}^\infty(\Omega)$, $\boldsymbol{u}_0^\varepsilon \in \mathrm{H}_0^1(\Omega)^d$ and, taking without loss of generality $\bar\rho = 1$, there exists C independent of ε such that:

$$\|\boldsymbol{u}_0^\varepsilon\|_{\mathrm{H}^1(\Omega)^d} + \frac{1}{\varepsilon}\, \|\operatorname{div} \boldsymbol{u}_0^\varepsilon\|_{\mathrm{L}^2(\Omega)} + \frac{1}{\varepsilon^2}\, \|\rho_0^\varepsilon - 1\|_{\mathrm{L}^\infty(\Omega)} \leq C. \tag{3}$$

Consequently, ρ_0^ε tends to 1 when $\varepsilon \to 0$; moreover, we suppose that $\boldsymbol{u}_0^\varepsilon$ converges in $\mathrm{L}^2(\Omega)^d$ towards a function $\bar{\boldsymbol{u}}_0 \in \mathrm{L}^2(\Omega)^d$ (the uniform boundedness of the sequence in the $\mathrm{H}^1(\Omega)^d$ norm already implies this convergence up to a subsequence).



primal cells: K, L.
dual cell for the y-component of the velocity: $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$.
primal and dual cells d-dimensional measures: $|K|$, $|D_\sigma|$, $|D_{K,\sigma}|$.

faces $(d-1)$-dimensional measures: $|\sigma|$, $|\varepsilon|$.

vector normal to σ outward K: $\boldsymbol{n}_{K,\sigma}$.

FIGURE 1. Notations for control volumes and faces.

2. The numerical scheme

Let $\mathcal{M}$ be a MAC mesh (see e.g. [1] and Figure 1 for the notations). The discrete density unknowns are associated with the cells of the mesh $\mathcal{M}$, and are denoted by $\{\rho_K,\ K \in \mathcal{M}\}$. We denote by $\mathcal{E}$ the set of the faces of the mesh, and by $\mathcal{E}^{(i)}$ the subset of the faces orthogonal to the i-th vector of the canonical basis of $\mathbb{R}^d$. The discrete i^{th} component of the velocity is located at the centre of the faces $\sigma \in \mathcal{E}^{(i)}$, so the whole set of discrete velocity unknowns reads $\{u_{\sigma,i},\ \sigma \in \mathcal{E}^{(i)}, 1 \le i \le d\}$. We define $\mathcal{E}_{\rm ext} = \{\sigma \in \mathcal{E}, \sigma \subset \partial\Omega\}$, $\mathcal{E}_{\rm int} = \mathcal{E} \setminus \mathcal{E}_{\rm ext}$, $\mathcal{E}^{(i)}_{\rm int} = \mathcal{E}_{\rm int} \cap \mathcal{E}^{(i)}$ and $\mathcal{E}^{(i)}_{\rm ext} = \mathcal{E}_{\rm ext} \cap \mathcal{E}^{(i)}$. The boundary conditions (1c) are taken into account by setting $u_{\sigma,i} = 0$ for all $\sigma \in \mathcal{E}^{(i)}_{\rm ext}$, $1 \le i \le d$. Let $\delta t > 0$ be a constant time step. The approximate solution $(\rho^n, \boldsymbol{u}^n)$ at time $t_n = n\delta t$ for $1 \le n \le N = \lfloor T/\delta t \rfloor$ is computed as follows: knowing $\{\rho_K^{n-1}, \rho_K^n, K \in \mathcal{M}\} \subset \mathbb{R}$ and $(u^n_{\sigma,i})_{\sigma \in \mathcal{E}^{(i)}_{\rm int}, 1 \le i \le d} \subset \mathbb{R}$, find $(\rho_K^{n+1})_{K \in \mathcal{M}} \subset \mathbb{R}$ and $(u^{n+1}_{\sigma,i})_{\sigma \in \mathcal{E}^{(i)}_{\rm int}, 1 \le i \le d} \subset \mathbb{R}$ by the following algorithm:

Pressure gradient scaling step:

$$\text{For } 1 \le i \le d,\ \forall \sigma \in \mathcal{E}^{(i)}_{\rm int}, \quad (\overline{\boldsymbol{\nabla} p})^n_{\sigma,i} = \Big(\frac{\rho^n_{D_\sigma}}{\rho^{n-1}_{D_\sigma}}\Big)^{1/2} (\boldsymbol{\nabla} p^n)_{\sigma,i}. \tag{4a}$$

Prediction step – Solve for $\tilde{\boldsymbol{u}}^{n+1}$:

For $1 \le i \le d$, $\forall \sigma \in \mathcal{E}^{(i)}_{\rm int}$,

$$\frac{1}{\delta t}\big(\rho^n_{D_\sigma} \tilde u^{n+1}_{\sigma,i} - \rho^{n-1}_{D_\sigma} u^n_{\sigma,i}\big) + {\rm div}(\rho^n \tilde u^{n+1}_i \boldsymbol{u}^n)_\sigma - {\rm div}\boldsymbol{\tau}(\tilde{\boldsymbol{u}}^{n+1})_{\sigma,i} + \frac{1}{\varepsilon^2}(\overline{\boldsymbol{\nabla} p})^n_{\sigma,i} = 0. \tag{4b}$$

Correction step – Solve for ρ^{n+1} and $\boldsymbol{u}^{n+1}$:

For $1 \le i \le d$, $\forall \sigma \in \mathcal{E}^{(i)}_{\rm int}$,

$$\frac{1}{\delta t}\ \rho^n_{D_\sigma}\ (u^{n+1}_{\sigma,i} - \tilde u^{n+1}_{\sigma,i}) + \frac{1}{\varepsilon^2}\,(\boldsymbol{\nabla} p^{n+1})_{\sigma,i} - \frac{1}{\varepsilon^2}\,(\overline{\boldsymbol{\nabla} p})^n_{\sigma,i} = 0, \tag{4c}$$

$$\forall K \in \mathcal{M}, \quad \frac{1}{\delta t}(\rho^{n+1}_K - \rho^n_K) + {\rm div}(\rho^{n+1}\boldsymbol{u}^{n+1})_K = 0, \tag{4d}$$

$$\forall K \in \mathcal{M}, \quad p^{n+1}_K = \wp(\rho^{n+1}), \tag{4e}$$

where the discrete densities and space operators are defined below (see also [3, 2]).

Mass convection flux – Given a discrete density field $\rho = \{\rho_K,\ K \in \mathcal{M}\}$, and a velocity field $\boldsymbol{u} = \{u_{\sigma,i},\ \sigma \in \mathcal{E}^{(i)},\ 1 \le i \le d\}$, the convection term in (4d) reads:

$$ {\rm div}(\rho\boldsymbol{u})_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}(\rho, \boldsymbol{u}), \qquad K \in \mathcal{M}, \tag{5}$$

2000 *Mathematics Subject Classification.* 35Q30,65N12,76M12.
Key words and phrases. Compressible Navier-Stokes equations, low Mach number flows, finite volumes, MAC scheme, staggered discretizations.

where $F_{K,\sigma}(\rho, \boldsymbol{u})$ stands for the mass flux across σ outward K. This flux is set to 0 on external faces to account for the homogeneous Dirichlet boundary conditions; it is given on internal faces by:

$$F_{K,\sigma}(\rho, \boldsymbol{u}) = |\sigma| \ \rho_\sigma \ u_{K,\sigma}, \qquad \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L, \tag{6}$$

where $u_{K,\sigma} = u_{\sigma,i}\ \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}^{(i)}$, with $\boldsymbol{e}^{(i)}$ the i-th vector of the orthonormal basis of $\mathbb{R}^d$. The density at the face $\sigma = K|L$ is approximated by the upwind technique, *i.e.* $\rho_\sigma = \rho_K$ if $u_{K,\sigma} \geq 0$ and $\rho_\sigma = \rho_L$ otherwise.

Pressure gradient term – In (4a) and (4c), the term $(\boldsymbol{\nabla} p)_{\sigma,i}$ stands for the i^{th} component of the discrete pressure gradient at the face σ. Given a discrete density field $\rho = \{\rho_K,\ K \in \mathcal{M}\}$, this term is defined as:

$$(\boldsymbol{\nabla} p)_{\sigma,i} = \frac{|\sigma|}{|D_\sigma|} (\wp(\rho_L) - \wp(\rho_K))\ \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}^{(i)}, \quad 1 \leq i \leq d,\ \sigma \in \mathcal{E}_{\rm int}^{(i)},\ \sigma = K|L. \tag{7}$$

Defining for all $K \in \mathcal{M}$, $({\rm div}\boldsymbol{u})_K = {\rm div}(1 \times \boldsymbol{u})_K$ (see (5)), the following discrete duality relation holds for all discrete density and velocity fields $(\rho, \boldsymbol{u})$:

$$\sum_{K \in \mathcal{M}} |K| \wp(\rho_K)\ ({\rm div}\boldsymbol{u})_K + \sum_{i=1}^{d} \sum_{\sigma \in \mathcal{E}_{\rm int}^{(i)}} |D_\sigma|\ u_{\sigma,i}\ (\boldsymbol{\nabla} p)_{\sigma,i} = 0. \tag{8}$$

The MAC scheme is *inf-sup* stable: there exists $\beta > 0$, depending only on Ω and the regularity of the mesh, such that, for all $p = \{p_K,\ K \in \mathcal{M}\}$, there exists $\boldsymbol{u} = \{u_{\sigma,i},\ \sigma \in \mathcal{E}^{(i)},\ 1 \leq i \leq d\}$ satisfying homogeneous Dirichlet boundary conditions with:

$$\|\boldsymbol{u}\|_{1,\mathcal{M}} = 1 \text{ and } \sum_{K \in \mathcal{M}} |K|\, p_K\, ({\rm div}\boldsymbol{u})_K \geq \beta\, \|p - \frac{1}{|\Omega|} \int_\Omega p\, {\rm d}\boldsymbol{x}\|_{L^2(\Omega)},$$

where $\|\boldsymbol{u}\|_{1,\mathcal{M}}$ is the usual discrete H^1-norm of $\boldsymbol{u}$ (see [1]).

Velocity convection operator – Given a density field $\rho = \{\rho_K,\ K \in \mathcal{M}\}$, and two velocity fields $\boldsymbol{u} = \{u_{\sigma,i},\ \sigma \in \mathcal{E}^{(i)},\ 1 \leq i \leq d\}$ and $\boldsymbol{v} = \{v_{\sigma,i},\ \sigma \in \mathcal{E}^{(i)},\ 1 \leq i \leq d\}$, we build for each $\sigma \in \mathcal{E}_{\rm int}$ the following quantities:

- an approximation of the density on the dual cell ρ_{D_σ} defined as:

$$|D_\sigma|\, \rho_{D_\sigma} = |D_{K,\sigma}|\, \rho_K + |D_{L,\sigma}|\, \rho_L, \qquad \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L, \tag{9}$$

- a discrete divergence for the convection on the dual cell D_σ:

$${\rm div}(\rho v_i \boldsymbol{u})_\sigma = \sum_{\varepsilon \in \bar{\mathcal{E}}(D_\sigma)} F_{\sigma,\varepsilon}(\rho, \boldsymbol{u})\ v_{i,\varepsilon}, \qquad \sigma \in \mathcal{E}_{\rm int}^{(i)},\ 1 \leq i \leq d.$$

  For $i \in \{1, .., d\}$, and $\sigma \in \mathcal{E}_{\rm int}^{(i)}$, $\sigma = K|L$,

  - If the vector $\boldsymbol{e}^{(i)}$ is normal to ε, ε is included in a primal cell K, and we denote by σ' the second face of K which, in addition to σ, is normal to $\boldsymbol{e}^{(i)}$. We thus have $\varepsilon = D_\sigma | D_{\sigma'}$. Then the mass flux through ε is given by:

$$F_{\sigma,\varepsilon}(\rho, \boldsymbol{u}) = \frac{1}{2} \big(F_{K,\sigma}(\rho, \boldsymbol{u})\ \boldsymbol{n}_{D_\sigma,\varepsilon} \cdot \boldsymbol{n}_{K,\sigma} + F_{K,\sigma'}(\rho, \boldsymbol{u})\ \boldsymbol{n}_{D_\sigma,\varepsilon} \cdot \boldsymbol{n}_{K,\sigma'} \big). \tag{10}$$

  - If the vector $\boldsymbol{e}^{(i)}$ is tangent to ε, ε is the union of the halves of two primal faces τ and τ' such that $\tau \in \mathcal{E}(K)$ and $\tau' \in \mathcal{E}(L)$. The mass flux through ε is then given by:

$$F_{\sigma,\varepsilon}(\rho, \boldsymbol{u}) = \frac{1}{2}\ \big(F_{K,\tau}(\rho, \boldsymbol{u}) + F_{L,\tau'}(\rho, \boldsymbol{u}) \big). \tag{11}$$

With this definition, the dual fluxes are locally conservative through dual faces $\varepsilon = D_\sigma | D_{\sigma'}$ (*i.e.* $F_{\sigma,\varepsilon}(\rho, \boldsymbol{u}) = -F_{\sigma',\varepsilon}(\rho, \boldsymbol{u})$), and vanish through a dual face included in the boundary of Ω. For this reason, the values $v_{\varepsilon,i}$ are only needed at the internal dual faces, and are chosen centered, *i.e.*, for $\varepsilon = D_\sigma | D_{\sigma'}$, $v_{\varepsilon,i} = (v_{\sigma,i} + v_{\sigma',i})/2$.

As a result, a finite volume discretization of the mass balance (1a) holds over the internal dual cells. Indeed, if $\rho^{n+1} = \{\rho_K^{n+1},\ K \in \mathcal{M}\}$, $\rho^n = \{\rho_K^n,\ K \in \mathcal{M}\}$ and $\boldsymbol{u}^{n+1} = \{u_{\sigma,i}^{n+1},\ \sigma \in \mathcal{E}^{(i)},\ 1 \leq i \leq d\}$ are density and velocity fields satisfying (4d), then, the dual quantities $\{\rho_{D_\sigma}^{n+1}, \rho_{D_\sigma}^n,\ \sigma \in \mathcal{E}_{\rm int}\}$ and the dual fluxes

$\{F_{\sigma,\varepsilon}(\rho^{n+1},\boldsymbol{u}^{n+1}),\ \sigma\in\mathcal{E}_{\rm int},\ \varepsilon\in\bar{\mathcal{E}}(D_\sigma)\}$ satisfy a finite volume discretization of the mass balance (1a) over the internal dual cells:

$$\frac{|D_\sigma|}{\delta t}(\rho^{n+1}_{D_\sigma}-\rho^n_{D_\sigma})+\sum_{\varepsilon\in\bar{\mathcal{E}}(D_\sigma)}F_{\sigma,\varepsilon}(\rho^{n+1},\boldsymbol{u}^{n+1})=0,\qquad \sigma\in\mathcal{E}_{\rm int}. \tag{12}$$

Diffusion term – The discrete diffusion term in (4b) is defined in [2] and is coercive in the following sense: for every discrete velocity field $\boldsymbol{u}$ satisfying the homogeneous Dirichlet boundary conditions, one has:

$$-\sum_{i=1}^{d}\sum_{\varepsilon\in\mathcal{E}^{(i)}_{\rm int}}|D_\sigma|\ u_{\sigma,i}\ {\rm div}\boldsymbol{\tau}(\boldsymbol{u})_{\sigma,i}\geq\mu\ \|\boldsymbol{u}\|^2_{1,\mathcal{M}}. \tag{13}$$

The initialization of the scheme (4) is performed by setting

$$\forall K\in\mathcal{M}, \rho^0_K=\frac{1}{|K|}\int_K\rho^\varepsilon_0(\boldsymbol{x})\,{\rm d}\boldsymbol{x}\ \text{and}\ \forall\sigma\in\mathcal{E}^{(i)}_{\rm int}, 1\leq i\leq d,\ u^0_{\sigma,i}=\frac{1}{|\sigma|}\int_\sigma\boldsymbol{u}^\varepsilon_0(\boldsymbol{x})\cdot\boldsymbol{e}^{(i)}\,{\rm d}\boldsymbol{x},$$

and computing ρ^{-1} by solving the backward mass balance equation (4d) for $n=-1$ where the unknown is ρ^{-1} and not ρ^0. This allows to perform the first prediction step with $\{\rho^0_{D_\sigma},\rho^{-1}_{D_\sigma},\ \sigma\in\mathcal{E}_{\rm int}\}$ and the dual mass fluxes $\{F_{\sigma,\varepsilon}(\rho^0,\boldsymbol{u}^0),\ \sigma\in\mathcal{E}_{\rm int},\ \varepsilon\in\bar{\mathcal{E}}(D_\sigma)\}$ satisfying the mass balance (12). Moreover, since $\rho^\varepsilon_0>0$, one clearly has $\rho^0_K>0$ for all $K\in\mathcal{M}$ and therefore $\rho^0_{D_\sigma}>0$ for all $\sigma\in\mathcal{E}_{\rm int}$. The positivity of ρ^{-1} is a consequence of the following Lemma.

Lemma 2.1. *If $(\rho^\varepsilon_0,\boldsymbol{u}^\varepsilon_0)$ satisfies (3), then there exists C, depending on the mesh but independent of ε such that:*

$$\frac{1}{\varepsilon^2}\max_{K\in\mathcal{M}}|\rho^0_K-1|+\frac{1}{\varepsilon^2}\max_{1\leq i\leq d}\max_{\sigma\in\mathcal{E}^{(i)}_{\rm int}}|(\boldsymbol{\nabla}p)^0_{\sigma,i}|+\frac{1}{\varepsilon}\max_{K\in\mathcal{M}}|\rho^{-1}_K-1|\leq C. \tag{14}$$

Proof. We sketch the proof. The boundedness of the first two terms is a straightforward consequence of (3). For the third term we remark that, again by (3):

$$\forall K\in\mathcal{M},\quad \rho^{-1}_K-1=\underbrace{\rho^0_K-1}_{=\mathcal{O}(\varepsilon^2)}+\underbrace{\delta t\,\rho^0_K({\rm div}\boldsymbol{u}^0)_K}_{=\mathcal{O}(\varepsilon)}+\underbrace{\delta t\sum_{\sigma\in\mathcal{E}(K)}\frac{|\sigma|}{|K|}(\rho^0_\sigma-\rho^0_K)\boldsymbol{u}^0_{K,\sigma}}_{=\mathcal{O}(\varepsilon^2)}.$$

□

3. Asymptotic analysis of the zero Mach limit

By the results of [3], there exists a solution $(\rho^n,\boldsymbol{u}^n)_{0\leq n\leq N}$ to the scheme (4) and any solution satisfies the following relations:

- a discrete kinetic energy balance: for all $\sigma\in\mathcal{E}^{(i)}_{\rm int}$, $1\leq i\leq d$, $0\leq n\leq N-1$:

$$\begin{aligned}\frac{1}{2\delta t}\Big(\rho^n_{D_\sigma}\,|u^{n+1}_{\sigma,i}|^2-\rho^{n-1}_{D_\sigma}\,|u^n_{\sigma,i}|^2\Big)+\frac{1}{2|D_\sigma|}\sum_{\substack{\varepsilon\in\bar{\mathcal{E}}(D_\sigma)\\ \varepsilon=D_\sigma|D_{\sigma'}}}F_{\sigma,\varepsilon}(\rho^n,\boldsymbol{u}^n)\,\tilde{u}^{n+1}_{\sigma,i}\,\tilde{u}^{n+1}_{\sigma',i}\\ -{\rm div}\boldsymbol{\tau}(\tilde{\boldsymbol{u}}^{n+1})_{\sigma,i}\,\tilde{u}^{n+1}_{\sigma,i}+\frac{1}{\varepsilon^2}(\boldsymbol{\nabla}p)^{n+1}_{\sigma,i}\,u^{n+1}_{\sigma,i}+\frac{\delta t}{\varepsilon^4}\Big(\frac{|(\boldsymbol{\nabla}p)^{n+1}_{\sigma,i}|^2}{2\,\rho^n_{D_\sigma}}-\frac{|(\boldsymbol{\nabla}p)^n_{\sigma,i}|^2}{2\,\rho^{n-1}_{D_\sigma}}\Big)\\ +R^{n+1}_{\sigma,i}=0,\qquad \text{with } R^{n+1}_{\sigma,i}=\frac{1}{2\delta t}\rho^{n-1}_{D_\sigma}(\tilde{u}^{n+1}_{\sigma,i}-u^n_{\sigma,i})^2.\end{aligned} \tag{15}$$

- a discrete renormalization identity: for all $K\in\mathcal{M}$, $0\leq n\leq N-1$:

$$\frac{1}{\delta t}\Big(\Pi_\gamma(\rho^{n+1}_K)-\Pi_\gamma(\rho^n_K)\Big)+{\rm div}\Big(b_\gamma(\rho^{n+1})\boldsymbol{u}^{n+1}-b'_\gamma(1)\rho^{n+1}\boldsymbol{u}^{n+1}\Big)_K+p^{n+1}_K\,{\rm div}(\boldsymbol{u}^{n+1})_K+R^{n+1}_K=0, \tag{16}$$

with $R^{n+1}_K\geq 0$, where the function b_γ is defined by $b_\gamma(\rho)=\rho\log\rho$ if $\gamma=1$, $b_\gamma(\rho)=\rho^\gamma/(\gamma-1)$ if $\gamma>1$ and satisfies $\rho b'_\gamma(\rho)-b_\gamma(\rho)=\rho^\gamma=\wp(\rho)$ for all $\rho>0$, and $\Pi_\gamma(\rho)=b_\gamma(\rho)-b_\gamma(1)-b'_\gamma(1)(\rho-1)$.

Summing (15) and (16) over the primal cells from one side, and over the dual cells and the components on the other side, and invoking the grad-div duality relation (8), we obtain a local-in-time discrete entropy inequality, for $0 \le n \le N-1$:

$$\begin{aligned}(17)\quad \frac{1}{2}\sum_{i=1}^{d}\sum_{\sigma\in\mathcal{E}_{\rm int}^{(i)}}|D_\sigma|\Big(\rho_{D_\sigma}^{n}\,|u_{\sigma,i}^{n+1}|^2-\rho_{D_\sigma}^{n-1}\,|u_{\sigma,i}^{n}|^2\Big)+\frac{1}{\varepsilon^2}\sum_{K\in\mathcal{M}}|K|\Big(\Pi_\gamma(\rho_K^{n+1})-\Pi_\gamma(\rho_K^{n})\Big)\\ +\mu\delta t\|\tilde{\boldsymbol{u}}^{n+1}\|_{1,\mathcal{M}}^2+\frac{1}{\varepsilon^4}\sum_{i=1}^{d}\sum_{\sigma\in\mathcal{E}_{\rm int}^{(i)}}|D_\sigma|\delta t^2\Big(\frac{|(\boldsymbol{\nabla}p)_{\sigma,i}^{n+1}|^2}{2\,\rho_{D_\sigma}^{n}}-\frac{|(\boldsymbol{\nabla}p)_{\sigma,i}^{n}|^2}{2\,\rho_{D_\sigma}^{n-1}}\Big)+\mathcal{R}^{n+1}\le 0\end{aligned}$$

where $\mathcal{R}^{n+1}=\displaystyle\sum_{i=1}^{d}\sum_{\sigma\in\mathcal{E}_{\rm int}^{(i)}}R_{\sigma,i}^{n+1}+\frac{1}{\varepsilon^2}\sum_{K\in\mathcal{M}}R_K^{n+1}\ge 0$.

The function Π_γ has some important properties:

(18a) • For all $\gamma\ge 1$ there exists C_γ such that: $\Pi_\gamma(\rho)\le C_\gamma\,|\rho-1|^2,\ \forall\rho\in(0,2)$.

(18b) • If $\gamma\ge 2$ then $\Pi_\gamma(\rho)\ge|\rho-1|^2,\ \forall\rho>0$.

(18c) • If $\gamma\in[1,2)$ then for all $R\in(2,+\infty)$, there exists $C_{\gamma,R}$ such that:
$$\Pi_\gamma(\rho)\ge C_{\gamma,R}|\rho-1|^2,\quad \forall\rho\in(0,R),$$
$$\Pi_\gamma(\rho)\ge C_{\gamma,R}|\rho-1|^\gamma,\quad \forall\rho\in[R,\infty).$$

Lemma 3.1 (Global discrete entropy inequality)**.** *Under assumption (3), there exists $C_0>0$ independent of ε such that the solution $(\rho^n,\boldsymbol{u}^n)_{0\le n\le N}$ to the scheme (4) satisfies, for ε small enough, and for $1\le n\le N$:*

$$\begin{aligned}(19)\qquad \frac{1}{2}\sum_{i=1}^{d}\sum_{\sigma\in\mathcal{E}_{\rm int}^{(i)}}|D_\sigma|\rho_{D_\sigma}^{n-1}\,|u_{\sigma,i}^{n}|^2+\mu\sum_{k=1}^{n}\delta t\,\|\tilde{\boldsymbol{u}}^k\|_{1,\mathcal{M}}^2\\ +\frac{1}{\varepsilon^2}\sum_{K\in\mathcal{M}}|K|\,\Pi_\gamma(\rho_K^n)+\frac{1}{\varepsilon^4}\sum_{i=1}^{d}\sum_{\sigma\in\mathcal{E}_{\rm int}^{(i)}}\frac{|D_\sigma|\,\delta t^2}{2\,\rho_{D_\sigma}^{n-1}}\,|(\boldsymbol{\nabla}p)_{\sigma,i}^n|^2\le C_0.\end{aligned}$$

Proof. Summing (17) over n yields the inequality (19) with

$$(20)\qquad C_0=\frac{1}{2}\sum_{i=1}^{d}\sum_{\sigma\in\mathcal{E}_{\rm int}^{(i)}}|D_\sigma|\rho_{D_\sigma}^{-1}\,|u_{\sigma,i}^{0}|^2+\frac{1}{\varepsilon^2}\sum_{K\in\mathcal{M}}|K|\,\Pi_\gamma(\rho_K^0)+\frac{1}{\varepsilon^4}\sum_{i=1}^{d}\sum_{\sigma\in\mathcal{E}_{\rm int}^{(i)}}\frac{|D_\sigma|\,\delta t^2}{2\,\rho_{D_\sigma}^{-1}}\,|(\boldsymbol{\nabla}p)_{\sigma,i}^0|^2.$$

By (14), for ε small enough, one has $\rho_K^{-1}\le 2$ for all $K\in\mathcal{M}$ and therefore $\rho_{D_\sigma}^{-1}\le 2$ for all $\sigma\in\mathcal{E}_{\rm int}^{(i)}$ and $1\le i\le d$. Hence, since $\boldsymbol{u}_0^\varepsilon$ is uniformly bounded in $\mathrm{H}^1(\Omega)^d$ by (3), a classical trace inequality yields the boundedness of the first term. Again by (14), one has $|\rho_K^0-1|\le C\varepsilon^2$ for all $K\in\mathcal{M}$. Hence, by (18a), the second term vanishes as $\varepsilon\to 0$. The third term is also uniformly bounded with respect to ε thanks to (14). □

Lemma 3.2 (Control of the pressure)**.** *Assume that $(\rho_0^\varepsilon,\boldsymbol{u}_0^\varepsilon)$ satisfies (3) and let $(\rho^n,\boldsymbol{u}^n)_{0\le n\le N}$ satisfy (4). Let $p^n=\wp(\rho^n)$ and define $\delta p^n=\{\delta p_K^n,\ K\in\mathcal{M}\}$ where $\delta p_K^n=(p_K^n-|\Omega|^{-1}\int_\Omega p^n\,\mathrm{d}\boldsymbol{x})/\varepsilon^2$. Then, one has, for all $1\le n\le N$:*

$$\|\delta p^n\|\le C_{\mathcal{M},\delta t},$$

where $C_{\mathcal{M},\delta t}\ge 0$ depends on the mesh and δt but not on ε, and $\|\cdot\|$ stands for any norm on the space of discrete functions.

Proof. By (19), the discrete pressure gradient is controlled in L^∞ by $C_{\mathcal{M},\delta t}\,\varepsilon^2$, so that $\boldsymbol{\nabla}(\delta p^n)$ is bounded in any norm independently of ε. Using the discrete $(\mathrm{H}^{-1})^d$-norm (see e.g. [1]), invoking the gradient divergence duality (8) and the *inf-sup* stability of the scheme, $\|\boldsymbol{\nabla}(\delta p^n)\|_{-1,\mathcal{M}}\le C_{\mathcal{M},\delta t}$ implies that $\|\delta p^n\|_{\mathrm{L}^2}\le\beta^{-1}C_{\mathcal{M},\delta t}$. □

Theorem 3.3 (Incompressible limit of the MAC pressure correction scheme)**.**
Let $(\varepsilon^{(m)})_{m\in\mathbb{N}}$ be a sequence of positive real numbers tending to zero, and let $(\rho^{(m)}, \boldsymbol{u}^{(m)})$ be a corresponding sequence of solutions of the scheme (4). Then the sequence $(\rho^{(m)})_{m\in\mathbb{N}}$ converges to the constant function $\rho = 1$ when m tends to $+\infty$ in $\mathrm{L}^\infty((0,T),\mathrm{L}^q(\Omega))$, for all $q \in [1, \min(\gamma, 2)]$.

In addition, the sequence $(\boldsymbol{u}^{(m)}, \delta p^{(m)})_{m\in\mathbb{N}}$ tends, in any discrete norm, to the solution $(\boldsymbol{u}, \delta p)$ of the usual MAC pressure correction scheme for the incompressible Navier-Stokes equations, which reads:

Prediction step – *Solve for $\tilde{\boldsymbol{u}}^{n+1}$:*

$$\text{For } 1 \le i \le d,\ \forall \sigma \in \mathcal{E}_{\rm int}^{(i)}, \qquad \frac{1}{\delta t}\,\bigl(\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^{n}\bigr) + \mathrm{div}(\tilde{u}_i^{n+1}\boldsymbol{u}^n)_\sigma - \mathrm{div}\boldsymbol{\tau}(\tilde{\boldsymbol{u}}^{n+1})_{\sigma,i} + (\boldsymbol{\nabla}(\delta p)^n)_{\sigma,i} = 0.$$

Correction step – *Solve for $(\delta p)^{n+1}$ and $\boldsymbol{u}^{n+1}$:*

$$\text{For } 1 \le i \le d,\ \forall \sigma \in \mathcal{E}_{\rm int}^{(i)}, \qquad \frac{1}{\delta t}\,(u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1}) + (\boldsymbol{\nabla}(\delta p)^{n+1})_{\sigma,i} - (\boldsymbol{\nabla}(\delta p)^n)_{\sigma,i} = 0,$$

$$\forall K \in \mathcal{M}, \qquad \mathrm{div}(\boldsymbol{u}^{n+1})_K = 0.$$

Proof. By (18b) and the global entropy estimate (19), one has for $\gamma \ge 2$,

$$\|\rho^{(m)}(t) - 1\|^2_{\mathrm{L}^2(\Omega)} \le \int_\Omega \Pi_\gamma(\rho^{(m)}(t)) \le C_0\,\varepsilon^2,\ \forall t \in (0,T).$$

For $1 \le \gamma \le 2$, invoking (18c) and estimate (19), we obtain for all $t \in (0,T)$ and for all $R \in (2, +\infty)$:

$$(i) \qquad \|(\rho^{(m)}(t) - 1)\mathbb{1}_{\{\rho^{(m)}(t)\le R\}}\|^2_{\mathrm{L}^2(\Omega)} \le \frac{1}{C_{\gamma,R}} \int_\Omega \Pi_\gamma(\rho^{(m)}(t)) \le C\,\varepsilon^2, \forall t \in (0,T),$$

$$(ii) \qquad \|(\rho^{(m)}(t) - 1)\mathbb{1}_{\{\rho^{(m)}(t)\ge R\}}\|^\gamma_{\mathrm{L}^\gamma(\Omega)} \le \frac{1}{C_{\gamma,R}} \int_\Omega \Pi_\gamma(\rho^{(m)}(t)) \le C\,\varepsilon^2, \forall t \in (0,T),$$

which proves the convergence of $(\rho^{(m)})_{m\in\mathbb{N}}$ to the constant function $\rho = 1$ as $m \to +\infty$ in $\mathrm{L}^\infty((0,T),\mathrm{L}^q(\Omega))$ for all $q \in [1, \min(\gamma, 2)]$. Using again (19), the sequence $(\boldsymbol{u}^{(m)})_{m\in\mathbb{N}}$ is bounded in any discrete norm and the same holds for the sequence $(\delta p^{(m)})_{m\in\mathbb{N}}$ by Lemma 3.2. By the Bolzano-Weiertrass theorem and a norm equivalence argument, there exists a subsequence of $(\boldsymbol{u}^{(m)}, \delta p^{(m)})_{m\in\mathbb{N}}$ which tends, in any discrete norm, to a limit $(\boldsymbol{u}, \delta p)$. Passing to the limit cell-by-cell in (4), one obtains that $(\boldsymbol{u}, \delta p)$ is a solution to (21). Since this solution is unique, the whole sequence converges, which concludes the proof. □

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