Calculus of Variations and Elliptic Equations
–
10th class

Vector measures and BV space

A vector measure λ on a set X is a function associating with every measurable subset $A \subset X$ a vector $\lambda(A) \in \mathbb{R}^n$, satisfying the usual properties for measures (additivity on disjoint sets and countable unions... in particular we need, whenever $A = \cup_i A_i$ is a disjoint countable union, to have $\sum_i |\lambda(A_i)| < +\infty$ and $\lambda(A) = \sum_i \lambda(A_i)$). For a vector measure λ, given a norm on $\mathbb{R}^n$ (we will always use the Euclidean one), we define the positive scalar measure

$$|\lambda|(A) := \sup\{\sum_i |\lambda(A_i)| \ : \ A = \cup_i A_i \text{ disjoint countable union } \}.$$

We can check that $|\lambda|$ is a measure and $\lambda \ll |\lambda|$ with a density of unit norm.

If X is acompact, the space of vector measures, denoted by $\mathcal{M}^n(X)$, is the topological dual of $C(X;\mathbb{R}^n)$ with the duality $< f, \lambda >:= \int f d\lambda = \sum_i f_i d\lambda_i$. If X is non-compact, then $\mathcal{M}^n(X)$ is the dual oc $C_0(X;\mathbb{R}^n)$, the space of continuous functions vanishing at infinity. The norm on $\mathcal{M}^n(X)$ is given by

$$||\lambda||_{\mathcal{M}} := |\lambda|(X) = \sup\{\int f d\lambda \ : \ |f| \leq 1\}.$$

Note that L^1 vector functions can be idenditifed with vector measures which are absolutely continuous wrt Lebesgue. Their L^1 norm coincides in this case with the norm in $\mathcal{M}^n(X)$

Once we know the space of vector measures on a domain $\Omega \subset \mathbb{R}^d$, we can define the space of functions with bounded variation, called $BV(\Omega)$: we define

$$BV(\Omega) := \{u \in L^1(\Omega) \ : \ \nabla u \in \mathcal{M}^d(\Omega)\},$$

where the gradient is to be intended in the sense of distributions. The norm on the space BV is given by $||u||_{BV} := ||u||_{L^1} + ||\nabla u||_{\mathcal{M}}$.

We can see that the space $W^{1,1}$, where gradients are in L^1, is a subset of BV, since when a gradient is an L^1 function it is also a measure.

$BV(\Omega)$ is a Banach space, which is coninuously injected in all L^p spaces for $p \leq d/(d-1)$. If Ω is bounded, the injection is compact for every $p < d/(d-1)$ and in particular in L^1.

Some non-trivial indicator functions may belong to the space BV, differently than what happens for sobolev spaces. For smooth sets A we can indeed see that we have

$$\nabla I_A = -n \cdot \mathcal{H}^{d-1}_{|\partial A},$$

where n is the exterior unit normal to A, and $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure (see, for instance, [1] or [2]).

We say that a set $A \subset \Omega$ is a set of finite perimeter if $I_A \in BV(\Omega)$, and we define its perimeter $\mathrm{Per}(A)$ as $||\nabla I_A||_{\mathcal{M}}$.

Note that the perimeter of A defined in this way depends on the domain Ω. More precisely, this perimeter corresponds to the part of the boundary of 1 which is not $\partial\Omega$.

Approximation of the perimeter

We consider the following sequence of functionals defined on $L^1(\Omega)$,

$$F_\varepsilon(u) := \begin{cases} \varepsilon \int |\nabla u|^2 + \frac{1}{\varepsilon} \int W(u) & \text{if } u \in H^1_0(\Omega), \\ +\infty & \text{if not,} \end{cases}$$

where $W : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function satisfying $W(0) = W(1) = 0$ and $W > 0$ on $\mathbb{R} \setminus \{0,1\}$. We denote by c_0 the constant given by $c_0 = 2\int_0^1 \sqrt{W}$.

Note that the condition $u \in H_0^1(\Omega)$ is equivalent to requiring that we have $u \in H^1(\mathbb{R}^d)$ when we extend u to 0 outside Ω. We also define

$$F(u) := \begin{cases} c_0 \operatorname{Per}(A) & \text{if } u = I_A \in BV(\mathbb{R}^d), \\ +\infty & \text{if not.} \end{cases}$$

In this case we stress that the perimeter of A is computed inside the whole space, i.e. also condiering $\partial A \cap \partial\Omega$.

We will prove the following result, due to Modica and Mortola [3].

Proposition 1. *Suppose that Ω is a bounded convex set in $\mathbb{R}^d$. Then $F_\varepsilon \xrightarrow{\Gamma} F$ in $L^1(\Omega)$, as $\varepsilon \to 0$.*

Proof. Let us start from the Γ-liminf inequality. Consider $u_\varepsilon \to u$ in L^1.

Note that we have the lower bound

$$F_\varepsilon(u_\varepsilon) \geq 2\int \sqrt{W(u_\varepsilon)}|\nabla u_\varepsilon| = 2\int |\nabla\Phi(u_\varepsilon)|,$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is the function defined by $\Phi(0) = 0$ and $\Phi' = \sqrt{W}$. Note that, W being bounded, we have $|\Phi(u_\varepsilon)| \leq C|u_\varepsilon|$. This means that $\Phi(u_\varepsilon)$ is bounded in $BV(\Omega)$ and, up to a subsequence, it converges strongly in L^1 to a function v. Up to another subsequence we also have pointwise convergence a.e., but we already had $u_\varepsilon \to u$ a.e., hence $v = \Phi(u)$. We then have $|\nabla\Phi(u_\varepsilon) \to |\nabla\Phi(u)$ in the sense of distributions and weakly as measures, and the lower semicontinuity of the norm implies

$$||\nabla\Phi(u)||_{\mathcal{M}} \leq \liminf_\varepsilon \int |\nabla\Phi(u_\varepsilon)| \leq \frac{1}{2}\liminf_\varepsilon F_\varepsilon(u_\varepsilon)$$

and the fact that we have $\Phi(u) \in BV(\Omega)$.

On the other hand, since we can of course assume $F_\varepsilon(u_\varepsilon) \leq C$, we also have $\int W(u_\varepsilon) \leq C\varepsilon$ and, by Fatou, $\int W(u) = 0$, i.e. $u \in \{0,1\}$ a.e. This means that we do have $u = I_A$ for a measurable set $A \subset \Omega$. Note that in this case we have $\Phi(u) = \Phi(1)u = \Phi(1)I_A$. Since we have $\Phi(1) = \int_0^1 \sqrt{W} > 0$, this implies $I_A \in BV(\Omega)$, and we finally have

$$F(u) = 2\Phi(1)\operatorname{Per}(A) \leq \liminf_\varepsilon F_\varepsilon(u_\varepsilon).$$

We now switch to the Γ-limsup inequality. We first consider the case $u = I_A$ with A smooth and $d(A, \partial\Omega) > 0$. We need to build a recovery sequence u_ε. Let us define sd_A the signed distance function to A given by

$$\mathrm{sd}_A(x) \begin{cases} d(x, A) & \text{if } x \notin A \\ -d(x, A^c) & \text{if} x \in A \end{cases}$$

Take a function $\phi : \mathbb{R} \to [0,1]$ such that there exist $L_\pm > 0$ with $\phi = 1$ on $(-\infty, L_-]$, $\phi = 0$ on $[L_+, +\infty)$, and $\phi \in C^1([-L_-, L_+])$. Define

$$u_\varepsilon = \phi\left(\frac{\mathrm{sd}_A}{\varepsilon}\right).$$

Note that $|\mathrm{sd}_A| = 1$ a.e. and hence we have $\varepsilon|\nabla u_\varepsilon|^2 = \frac{1}{\varepsilon}|\phi'|^2\left(\frac{\mathrm{sd}_A}{\varepsilon}\right)$ and

$$F_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon}\int \left(|\phi'|^2 + W\right)\left(\frac{\mathrm{sd}_A}{\varepsilon}\right).$$

We now use the co-area formula (see [2]) which provides the following equality, valid at least for smooth functions $f, g : \Omega \to \mathbb{R}$:

$$\int f|\nabla g| = \int_{\mathbb{R}} dt \int_{\{g=t\}} f d\mathcal{H}^{d-1}.$$

We apply it to the case $g = \mathrm{sd}_A$, for which the norm of the gradient is always 1. We then have

$$F_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon}\int_{\mathbb{R}} \left(|\phi'|^2 + W(\phi)\right)\left(\frac{t}{\varepsilon}\right)\mathcal{H}^{d-1}(\{\mathrm{sd}_A = t\})dt = \int_{\mathbb{R}} \left(|\phi'|^2 + W(\phi)\right)(r)\mathcal{H}^{d-1}(\{\mathrm{sd}_A = \varepsilon r\})dr,$$

where the second equality comes from the change of variable $t = \varepsilon r$.

Since A is smooth we have, for every r, the convergence $\mathcal{H}^{d-1}(\{\mathrm{sd}_A = \varepsilon r\}) \to \mathrm{Per}(A)$. We can restrinct the integral to $r \in [-L_-, L_+]$ which allows to apply dominated convergence and obtain

$$\lim_\varepsilon F_\varepsilon(u_\varepsilon) = \mathrm{Per}(A)\int_{\mathbb{R}} \left(|\phi'|^2 + W\right)(r)dr.$$

Moreover, it is clear that $u_\varepsilon \to I_A$ in L^1 because of dominated convergence.

We have now to choose ϕ. Choose a funcion $\tilde\phi$ such that $\tilde\phi(0) = 1/2$ and $\tilde\phi' = -\sqrt{W(\tilde\phi)}$. We necessarily have $\lim_{r\to-\infty}\tilde\phi(r) = 1$ and $\lim_{r\to+\infty}\tilde\phi(r) = 0$. The function $\tilde\phi$ is C^1 and strictly monotone. Fix $\delta > 0$ and let $r_\pm$ be defined via $\tilde\phi(r_-) = 1-\delta$ and $\tilde\phi(r) = \delta$. We then take $\phi = \phi_\delta$ a function such that $\phi_\delta = \tilde\phi$ on $[r_-, r_+]$, $\phi_\delta \in C^1([r_- - 2\delta, r_+ + 2\delta])$, and $|\phi'_\delta| \le 1$ on $[r_- - 2\delta, r_-] \cup [r_+, r_+ + 2\delta]$. We have

$$\int_{\mathbb{R}} \left(|\phi'_\delta|^2 + W(\phi_\delta)\right)(r)dr \le (1 + \sup W)2\delta + \int_{r_-}^{r_+} \left(|\tilde\phi'|^2 + W(\tilde\phi)\right)(r)dr \le C\delta + \int_{\mathbb{R}} \left(|\tilde\phi'|^2 + W(\tilde\phi)\right)(r)dr.$$

Note that we have $\int_{\mathbb{R}} \left(|\tilde\phi'|^2 + W(\tilde\phi)\right)(r)dr = 2\int_{\mathbb{R}} \sqrt{W(\tilde\phi)}|\tilde\phi'| = 2|\Phi(\tilde\phi(+\infty) - Phi(\tilde\phi(-\infty)| = 2\Phi(1) = c_0$, which means

$$\lim_\varepsilon F_\varepsilon(u_\varepsilon) \le (c_0 + C\delta)\,\mathrm{Per}\,A,$$

for arbitrary $\delta > 0$, and hence

$$(\Gamma - \limsup F_\varepsilon)(u) \le c_0\,\mathrm{Per}\,A = F(u).$$

We now need to extend our Γ-limsup inequality to other functions u which are not of the form $u = I_A$ with A smooth and far from the boundary, but only $u = I_A$ with A of finite perimeter. We need hence to show that the class $\mathcal{S}$ of indicators of smooth sets far from the boundary is dense in energy.

We start from a set A fo finite perimeter with $d(A, \partial\Omega) > 0$ and set $u = I_A$. Take a smooth and compactly supported convolution kernel $\eta_n \rightharpoonup \delta_0$ defined by reslacing of a fxed kernel η_1, and define $v_n := \eta_n * u$. The function v_n is smooth and, for large n, compactly supported in Ω. We have $\int |\nabla v_n| \le ||\nabla u||_{\mathcal{M}}$ since the norm of the gradient is a convex functional invariant by translations, hence it decreases under convolution. We use again the coarea formula to write

$$\int |\nabla v_n| = \int_0^1 dr\mathcal{H}^{d-1}(\{v_n = r\}).$$

If we fix $\delta > 0$ we can then choose a number $r_n \in [\delta, 1-\delta]$ such that

$$\mathrm{Per}(\{v_n \ge r_n\}) \le \mathcal{H}^{d-1}(\{v_n = r_n\}) \le \frac{\mathrm{Per}\,A}{1-2\delta}$$

and r_n is not a critical value for v_n (since, by Sard's lemma, a.e. value is not critical). In particular, the set $A_n := \{v_n \ge r_n\}$ is a smooth set, and we then take $u_n = I_{A_n}$. By construction the perimeter of A_n is bounded by $\frac{\mathrm{Per}\,A}{1-2\delta}$ and u_n is bounded in BV. We want to prove that we have $u_n \to u$ in L^1. Up to subsequences, we do have $u_n \to w$ strongly in L^1 and a.e. Since $u_n \in \{0,1\}$. also $w \in \{0,1\}$ a.e. Considera point x which is a Lebesgue point for u. In particular, this implies $v_n(x) \to u(x)$ as $n \to \infty$. Consider the case $w(x) = 1$. Then $u_n(x) = 1$ for large n, i.e. $v_n(x) \geq r_n \geq \delta$. Then $u(x) \geq \delta$, which means $u(x) = 1$. Analogously, $w(x) = 0$ implies $u(x) = 0$. Finally we have $w = u$ and $u_n \to u$ strongly in L^1. This is not yet the desired sequence, since we have $F(u_n) \leq (1-2\delta)^{-1}F(u)$ instead of $\limsup_n F(u_n) \leq F(u)$ but this can be fixed easily. Indeed, this allows for every δ to find $\tilde{u} \in \mathcal{S}$ with $||\tilde{u} - u||_{L^1}$ arbitrarily small and $F(\tilde{u}) \leq (1-2\delta)^{-1}F(u)$, which can be turned, using $\delta_n \to 0$, into a sequence which shows that $\mathcal{S}$ is dense in energy.

We have no tget rid of the assumption $d(A, \partial\Omega) > 0$. Using the fact that Ω is convex, and supposing without loss of generality that the origin 0 belongs to the interior of Ω, we can take $u = I_A$ and define $u_n = I_{t_n A}$ for a sequence $t_n \to 1$. The sets $t_n A$ are indeed far from the boundary. Moreover, we have $F(u_n) = c_0 \operatorname{Per}(t_n A) = t_n^{d-1} F(u) \to F(u)$. We are just left to prove $u_n \to u$ strongly in L^1. Because of the BV bound which provides compactness in L^1 we just need to prove any kind of weak convergence. Take a test function φ and compute $\int \varphi u_n = \int_{t_n A} \varphi = (t_n)^d \int_A \varphi(t_n y) dy \to \int_A \varphi$, as soon as φ is continuous. This shows weak convergence in the sense of measures $u_n \rightharpoonup u$ and, thanks to the BV bound, strong L^1 convergence and concludes the proof. □

References

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