Calculus of Variations and Elliptic Equations
–
5th class

Smooth Harmonic functions

A well-known fact abouth harmonic functions is the mean value property

Proposition 1. *Consider $u \in C^\infty(\Omega)$ a function satisfying $\Delta u = 0$ in Ω, and take x_0, R such that $\overline{B(x_0,R)} \subset \Omega$. Then we have*

$$u(x_0) = \fint_{B(x_0,R)} u = \fint_{\partial B(x_0,R)} u,$$

where the integral on the ball is performed w.r.t. the Lebesgue measure, the one on the sphere w.r.t. the boundary measure.

Proof. It is enough to prove the equality with the average on the sphere, since this one for every radius implies the one with averages on the ball. Indeed, if we have $u(x_0) = \fint_{\partial B(x_0,r)} u$,, then we also have

$$\int_{B(x_0,R)} u = \int_0^R d\omega_d r^{d-1} \fint_{\partial B(x_0,r)} u = u(x_0) \int_0^R d\omega_d r^{d-1} = u(x_0)\omega_d R^d,$$

which gives the result (here ω_d is the volume of the unit ball in $\mathbb{R}^d$).

It it then enough to prove that $R \mapsto g(R) := \fint_{\partial B(x_0,R)} u = \fint_{\partial B(0,1)} u(x_0 + Rx)dx$ is constant, since its limit as $R \to 0$ is $u(x_0)$. We differentiate it, and obtain

$$g'(R) = \fint_{\partial B(0,1)} \nabla u(x_0 + Rx) \cdot x dx = \fint_{\partial B(0,1)} \nabla v \cdot n = \frac{1}{d\omega_d} \int_{B(0,1)} \Delta v = 0,$$

where $v(x) = R^{-1}u(x_0 + Rx)$ and $\Delta v = R\Delta u(x_0 + Rx)$. We used the fact that, on the unit ball, the vector x coincides on the boundary with the normal vector, and applied the divergence theorem. □

Proposition 2. *Consider $u \in C^\infty(\Omega)$ a function satisfying $\Delta u = 0$ in Ω. Then u is analytic in Ω.*

Proof. This requires to obtain a bound of the form $|D^k u(x_0)| \leq \frac{d^k}{R^k} k! \sup_{B(x_0,R)} |u|$, whenever $\overline{B(x_0,R)} \subset \Omega$. Adapting the exact bound to the case of multi-indices is delicate, see the proof in [**?**]. □

Caccioppoli inequality

Proposition 3. *Suppose that $u \in C^\infty(\Omega)$ is harmonic in Ω, and take $x_0, r < R$ such that $\overline{B(x_0,R)} \subset \Omega$. Then we have*

$$\int_{B(x_0,r)} |\nabla u|^2 \leq \frac{4}{(R-r)^2} \int_{B(x_0,R)} |u|^2.$$

Proof. From the harmonicity of u we get $\int \nabla u \cdot \nabla \varphi = 0$ for every $\varphi \in C_c^\infty(\Omega)$. We choose a cut-off function $\eta \in C^\infty(\Omega)$ with $\eta = 1$ on $B(x_0,r)$ and $\eta = 0$ outside of $B(x_0,R)$, and use $\varphi = u\eta^2$. We then get

$$\int |\nabla u|^2 \eta^2 = -2 \int \nabla u \cdot \nabla \eta u \eta \leq 2 \left(\int |\nabla u|^2 \eta^2 \right)^{1/2} \left(\int |\nabla \eta|^2 |u|^2 \right)^{1/2},$$

which implies

$$\int |\nabla u|^2 \eta^2 \leq 4 \int |\nabla \eta|^2 |u|^2.$$

Hence,

$$\int_{B(x_0,r)} |\nabla u|^2 \leq 4||\nabla \eta||^2_{L^\infty} \int_{B(x_0,R)} |u|^2$$

and the L^∞ norm of the gradient of η can be takes as close as we want to $(R-r)^{-1}$. □

We can state a series of generalizations of the above result

- u does not need to be a smooth harmonic function, but $u \in H^1_{loc}$ and $\Delta u = 0$ in the sense of distributions (which implies $\int \nabla u \cdot \nabla \varphi = 0$ for every $\varphi \in H^1_c(\Omega)$) is enough.
- actually, the inequality $\Delta u \geq 0$ together with $u \geq 0$ is enough (since we have $\int \nabla u \cdot \nabla \varphi \leq 0$ for every $\varphi \in H^1_c(\Omega)$ with $\varphi \geq 0$).
- more general equations can be considered. If a point-dependent symmetric matrix $a(x)$ satisfying $\lambda I \leq a(x) \leq \Lambda I$ for every x is given the result for an H^1 function satisfying $\nabla \cdot (a \nabla u) = 0$ (in the sense $\int a \nabla u \cdot \nabla \varphi = 0$ for every $\varphi \in H^1_c(\Omega)$) is the following variant :

$$\int_{B(x_0,r)} |\nabla u|^2 \leq \frac{4\Lambda/\lambda}{(R-r)^2} \int_{B(x_0,R)} |u|^2.$$

- of course this also generalizes to non-negative H^1 functions $u \geq 0$ satisfying $\nabla \cdot (a \nabla u) \geq 0$.

A corollary of the Caccioppoli inequality is the following:

Proposition 4. *Suppose that $u \in C^\infty(\mathbb{R}^d)$ is harmonic and has polyunomial growth ($|u(x)| \leq C(1+|x|^p)$). Then u is a polynomial.*

Proof. Applying Caccioppoli's inequality with $R = 2r$ we obtain

$$\int_{B(x_0,r)} |\nabla u|^2 \leq C r^{-2+d}(1+r^p)^2.$$

Applying the same inequality to te derivatives of u, which are also harmonic, we obtain

$$\int_{B(x_0,r)} |D^2 u|^2 \leq C_1 r^{-4+d}(1+r^p)^2$$

for a new constant C_1. Iterating, we obtain

$$\int_{B(x_0,r)} |D^k u|^2 \leq C_k r^{-2k+d}(1+r^p)^2.$$

Selecting k such that $2k > d + 2p$ we obtain

$$\lim_{r\to\infty} \int_{B(x_0,r)} |D^k u|^2 = 0,$$

which implies $D^k u = 0$ everywhere. Then u is a polynomial of degree at most $k-1$ (but actually of degree at most p, because of its growth condition). □

L^2_{loc} harmonic functions

We can now state the following result.

Proposition 5. *Suppose that $u \in L^2_{loc}(\Omega)$ is harmonic in Ω in the sense of distributions. Then u is analyitic.*

Proof. We will prove that u is C^∞, and analyticity will just be a consequence. We fix a ball $B(x_0,R)$ which is compactly contained in Ω and a dcreasing sequence of radii $r_k \geq r_{k+1}$ with $r_0 = R$ and $r_k \geq R/2$. We take a sequence of standard mollifiers η_ε supported in $B(0,\varepsilon)$ with $\varepsilon < d(B(x_0,R), \partial\Omega)$. We define $u_\varepsilon := \eta_\varepsilon * u$. These functions are harmonic in a neighborhood of $\overline{B(x_0,R)}$ and smooth. The norms $||u_\varepsilon||_{L^2(B(x_0,R)}$ are bounded since $u \in L^2_{loc}(\Omega)$ and $u_\varepsilon \to u$ in $L^2(B(x_0,R)$. Yet, Caccioppoli's inequality implies a bound on the H^1 norm of u_ε on $B(x_0,r_1)$. Applying the same inequality to the derivatives of u (which are also harmonic) we obtain a bound on the H^2 norm of u_ε on $B(x_0,r_2)$ and, iterating, on the H^k norm of u_ε on $B(x_0,r_k)$. On $B(x_0,R/2)$, the sequence u_ε is bounded in all the spaces H^k. Extracting a subsequence, we have for every k weak convergence in H^k of u_ε to its L^2 limit u. Hence $u\in\cap_k H^k=C^\infty$. □

Solutions of $\Delta u = f$

Let us consider the fundamental solution of the Laplacian

$$\Gamma(x)=\begin{cases}\frac{1}{2\pi}\log|x| & \text{if } d=2,\\ -\frac{1}{d(d-2)\omega_d}|x|^{2-d} & \text{if } d>2.\end{cases}$$

We can see that we have

- $\Gamma\in L^1_{loc}$, $\nabla\Gamma\in L^1_{loc}$, but $D^2\Gamma\notin L^1_{loc}$.
- $\int_{\partial B(0,R)}\nabla\Gamma\cdot n=1$ for every R.
- $\Gamma\in C^\infty(\mathbb{R}^d\setminus\{0\})$ and $\Delta\Gamma=0$ on $\mathbb{R}^d\setminus\{0\}$.
- $\Delta\Gamma=\delta_0$ in the sense of distributions.

As a consequence, for every $f\in C^\infty_c(\mathbb{R}^d)$, the function $u=\Gamma*f$ is a smooth function solving $\Delta u=f$ in the classical sense. It is of course not the unique solution to this equation, since we can add to u arbitrary harmonic functions.

For this solution, we have the following estimate

Proposition 6. *Given $f\in C^\infty_c(\mathbb{R}^d)$, let u be given by $u=\Gamma*f$. Then we have $\int_{\mathbb{R}^d}|D^2u|^2=\int_{\mathbb{R}^d}|f|^2$, where $|D^2u|^2$ denotes the squared Frobenius norm of the Hessian matricx, given by $|A|^2=\mathrm{Tr}(A^tA)=\sum_{i,j}A^2_{ij}$.*

Proof. We consider a ball $B(0,R)$ and obtain, by integration by parts

$$\int_{B(0,R)}f^2=\int_{B(0,R)}|\Delta u|^2=\sum_{i,j}\int_{B(0,R)}u_{ii}u_{jj}=-\sum_{i,j}\int_{B(0,R)}u_{iij}u_j+\sum_{i,j}\int_{\partial B(0,R)}u_{ii}u_jn^j.$$

If the support of f is compactly contained in $B(0,R)$ then the last boundary term vanishes since it equals $\int_{\partial B(0,R)}f\nabla u\cdot n$. Going on with the integration by parts we have

$$\int_{B(0,R)}f^2=\sum_{i,j}\int_{B(0,R)}u_{ij}u_{ij}-\sum_{i,j}\int_{\partial B(0,R)}u_{ij}u_jn^i,$$

hence

$$\int_{B(0,R)}f^2=\int_{B(0,R)}|D^2u|^2-\int_{\partial B(0,R)}D^2u\nabla u\cdot n.$$

We then note that, for $R>>1$, we have $|\nabla u(x)|\le C|x|^{1-d}$ and $|D^2u(x)|\le C|x|^{-d}$, as a consequence of the shape of Γ and the compact support of f, so that we have

$$|\int_{\partial B(0,R)}D^2u\nabla u\cdot n\le CR^{d-1}.R^{-d}.R^{1-d}=CR^{-d}\to 0\text{ as }R\to\infty,$$

which proves the claim. □

Applying the same result to the derivatives of f we immediately obtain

Corollary 1. *Given $f \in C_c^\infty(\Omega)$ for a bounded set Ω, let u be given by $u = \Gamma * f$. Then for every $k \geq 0$ we have*

$$||u||_{H^{k+2}(\Omega)} \leq C(k,\Omega)||f||_{H^k(\Omega)}.$$

Harmonic distributions

Lemma 1. *Suppose $u \in C^\infty(\Omega)$ is harmonic in Ω and take $x_0, r < R$ such that $\overline{B(x_0,R)} \subset \Omega$. Then, for every integer $k \geq 1$, we have*

$$||u||_{H^{1-k}(B(x_0,r)} \leq C(k,r,R)||u||_{H^{-k}(B(x_0,R)}.$$

Proof. We want to take $\varphi \in C^\infty$ with $\mathrm{spt}(\varphi) \subset B(x_0,r)$ and estimate $\int u\varphi$ in terms of $||\varphi||_{H^{k-1}}$ and $||u||_{H^{-k}(B(x_0,R))}$. To do this, we first consider $v = \Gamma * \varphi$ and a cutoff function $\eta \in C^\infty(\Omega)$ with $\eta = 1$ on $B(x_0,r)$ and $\eta = 0$ outside of $B(x_0,R)$. We write

$$0 = \int u\Delta(v\eta) = \int u\varphi\eta + \int uv\Delta\eta + 2intu\nabla v \cdot \nabla\eta.$$

Using $\varphi\eta = \varphi$ (since $\eta = 1$ on $\mathrm{spt}(\varphi)$), we obtain

$$\int u\varphi = -\int uv\Delta\eta - 2\int u\nabla v \cdot \nabla\eta \leq ||u||_{H^{-k}(B(x_0,R)} \left(||v\Delta\eta||_{H^k(B(x_0,R)} + 2||\nabla v \cdot \nabla\eta||_{H^k(B(x_0,R)}\right).$$

Since η is smooth and fixed, and its norms only depend on r, R, we obtain

$$||v\Delta\eta||_{H^k(B(x_0,R)}, ||\nabla v \cdot \nabla\eta||_{H^k(B(x_0,R)} \leq C(k,r,R)||\nabla v||_{H^k(B(x_0,R)}.$$

Applying the Corollary 1 we obtain $||\nabla v||_{H^k} \leq ||v||_{H^{k+1}} \leq C(k,r,R)||\varphi||_{H^{k-1}}$, which provides the desired result. ☐

We can then obtain

Proposition 7. *Suppose that $u \in H^{-k}_{loc}(\Omega)$ is harmonic in Ω in the sense of distributions. Then u is an analyitic funtion.*

Proof. The proof is the same as in Proposition 5: we regularize by convolution and apply the bounds on the Sobolev norms. Lemma 1 allows to pass from H^{-k} to H^{1-k} and, iterating, arrive to L^2. Once we know that u is in L^2_{loc} we directly apply Proposition 5. ☐

Finally, we have

Proposition 8. *Suppose that u is harmonic distribution in Ω. Then u is an analyitic funtion.*

Proof. We just need to show that u locally belongs to a space H^{-k}. This is a consequence of the definition of distributions. Indeed, we have the following: for every distribution u and every compact set $K \subset \Omega$ there exist n, C such that $< u, \varphi > \leq C||\varphi||_{C^n}$ for very $\varphi \in C^\infty$ with $\mathrm{spt}(\varphi) \subset K$. to prove this we just need to act by contradiction: if it is not true, then there exists a distribution u and a compact set K such that for every n we find φ_n with

$$< u, \varphi_n >= 1, \quad ||\varphi_n||_{C^n} \leq \frac{1}{n}, \quad \mathrm{spt}(\varphi_n) \subset K.$$

Note that we define the C^n norm as the sup of all derivatives up to order n; so that $||\varphi||_{C^{n+1}} \geq ||\varphi||_{C^n}$. Yet, this is a contradiction since the sequence φ_n tends to 0 in the space of C_c^∞ functions and u should be continuous for this convergence. So we have the inequality $< u, \varphi > \leq C||\varphi||_{C^n}$ which can be turned into $< u, \varphi > \leq C||\varphi||_{H^k}$ because of the continuous embedding of Sobolev spaces into C^n spaces (take $k > n + d/2$). ☐

References

[1] Q. Han, F. Lin *Elliptic Partial Differential Equations* Second Edition, American Mathematical Society, 2011