Calculus of Variations and Elliptic Equations
–
5th class

From local to global regularity – reflections

Let $\Omega \subset \mathbb{R}^d$ be a compact domain, closure of an open set, with a flat part in $\partial\Omega$ and $R : \mathbb{R}^d \to \mathbb{R}^d$ a reflection which fixes this part. For instance, if $\Omega = [0,1]^d$, we can consider $R(x_1, \dots, x_d) = (-x_1, \dots, x_d)$. The map R is linear, self-adjoint, with determinant equal to -1, and $R^2 = id$.

Suppose that $u \in H^1_0(\Omega)$ solves in the weak sense $\Delta u = f$. Define $\tilde\Omega := \Omega \cup R(\Omega)$ and

$$\tilde u(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ -u(Rx) & \text{if } x \in R(\Omega), \end{cases}.$$

We define in analoguous way $\tilde f$. Note that the condition $u \in H^1_0$ (i.e., zero on the boundary) makes $\tilde u$ well-defined on $\Omega \cap R(\Omega)$ and we have $\tilde u \in H^1_0(\tilde\Omega)$. This is not the case for f if we do not suppose any boundry condition on it, but $f \in L^p(\Omega)$ clearly implies $\tilde f \in L^p(\tilde\Omega)$.

We want to prove that we also have $\Delta \tilde u = \tilde f$ in $\tilde\Omega$. To do so, we take a test function $\varphi \in H^1_0(\tilde\Omega)$ and we compute

$$\int_{\tilde\Omega} \nabla\tilde u \cdot \nabla\varphi = \int_\Omega \nabla u \cdot \nabla\varphi + \int_{R(\Omega)} -R\nabla u \circ R \cdot \nabla\varphi.$$

Now, we have

$$\int_{R(\Omega)} R\nabla u \circ R \cdot \nabla\varphi = \int_{R(\Omega)} \nabla u \circ R \cdot R\nabla\varphi = \int_\Omega \nabla u \cdot R\nabla\varphi \circ R = \int_\Omega \nabla u \cdot \nabla(\varphi \circ R),$$

where the first equality comes from R being self-adjoint and the second is a change-of-variable. If we now set $\hat\varphi := \varphi - \varphi \circ R$ we have

$$\int_{\tilde\Omega} \nabla\tilde u \cdot \nabla\varphi = \int_\Omega \nabla u \cdot \nabla\hat\varphi.$$

We just need to observe that we have $\hat\varphi \in h^1_0(\Omega)$ since it vanishes both on $\partial\Omega \setminus R(\Omega)$ (as a consequence of $\varphi \in H^1_0(\tilde\omega)$) and on $\partial\Omega \cap R(\Omega)$ (because here R is the identity). We then have

$$\int_{\tilde\Omega} \nabla\tilde u \cdot \nabla\varphi = \int_\Omega \nabla u \cdot \nabla\hat\varphi = -\int_\Omega f\hat\varphi = -\int_\Omega f\varphi + \int_\Omega f\,\varphi \circ R.$$

In the very last integral we can use a change of variable and obtain $\int_\Omega f\,\varphi \circ R = \int_{R(\Omega)} f \circ R\,\varphi$, so that we finally have

$$\int_{\tilde\Omega} \nabla\tilde u \cdot \nabla\varphi = -\int_{\tilde\Omega} \tilde f\varphi,$$

which means $\Delta\tilde u = \tilde f$.

In the case of Neumann boundary conditions instead of Dirchlet ones it is possible to use a different symmetrization, namely

$$\tilde u(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ u(Rx) & \text{if } x \in R(\Omega), \end{cases}$$

and the same for $\tilde f$ (no change of sign in $R(\Omega)$). This is useful because if u does not vanish on the boundary the extension with the cange of sign does not belong to H^1. On the other hand, the function $\hat\varphi$ will be now defined as $\hat\varphi := \varphi + \varphi \circ R$ and does not vanish anymore on $\partial\Omega \cap R(\Omega)$, but since u solves $\Delta u = f$ with Neumann boundary conditions, the relation $\int \nabla u \cdot \nabla\phi = -\int f\phi$ is true for any $\phi \in H^1(\Omega)$, with no conditions on the values on the boundary.

Proposition 1. *Suppose that* $u \in H^1_0(\Omega)$ *is a weak solution of* $\Delta u = f$ *(i.e.* $\int \nabla u \cdot \nabla \phi = -\int f\phi$ *for every* $\phi \in C^1_c(\Omega)$*) with* $f \in L^p(\Omega)$ *and* Ω *is a cube. Then* $u \in W^{2,p}(\Omega)$*.*

Suppose that $u \in H^1(\Omega)$ *is weak solution of* $\Delta u = f$ *with Neumann boundary conditions (i.e.* $\int \nabla u \cdot \nabla \phi = -\int f\phi$ *for every* $\phi \in C^1(\Omega)$*) with* $f \in L^p(\Omega)$ *and* Ω *is a cube. Then* $u \in W^{2,p}(\Omega)$*. If moreover* $f \in W^{1,p}(\Omega)$ *then* $u \in W^{3,p}(\Omega)$*.*

Proof. Both for the Dirichlet and for the Neumann case we can iterate reflections R_k so that we iterately extend u into functions u_k, defined on $\Omega_k := \Omega_{k-1} \cup R_{k-1}(\Omega_{k-1})$, with $\Omega_0 = \Omega$, and extanding f to f_k on the same domains, with $\Delta u_k = f_k$, according to the construction that we previously detailed. After a suitable number of iterations if the reflections are well-chosen then Ω is compactly contained in Ω_k and the local regularity result coming from the L^p theory for the Laplacian (see [1]) implies $u \in W^{2,p}$ if $f_k \in W^{2,p}$. Since reflections preserve the L^p summability, we do have $f_k \in L^p(\Omega_k)$ and we can apply this to $u \in W^{2,p}(\Omega)$ in both cases. Should we have $f_k \in W^{1,p}$, by differentiating the equation we would also have $u \in W^{3,p}(\Omega)$ (note that we need to first reflect and then differentiate, since if we first differentiate the equation we lose the boundary data). On the other hand, the reflection defined with a minus sign in the Dirichlet case in general destroy the $W^{1,p}$ behavior as they create discontinuities at the common boundary between Ω and $R(\Omega)$, so we cannot conclude the $W^{3,p}$ regularity in the Dirichlet case. Yet, this can be done in the Neumann case as the reflection without changing the sign preserves the $W^{1,p}$ behavior. $\square$

References

[1] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, Berlin, 1977.