Calculus of Variations and Elliptic Equations
–
9th-10th class

Asymptotics of an optimal location problem

Let us consider $f \in C^0(\Omega)$ a strictly positive probability density on a compact domain $\Omega \subset \mathbb{R}^d$. We consider

$$\min\{\int d(x,S)f(x)dx \;\; S \subset \Omega, \#S = N\} \tag{1}$$

and associate with every set S with $\#S = N$ the uniform probability measure on S, i.e. $\mu_S = \frac{1}{N}\sum_{y \in S}\delta_y \in \mathcal{P}(\Omega)$. Our question is to identify the limit as $N \to \infty$ of the measures μ_{S_N} where S_N is optimal.

We define the functionals $F_N : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ through

$$F_N(\mu) := \begin{cases} N^{1/d}\int d(x,S)f(x)dx & \text{if } \mu = \mu_S \text{ with } \#S = N, \\ +\infty & \text{otherwise.} \end{cases}$$

We denote by I^d the unit cube $I^d = [0,1]^d$. Let us define the constant

$$\theta := \inf\{\liminf_N N^{1/d}\int_{I^d} d(x,S_N)dx, \; \#S_N = N, \}$$

as well as, for technical reasons, the similar constant

$$\tilde\theta := \inf\{\liminf_N N^{1/d}\int_{I^d} d(x,S_N \cup \partial I^d)dx, \; \#S_N = N, \}.$$

Proposition 1. *We have $\theta = \tilde\theta$ and $0 < \theta < \infty$.*

Proof. We have of course $\theta \geq \tilde\theta$. To prove the opposite inequality, fix $\varepsilon > 0$ and select a sequences of uniform grids on ∂I^d: decompose the boundary into $2dM^{d-1}$ small subes, each of size $1/M$, choosing M such that $M^{-1} < \varepsilon N^{-1/d}$. We call such a grid G_N. Take a sequence S_N which almost realizes the infimum in the definition of $\tilde\theta$, i.e. $\#S_N = N$ and $\liminf_N N^{1/d}\int_{I^d} d(x,S_N \cup \partial I^d)dx \leq (1+\varepsilon)\tilde\theta$. We then use

$$d(x,S_N \cup G_N) \leq d(x,S_N \cup \partial I^d) + \frac{\sqrt{d-1}}{M}$$

to obtain

$$\liminf_N N^{1/d}\int_{I^d} d(x,S_N \cup G_N)dx \leq (1+\varepsilon)\tilde\theta + \limsup_N N^{1/d}\frac{1}{M} \leq (1+\varepsilon)\tilde\theta + \varepsilon.$$

If we use $\#(S_N \cup G_N) \leq N + 2dM^{d-1} = N + O(N^{(d-1)/d}) = N + o(N)$ we obtain a sequence of sets $\tilde S_N := S_N \cup G_N$ such that

$$\liminf_N (\#\tilde S_N)^{1/d}\int_{I^d} d(x,\tilde S_N)dx \leq (1+\varepsilon)\tilde\theta + \varepsilon,$$

hence $\theta \leq \tilde\theta$.

In order to prove $\theta < +\infty$, just use a sequence of sets on a uniform grid in I^d: we can decompose the whole cube into M^d small subes, each of size $1/M$, choosing M such that $M^{\approx}N^{1/d}$.

In order to prove $\theta > 0$ we also use a uniform grid, but choosing M such that $M^d > 2N$.Then we take an arbitrary S_N with N points: in this case at least half of the cubes of the grid do not contain points of S_N. An empty cube of size δ contributes for at least $C\delta^{d+1}$ in the integral, i.e. $M^{-(d+1)}$. Nice at least N cubes are empty we obtain $\theta \geq N^{1/d}.N.M^{-(d+1)} = O(1)$. □

We then define the functional $F : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ through

$$F(\mu) := \theta \int_\Omega \frac{f}{(\mu^{ac})^{1/d}},$$

where μ^{ac} is the density of the absolutely continuous part of μ.

We will prove the following.

Proposition 2. *Suppose that Ω is a cube and f is strictly positive and condinuous. Then we have $F_N \xrightarrow{\Gamma} F$ in $\mathcal{P}(\Omega)$ (endowed with the weak-* convergence) as $N \to \infty$.*

Proof. Let us start from the Γ-liminf inequality. Consider $\mu_N \rightharpoonup \mu$and suppose $F_N(\mu_N) \leq C$. In particular, we have $\mu_N = \mu_{S_N}$ for a sequence of sets S_N with $\#S_N = N$. Let us define the functions $\lambda_N := N^{1/d} f d(s, S_N)$. This sequence of functions is bounded in L^1, so we can assume that it converges weakly-* as positive measures to a measure λ up to a subsequence. Choosing a subsequence which realizes the liminf we will have $\liminf_N F_N(\mu_N) = \lambda(\Omega)$.

In order to estimate λ from below we fix a closed cube $Q \subset \Omega$. Let us call δ the size of this cube (its side, so that $|Q| = \delta^d$). We write

$$\lambda_N(Q) = N^{1/d} \int_Q f d(s, S_N) \geq \min_Q f \left(\frac{1}{\mu_N(Q)}\right)^{1/d} (\#S_N \cap Q)^{1/d} \int_Q d(x, S_N \cup \partial Q) dx.$$

We note that the last part of the right-hand side recalls the definition of $\tilde{\theta}$. We also note that if we want to bound from below $\lambda_N(Q)$ we can assume $\lim_N \#S_N \cap Q = \infty$, otherwise if the number of points in Q stays bounded we necessarily have $\lambda_N(Q) \to \infty$. So, the sequence of sets $S_N \cap Q$ is admissible in the definition of $\tilde{\theta}$,but we need to scale: indeed, if the unit cube in the definiiton of $\tilde{\theta}$ is replaced by a cube of size δ, the values of the integrals are multiplied times δ^{d+1}. We then have

$$\liminf_N (\#S_N \cap Q)^{1/d} \int_Q d(x, S_N \cup \partial Q) dx \geq \delta^{d+1} \tilde{\theta}$$

and hence

$$\liminf_N \lambda_N(Q) \geq \min_Q f \liminf_N \left(\frac{1}{\mu_N(Q)}\right)^{1/d} \delta^{d+1} \tilde{\theta}.$$

We now use the fact that, for closed sets, when a sequence of measures weakly converges the mass given by the limit measure is larger thatn the limusp of the masses:

$$\lambda(Q) \geq liminf_N \lambda_N(Q) \geq \min_Q f \left(\frac{1}{\mu(Q)}\right)^{1/d} \delta^{d+1} \tilde{\theta}.$$

This can be re-written as

$$\frac{\lambda(Q)}{|Q|} \geq \min_Q f \left(\frac{|Q|}{\mu(Q)}\right)^{1/d} \theta,$$

where we aso used $\theta = \tilde{\theta}$. We now choose a sequence of cubes shrinking around a point $x \in \Omega$ and we use the fact that, for a.e. x, the ratio between the mass a measure gives to the cube and the volume of the cube tends to the density of the absolutely continuous part, thus obtaining (also using the continuity of f)

$$\lambda^{ac}(x) \geq \theta f(x) \left(\frac{1}{\mu^{ac}(x)}\right)^{1/d}.$$

This implies

$$\liminf_N F_N(\mu_N) = \lambda(\Omega) \geq \int_\Omega \lambda^{ac}(x) dx \geq F(\mu).$$

We now switch to the Γ-limsup inequality. Let us start from the case $\mu = \sum_i a_i I_{Q_i}$, i.e. μ is absolutely continuous with piecewise constant density $a_i > 0$ on the cubes of a regular grid. In order to have a probability measure, we suppose $\sum_i a_I |Q_i| = 1$. Fix $\varepsilon > 0$. Using the definition of θ we can find a finite set $S_0 \subset I^d$ with $\# S_0 = N_0$ such that $N_0^{1/d} \int_{I^d} d(x, S_0) dx < \theta(1+\varepsilon)$. We then divide each cube Q_i into M_i^d subcubes $Q_{i,j}$ of size δ_i on a regular grid, and on each subcube we put a scaled copy of S_0. We have $M_i^d \delta_i^d = |Q_i|$. We choose M_i such that $N_0 M_i^d \approx a_i |Q_i| N$ so that, for $N \to \infty$, we have indeed $\mu_N \rightharpoonup \mu$ (where μ_N is the the uniform measure on the set S_N obtained by the union of all these scaled copies). We now estimate

$$F_N(\mu_N) \le N^{1/d} \sum_{i,j} \delta_i^{d+1} \theta(1+\varepsilon) N_0^{-1/d} \max_{Q_{i,j}} f.$$

For N large enough, the cubes $Q_{i,j}$ are small and we have $\max_{Q_{i,j}} f \le (1+\varepsilon) \fint_{Q_{i,j}} f = (1+\varepsilon)\delta_i^{-d} \int_{Q_{i,j}} f$, hence we get

$$F_N(\mu_N) \le N^{1/d} \theta(1+\varepsilon) N_0^{-1/d} \sum_{i,j} \delta_i \int_{Q_{i,j}} f.$$

Note that we have $\delta_i = |Q_i|^{1/d}/M \approx N_0^{1/d} N^{-1/d} a_i^{-1/d}$, whence

$$\limsup_N F_N(\mu_N) \le \theta(1+\varepsilon) \sum_{i,j} \int_{Q_{i,j}} \frac{f}{a_i^{1/d}} = (1+\varepsilon) F(\mu).$$

This shows, ε being arbitrary, the Γ-limsup inequality in the case $\mu = \sum_i a_i I_{Q_i}$.

We now need to extend our Γ-limsup inequality to other measures μ which are not of the form $\mu = \sum_i a_i I_{Q_i}$. We need hence to show that this class of measures is dense in energy.

Take now anarbitrary probability μ with $F(\mu) < /\infty$. Since f is supposed to be stritly positive, this implies $\mu^{ac} > 0$ a.e. Take a regular grid of size $\delta_k \to 0$, composed of k^d disjoint cubes Q_i and define $\mu_k := \sum_i a_i I_{Q_i}$ with $a_i = \mu(Q_i)$ (one has to define the subcubes in a disjoint way, for instance as products of semi-open intevals, of the form $[0, \delta_k)^d$). It is clear that we have $\mu_k \rightharpoonup \mu$ since the mass is preserved in every cube, whose diameter tends to 0.

We then compute $F(\mu_k)$. We have

$$F(\mu_k) \le \sum_i \max_{Q_i} f |Q_i| \left(\frac{\mu(Q_i)}{|Q_i|} \right)^{-1/d}.$$

We use the function $U(s) = s^{-1/d}$, which is decreasin and convex, with a Jensen's inequality to obtain

$$\left(\frac{\mu(Q_i)}{|Q_i|} \right)^{-1/d} = U\left(\frac{\mu(Q_i)}{|Q_i|}\right) \le U\left(\fint_{Q_i} \mu^{ac}\right) \le \fint_{Q_i} g(\mu^{ac}).$$

This allows to write

$$F(\mu_k) \le \sum_i \max_{Q_i} f |Q_i| \fint_{Q_i} U(\mu^{ac}) = \sum_i (\max_{Q_i} f) \int_{Q_i} U(\mu^{ac}).$$

We finish by noting that, for $k \to \infty$, we have $(\max_{Q_i} f) \int_{Q_i} U(\mu^{ac}) \le (1+\varepsilon_k) \int_{Q_i} f U(\mu^{ac})$ for $\varepsilon_k \to 0$ (depending on the modulus of continuity of f), and hence

$$F(\mu_k) \le (1+\varepsilon_k) F(\mu),$$

which concludes the proof. □

Note that the assumption that Ω is a cube is just done for simplicity in the Γ-limsup, and that it is possible to get rid of it by suitably considering the "rests" after filling Ω with cubes.

The above proof is a simplified version of that in [?] where f was only supposed to be lsc. Actually, in [?] (which deals with a similar but different problem) even this assumption is removed, and f is only supposed to be L^1.

A consequence is the following

Proposition 3. *Suppose that S_N is a sequence of optimizers for* (1) *with $N \to \infty$. Then he sequence μ_N weakly-* converges to the measure μ which is absolutely continuous with density ρ equal to $cf^{d/(d+1)}$, where c is a normalization constant such that $\int \rho = 1$.*

Proof. We just need to prove that this measure μ is the unique optimizer of F. First note that F can only be minimized by an absolutely continuous measure, as singular part do not affect the value of the functional, so it is better to remove a possible singular part and use the same mass to increase the absolutely continuous part.

Then, using again the notation $U(s) = s^{-1/d}$. Also write $\rho = cf^{d/(d+1)}$ as in the statement. Then we have, for $\mu \ll \mathcal{L}^d$,

$$F(\mu) = \int fU(\mu^{ac}) = c \fint U\left(\frac{\mu^{ac}}{\rho}\right) \rho \geq cU\left(\fint \left(\frac{\mu^{ac}}{\rho}\right) \rho\right) = cU(\int \mu^{ac}) = cU(1).$$

The inequality is due to Jensen's inequality and is an equality if and only if μ^{ac}/ρ is constant, which proves the claim. □

References

[1] G. Bouchitté, C. Jimenez, M. Rajesh: Asymptotique d'un problème de positionnement optimal, *C. R. Acad. Sci. Paris* Ser. I, **335** (2002) 1–6.

[2] S. Mosconi and P. Tilli: Γ−Convergence for the Irrigation Problem, 2003. *J. of Conv. Anal.* **12**, no.1 (2005), 145–158.