Calculus of Variations and Elliptic Equations
–
9th class

Γ-convergence

Let us fix a metric space (X, d). Given a sequence of functionals $F_n : X \to \mathbb{R} \cup \{+\infty\}$ we define their Γ-liminf and their Γ-limsup as two other functionals on X, in the following way.

$$(\Gamma - \liminf F_n)(x) := \inf\{\liminf_n F_n(x_n) \, : \, x_n \to x\}, \quad (\Gamma - \limsup F_n)(x) := \inf\{\limsup_n F_n(x_n) \, : \, x_n \to x\}.$$

We say that F_n Γ-converges to F, and we write $F_n \xrightarrow{\Gamma} F$, if $(\Gamma - \liminf F_n) = (\Gamma - \limsup F_n) = F$. Very often we will write F^- instead of $\Gamma - \liminf F_n$ and F^+ instead of $\Gamma - \limsup F_n$.

In practive, proving $F_n \xrightarrow{\Gamma} F$ requires to prove two facts:

- *(Γ−liminf inequality)*: we need to prove $F^- \geq F$, i.e. we need to prove $\liminf_n F_n(x_n) \geq F(x)$ for any approximating sequence $x_n \to x$; of course it is sufficient to prove it when $F_n(x_n)$ is bounded;
- *(Γ−limsup inequality)*: we need to prove $F^+ \leq F$, i.e. we need to find, for every x, a sequence $x_n \to x$ such that $\limsup_n F_n(x_n) \leq F(x)$ or at least, for every $\varepsilon > 0$, find a sequence with $\limsup_n F_n(x_n) \leq F(x) + \varepsilon$. We will see later that this inequality can be restricted to suitable dense subsets. In any cases it can be obviously restricted to the set $\{F < +\infty\}$, otherwise the inequality is trivial.

We will prove the following propositions

Among the properties of Γ−convergence we have the following:

- if there exists a compact set $K \subset X$ such that $\inf_X F_n = \inf_K F_n$ for any n, then F attains its minimum and $\inf F_n \to \min F$;
- if $(x_n)_n$ is a sequence of minimizers for F_n admitting a subsequence converging to x, then x minimizes F
- if F_n is a sequence Γ−converging to F, then $F_n + G$ will Γ−converge to $F + G$ for any continuous function $G : X \to \mathbb{R} \cup \{+\infty\}$.

Proposition 1. *If $F_n \xrightarrow{\Gamma} F$ then for every subsequence we still have $F_{n_k} \xrightarrow{\Gamma} F$.*

Proof. Let us take a sequence $x_{n_k} \to x$ corresponding to the indices of this subsequence and let us complete it to a full sequence using $x_n = x$ for all the other indices. We have

$$\liminf_k F_{n_k}(x_{n_k}) \geq \liminf_n F_n(x_n) \geq (\Gamma - \liminf F_n)(x),$$

which proves that the Γ-liminf of the subsequence is larger than that of the full sequence.

For the Γ-limsup, let us take a recovery sequence $x_n \to x$ for the full sequence. We then have

$$\limsup_k F_{n_k}(x_{n_k}) \leq \limsup F_n(x_n) \leq (\Gamma - \limsup F_n)(x) + \varepsilon,$$

thus proving $(\Gamma - \limsup F_{n_k}) \leq (\Gamma - \limsup F_n) + \varepsilon$.

this proves that passing to a subsequence increases the Γ-liminf and reduces the Γ-limsup, but when they coincide they cannot change, thus proving the claim. □

Proposition 2. *Both F^- and F^+ are lsc functionals.*

Proof. Let us take $x \in X$ and a sequence $x_k \to x$ as in the statement. We want to prove the semicontinuity of F^-. For each k there is a sequence $x_{n,k}$ such that $\lim_n x_{n,k} = x_k$ and $\liminf_n F_n(x_{n,k}) < F^-(x_k) + 2^{-k}$. We can hence choose $n = n(k)$ arbitrarily large such that $d(x_{n,k}, x_k) < 2^-k$ and $F_n(x_{n,k}) < F^-(x_k) + 2^{-k+1}$. We can assume $n(k+1) > n(k)$ which means that the sets of indices n that we use defines a subsequence. Consider a sequence $\tilde{x}_n$ defined as

$$\tilde{x}_n = \begin{cases} x_{n,k} & \text{if } n = n(k), \\ x & \text{if } n \neq n(k) \text{ for all } k \end{cases}$$

We have $\lim_n \tilde{x}_n = x$ and

$$F^-(x) \leq \liminf_n F_n(\tilde{x}_n) \leq \liminf_k F_{n(k)}(x_{n(k),k}) \leq \liminf F^-(x_k),$$

which proves that F^- is lsc.

We want to prove the semicontinuity of F^+. We can assume, up to subsequences, that $\lim F^-(x_k)$ exists. For each k there is a sequence $x_{n,k}$ such that $\lim_n x_{n,k} = x_k$ and $\limsup_n F_n(x_{n,k}) < F^+(x_k) + 2^{-k}$. This means that we can hence choose $n = n(k)$ such that for every $n \geq n(k)$ we have $d(x_{n,k}, x_k) < 2^-k$ and $F_n(x_{n,k}) < F^+(x_k) + 2^{-k+1}$. We can choose the sequence $n(k)$ such that $n(k+1) > n(k)$. We now define a sequence $\tilde{x}_n$ defined as

$$\tilde{x}_n = x_{n,k} \text{ if } n(k) \leq n < n(k+1).$$

We have $\lim_n \tilde{x}_n = x$ and

$$F^+(x) \leq \limsup_n F_n(\tilde{x}_n) \leq \lim_k F^+(x_k),$$

which proves that F^+ is lsc.

□

Proposition 3. *If $F^+ \leq F$ on a set D with the following property: for every $x \in X$ there is a sequence $x_n \to x$ contained in D and such that $F(x_n) \to F(x)$, then $F^+ \leq F$ on X.*

Proof. Let us take $x \in X$ and a sequence $x_n \to x$ as in the statement. We then write $F^+(x_n) \leq F(x_n)$ and pass to the limit, using the semicontinuity of F^+ and the assumption $F(x_n) \to F(x)$. This provides $F^+(x) \leq F(x)$, as required. □

Proposition 4. *If $F_n \xrightarrow{\Gamma} F$ and there exists a compact set $K \subset X$ such that $\inf_X F_n = \inf_K F_n$ for any n, then F attains its minimum and $\inf F_n \to \min F$;*

Proof. Let us first prove $\liminf_n \inf F_n \geq \inf F$. Up to extracting a subsequence, we can suppose that this liminf is a limit, and Γ-convergence is preserved. Now, take $x_n \in K$ such that $F_n(x_n) \leq \inf F_n + \frac{1}{n}$. Extract a subsequence $x_{n_k} \to x_0$ and look at the sequence of functionals F_{n_k} whoch also Γ-converge to F. We then have $\inf F \leq F(x_0) \leq \liminf F_{n_k}(x_{n_k}) = \liminf \inf F_n$.

Take now a point x, and $\varepsilon > 0$. There exists $x_{\to} x$ such that $\limsup F_n(x_n) \leq F(x) + \varepsilon$. In particular, $\limsup \inf F_n \leq F(x) + \varepsilon$ and, $\varepsilon > 0$ and $x \in X$ being arbitrary, we get $\limsup \inf F_n \leq \inf F$.

We then have

$$\limsup \inf F_n \leq \inf F \leq F(x_0) \leq \mathit{liminf} \inf F_n \leq \limsup \inf F_n.$$

Hence, all inequalities are equalities, which proves at the same $\inf F_n \to \inf F$ and $\inf F = F(x_0)$, i.e. the min is attained by x_0. □

Proposition 5. *If $F_n \xrightarrow{\Gamma} F$ and $x_n \in \operatorname{argmin} F_n$ and $x_n \to x$, then $x \in \operatorname{argmin} F$.*

Proof. First notice that $\bigcup_n \{x_n\} \cup \{x\}$ is a compact set on which all the functionals F_n attain their minimum, so we can apply the previous proposition and get $\inf F_n \to \inf F$. We then apply the Γ-liminf property to this sequence, thus obtaining

$$F(x) \leq \liminf F_n(x_n) = \liminf \inf F_n = \inf F,$$

which proves at the same time $x \in \operatorname{argmin} F$ and $\operatorname{argmin} F \neq \emptyset$. □

Proposition 6. *If $F_n \xrightarrow{\Gamma} F$, then $F_n + G \xrightarrow{\Gamma} F + G$ for any continuous function $G : X \to \mathbb{R} \cup \{+\infty\}$.*

Proof. To prove the Γ-liminf inequality, take $x_n \to x$: we have $\liminf(F_n(x_n)+G(x_n)) = (\liminf F_n(x_n))+G(x) \geq F(x) + G(x)$. To prove the Γ-limisupinequality, the same recovery sequence $x_n \to x$:can be used: indeed, $\limsup(F_n(x_n) + G(x_n)) = (\limsup F_n(x_n)) + G(x) \leq F(x) + \varepsilon + G(x)$. □

We then consider some examples

Example 1 Consider $X = L^2(\Omega)$ and some functions $f_n \in L^2(\Omega)$. Consider

$$F_n(u) := \begin{cases} \int \frac{1}{p}|\nabla u|^p + f_n u & \text{if } u \in W_0^{1,p}(\Omega), \\ +\infty & \text{if not.} \end{cases}$$

Suppose $f_n \rightharpoonup f$. Then we have $F_n \xrightarrow{\Gamma} F$ where F is defined by replacing f_n with f.

This can be proven in the following way. For the Γ-liminf inequality, suppose $u_n \to u$ in L^2 and $F_n(u_n) \leq C$. Using the L^2 bounds on u_n and f_n we see that the $W^{1,p}$ norm of u_n is bounded, so that we can assume $\nabla u_n \rightharpoonup \nabla u$ in L^p and, by semicontinuity $\int \frac{1}{p}|\nabla u|^p \leq \liminf \int \frac{1}{p}|\nabla u_n|^p$. In thaw concerns the second part of the integral we have $\int f_n u_n \to \int fu$ since we have weak convergence of f_n and strong of u_n. We tthen obtain $\liminf F_n(u_n) \geq F(u)$. For the Γ-limsup inequality it is enough to choose u with $F(u) < +\infty$ and take $u_n = u$.

This Γ-convergence result implies the convergence of the minimizers as soon as they are compact for the L^2 strong convergence. This is true for $p \geq \frac{2d}{d-2}$ so that $W^{1,p}$ compactly embeds into L^2. We then deduce the L^2 convergence of the solutions u_n of $\Delta_p u_n = f_n$ to the solution of $\Delta_p u = f$. Note that the linear case $p = 2$ could be easy studied by bounding the norm $||u_n - u||_{H^1}$ in termes of $||f_n - f||_{H^{-1}}$ (which tends to 0 since L^2 compactly embeds into H^{-1}) but the situation is more complicated in the non-linear case and Γ-convergence is a useful tool.

As a last remark, we observe that for $p \geq \frac{2d}{d-2}$ the choice of the strong L^2 convergence to establish the Γ-convergence is irrelevant. We could have for instance chosen the weak L^2 convergence and deduce the strong L^2 convergence in the Γ-liminf inequality from the $W^{1,p}$ bound. We could have chosen other norms, but in this case it would have been necessary to first obtain a bound on $||u_n||_{W^{1,p}}$.

Example 2 We now consider a more surprising example. We consider a sequence of functions $a_n : [0, 1] \to \mathbb{R}$ such that $\lambda \leq a_n \leq \Lambda$ for two strictly positive constants λ, Λ. We then define a functional on $L^2([0, 1])$ via

$$F_n(u) := \begin{cases} \int_0^1 a_n(x)\frac{|u'(x)|^2}{2}dx & \text{if } u \in W^{1,2}, u(1) = 0, \\ +\infty & \text{if not.} \end{cases}.$$

We now wonder what could be the Γ-limit of F_n. A natural guess is to suppose, up to subsequences, that we have $a_n \rightharpoonup a$ (weak-* convergence in L^∞) and hope to prove that the functional where we replace a_n with a is the Γ-limit. If this was true, then we would also have Γ-convergence should we add $\int fu$ to the functional, for fixed $f \in L^2$, and convergence of the minimizers. The minimizers u_n would be characterized by $(a_n u_n')' = f$ with a transversality condition $u_n'(0) = 0$, so that we have $a_n(x)u_n'(x) = \int_0^x f(t)dt$ and hence $u_n'(x) = (a_n(x))^{-1} \int_0^x f(t)dt$. We then see that the weak convergence of a_n is not the good assumption, but we should rather require $a_n^{-1} \rightharpoonup a^{-1}$!!

We can now prove the following. Suppose that a_n is such that $a_n^{-1} \rightharpoonup a^{-1}$, then $F_n \xrightarrow{\Gamma} F$, where F is defined as F_n by replacing a_n with a.

To prove the Γ-liminf inequality, we write $F_n(u) = \int L(u'_n,{}_n^{-1})dx$, where $L : \mathbb{R} \times \mathbb{R}_+$ is given by $L(v, s) = \frac{|v|^2}{2s}$. We take a sequence $u_n \to u$ with $F_n(u_n) \leq C$. This boiund implies that u_n is bounded in H^1 and then $u'_n \rightharpoonup u'$ in L^2. We can easily check that L is a convex function of two variables, for instance by computing its Hessiand, which is given by

$$D^2L(v, s) = \begin{pmatrix} \frac{1}{s} & -\frac{v}{s^2} \\ \frac{v}{s^2} & \frac{|v|^2}{s^3} \end{pmatrix} \geq 0.$$

Hence, by semicontinuity, we deduce from $a_n^{-1} \rightharpoonup a^{-1}$ and $u'_n \rightharpoonup u'$ that we have $\liminf F_n(u_n) \geq F(u)$.

To prove the Γ-limsup inequality, given u with $F(u) < +\infty$ (i.e. $u \in H^1$), we define u_n via $u'_n = \frac{a}{a_n}u'$ and $u_n(1) = 0$. We see that u'_n is bounded in L^2 and hence $u'_n \rightharpoonup v$. Integrating against a test function ans using $a_n^{-1} \rightharpoonup a^{-1}$ we see $v = \frac{a}{a}u' = u'$, so that we have $u'_n \rightharpoonup u'$ and, thanks to the final value $u_n(1) = u(1) = 0$, we deduce weak convergence of u_n to u in H^1, and strong in L^2. We then have

$$F_n(u_n) = \int \frac{1}{2}a_n u'_n \cdot u'_n = \int \frac{1}{2}au' \cdot u'_n \to \int \frac{1}{2}au' \cdot u' = F(u),$$

which proves the Γ-limsup inequality.