ENS Lyon and Université Claude Bernard Lyon 1, M2A
November 7th, 2025

Calculus of Variations and Elliptic PDEs
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Mid-Term Examination
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All kind of documents (notes, books...) are authorized. The total number of points is much larger than 20, which means that attacking only some exercises could be a reasonable option.
The exercises are not necessarily ordered by difficulty.

Exercice 1 (8 points). Given $\varepsilon > 0$ and a function $f : [0, \pi] \to \mathbb{R}$ consider the problem

$$\min \left\{ \int_0^\pi \frac{\varepsilon}{2}|u'(t)|^2 + \frac{1}{2\varepsilon}|u(t) - f(t)|^2 dt : \quad u \in C^1([0, \pi]) \right\}.$$

We consider two cases.
In case a) the function f is given by $f(t) = \cos(t)$.
In case b) the function f is given by

$$f(t) = \begin{cases} 1 & \text{if } t < \pi/2, \\ 0 & \text{if } t = \pi/2, \\ -1 & \text{if } t > \pi/2. \end{cases}$$

In both cases find the solution u_ε of the problem, properly justifying its minimality and its uniqueness, and prove that we have $\lim_{\varepsilon \to 0} u_\varepsilon = f$, at least pointwisely. In which cases is this convergence uniform?

Answer Let us write the Euler-Lagrange equation of the problem. Using

$$L(t, x, v) = \frac{\varepsilon}{2}|v|^2 + \frac{1}{2\varepsilon}|x - f(t)|$$

we see that the equation, together with its boundary conditions, is

$$\begin{cases} \varepsilon^2 u'' = u - f \\ u'(0) = 0, \\ u'(\pi) = 0. \end{cases}$$

Let us solve it in the two cases. For simplicity, we set $n = \varepsilon^{-1}$

In case a) we have $u(t) = Ae^{nt} + Be^{-nt} + C \cos t$, where the last term comes from a particular solution. In order to satisfy the equation we need $C \cos t$ to be indeed a particular solution, so we need $-\varepsilon^2 C = C - 1$, i.e. $C = (1 + \varepsilon^2)^{-1}$. In order to satisfy the boundary conditions, using that the derivative of the cosinus vanishes at $t = 0, \pi$, we need $nA - nB = 0$, i.e. $A = B$, and then $nAe^{n\pi} - nBe^{-n\pi} = 0$, which finally provides $A = B = 0$. The solution is then $u_\varepsilon(t) = \cos(t)/(1 + \varepsilon^2)$ and it is indeed C^1, and converges uniformly to f as $\varepsilon \to 0$. The optimality of u_ε is guaranteed by the convexity of the functional. Also note that it was possible to guess the solution, by directly putting $A = B = 0$, and the strict convexity of the functional would have automatically guaranteed that it is the only solution.

In case b) we have to solve $\varepsilon^2 u'' = u - 1$ on $(0, \pi/2)$ and $\varepsilon^2 u'' = u + 1$ on $(\pi/2, \pi)$. The solution will maybe not be very smooth, but we want to construct it so that it is C^1. The solution in the first interval is of the form $u(t) = Ae^{nt} + Be^{-nt} + 1$ and in the second of the form $u(t) = Ce^{n(\pi - t)} + De^{n(t - \pi)} - 1$.

Imposing $u'(0)=0$ means $A=B$. Imposing $u'(\pi)=0$ means $C=D$. We still need two other conditions, which are the continuity of u and of u' at $\pi/2$. The continuity of u' guarantees that the equation is indeed satisfied on the whole interval $(0,\pi)$ and not only on the two subintervals separately. If we set $\kappa_n := e^{n\pi/2}+e^{-n\pi/2}$ the continuity of u means $A\kappa_n+1=C\kappa_n-1$, while the continuity of the derivative provides $nA(e^{n\pi/2}-e^{-n\pi/2})=-nC(e^{n\pi/2}-e^{-n\pi/2})$. We then deduce $A=-C$ and $2C\kappa_n=2$. Finally we obtain

$$u_\varepsilon(t)=-\frac{1}{e^{n\pi/2}+e^{-n\pi/2}}(e^{nt}+e^{-nt})+1 \quad \text{for } t<\frac{\pi}{2},$$

as well as

$$u_\varepsilon(t)=\frac{1}{e^{n\pi/2}+e^{-n\pi/2}}(e^{nt}+e^{-nt})-1 \quad \text{for } t>\frac{\pi}{2}.$$

We observe that both expressions give $u_\varepsilon(\pi/2)=0$ and that, since as $\varepsilon\to 0$ we have $n\to\infty$ and $e^{n\pi/2}+e^{-n\pi/2}\to\infty$, we have pointwise convergence to f. Of course the convergence cannot be unifrm since the u_ε are continuous functions and f is not.

Exercice 2 (5 points)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the minimization problem

$$\min\left\{\int_\Omega\left(\sqrt{v^4+|\nabla u|^4}+v\cos(u-g)\right)dx \;:\; u\in H^1(\Omega), v\in L^2(\Omega)\right\},$$

where g is a given measurable function. Prove that the problem has a solution. Assuming that g is continuous but not constant, prove that no solution $(\bar u,\bar v)$ is such that the $\bar v$ is identically zero.

Answer We observe that the functional to be minimized is larger than $\int(v^2-v)$. This implies that any minimizing sequence (u_n,v_n) is such that v_n is bounded in L^2. The functional is also larger than $\int|\nabla u|^2-v$ and, using the previous bound on v, this implies the boundedness of ∇u_n in L^2 as well. By Poincaré-Wirtinger, u_n minus its average is also bounded in L^2. We then observe that the functional is invariant if we add $2k\pi$ for $k\in\mathbb{Z}$ to u, so we can also assume that the average of u_n is in $[-\pi,\pi)$. Hence, we can obtain a minimizing sequence (u_n,v_n) such that v_n is bounded in L^2 and u_n in H^1. We then extract a subsequence weakly converging to a limit $(\bar u,\bar v)$. The first part of the integrand is convex in v and ∇u and hence the corresponding integral is lower-semicontinuous. For the second, we have $u_n\to\bar u$ a.e. (up to a subsequence). and hence $\cos(u_n-g)\to\cos(\bar u-g)$ strongly in L^2 because of dominated convergence. Then, $\int v_n\cos(u_n-g)\to\int\bar v\cos(\bar u-g)$, which proves the continuity of this part of the functional. This proves that $(\bar u,\bar v)$ is a minimizer.

Suppose now $\bar v=0$ and let S be a set of positive measure (if it exists) where $\cos(\bar u-g)>0$. We see then the replacing $\bar v$ with the value $-\varepsilon$ on S improves the functional. Analoguously, if there is a set S' of positive measure where $\cos(\bar u-g)<0$, replacing $\bar v$ with the value ε on S' improves the functional. Hence from $\bar v=0$ a.e. we deduce $\cos(\bar u-g)=0$ a.e. Then, observe that $\bar u$ is necessarily constant, since for $v=0$ the functional in u becomes just $\int|\nabla u|^2$. Hence $\cos(c-g)=0$ for a certain constant c. If g is continuous bbut not constant then $c-g$ is also continuous but not constant and it is not possible that its cosinus is constant. Finally, we deduce that $\bar v$ is not the zero function.

Exercice 3 (4 points)**.** Consider the function $H:\mathbb{R}^n\to\mathbb{R}$ given by $H(x)=\sqrt{1+|x|^2}$. Compute H^* and prove that we have

$$H(x)+H^*(y)\geq x\cdot y+\frac{1}{2}\left|\frac{x}{H(x)}-y\right|^2.$$

Answer We first compute H^*. We have

$$H^*(y)=\sup_x x\cdot y-\sqrt{1+|x|^2}.$$

If $|y|>1$ taking $x=ny$ we obtain $n|y|^2-\sqrt{1+n^2|y|^2}>n|y|^2-1-n|y|$ and this quantity tends to $+\infty$ since $|y|^2-|y|>0$. Hence the sup is $+\infty$.

If $|y|=1$ we observe $\sqrt{1+|x|^2}\geq|x|\geq x\cdot y$ so that the sup is non-positive. Yet, taking again $x=ny$ we obtain $n-\sqrt{1+n^2}$ which becomes $\frac{n^2-(1+n^2)}{n+\sqrt{1+n^2}}=-\frac{1}{n+\sqrt{1+n^2}}\to 0$, so that the sup is 0.

If $|y|<1$ then the function $x\mapsto -x\cdot y+\sqrt{1+|x|^2}$ that we need to minimize is coercive, so a minimizer exists,. We can differentiate in order to find it, and we have $y=\frac{x}{\sqrt{1+|x|^2}}$. This implies that x and y are colinear (and with the same orientation), and we can llok at their norms. We square and obtain $(1+|x|^2)|y|^2=|x|^2$, i.e. $|x|=|y|/\sqrt{1-|y|^2}$. This provides $H^*(y)=|x||y|-\sqrt{1+|x|^2}=-\sqrt{1-|y|^2}$. This expression is also valid for $|y|=1$.

We observe that we have, inside the ball $B(0,1)$, $D^2H^*\geq I$. Indeed we can compute

$$\nabla H^*(y)=\frac{y}{\sqrt{1-|y|^2}},\quad D^2H^*(y)=\frac{I}{\sqrt{1-|y|^2}}+\frac{y\otimes y}{(1-|y|^2)^{3/2}}\geq\frac{I}{\sqrt{1-|y|^2}}\geq I,$$

where the inequalities are in the sense of symmetric matrices and we use $y\otimes y\geq 0$ and $\sqrt{1-|y|^2}\leq 1$. Hence we have, for every y,y_0

$$H^*(y)\geq H^*(y_0)+\nabla H^*(y_0)\cdot(y-y_0)+\frac{1}{2}|y-y_0|^2.$$

We take $y_0=\nabla H(x)=\frac{x}{H(x)}$, and this gives

$$H^*(y)\geq H^*(\nabla H(x))+x\cdot(y-\nabla H(x))+\frac{1}{2}|y-\frac{x}{H(x)}|^2.$$

We conclude by adding $H(x)$ on both sides and using $H(x)+H^*(\nabla H(x))=x\cdot\nabla H(x)$.

Exercice 4 (6 points)**.** Consider a function $u\in H^1_0(\Omega)$, where $\Omega=(0,1)^d\subset\mathbb{R}^d$ is a cube. Suppose that u solves $\Delta u=|\nabla u|+1$ in Ω. Prove that we have $u\in W^{2,p}(\Omega)\cap W^{3,p}_{loc}(\Omega)$ for every $p<\infty$. Also prove that u is a C^∞ function outside a closed set of zero Lebesgue measure.

Answer

We start from $|\nabla u|+1\in L^2$ and obtain $u\in W^{2,2}$. We want to improve this and we do the following: whenever we know $u\in W^{2,p}$ and $p<d$ we deduce $\nabla u\in L^{p*}$ where $p^*=\frac{dp}{d-p}$, which also guarantees $u\in W^{2,p^*}$. This allows to obtain $u\in W^{2,p_k}$ for a sequence of exponents starting from $p_0=2$. This sequence is strictly increasing and reaches either $p_k=d$ (in which case we can take $p_{k+1}=d+1$) or $p_k>d$. In all cases, at a moment we obtain $u\in W^{2,p}$ for some $p>d$, i.e. $u\in C^{1,\alpha}$. Hence ∇u is bounded and $|\nabla u|+1\in L^\infty$. Since elliptic regularity does not work for $p=\infty$ we can then only deduce $u\in W^{2,p}$ for all p. These estimates hold up to the boundary thanks to the reflection method for Dirichlet boundary conditions.

We then observe that the right-hand side is better than just integrable, it also belons to $W^{1,p}$ for every p (since the norm is a Lipschitz functiona and does not destroy the Sobolev regularity). So we deduce, differetiating the equation, $u\in W^{3,p}$. But htis result is only local, because ther eflection does not preserve the Sobolev regularity, as we change the sign acress the boundary.

Finally, we observe that where $\nabla u\neq 0$ we can go on withthe argument, have $\nabla u\in W^{2,p}$, hence $u\in W^{4,p}\dots$ since the only point where the norm reduces the regularit is at the origin. We observe that ∇u is continuous (even up to the boundary, since $\nabla u\in W^{1,p}$ for $p>d$). Hence we can choose a represenative so that $S=\{\nabla u=0\}$ is closed. In the interior of Ω,the function u is also C^2 (since it is $W^{3,p}$ for lage p), and on $\{\nabla u=0\}$ we have $\Delta u=1$. So, in the points of S there exists at least a non-zero eigevalue of D^2u, which means that S will be locally contained in a hypersurface of at most dimension $d-1$, and hence S is of zero measure (an alternative argument is: almost any point of S is a Lebesgue point of S, and $u\in C^2$ so that $D^2u=0$ a.e. on S, but $\Delta u=0$ on S: the only possibility is that the measure of S is zero).

Exercice 5 (9 points)**.** Let $\mathbb{T}^d$ be the d-dimensional torus. Consider the following minimization problem

$$\inf\left\{\int_{\mathbb{T}^d}-\sqrt{1-|v(x)|^2}dx\ :\ v\in L^\infty(\mathbb{T}^d),\,|v|\leq 1\,a.e.,\,\nabla\cdot v=f\right\},$$

where f is a given distribution on $\mathbb{T}^d$ such that at least one admissible v exists.

1. Prove that the problem has a solution.
2. Formally find its dual as an optimization problem in the space $W^{1,1}$, via an inf-sup exchange. Explain why it is not clear whether the dual has a solution. Also explain why we should rather call dual the above problem and primal the other one.
3. Prove the duality result in this case, explaining why it does not fit the result we saw in class.
4. **(More difficult)** Adapt the regularity-via-duality proof to this case so as to prove that if f is a Lipschitz function such that there exists an admissible v_0 with $||v_0||_{L^\infty} < 1$, then the optimal v is H^1.

Answer

1. Take a minimizing sequence. It is bounded in L^∞, so one can extract a weakly-* converging subsequence. The function $y \mapsto -\sqrt{1-|y|^2}$ (which is by the way equal to H^* from Exercise 3) is convex, so that the functional is lower semicontinuous. The constraint $\int v \cdot \nabla \phi + \int f\phi = 0$ for all $\phi \in C^1$ passes to the limit via weak convergence, and the limit is a minimizer.

2. We write
$$\inf_v \int H^*(v) + \sup_\phi - \int v \cdot \nabla\phi - \int f\phi$$
(the choice of the minus sign is arbitrary) and, excahanging, we obtain
$$\sup_\phi - \int f\phi + \inf_v \int (H^*(v) - v \cdot \nabla\phi),$$
i.e.
$$\sup_\phi - \int f\phi - \sup v \int (-H^*(v) + v \cdot \nabla\phi).$$
The internal sup is taken among L^∞ functions bounded by 1, and it canbe realized pointwisely, so that we have
$$\sup_\phi - \int f\phi - \int H^{**}(\nabla\phi) = \sup_\phi - \int f\phi - \int H(\nabla\phi).$$
The sup can be taken over smooth function, but we know $\int H(\nabla\phi) = \int \sqrt{1+|\nabla\phi|^2}$, which is wel-defined for $\phi \in W^{1,1}$. Since smooth functions are strongly dense in $W^{1,1}$, the result is the same if we optimize over $W^{1,1}$.

   There are two reasons for maybe not finding a solution to this dual problem, which is equivalent to
$$\inf \int f\phi + \int \sqrt{1+|\nabla\phi|^2}.$$
The first is the factthat L^1 and $W^{1,1}$ are not reflexive. Even if a minimizing sequence satisfied $||\nabla\phi_n||_{L^1} \le C$ it would not be possible to extract a subsequence converging in a weak sense to another function in the same space (we would actually need to estend the problem to BV). The second is that, for general f, we could have lack of coercivity, insce both terms grow linearly. If $f = \nabla v_0$ for a certain v_0 we can re-write the functional as $\int \sqrt{1+|\nabla\phi|^2} - v_0 \cdot \nabla\phi$, and we can obtain coercivity in terms of the L^1 norm of the gradient if $||v_0||_{L^\infty} < 1$, but not if $||v_0||_{L^\infty} = 1$.

   Note that our dual problem uses the space L^1 and our primal the space L^∞, but $(L^1)' = L^\infty$ and not $(L^\infty)' = L^1$. This is why the primal should be the one on ϕ. Also, in the general theory it is often the case that the primal has no solution while the dual has one.

3. We consider
$$\mathcal{F}(p) := \inf \left\{ \int H^*(v) \,:\, ||v||_{|\infty} \le 1,\, \nabla \cdot v = f + p \right\}$$
and apply the usual strategy. $\mathcal{F}$ is defined on the space of divergences of L^∞ functions but in class we assumed all the spaces to be reflexive, so we have to pay attention. We compute $\mathcal{F}^*$ and

$\mathcal{F}^{**}$. The only difficult point is to prove that $\mathcal{F}$ is l.s.c., where we used coercivity and extracted a weakly converging subsequence from the minimizers v_n corresponding to a sequence p_n. This can also be done here, since v_n i sbounded in L^∞. There is no additional difficulty.

4. Let us write the two dual problems as $\min F(v)$ and $\inf G(\phi)$, where F also includes the constraint. From the duality we have $\min F + \inf G = 0$. The difficulty comes from the fact that the inf of G is maybe not attained. However, the assumption $f = \nabla \cdot v_0$ with $||v_0||_{L^\infty} < 1$, allows to obtain coercivity, so that for every $\varepsilon > 0$ we can find a function $\phi^\varepsilon \in W^{1,1}$ (actually, we could even have $\phi \in C^\infty$) such that $G(\phi^\varepsilon) < \inf G + \frac{\varepsilon^2}{2}$. We use Exercise 3 which provides, together with the usual computations,

$$\frac{1}{2}||v - j(\nabla\phi)||^2 \leq F(v) + G(\phi) = G(\phi) - \inf G,$$

where the norm is the L^2 norm, the function j is given by $j(w) = w/H(w)$, and we use the optimality of v. If we take $\phi^v e$ we obtain $||v - j(\nabla\phi^\varepsilon)|| \leq \varepsilon$ (which explains the choice of the parameter $\frac{\varepsilon^2}{2}$).

For a given function ϕ, consider its translations $\phi_h := \phi(\cdot + h)$ and the function $h \mapsto G(\phi_h)$. The usual computations in regularity-via-duality prove that this function is C^2 with a Hessian bounded by $||\nabla f||_{L^\infty}||\nabla\phi||_{L^1}$. In Particular since it is continuous, when choosing ϕ^ε we can decide to optimaize among its translations, and assume that $h \mapsto G(\phi^\varepsilon_h)$ is optmal at $h = 0$. But this implies that its gradient (in h) vanishes at $h = 0$ and hence $G(\phi^\varepsilon_h) - G(\phi^\varepsilon) \leq C|h|^2$ where $C = ||\nabla f||_{L^\infty}||\nabla\phi^\varepsilon||_{L^1}$. This constant is bounded as $\varepsilon \to 0$ since, by coercivity, the L^1 norm of $\nabla\phi^\varepsilon$ is bounded, and we assumed f to be Lipschitz. Now we compute

$$||v_h - v|| \leq ||v_h - (j(\nabla\phi^\varepsilon))_h|| + ||(j(\nabla\phi^\varepsilon))_h - v|| = ||v - j(\nabla\phi^\varepsilon)|| + ||(j(\nabla\phi^\varepsilon))_h - v|| \leq \varepsilon + ||(j(\nabla\phi^\varepsilon))_h - v||.$$

Then,we use

$$\frac{1}{2}||(j(\nabla\phi^\varepsilon))_h - v||^2 \leq G(\phi^\varepsilon_h) - \inf G = G(\phi^\varepsilon_h) - G(\phi^\varepsilon) + G(\phi^\varepsilon) - \inf G \leq C|h|^2 + \frac{\varepsilon^2}{2}$$

and we obtain

$$||v_h - v|| \leq \varepsilon + \sqrt{2C|h|^2 + \varepsilon^2}.$$

Sending $\varepsilon \to 0$ we have

$$||v_h - v|| \leq C'|h|,$$

which means $v \in H^1$.