ENS Lyon and Université Claude Bernard Lyon 1, M2A
November 7th, 2025

Calculus of Variations and Elliptic PDEs
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Mid-Term Examination
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All kind of documents (notes, books...) are authorized. The total number of points is much larger than 20, which means that attacking only some exercises could be a reasonable option.
The exercises are not necessarily ordered by difficulty.

Exercice 1 (8 points)**.** Given $\varepsilon > 0$ and a function $f : [0, \pi] \to \mathbb{R}$ consider the problem

$$\min \left\{ \int_0^\pi \frac{\varepsilon}{2}|u'(t)|^2 + \frac{1}{2\varepsilon}|u(t) - f(t)|^2 dt : \quad u \in C^1([0, \pi]) \right\}.$$

We consider two cases.
In case a) the function f is given by $f(t) = \cos(t)$.
In case b) the function f is given by

$$f(t) = \begin{cases} 1 & \text{if } t < \pi/2, \\ 0 & \text{if } t = \pi/2, \\ -1 & \text{if } t > \pi/2. \end{cases}$$

In both cases find the solution u_ε of the problem, properly justifying its minimality and its uniqueness, and prove that we have $\lim_{\varepsilon \to 0} u_\varepsilon = f$, at least pointwisely. In which cases is this convergence uniform?

Exercice 2 (5 points)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the minimization problem

$$\min \left\{ \int_\Omega \left(\sqrt{v^4 + |\nabla u|^4} + v \cos(u - g) + v^2|\nabla u| \right) dx \; : \; u \in H^1(\Omega), v \in L^2(\Omega) \right\},$$

(the term above was a mistake and has been removed during the examination) where g is a given measurable function. Prove that the problem has a solution. Assuming that g is continuous but not constant, prove that no solution $(\bar{u}, \bar{v})$ is such that the $\bar{v}$ is identically zero.

Exercice 3 (4 points)**.** Consider the function $H : \mathbb{R}^n \to \mathbb{R}$ given by $H(x) = \sqrt{1 + |x|^2}$. Compute H^* and prove that we have

$$H(x) + H^*(y) \geq x \cdot y + \frac{1}{2}\left| \frac{x}{H(x)} - y \right|^2.$$

Exercice 4 (6 points)**.** Consider a function $u \in H_0^1(\Omega)$, where $\Omega = (0, 1)^d \subset \mathbb{R}^d$ is a cube. Suppose that u solves $\Delta u = |\nabla u| + 1$ in Ω. Prove that we have $u \in W^{2,p}(\Omega) \cap W^{3,p}_{loc}(\Omega)$ for every $p < \infty$. Also prove that u is a C^∞ function outside a closed set of zero Lebesgue measure.

Exercice 5 (9 points)**.** Let $\mathbb{T}^d$ be the d-dimensional torus. Consider the following minimization problem

$$\inf \left\{ \int_{\mathbb{T}^d} -\sqrt{1 - |v(x)|^2} dx \; : \; v \in L^\infty(\mathbb{T}^d), \, |v| \leq 1 \, a.e., \, \nabla \cdot v = f \right\},$$

where f is a given distribution on $\mathbb{T}^d$ such that at least one admissible v exists.

1. Prove that the problem has a solution.
2. Formally find its dual as an optimization problem in the space $W^{1,1}$, via an inf-sup exchange. Explain why it is not clear whether the dual has a solution. Also explain why we should rather call dual the above problem and primal the other one.
3. Prove the duality result in this case, explaining why it does not fit the result we saw in class.
4. **(More difficult)** Adapt the regularity-via-duality proof to this case so as to prove that if f is a Lipschitz function such that there exists an admissible v_0 with $||v_0||_{L^\infty} < 1$, then the optimal v is H^1.