Université Paris-Saclay
M2 AMS and Optimization
January 18th, 2018

Calculus of Variations
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Final Examination
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Duration : 3h ; all kind of paper documents (notes, books...) are authorized.

The total score of this exam is much more than 20 : you are not expected to deal with all the exercises (but of course you can). The grade will just be truncated at 20.

Exercice 1 (6 points)**.** Consider the minimization problem

$$\min\left\{\int_0^1 e^{-2t}\left(\frac{1}{2}u'(t)^2+\frac{3}{2}u(t)^2+\frac{5}{2}u(t)\right)dt \quad : \quad u\in C^1([0,1]),\ u(0)=u(1)=a\right\}$$

and prove that it admits a minimizer, that it is unique, and find it, in the two cases $a=-5/6$ and $a=5/6$.

Exercice 2 (5 points)**.** Let Ω be an open and bounded subset of $\mathbb{R}^d$, $p>1$ and $h:\mathbb{R}\to\mathbb{R}_+$ a continuous function. Consider the following minimization problem

$$\min\left\{\int_\Omega \sqrt{h(u(x))+|\nabla u(x)|^{2p}}\,dx\ :\ u\in W_0^{1,p}(\Omega)\right\}.$$

Prove that it admits a solution. Also prove that its minimal value is strictly positive if $h(0)>0$.

Consider now

$$\inf\left\{\int_\Omega \sqrt{\frac{1+|\nabla\varphi(x)|^{2p}}{1+|\varphi(x)|^{2p}}},dx\ :\ \varphi\in C_c^\infty(\Omega)\right\}.$$

Prove that the value of this infimum is strictly positive.

Exercice 3 (6 points)**.** Let Ω be a given bounded $d-$dimensional domain, $f\in L^2(\Omega)$ with $\int_\Omega f(x)dx=0$, and $L\le\pi/2$ a given constant. Consider the following minimization problem

$$\min\left\{\int_\Omega [1-\cos(|\nabla u(x)|)+f(x)u(x)]dx\ :\ u\in \mathrm{Lip}(\Omega),\ |\nabla u|\le L \text{ a.e.},\right\}.$$

1. Preliminarly, justify that the function $h:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ defined by $h(s)=1-\cos(s)$ for $|s|\le L$ and $h(s)=+\infty$ for $|s|>L$ and the function $H:\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$ defined by $H(w)=h(|w|)$ are convex, and find their transforms h^* and H^*.
2. Prove that this problem admits a solution.
3. Prove that the solution is unique up to additive constants.
4. Formally write the dual of this problem ("formally" means that the proof of the duality result is not required, as the growth conditions assumed in class are not satisfied).
5. Assuming that duality holds, that Ω is the d-dimensional torus, that $L<\pi/2$ and that $f\in W^{1,1}(\Omega)$, prove that the solution u of the above problem belongs to $H^2(\Omega)$. Does it work also if $f\in BV(\Omega)$?

Look at the back of the paper for the last two exercises

Exercice 4 (7 points)**.** Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain. On the space $H^1_0(\Omega)$ consider the sequence of functionals

$$F_\varepsilon(u) = \int_\Omega \left[\frac{|\nabla u(x)|^2}{2} + \frac{\sin(\varepsilon u(x))}{\varepsilon} \right] dx.$$

1. Prove that, for each $\varepsilon > 0$, the functional F_ε admits at least a minimizer u_ε.
2. Prove that the minimizers u_ε satisfy $||\nabla u_\varepsilon||^2_{L^2} \leq 2||u_\varepsilon||_{L^1}$ and that the norm $||u_\varepsilon||_{H^1_0}$ is bounded by a constant independent of ε.
3. Find the Γ-limit F_0, in the weak H^1_0 topology, of the functionals F_ε as $\varepsilon \to 0$.
4. Characterize via a PDE the unique minimizer u_0 of the limit functional F_0.
5. Prove $u_\varepsilon \rightharpoonup u_0$ in the weak H^1_0 topology.
6. Prove that the convergence $u_\varepsilon \to u_0$ is actually strong in H^1_0.
7. Prove that all minimizers u_ε satisfy $-\frac{\pi}{2\varepsilon} \leq u_\varepsilon \leq 0$, and that for each ε the minimizer is unique.

Exercice 5 (7 points)**.** Let $\Omega \subset \mathbb{R}^2$ be the ball $B(0, 2)$. On the space $L^1(\Omega)$ consider the sequence of functionals

$$F_\varepsilon(u) = \begin{cases} \int_\Omega \left[\frac{\varepsilon}{2}|\nabla u(x)|^2 + \frac{1}{2\varepsilon}\left(\frac{1}{1+(2u(x)-1)^2} - \frac{1}{2} \right)^2 + 2(|x|-1)u(x) \right] dx & \text{if } u \in H^1_0(\Omega), 0 \leq u \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

1. Find the Γ-limit, in the strong L^1 topology, of the functionals F_ε as $\varepsilon \to 0$.
2. Prove that the unique minimizer of the limit functional is the indicator function of a ball, and find it.
3. Prove that, for each $\varepsilon > 0$, the functional F_ε admits at least a minimizer u_ε, and prove that u_ε admits a strong L^1 limit as $\varepsilon \to 0$, and find it.
4. Prove that, for each $\varepsilon > 0$, the functional F_ε admits at least a radially decreasing minimizer u_ε.