Université Paris-Saclay
M2 AMS and Optimization
January 18th, 2018

Calculus of Variations

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Final Examination

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Duration : 3h ; all kind of paper documents (notes, books. . .) are authorized.

The total score of this exam is much more than 20 : you are not expected to deal with all the exercises (but of course you can). The grade will just be truncated at 20.

Exercice 1 (6 points)**.** Consider the minimization problem

$$\min\left\{\int_0^1 e^{-2t}\left(\frac{1}{2}u'(t)^2+\frac{3}{2}u(t)^2+\frac{5}{2}u(t)\right)dt \quad : \quad u\in C^1([0,1]),\ u(0)=u(1)=a\right\}$$

and prove that it admits a minimizer, that it is unique, and find it, in the two cases $a=-5/6$ and $a=5/6$.

Solution

The minimization problem above is convex, and even strictly convex. Hence, it admits at most one soluton, and it is enough to write the Euler-Lagrange equation with its boundary conditions, and solve it : the solution of the equation will also be the unique solution of the minimization problem.

From $L(t,x,v)=e^{-2t}(\frac{|v|^2}{2}+\frac{3|x|^2}{2}+\frac{5x}{2}$ we find the Euler-Lagrange equation $(\partial_v L(t,u,u'))'=\partial_x L(t,u,u')$, which, after simplifying e^{-2t}, reads $u''-2u'=3u+5/2$.

First notice that the constant $u=-5/6$ is a solution of the equation, so, in case $a=-5/6$, the answer is just $u(t)=-5/6$, which is a C^1 function and solves the problem.

For $a=5/6$ we have to solve the equation. The solution is of the form

$$u(t)=Ae^{-t}+Be^{3t}-\frac{5}{6},$$

which is found by using the particular solution $-5/6$ and adding arbitrary solutions of the homogeneous equation $u''-2u'-3u=0$ (a basis of the space of solutions is given by the functions of the form $e^{\lambda t}$ for λ solving $\lambda^2-2\lambda=3=0$, i.e. $\lambda=-1$ and $\lambda=3$).

Imposing $u(0)=u(1)=5/6$ we can find

$$A=\frac{5}{3}\cdot\frac{e^4-e}{e^4-1},\quad B=\frac{5}{3}\cdot\frac{e-1}{e^4-1}.$$

Exercice 2 (5 points)**.** Let Ω be an open and bounded subset of $\mathbb{R}^d$, $p>1$ and $h:\mathbb{R}\to\mathbb{R}_+$ a continuous function. Consider the following minimization problem

$$\min\left\{\int_\Omega\sqrt{h(u(x))+|\nabla u(x)|^{2p}}\,dx\ :\ u\in W_0^{1,p}(\Omega)\right\}.$$

Prove that it admits a solution. Also prove that its minimal value is strictly positive if $h(0)>0$.

Consider now

$$\inf\left\{\int_\Omega\sqrt{\frac{1+|\nabla\varphi(x)|^{2p}}{1+|\varphi(x)|^{2p}}},dx\ :\ \varphi\in C_c^\infty(\Omega)\right\}.$$

Prove that the value of this infimum is strictly positive.

Solution

For the first part, notice that by $h \geq 0$ any minimizing sequence u_n will be such that $\int \sqrt{|\nabla u_n|^{2p}} = ||\nabla u_n||_{L^p}^p$ will be bounded and, using the Poincaré inequality (since we are in $W_0^{1,p}$), any minimizing sequence is bounded in $W_0^{1,p}$. We can extract a weakly converging subsequence. The functional is of the form $u \mapsto \int L(u, \nabla u)$ with L continuous in the first variable and convex in the second. Hence it is l.s.c. for the weak $W^{1,p}$ convergence, and the limit of the sequence is a minimizer. **Warning :** since the functional is not the sum of a part with u and a part with ∇u, the semicontinuity cannot be discussed by separating the two parts.

The minimum is for sure not negative, and could only be zero if the minimizer u satisfied both $|\nabla u| = 0$ and $h(u) = 0$ a.e. But the first condition implies that it is constant equal to 0 (because it is 0 on the boundary), and if $h(0) > 0$ then the minimum is strictly positive. . **Warning :** unless you prove continuity of the minimizers up to $\partial\Omega$ (which is not a consequence of $u \in W^{1,p}$), saying that $h(u)$ is strictly positive on $\partial\Omega$ and hence must be strictly positive on a neighborhood of the boundary does not work.

For the second part, define $g : \mathbb{R} \to \mathbb{R}$ by setting $g(0) = 0$ and $g'(t) = (1 + t^{2p})^{1/2p}$. The function g is C^1 and strictly increasing. Then we have

$$\sqrt{\frac{1 + |\nabla\varphi|^{2p}}{1 + |\varphi|^{2p}}} = \sqrt{\frac{1}{1 + |\varphi|^{2p}} + |\nabla(g \circ \varphi)|^{2p}}$$

and

$$\frac{1}{1 + |\varphi|^{2p}} = \frac{1}{1 + |g^{-1}(g \circ \varphi)|^{2p}} = h(g \circ \varphi),$$

for a certain continuous function $h : \mathbb{R} \to \mathbb{R}_+$ with $h(0) = 1/(1 + |g^{-1}(0)|^{2p} = 1 > 0$.

Hence, the values in the inf below are all larger than the minimum above (by using $u = g \circ \varphi$, and not $u = \varphi$), which is strictly positive.

Exercice 3 (6 points)**.** Let Ω be a given bounded $d-$dimensional domain, $f \in L^2(\Omega)$ with $\int_\Omega f(x)dx = 0$, and $L \leq \pi/2$ a given constant. Consider the following minimization problem

$$\min \left\{ \int_\Omega [1 - \cos(|\nabla u(x)|) + f(x)u(x)]dx \ : \ u \in \text{Lip}(\Omega), \ |\nabla u| \leq L \text{ a.e.}, \right\}.$$

1. Preliminarly, justify that the function $h : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by $h(s) = 1 - \cos(s)$ for $|s| \leq L$ and $h(s) = +\infty$ for $|s| > L$ and the function $H : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ defined by $H(w) = h(|w|)$ are convex, and find their transforms h^* and H^*.
2. Prove that this problem admits a solution.
3. Prove that the solution is unique up to additive constants.
4. Formally write the dual of this problem ("formally" means that the proof of the duality result is not required, as the growth conditions assumed in class are not satisfied).
5. Assuming that duality holds, that Ω is the d-dimensional torus, that $L < \pi/2$ and that $f \in W^{1,1}(\Omega)$, prove that the solution u of the above problem belongs to $H^2(\Omega)$. Does it work also if $f \in BV(\Omega)$?

Solution

1. The function h is finite and C^2 on an interval, and its second derivative is non-negative on this interval : hence, it is convex. Moreover, h is increasing on $\mathbb{R}_+$: when we compose it with $w \mapsto |w|$, which is convex and non-negative, the composition H is convex.

   To compute h^* we write

$$h^*(t) = \sup_s ts - h(s) = \sup_{|s| \leq L} st - 1 + \cos(s).$$

The function to maximize is concave in s and its derivative is given by $t - \sin(s)$. Hence, if there is $s \in [-L, L]$ with $\sin(s) = t$ (which means, if $|t| \leq \sin L$), the maximizer is such a point. Otherwise it is $s = \pm L$, depending on the sign of t (same sign as t, in order to maximize the term ts). So we have

$$h^*(t) = \begin{cases} t\arcsin(t) - 1 + \cos(\arcsin(t)) = t\arcsin(t) - 1 + \sqrt{1-t^2} & \text{if } |t| \leq \sin(L), \\ tL - 1 + \cos(L) & \text{if } t > \sin(L), \\ -tL - 1 + \cos(L) & \text{if } t < -\sin(L) \end{cases}.$$

One can check that this function is C^1 and convex.

As for H^*, we have $H^*(v) = \sup_w v \cdot w - h(|w|)$, and it is optimal to take v and w in the same direction, so that we have $H^*(v) = h^*(|v|)$.

2. Take a minimizing sequence u_n. Because of $\int f = 0$, we can assume $\int u_n = 0$ (adding a constant does not change the value of the functional). The sequence (u_n) is uniformly Lipschitz and uniformly bounded (because u_n vanishes somewhere, and is L-Lipschitz, so that we have $|u_n| \leq L\text{diam}(\Omega)$). We can extract a subsequence which converges uniformly, and also weakly in $W^{1,p}$, for any p. The limit will also have the same Lipschitz constant, and the functional is l.s.c.. So, the limit is admissible and minimizes the functional.
3. The functional is strictly convex w.r.t. ∇u : any two minimizers mus thave the same gradient. Hence, they coincide up to additive constants. **Warning :** checking that the value for $u + c$ is the same as that for u is not a valid answer, it only proves that you can add constants to minimizers, not that you can ONLY add constants to minimizers.
4. From the formulas we know the dual is given by

$$\min \left\{ \int H^*(v) \, : \, \nabla \cdot v = f \right\},$$

where H is the function of Question 1. Hence, here we get the expression of H^* that we computed above. Note that this functional has lineargrowth in v.

5. The usual argument from "regularity via duality" is the following : suppose $H(w) + H^*(v) \geq v \cdot w + c|J_*(v) - J(w)|^2$, and denote by u_h the translation of u ($u_h(x) = u(x+h)$) ; let F be the functional we minimize in the primal problem, then we have

$$c \int |J(\nabla u_h) - J(\nabla u)|^2 = c \int |J(\nabla u_h) - J_*(v)|^2 \leq F(u_h) - F(u).$$

Here $D^2 H \geq cI$ (this is why we suppose $L < \pi/2$, since the second derivative of the cosinus vanishes at $\pi/2$), so that we know that we can take $J(w) = w$ and $J_*(v) = \nabla H^*(v)$. We are just left to prove that $F(u_h) - F(u) = o(|h|^2)$, which would give $\nabla u \in H^1$, hence $u \in H^2$. We know that it is enough to prove that $h \mapsto F(u_h)$ is $C^{1,1}$, and we know that we just need to consider $h \mapsto \int f u_h$, since the first part of the functional, by change-of-variable, does not depend on h. The Hessian if this quantity (standard computations) is given by

$$\int \nabla f \otimes \nabla u_h$$

and we just need $f \in W^{1,1}$ and $u \in W^{1,\infty}$ (which is the case) in order to bound it by a constant. The case $f \in BV$ can be justified, for instance, by approximation (it has no meaning to integrate ∇f times ∇u_h if one is a measure and the other L^∞).

Exercice 4 (7 points)**.** Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain. On the space $H_0^1(\Omega)$ consider the sequence of functionals

$$F_\varepsilon(u) = \int_\Omega \left[\frac{|\nabla u(x)|^2}{2} + \frac{\sin(\varepsilon u(x))}{\varepsilon} \right] dx.$$

1. Prove that, for each $\varepsilon > 0$, the functional F_ε admits at least a minimizer u_ε.
2. Prove that the minimizers u_ε satisfy $||\nabla u_\varepsilon||^2_{L^2} \leq 2||u_\varepsilon||_{L^1}$ and that the norm $||u_\varepsilon||_{H_0^1}$ is bounded by a constant independent of ε.
3. Find the Γ-limit F_0, in the weak H_0^1 topology, of the functionals F_ε as $\varepsilon \to 0$.
4. Characterize via a PDE the unique minimizer u_0 of the limit functional F_0.
5. Prove $u_\varepsilon \rightharpoonup u_0$ in the weak H_0^1 topology.
6. Prove that the convergence $u_\varepsilon \to u_0$ is actually strong in H_0^1.
7. Prove that all minimizers u_ε satisfy $-\frac{\pi}{2\varepsilon} \leq u_\varepsilon \leq 0$, and that for each ε the minimizer is unique.

Solution

1. Using the lower bound $\sin(\varepsilon u) \geq -1$ we see that any minimizing sequence is bounded in H_0^1. We extract a weakly converging subsequence, and the functional is l.s.c., since the integrand is convex in the gradient part and continuous in u. Hence, the limit minimizes.
2. The estimate can be obtained by comparing with $u = 0$: we have $F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(0) = 0$. This gives $||\nabla u_\varepsilon||^2_{L^2} \leq 2\int -\frac{\sin(\varepsilon u_\varepsilon(x))}{\varepsilon} dx \leq \int |u_\varepsilon|$ (by the way, using the Euler-Lagrange equation it is also possible to obtain the same estimate without the factor 2). Bounding the L^1 norm with the L^2 norm, and using Poincaré, we get

$$||\nabla u_\varepsilon||^2_{L^2} \leq C||\nabla u_\varepsilon||_{L^2},$$

which gives a bound on $||\nabla u_\varepsilon||_{L^2}$ and the sequence is bounded in H_0^1.
3. We can guess the Γ-limit by looking at the pointwise limit. If we fix u, we have $\sin(\varepsilon u)/\varepsilon \to u$, hence we guess

$$F_0(u) = \int_\Omega \left[\frac{|\nabla u(x)|^2}{2} + u(x) \right] dx.$$

Since it is a pointwise limit, the Γ-limsup part is easy : just take the constant sequence $u_\varepsilon = u$. For the Γ-liminf, we take $u_\varepsilon \rightharpoonup u$ (weak convergence in H_0^1, hence strong in L^2) and write, using $\sin(s) \geq s - Cs^2$ (Taylor expansion)

$$F_\varepsilon(u_\varepsilon) \geq F_0(u_\varepsilon) - C\varepsilon \int u_\varepsilon^2.$$

We then use the semicontinuity of F_0 and the fact that u_ε is bounded in L^2 to get that the liminf is at least $F_0(u)$.
4. The solution u_0 of $\min\{F_0(u) : u \in H_0^1(\Omega)\}$ is the solution of

$$\begin{cases} \Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on} \partial\Omega. \end{cases}$$

5. The sequence of minimizers u_ε is bounded in H_0^1, hence compact for the weak convergence. Any limit must minimize F_0, but the minimizer is unique, so the whole sequence converges to u_0.
6. Since the minimizers of F_ε stay in a same compact set (a bounded set in H_0^1), we have the compactness assumption (equicoercivity) which guarantees $\min F_\varepsilon \to \min F_0$. But this means $F_\varepsilon(u_\varepsilon) \to F_0(u_0)$ and implies $||\nabla u_\varepsilon||_{L^2} \to ||\nabla u_0||_{L^2}$. Together with the weak converge,ce this gives strong convergence.

7. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = -|s|$ for $|s| \leq \pi/2$, and extended by periodicity on $\mathbb{R}$. This function is Lipschitz with constant 1, and we can check that we have $\sin(f(s)) \leq \sin(s)$. Hence, if we define $\tilde{u}_\varepsilon = \varepsilon^{-1} f(\varepsilon u_\varepsilon)$, we have $F_\varepsilon(\tilde{u}_\varepsilon) \leq F_\varepsilon(u_\varepsilon)$. Moreover, as soon as there is a non-negligible set where $\varepsilon u_\varepsilon$ belongs to $]-2\pi, -\pi[$ or $]0, \pi[$, the inequality in the sinus is strict, hence u_ε could not be a minimizer. This proves that u_ε cannot take values outside $[-\pi/\varepsilon, 0]$ (we can prove, by using the regularity associated with the Euler-Lagrange equation, that u_ε is continuous, so that if $\varepsilon u_\varepsilon$ takes values outside $[-\pi/\varepsilon, 0]$ then it takes values in $]-2\pi, -\pi[$ or $]0, \pi[$ on a non-negligible set). In order to prove that it actually takes values in $[-\pi/(2\varepsilon), 0]$, we can define $\hat{u}_\varepsilon = \max\{-\pi/(2\varepsilon), u_\varepsilon\}$ and see that also in this case we would have a strict inequality if $\varepsilon u_\varepsilon$ takes values smaller than $-\pi/2$.

   Once that we know that the minimizers take value in $[-\pi/(2\varepsilon), 0]$, we see that the functional is strictly convex on these functions, and the minimizer is unique.

Exercice 5 (7 points)**.** Let $\Omega \subset \mathbb{R}^2$ be the ball $B(0, 2)$. On the space $L^1(\Omega)$ consider the sequence of functionals

$$F_\varepsilon(u) = \begin{cases} \int_\Omega \left[\frac{\varepsilon}{2}|\nabla u(x)|^2 + \frac{1}{2\varepsilon}\left(\frac{1}{1+(2u(x)-1)^2} - \frac{1}{2}\right)^2 + 2(|x|-1)u(x)\right] dx & \text{if } u \in H^1_0(\Omega), 0 \leq u \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

1. Find the Γ-limit, in the strong L^1 topology, of the functionals F_ε as $\varepsilon \to 0$.
2. Prove that the unique minimizer of the limit functional is the indicator function of a ball, and find it.
3. Prove that, for each $\varepsilon > 0$, the functional F_ε admits at least a minimizer u_ε, and prove that u_ε admits a strong L^1 limit as $\varepsilon \to 0$, and find it.
4. Prove that, for each $\varepsilon > 0$, the functional F_ε admits at least a radially decreasing minimizer u_ε.

Solution

1. The first part of the functional is a Modica-Mortola term, with a double-well function given by

   $$W(s) = \left(\frac{1}{1+(2s-1)^2} - \frac{1}{2}\right)^2,$$

   which only vanishes at $s = 0, 1$. We know that it Γ-converges, for the strong L^1 topology, to the functional

   $$F(u) = \begin{cases} c\text{Per}(A) & \text{if } u = I_A \in BV(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

   where $c = \int_0^1 \sqrt{W(s)} ds$. Notice that we can include the constraint $0 \leq u \leq 1$ in the Γ-convergence since the construction of the recovery sequence in the $\Gamma - \limsup$ preserves it. Also notice that the constraint $u \in H^1_0$ means that, in the limit, the perimeter also counts the part of boundary of A which is on the boundary of Ω (even if $u = 1$ close to the boundary, one has to go down to 0... the recovery sequence in the $\Gamma - \limsup$ is built by first supposing $d(A, \partial\Omega) > 0$, and the proving that we have a dense-in-energy sequence).

   The other part of the functional is continuous for the convergence we use, so it canbe added to the Γ-convergence result. Hence we have a Γ-limit F_0 given by

   $$F(u) = \begin{cases} c\text{Per}(A) + \int_A 2(|x|-1)dx & \text{if } u = I_A \in BV(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

   In our case we can compute the constant c

   $$c = \int_0^1 \left(\frac{1}{1+(2s-1)^2} - \frac{1}{2}\right) ds = \int_{-1}^1 \left(\frac{1}{1+t^2} - \frac{1}{2}\right) \frac{dt}{2} = \frac{\pi - 2}{4} \approx 0.27.$$

2. By symmetrization, any set A can be replaced by a ball with the same volume, and this reduces the perimeter. Moreover, if we choose to center this ball at the origin and we call it B, we also have $\int_A 2(|x|-1) \geq \int_B 2(|x|-1)$ since the values of the function $2(|x|-1)$ are radially increasing, and concentrating the same measure where it is minimal decreases the integral. By the way, the integral strictly decreases unless A was already equal to B. Hence, the minimum is a given by $u = I_A$, with A a ball centered at the origin. Set $A = B(0,R)$, and compute the value fo the functional : we have
$$F(I_{B(0,R)}) = c2\pi R + 4\pi(\frac{R^3}{3} - \frac{R^2}{2}).$$
This function is minimized on the positive values of R by
$$R = \frac{1}{2} + \sqrt{\frac{1-2c}{4}}$$
(the derivative vanishes in two points, but $\frac{1}{2} - \sqrt{\frac{1-2c}{4}}$ is a local maximum). We have found the unique minimizer of the limit functional.

3. Using $W \geq 0$ and $2(|x|-1)u \geq -2$ we see that any minimizing sequence is bounded (for fixed $\varepsilon > 0$) in H_0^1, and the functional is l.s.c. for the weak H_0^1 convergence. We deduce the existence of a minimizer. From the proof of the lower bounds in the Γ-convergence, we have a strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ (the anti-derivative of $\sqrt{W}$) such that $\int |\nabla(\phi \circ u_\varepsilon)|$ is bounded. This proves that, up to subsequences, $\phi \circ u_\varepsilon$ converges strongly in L^1 (we use the compact injection of BV into L^1) to something, and in particular it converges a.e. Composing with ϕ^{-1}, also u_ε converges a.e. to something and, because of the bounds $0 \leq u_\varepsilon \leq 1$, it converges strongly in L^1 since it is domainated. The limit can only by a minimizer of F_0, i.e. the indicator of the ball of the radius we just found.

4. We can symmetrize the minimizers u_ε. The symmetrization will provide a better result (and hence a contradiction, unless u_ε is already symmetric, i.e. radially diecreasing) provided we can prove
$$\int fu \geq \int fu^*$$
as soon as f is a radially increasing function (here $f(x) = 2(|x|-1)$; we also need strict inequality as soon as $u \neq u_*$. To do this, use u and u^* as measures (they are positive, and have the same mass) and remember $\int f d\mu = \int \mu(\{f > t\})dt$. Now, for each t, we have $\int_{\{f>t\}} u \geq \int_{\{f>t\}} u^*$ via a similar argument as before : u^* brings more mass closer to the origin. Indeed,
$$\int_{\{f>t\}} u = \int |\{u > s\} \cap \{f > t\}| ds$$
and $|\{u > s\} \cap \{f > t\}| \geq |\{u^* > s\} \cap \{f > t\}|$ since $\{f > t\}$ is the complement of a centered ball, and $\{u^* > s\}$ has the same volume as $\{u > s\}$, but is more contained in the ball $\{f \leq t\}$. We also see that the inequality is strict as soon as the sublevel sets of u and u^* have not the same measure in all the balls.

   Actually, there is a more general inequality that one could prove : $\int u^* v^* \geq \int uv$ for every u, v (but in our case, one of the two functions is already radial).