ENS Lyon and Université Claude Bernard Lyon 1, M2A
January 11th, 2020

Calculus of Variations and Elliptic PDEs

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Final Exam

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Duration: 3h. All kind of documents (notes, books...) are authorized, but you cannot collaborate with anyone else. The total number of points is much larger than 20, which means that attacking two or three exercises could be a reasonable option.

Exercice 1 (6 points)**.** Consider the problem

$$\min\left\{\int_0^{2\pi}\frac{|u'(t)|^2}{2+\sin t}dt \quad : \quad u\in C^1([0,2\pi]),\ u(0)=0,u(2\pi)=1\right\}.$$

Find the minimizer as well as its minimal value.

More generally, given a function $h\in C^0([a,b])$, $h>0$, characterize the solution and the find the minimal value of

$$\min\left\{\int_a^b\frac{|u'(t)|^2}{h(t)}dt \quad : \quad u\in C^1([a,b]),\ u(a)=A,u(b)=B\right\}$$

in terms of h, A and B.

Solution: We directly look at the general case with the function h. Let us write the Euler-Lagrange equation of the problem, which is

$$(2\frac{u'}{h})'=0,$$

i.e. $u'=ch$ for a suitable cnstant c. The constant can be found by imposing $B-A=\int_a^b u'=c\int_a^b h$. The solution of the Euler-Lagrange equation with the given Dirichlet data is then given by $u=A+cH$, where H is the antiderivative of h, i.e. $H'=h$ and $H(a)=0$. This function is C^1, and it minimizes the functional since the functional is convex. The minimal value is given by

$$\int_a^b\frac{|ch|^2}{h}=c^2\int_a^b h=\frac{|B-A|^2}{\int_a^b h}.$$

In the precise case where $a=0,b=2\pi,A=0,B=1,h(t)=2+\sin t$ we get $u(t)=\frac{1}{4\pi}(2t+1-\cos t)$ and the minimal value is $\frac{1}{4\pi}$.

Exercice 2 (8 points)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$ and $a,f:\Omega\to\mathbb{R}$ two Lipschitz functions. Suppose $\inf a>0$. Consider the following minimization problem

$$\min\left\{F(u):=\int_\Omega a\left(\frac{|\nabla u|^2}{2}+\frac{u^2}{2}+fu\right)dx\ :\ u\in C^1(\Omega)\cap L^2(\Omega)\right\},$$

where the condition $u\in C^1(\Omega)$ is to be intended locally in the interior of Ω (no behavior or regularity on the boundary is imposed). Prove that the problem admits a solution. It can be useful to treat separately the cases of dimension $d=1$, $d=2,3$, and $d>3$ and to prove the following lemmas

1. If we have $d\leq 3$ and a function v satisfying $v\in H^1(\Omega)$, then v can be written as $v=\nabla\cdot F$ where F is locally Hölder continuous.

2. If u solves an equation of the form $\nabla \cdot (a\nabla u) = g$, where a is Lipschitz continuous and bounded from below by a strictly positive constant and $g \in L^1_{loc}$, then we can also deduce, in the weak sense, the equation $\Delta u = \frac{g}{a} - \nabla(\log a) \cdot \nabla u \in L^1_{loc}$.

Solution: We will prove the existence of a soluzion by first minimizing in H^1 and then proving that the solution is C^1 by regularity (being L^2 is included in being H^1). It is clear that we have $F(u) \geq c||u||^2_{H^1} - C||u||_{L^2} \geq c||u||^2_{H^1} - C||u||_{H^1}$, which proves that any minimizing sequence is bounded in H^1. We can then extract a weakly converging subsequence and, since the integrand is convex in ∇u, continuous in u, and measurable in x, the functional is l.s.c. and the limit minimizes. We write the Euler-Lagrange equation to study its regularity. We have

$$\nabla \cdot (a\nabla u) = a(u + f).$$

From $u \in H^1$ and $a, f \in \mathrm{Lip}$ we deduce that the right had side is H^1. In dimension 1 this means $au' \in H^2 \subset C^1$, and using again a Lipschitz, as well as $\inf a > 0$, we find $u' \in \mathrm{Lip}$. Thus $u \in C^{1,1}$, and the result is even global.

In dimension 2 and 3 we use point 1. Assuming we prove it, we have $\nabla \cdot (\nabla u) = \nabla \cdot F$ with $a, F \in C^{0,\alpha}$. Then $\nabla \in C^{0,\alpha}$, this result being true locally. It proves $u \in C^1$.

In higher dimension we use point 2. The equation tells us $\Delta u = u + f - \nabla(\log a) \cdot \nabla u$. In order to prove $u \in C^1$ it is enough to prove $u \in W^{2,p}_{loc}$ with p large (larger than the dimension. Hence, it is enough to prove that the right hand side is L^p_{loc} for p large. We first know that it is L^2. Then, we observe that if we have a right hand side in L^p we deduce $u \in W^{2,p}_{loc}$ hence $\nabla u \in W^{1,p}_{loc} \subset L^{p^*}_{loc}$ and this improves the summability of the right hand side. We can then prove by induction that the right hand side belong to $L^{p_k}_{loc}$ where $p_{k+1} = (p_k)^* = \frac{dp_k}{d-p_k} > p_k$ if $p_k < d$. In finitely many steps we obtain $p_k > d$ (otherwise we have an increasing sequence with a limit $\ell \in [2, d]$ such that $\ell = \frac{d\ell}{d-\ell}$, which is impossible). Hence we have $u \in W^{2,p}_{loc}$ with $p > d$.

We need now to prove points 1. and 2.

For point 1. we solve $\Delta\varphi = v$ and elliptic regularity tells us $v \in H^1 \Rightarrow \varphi \in H^3$. Hence $v = \nabla \cdot F$ with $F = \nabla\varphi \in H^2$. In dimension 2 we have $H^2 \subset W^{1,p}$ for every p, and if we choose $p > 2$ we have $H^2 \subset C^{0,\alpha}$. In dimension 3 we have $H^2 \subset W^{1,6}$ and again, using $6 > 3$, we have $H^2 \subset C^{0,\alpha}$.

For point 2. we know

$$\int a\nabla u \cdot \nabla\varphi = -\int g\varphi$$

for every test function in H^1_0. If we take $\psi \in H^1_0$ and we use $\varphi = a^{-1}\psi$, we also have $\varphi \in H^1_0$ as the product of an H^1_0 function and a Lipschitz function. We then obtain

$$-\int \frac{g}{a}\psi = -\int g\varphi = \int a\nabla u \cdot \nabla\varphi = \int \nabla u \cdot \nabla\psi - \int a\nabla u \cdot \frac{\nabla a}{a^2}\psi,$$

which is the weak formuation of the desired equation $\Delta u = \frac{g}{a} - \nabla(\log a) \cdot \nabla u$.

Exercice 3 (8 points)**.** Let $H : \mathbb{R}^d \to \mathbb{R}$ be given by $H(z) = \sqrt{1 + |z|^2 + |z|^4}$ and Ω be the d-dimensional torus. Consider the equation

$$\nabla \cdot \left(\frac{(1 + 2|\nabla u|^2)\nabla u}{H(\nabla u)} \right) = f.$$

1. Given $f \in H^{-1}(\Omega)$ such that $\langle f, 1 \rangle = 0$ (i.e. f has zero average), prove that there exists a solution $u \in H^1(\Omega)$ to this equation, which is unique up to additive constants.

2. If $f \in C^\infty$, prove that the solution u is also C^∞.

Solution: First let us note that H is a convex function which satisfies $\max\{1, |z|^2\} \leq H(z) \leq 1 + |z| + |z|^2 \leq 2 + 2|z|^2$, and let us compute its gradient and Hessian. We have

$$\nabla H(z) = \frac{1}{2} \frac{2z + 4|z|^2 z}{H(z)} = \frac{(1 + 2|z|^2)z}{H(z)}$$

and

$$D^2H(z) = \frac{(1+2|z|^2)}{H(z)}I + \frac{4z\otimes z}{H(z)} - \frac{(1+2|z|^2)z}{H^2(z)}\otimes\nabla H(z),$$

hence, finally

$$D^2H(z) = \frac{(1+2|z|^2)}{H(z)}I + \frac{4H^2(z)-(1+2|z|^2)^2}{H^3(z)}z\otimes z.$$

From $4H^2(z) = 4+4|z|^2+4|z|^4$ and $(1+2|z|^2)^2 = 1+4|z|^2+4|z|^4$ we then simplify this into

$$D^2H(z) = \frac{(1+2|z|^2)}{H(z)}I + \frac{3}{H^3(z)}z\otimes z.$$

This allows to see that we have $D^2H(z) \geq \frac{(1+2|z|^2)}{2+2|z|^2}I \geq \frac{1}{2}I$ and $D^2H(z) \leq \frac{(1+2|z|^2)}{H(z)}I + \frac{3}{H^2(z)}\frac{z\otimes z}{H(z)} \leq 6I$. Hence, H is a convex function which is both elliptic and $C^{1,1}$.

The PDE that we are considering is of the form $\nabla\cdot\nabla H(\nabla u) = f$, which is the Euler-Lagrange equation of $\min\int H(\nabla u)+fu$. Since f has zero average, the minimization can be performed among zero-average functions u, since adding a constant does not change the value of the functional. Since H grows quadratically while the term $u\mapsto\int fu$ has linear growth in $||u||_{H^1}$, any minimizing sequence with zero average is bounded in H^1 and admits a weakly converging subsequence. The functional is l.s.c. for this convergence, hence a minimizer exists, which is also a solution of the equation.

The uniqueness of the solution up to additive constants comes the uniqueness of the minimizers up to the same constants, and is due to the strict convexity of the function w.r.t. ∇u.

We know from the regularity via duality that $f\in W^{1,2}$ is enough to guarantee $\nabla u\in H^1$ (here, since H is elliptic, we can obtain $H(z)+H^*(v)\geq z\cdot v + c|z-\nabla H^*(v)|^2$, which provides H^1 regularity on ∇u). This allows to differentiate the equation and obtain, if u' denotes the derivative of u in a certain direction, $\nabla\cdot D^2H(\nabla u)\nabla u') = f'$. To rigorously prove this differentiation one can write

$$\int \nabla H(\nabla u)\cdot\nabla\varphi = \int f\varphi,$$

which is the weak formulation of the PDE and is valid for arbitrary $\varphi\in C^\infty$, and apply it to $\varphi=\psi'$. The, using the fact that ∇H is Lipschitz continuous (since $H\in C^{1,1}$) and $\nabla u\in H^1$, we can integrate by parts putting the derivatives on $\nabla H(\nabla u)$ and f, thus obtaining the weak formulation of the differentiated equation.

We now have a function $u'\in H^1$ which solves a PDE of the form $\nabla\cdot(a(x)\nabla u') = g$, with $g\in C^\infty$. From the bounds on $a = D^2H(\nabla u)$ we can adapt the De Giorgi-Nash-Moser theorem (that we saw for the case $g=0$) and obtain $u'\in C^{0,\alpha}$. The adaptation of the proof is long, but works for much more general g (here we have $g\in C^\infty$, but it works for $g=\nabla\cdot G$, $G\in L^p$, $p>d$).

Since $u'\in C^{0,\alpha}$ is true for arbitrary derivative directions, we obtain $u\in C^{1,\alpha}$. We can then apply the Hölder regularity theory with Campanato spaces to prove by induction $u\in C^{k,\alpha}$ and finally $u\in C^\infty$.

Exercice 4 (8 points)**.** The goal is to find the limit as $\varepsilon\to 0$ of the following minimal value

$$\min\left\{\frac{\varepsilon}{2}\int_{B(0,R_0)}|\nabla u|^2 + \frac{1}{2\varepsilon}\int_{B(0,R_0)}u^2(1-u)^2 + \left(\int_{B(0,R_0)}u\right)^{-1} \;:\; u\in H^1_0(B(0,R_0)),\, 0\leq u\leq 1\right\}.$$

In order to find and justify this limit one can go through the following steps.

1. Prove that the functionals F_ε defined on $L^1(B(0,R_0))$ through

$$F_\varepsilon(u) := \begin{cases}\frac{\varepsilon}{2}\int_{B(0,R_0)}|\nabla u|^2 + \frac{1}{2\varepsilon}\int_{B(0,R_0)}u^2(1-u)^2 & \text{if} u\in H^1_0(B(0,R_0)),\, 0\leq u\leq 1\\ +\infty & \text{if not}\end{cases}$$

   Γ-converge for the strong L^1 topology to F given by

$$F(u) := \begin{cases}c\mathrm{Per}(A) & \text{if} u = I_A,\, A\subset B(0,R_0)\\ +\infty & \text{if not}\end{cases}$$

where c is a suitable constant (to be found) and $\mathrm{Per}(A)$ is the perimeter in the BV sense of the set $A \subset \mathbb{R}^d$ viewed as a subset of the whole space. For this proof, it is enough to explain which arguments have to be modified compared to the result seen in class.

2. Find and characterize the minimizers of $u \mapsto F(u) + (\int u)^{-1}$ making use, if needed, of the isoperimetric inequality.

3. Conclude and compute the limit value, which can depend on the dimension d and on R_0.

Solution:

1. The standard proof of the Modica-Mortola Γ-convergence result ignores the constraints $u \in H_0^1(B(0, R_0))$, $0 \le u \le 1$. The only point where we need to enforce them is in the Γ-limsup inequality. We usually take a smooth set A and define $u_\varepsilon = \phi(\frac{sd_A}{\varepsilon})$. If $d(A, \partial\Omega) > 0$ and ϕ is a function which arrives in finite time to the value 0 and takes value in $[0,1]$, the constructed sequence satisfies the constraints. It is not a problem to choose ϕ in this way (the Γ-limsup inequality can be obtained by taking the infimum among those functions), but the class of smooth sets far from the boundary has to be proven to be dense in energy. This can be easily proven, in the case where Ω is the ball (or convex), by scaling: any set A can be approximated by sets of the form $A_k = (1 - 1/k)A$.

   The value of the constant c is known to be $\int_0^1 \sqrt{W(s)}ds$, where here $W(s) = s^2(1-s)^2$, hence $c = 1/6$.

2. When we want to minimize $u \mapsto F(u) + G(u)$, where $G(u) = (\int u)^{-1}$, we can first take a function u (of the form $u = I_A$) and replace it with the bsest function having the same integral (i.e. an indicator function of a set with the same volume). Thanks to the isoperimetric inequality, we can reduce to the minimization among functions of the form I_{B_R}.

   We have
$$F(I_{B_R}) = \frac{1}{6} d\omega_d R^{d-1} + (\omega_d R^d)^{-1},$$
   where ω_d is the volume of the unit ball in dimension d. The minimizer has hence to be a ball with a radius R which minimizes this expression in $[0, R_0]$. This function is convex in R and its minimum is realized at a unique R_1 such that
$$\frac{1}{6} d(d-1)\omega_d R_1^{d-2} = \frac{d}{\omega_d} R_1^{-d-1}$$
   If $R_0 \le R_1$ then the optimum is realized taking $R = R_0$, otherwise we take $R = R_1$. We can also compute the value of R_1, which is given by
$$R_1 = \left(\frac{6}{\omega_d^2(d-1)}\right)^{\frac{1}{2d-1}}.$$

3. We first need to prove that the limit of the minimal value is the minimal value of the limit. We have Γ-convergence of F_ε to F and G is continuous for the L^1 convergence, hence we preserve the Γ-convergence. We then need to prove that a sequence of minimizers u_ε is compact in L^1. Since we have $0 \le u_\varepsilon \le 1$, strong convergence in L^1 is equivalent to a.e. convergence (by dominated convergence) and we know from the estimates used in the Γ-convergence proof that $\Phi(u_\varepsilon)$ is bounded in BV where $\Phi' = \sqrt{W}$. In particular, up to subsequences, using the compact embedding of BV into L^1 (on a bounded set, the ball), we can assume strong L^1 convergence for $\Phi(u_\varepsilon)$ and, extracting again, also a.e. convergence. Since $\Phi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism as it is continuous and strictly incresing we can compose with Φ^{-1} and preserve the a.e. convergence. This shows compactness of the minimizers and hence the minimal value passes to the limit.

   The limit minimal value is thus given by $\frac{1}{6} d\omega_d R^{d-1} + (\omega_d R^d)^{-1}$ with $R = R_0$ or R_1 as explained above. It indeed depends on d and R_0