Université Paris-Saclay
M2 AMS and Optimization
January 27th, 2017

Calculus of Variations
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Final Examination
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Duration : 3h ; all kind of paper documents (notes, books...) are authorized.

The total score of this exam is much more than 20 : you are not expected to deal with all the exercises (but of course you can). The grade will just be truncated at 20.

Exercice 1 (6 points)**.** Consider the problem

$$\min\left\{\int_0^1 e^{-t}\left(u'(t)^2+6u(t)^2+36u(t)\right)dt \quad : \quad u\in C^1([0,1]),\ u(1)=2e^3+3(e^{-2}-1)\right\}.$$

Prove that it admits a minimizer, that it is unique, and find it.

Exercice 2 (6 points)**.** Let Ω be an open and bounded subset of $\mathbb{R}^d$. Consider the following minimization problem

$$\min\left\{\int_\Omega\left(\frac{1}{2}|\nabla u(x)|^2+\cos(u(x))\right)dx\ :\ u\in H_0^1(\Omega)\right\}.$$

1. Prove that it admits a solution, which can be taken positive
2. Write the PDE satisfied by the solutions.
3. Prove that the solutions are C^∞ in the interior of the domain Ω.
4. Prove that if $\lambda_1(\Omega)\le 1$ then the only solution is $u=0$.
5. Prove that if $\lambda_1(\Omega)>1$ then $u=0$ is not a solution.
6. Prove that, if Ω is a ball, there exists a radial solution.

Exercice 3 (7 points)**.** Given a function $g\in L^2([0,L])$, consider the problem

$$\min\left\{\int_0^L\frac{1}{2}|u(t)-g(t)|^2dt\ :\ u(0)=u(L)=0, u\in \mathrm{Lip}([0,L]), |u'|\le 1\,\text{a.e.}\right\}.$$

1. Prove that this problem admits a solution.
2. Prove that the solution is unique.
3. Find the optimal solution in the case where g is the constant function $g=1$ in the terms of the value of L, distinguishing $L>2$ and $L\le 2$.
4. Computing the value of

   $$\sup\left\{-\int_0^L(u(t)z'(t)+|z(t)|)dt\ :\ z\in H^1([0,L])\right\}$$

   find the dual of the previous problem by means of a formal inf-sup exchange.
5. Assuming that the equality $\inf\sup=\sup\inf$ in the duality is satisfied, write the necessary and sufficient optimality conditions for the solutions of the primal and dual problem. Check that these conditions are satisfied by the solution found in the case $g=1$.
6. Prove the the equality $\inf\sup=\sup\inf$ *(more difficult)*.

Look at the back of the paper for the last two exercises

Exercice 4 (5 points)**.** Let $\Omega \subset \mathbb{R}^d$ be the unit cube $\Omega = (0,1)^d$. Prove that for any measurable set $A \subset \Omega$ there exists unique a solution to the problem

$$\min \left\{ \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx + \int_A u(x) dx - \int_{\Omega \setminus A} u(x) dx \ : \ u \in H^1_0(\Omega) \right\}.$$

Let us call such a solution u_A. Write the Euler-Lagrange equation satisfied by u_A. Prove that if $A_n \to A$ (in the sense that the Lebesgue measure of the symmetric difference $|A_n \setminus A| + |A \setminus A_n|$ tends to 0) then $u_{A_n} \to u_A$ in the strong H^1_0 sense.

Prove that the problem

$$\min \left\{ \int_\Omega |u_A(x)|^2 dx + \mathrm{Per}(A) \ : \ A \subset \Omega \right\}$$

admits a solution. Removing the perimeter penalization, prove that we have

$$\inf \left\{ \int_\Omega |u_A(x)|^2 dx \ : \ A \subset \Omega \right\} = 0$$

and that the infimum is not attained.

Exercice 5 (6 points)**.** Let $\Omega \subset \mathbb{R}^d$ be the unit square $\Omega = (0,1)^2$ and $\Gamma = \{0\} \times (0,1) \subset \partial\Omega$. Consider the functions $a_n : \Omega \to \mathbb{R}$ defined via

$$a_n(x,y) = \begin{cases} A_0 & \text{if } x \in [\frac{2k}{2n}, \frac{2k+1}{2n}) \text{ for } k \in \mathbb{Z} \\ A_1 & \text{if } x \in [\frac{2k+1}{2n}, \frac{2k+2}{2n}) \text{ for } k \in \mathbb{Z}, \end{cases}$$

where $0 < A_0 < A_1$ are two given values. On the space $L^2(\Omega)$ (endowed with the strong L^2 convergence ; every time that we write $\to$ here below we mean this kind of convergence), consider the sequence of functionals

$$F_n(u) = \begin{cases} \int_\Omega a_n |\nabla u|^2 & \text{if } u \in X, \\ +\infty & \text{if not,} \end{cases}$$

where $X \subset L^2(\Omega)$ is the space of functions $u \in H^1(\Omega)$ satisfying $u = 0$ on Γ (which can be defined via the condition $u\eta \in H^1_0(\Omega)$ for every cut-off function $\eta \in \mathbb{C}^\infty(\mathbb{R}^2)$ with $\mathrm{spt}(\eta) \cap (\partial\Omega \setminus \Gamma) = \emptyset$). The goal is to find the Γ-limit of the sequence $(F_n)_n$. Set

$$A_* := \left(\frac{\frac{1}{A_0} + \frac{1}{A_1}}{2} \right)^{-1} \quad \text{and} \quad A_\diamond := \frac{A_0 + A_1}{2}.$$

1. Given a sequence $(u_n)_n$ with $F_n(u_n) \leq C$, prove $u \in X$ and $\liminf_n \int_\Omega a_n |\partial_x u_n|^2 \geq A_* \int_\Omega |\partial_x u|^2$.
2. For the same sequence, also prove $\liminf_n \int_\Omega a_n |\partial_y u_n|^2 \geq A_\diamond \int_\Omega |\partial_y u|^2$.
3. For any $u \in X$, find a sequence u_n such that $u_n \to u$, $a_n \partial_x u_n \to A_* \partial_x u$ and $\partial_y u_n \to \partial_y u$.
4. Conclude by finding the Γ-limit of F_n. Is it of the form $F(u) = \int a |\nabla u|^2$ for $u \in X$?