Université Paris-Saclay
M2 AMS and Optimization
January 28th, 2016

Calculus of Variations

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Final Examination

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Duration : 3h ; all kind of paper documents (notes, books...) are authorized.

Exercice 1 (6 points)**.** Consider the problem

$$\min\left\{\int_0^1 e^{2t}\left(u'(t)^2+3u(t)^2\right)dt \quad : \quad u\in C^1([0,1]),\ u(1)=1\right\}.$$

Prove that it admits a minimizer, that it is unique, and find it.

Exercice 2 (6 points)**.** Let Ω be an open connected subset in $\mathbb{R}^d$, $a\in L^\infty(\Omega)$ be a function with $a\geq a_0$ where $a_0>0$ is a positive constant, and $b\in L^2(\Omega)$ be another function, which is not identically zero. Prove that the following minimization problem admits a solution

$$\min\left\{\frac{\int_\Omega a|\nabla u|^2dx}{|\int_\Omega bu\,dx|^2}\ :\ u\in H_0^1(\Omega)\ :\ \int_\Omega bu\,dx\neq 0\right\},$$

and write the PDE that such a solution satisfies. Finally, compute the value of the above minimum in the case $\Omega=B(0,1)\subset\mathbb{R}^2$, $a(x)=1$ and $b(x)=|x|$.

Exercice 3 (7 points)**.** Let Ω be an open connected subset of $\mathbb{R}^d$ and take $f\in L^2(\Omega)$ with $\int_\Omega f(x)dx=0$. Consider the following minimization problem

$$\min\left\{\int_\Omega\left(\frac{1}{2}|\nabla u|^2+|\nabla u|+fu\right)dx\ :\ u\in H^1(\Omega)\right\}.$$

1. Prove that it admits a solution, which is unique up to an additive constant.
2. Write the PDE satisfied by the solution $\bar{u}$ supposing $\nabla\bar{u}\neq 0$.
3. Find the dual of this problem.
4. Find a re-writing of the previous PDE which is also valid when $\nabla\bar{u}=0$ (in the form "there exists v with $\nabla\cdot v=f$ and such that v is linked to $\nabla\bar{u}$ in some way").
5. Prove that any optimal u is $H^2_{loc}(\Omega)$ (or H^2 if we replace the domain Ω with a torus).

Exercice 4 (4 points)**.** Given an open bounded set $\Omega\subset\mathbb{R}^d$ with volume $|\Omega|>2$, prove that the following problem has a solution

$$\min\left\{Per(A)+Per(B)+\int_\Omega|\nabla u|^2dx\ :\ A,B\subset\Omega,\ |A|=|B|=1, u\in H^1(\Omega),\ u=1\,\text{on }A,\ u=0\,\text{on }B\right\}.$$

$Per(A)$ and $Per(B)$ stand for the perimeters in the BV sense, i.e. the total variation of the corresponding indicator functions inside the open set Ω. Also compute the minimal value above when $\Omega=]0,L[\subset\mathbb{R}$ for $L>2$.

Look at the back of the paper for the last exercise

Exercice 5 (7 points)**.** Let X be the metric space $X = \{u \in L^2([0,1]) \ : \ |u| \leq M\}$, where $M > 0$ is a given constant, endowed with any distance inducing the weak convergence in L^2. Let $W, U : \mathbb{R} \to \mathbb{R}$ be given by $W(t) = (1-t^2)^2$ and $U(t) = ((1-t^2)_-)^2$ (i.e. $U = W$ outside $[-1,1]$ and $U = 0$ on $[-1,1]$). Consider the functionals $F_n : X \to \mathbb{R} \cup +\infty$ defined through

$$F_n(u) := \begin{cases} \int_0^1 W(u(s))ds + \frac{1}{n}\int_0^1 |u'(s)|^2 ds & \text{if } u \in H^1([0,1]), \\ +\infty & \text{otherwise} \end{cases}$$

and the functional F simply given by $F(u) = \int_0^1 U(u(s))ds$.

1. Justify why $U = W^{**}$.
2. Prove that for any sequence u_n with $u_n \rightharpoonup u$ we have $\liminf_n F_n(u_n) \geq F(u)$.
3. Suppose that u is a piecewise constant (the pieces being intervals) taking only the values ± 1, which implies $F(u) = 0$. Prove that there exists a sequence u_n such that $\lim_n F_n(u_n) = 0$ and $u_n \rightharpoonup u$.
4. Set $Y_1 = \{u \in X \ : \ u$ is a piecewise constant function taking only values in $[-M, 1] \cup [1, M]\}$ and $Y_2 = \{u \in X \ : \ u$ is a piecewise constant function$\}$. Prove that the same conclusion as before is true for every $u \in Y_1$.
5. Prove that Y_1 is dense in energy in Y_2 (i.e for every $u \in Y_2$ there is a sequence $u_k \in Y_1$ s.t. $u_k \rightharpoonup u$ and $F(u_k) \to F(u)$).
6. Prove that Y_2 is dense in energy in X.
7. Conclude $F_n \xrightarrow{\Gamma} F$.