**Université Paris-Saclay**
**M2 AMS and Optimization**
**March 22nd, 2016**

# Calculus of Variations

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# 2nd round Examination

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Duration : 3h ; all kind of paper documents (notes, books…) are authorized.

**Exercice 1** (7 points)**.** Consider the problem

$$\min\left\{\int_0^{\pi/2}\Big(u'(t)^2+u(t)^2+2\sin(t)u(t)\Big)\,dt\quad:\quad u\in C^1([0,\pi/2]),\ u(0)=0\right\}.$$

Prove that it admits a minimizer, that it is unique, find it, and compute the value of the minimum.

**Exercice 2** (5 points)**.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$. Consider the following minimization problem

$$\min\left\{\int_\Omega(2+\arctan(u))(1+|\nabla u|^2)dx\ :\ u\in X\right\}.$$

1. If $X=H^1(\Omega)$, prove that the problem has no solution.
2. If $X=H^1_0(\Omega)$, prove that the problem admits at least a solution $\bar u$, prove that $\bar u\le 0$, find the Euler-Lagrange equation solved by $\bar u$, and prove $\Delta\bar u\in L^1$.

**Exercice 3** (8 points)**.** Let $\Omega$ be the $d-$dimensional flat torus (just to avoid boundary conditions, think at a cube), $p,q>1$ two given exponents, $a>0$ and $f:\Omega\to\mathbb{R}$ a given Lipschitz continuous function. Consider the following minimization problem

$$\inf\left\{\int_\Omega\left(\frac{1}{p}|\nabla u|^p-\frac{a}{q}|u|^q+fu\right)dx\ :\ u\in W^{1,p}(\Omega)\cap L^q(\Omega)\right\}.$$

1. Prove that, if $q>p$, the inf is $-\infty$ and the minimization problem has no solution.
2. Prove that, if $q<p$, the infimum is attained.
3. Prove that, if $q=p$, the infimum is attained, provided $a$ is small enough.
4. In the cases where the infimum is attained, write the Euler-Lagrange equation solved by the minimizers.
5. Recall the condition on $f$ which guarantee that solutions of $\Delta_p u=f$, satisfy $(\nabla u)^{p/2}\in H^1$ (remember that, for a vector $v$, the expression $v^\alpha$ is to be intended as equal to a vector $w$ with $|w|=|v|^\alpha$ and $w\in\mathbb{R}_+v$).
6. For $p\ge 2$ and $2\le q\le p$, prove that the solution $\bar u$ satisfies $(\nabla\bar u)^{p/2}\in H^1$.

**Exercice 4** (4 points)**.** Given an open bounded set $\Omega\subset\mathbb{R}^d$ with volume $|\Omega|>2\omega_d$ ($\omega_d$ denoting the volume of the unit ball in $\mathbb{R}^d$) and a function $f\in L^1(\Omega)$, prove that the following problem has a solution

$$\min\left\{Per(A)+Per(B)+\int_{A\cap B}f(x)dx\ :\ A,B\subset\Omega,\ |A|=|B|=\omega_d\right\}.$$

$Per(A)$ and $Per(B)$ stand for the perimeters in the BV sense, i.e. the total variation of the corresponding indicator functions (considered as BV functions on the whole $\mathbb{R}^d$, not on $\Omega$).

Also compute the minimal value above when $\Omega$ is the smallest convex domain containing $B(e,1)$ and $B(-e,1)$, where $e$ is an arbitrary unit vector $e\in\mathbb{R}^d$ with $|e|=1$, and $f=1$ on $\Omega$.

**Look at the back of the paper for the last exercise**

**Exercice 5** (6 points)**.** Let $u_n, v_n$ be two sequences of functions belonging to $H^1([0,1])$. Suppose

$$u_n \rightharpoonup u_0, \quad v_n \rightharpoonup v_0, \quad E_n(u_n, v_n, [0,1]) \le C$$

where $E_n$ is the energy defined for every interval $J \subset [0,1]$ through

$$E_n(u,v,J) := \int_J \left( v(t)|u'(t)|^2 + \frac{1}{2n}|v'(t)|^2 + \frac{n}{2}|1-v(t)|^2 \right) dt,$$

$C \in \mathbb{R}$ is a given constant, the weak convergence of $u_n$ and $v_n$ occurr in $L^2([0,1])$, and $u_0$ is a function which is piecewise $C^1$ on $[0,1]$ (i.e. there exists a partition $0 = t_0 < t_1 < \cdots < t_N = 1$ such that $u_0 \in C^1(]t_i, t_{i+1}[)$, and the limits of $u_0$ exist finite at $t = t_i^{\pm}$ but $u_0(t_i^-) \neq u_0(t_i^+)$ for $i = 1, \ldots, N-1$).

Denote by $\mathcal{J}$ the family of all the intervals $J$ compactly contained in one of the open intervals $]t_i, t_{i+1}[$. Also suppose, for simplicity, that $u_n \rightharpoonup u_0$ in $H^1(J)$ for every interval $J \in \mathcal{J}$.

Prove that we necessarily have

1. $v = 1$ a.e. and $v_n \to v$ strongly in $L^2$.
2. $\liminf_{n\to\infty} E_n(u_n, v_n, J) \ge \int_J |u_0'(t)|^2 dt$ for every interval $J \in \mathcal{J}$.
3. $\liminf_{n\to\infty} E_n(u_n, v_n, J) \ge 1$ for every interval $J$ containing one of the points $t_i$.
4. $C \ge \liminf_{n\to\infty} E_n(u_n, v_n, [0,1]) \ge \int_0^1 |u_0'(t)|^2 dt + (N-1)$.