M2 Maths en Action et M2 Maths Avancées
March 29th, 2023

Transport Optimal pour l'Apprentissage – Final Exam

Duration: 1h30. All kind of documents (notes, books. . .) are authorized.
Each exercise is worth 10 points, so that solving two of them could be enough (but you can of course attack thel all).

Exercice 1 (10 points)**.** Let μ_ε be the uniform probability measure on the rectangle $[-1, 1] \times [-\varepsilon, \varepsilon] \subset \mathbb{R}^2$ and ν_ε the uniform probability measure on the rectangle $[-\varepsilon, \varepsilon] \times [-1, 1] \subset \mathbb{R}^2$. Fid the unique optimal transport map (for the quadratic cost $c(x, y) = ||x - y||^2$) from μ_ε to ν_ε.

As $\varepsilon \to 0$ the measures μ_ε and ν_ε weakly-* converge to two measures μ_0 and ν_0. Determine those limit measures and find all the optimal transport plans between them (for the same cost).

Answer: The map $T_\varepsilon(x_1, x_2) := (\varepsilon x_1, \varepsilon^{-1} x_2)$ maps μ_ε onto ν_ε. Moreover, $T_\varepsilon = \nabla u_\varepsilon$, where $u_\varepsilon(x_1, x_2) = \varepsilon\frac{x_1^2}{2} + \varepsilon^{-1}\frac{x_2^2}{2}$, which is a convex function, so T_ε is the optimal map.

The limit as $\varepsilon \to 0$ of μ_ε nd ν_ε are the uniform measures on the segments $[-1, 1] \times \{0\}$ and $\{0\} \times [-1, 1]$, respectively. For those measures, we have the following property: $x \in \operatorname{spt}\mu_0$, $y \in \operatorname{spt}\nu_0$ implies $x \cdot y = 0$. So, on the product of the two supports, we have $c(x, y) = ||x - y||^2 = ||x||^2 + ||y||^2$. Since the cost coincides with a separable function, sum of a function of x and a function of y, we have $\int c(x, y)d\gamma = \int ||x||^2 d\mu_0 + \int ||y||^2 d\nu_0$ for any admissible γ, a result which is independent of the transport plan γ. Hence, any admissible transport plan s optimal.

Exercice 2 (10 points)**.** Let μ_0 be the uniform probability measure on the square $[-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ and μ_1 the uniform probability measure on the union of the two segments $\{-1\} \times [-1, 1]$ and $\{+1\} \times [-1, 1]$ (i.d. μ_1 is a measure with constant density w.r.t. to the length measure on these segments). Find the constant-speed geodesic μ_t connecting μ_0 to μ_1 in the Wasserstein space W_2. Can we say that, for each $t > 0$, the measure μ_t is the uniform measure on a certain set? which set?

Answer: The optimal transport map from μ_0 to μ_1 is given by $T(x_1, x_2) = (\text{sign}(x_1), x_2)$. It transports μ_0 to μ_1 and it is the gradient of the convex function $u(x_1, x_2) := |x_1| + \frac{x_2^2}{2}$, hence it is optimal. The geodesic μ_t is then obtained as $\mu_t := (1 - t)id + tT)_{\#}\mu_0$. On $[0, 1] \times [-1, 1]$ we have $T_t(x_1, x_2) = (1 - tx_1 + t, x_2)$ and on $[-1, 0] \times [-1, 1]$ we have $T_t(x_1, x_2) = (1 - tx_1 - t, x_2)$. In both cases T_t is affine and hence its transforms uniform measures into uniform measures. The ratio between the densities is given by the Jacobian of this map, which is the same on the two sets. The measure μ_t is hence uniform on $T_t([-1, 1] \times [-1, 1]) = ([t, 1] \cup [-1, -t]) \times [-1, 1]$, with density equal to $\frac{1}{4(1-t)}$.

Exercice 3 (10 points)**.** Given $a, b \in [0, 1]$ with $a + b = 1$ and $t < s$ consider $\mu := a\delta_t + b\delta_s$ as a probability measure on $\mathbb{R}$, and let ν be the uniform probability measure on the interval $[-1, 1]$. Compute $W_2(\mu, \nu)$ and find the values of a, b, t, s for which this distance is minimal.

Consider ow a two-dimensional analogue of the previous situation: $\mu := a\delta_{(t,0)} + b\delta_{(s,0)}$ is a probability measure on $\mathbb{R}^2$, and let ν is the uniform probability measure on the square $[-1, 1]^2$. Again, compute $W_2(\mu, \nu)$ and find the values of a, b, t, s for which this distance is minimal.

Answer: The optimal map from ν to μ is an increasing map taking values t and s. Hence, it is equal to t on a first part of the interval $[-1, 1]$ with measure (according to ν) equal to a and to s on the rest. We then have $T(x) = t$ for $x \in [-1, -1 + 2a]$ and $T(x) = s$ on $[-1 + 2a, 1]$. Then, we compute

$$W_2^2(\mu, \nu) = \frac{1}{2}\int_{-1}^{-1+2a} |x - t|^2 dx + \frac{1}{2}\int_{-1+2a}^{1} |x - s|^2 dx = \frac{a^3}{3} + a| - 1 + a - t|^2 + \frac{b^3}{3} + b|1 - b - s|^2,$$

where we used $a + b = 1$. It is clear that this result can be minimized by taking $t = -1 + a$ and $s = 1 - b$ (note that we have $s - t = 2 - (a + b) = 1 > 0$, so that we do have $t < s$), so that it becomes equal to $\frac{a^3+b^3}{3}$, a quantity which is minimal (under the constraint $a + b = 1$) when $a = b = 1/2$.

In the two dimensional situation the optimal map will be given by $T(x1, x_2) = (\tilde{T}(x_1, 0))$, where $\tilde{T}$ is the optimal map of the first part of the exercise. In the computation of the Wasserstein distance we then have to add the integral $\int |x_2|^2 d\nu = \frac{1}{2}\int_{-1}^{1} |x_2|^2 dx_2 = \frac{1}{3}$. We then have

$$W_2^2(\mu, \nu) = \frac{a^3 + b^3 + 1}{3} + a|-1 + a - t|^2 + b|1 - b - s|^2,$$

and the values of (a, b, t, s) which minimize this are the same as before.