Calculus of Variations and Elliptic PDEs
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Exercises

Exercise 1. Solve the problem

$$\min\{J(f) := \int_0^1 \left[\frac{1}{2}f'(t)^2 + tf(t) + \frac{1}{2}f(t)^2\right] dt\,;\ f \in \mathcal{A}\}, \quad \text{où} \quad \mathcal{A} := \{f \in C^1([0,1])\ :\ f(0) = 0\}.$$

Find the minimal value of J on $\mathcal{A}$ and the function(s) f which attain it, proving that they are actually minimizers

Solution. Take f_n a minimizing sequence in $\mathcal{A}' := \{f \in H^1([0,1]) : f(0) = 0\}$ – it makes sense since H^1 functions on $[0,1]$ are continuous. We may assume that $J(f_n) \leq 0$ since $J(0) = 0$. We have

$$J(f) \geq \frac{1}{2}\|f'\|_{L^2}^2 - C\|f\|_{L^2} + \frac{1}{2}\|f\|_{L^2}^2 \geq C\|f\|_{H^1}^2 - C'\|f\|_{H^1}$$

so that f_n is bounded in $H^1([0,1])$. We may extract a subsequence such that $f_n \xrightarrow{H^1} f$ hence $f_n \to f$ in $\mathcal{C}([0,1])$ by compact injection. Hence $f(0) = 0$. The function J is lsc for the H^1 weak convergence (for example one recognizes the squared H^1 norm and a continuous linear part), so that $J(f) \leq \liminf_n J(f_n) = \inf_{\mathcal{A}'} J$. One needs to check that f is $\mathcal{C}^1$. Let us write the Euler-Lagrange equation. Take any test function $\phi \in \mathcal{C}_c^1(]0,1[)$. One has

$$0 = \frac{d}{d\epsilon}_{|\epsilon=0} J(f + \epsilon\phi) = \int_0^1 f'\phi' + t\phi + f\phi,$$

so that in the distribution sense

$$f'' = t + f.$$

Consequently $f \in C^k \implies f \in C^{k+2}$ and because $f \in H^1 \subset C^0$, it is actually in all the C^k spaces and is $\mathcal{C}^\infty$. Hence

$$J(f) = \inf_{\mathcal{A}} J = \inf_{\mathcal{A}'} J.$$

Uniqueness is clear by the strict convexity of J. Now the Euler-Lagrange equation allows us to find the solution. We know that f is of the form $f(t) = A\mathrm{ch}(t) + B\mathrm{sh}(t) - t$. The condition $f(0) = 0$ yields $A = 0$. An extra condition is given by the implicit Neumann condition at $t = 1$. Indeed one may take any test function $\phi \in \mathcal{A}$ and write as before

$$0 = \frac{d}{d\epsilon}_{|\epsilon=0} J(f + \epsilon\phi) = \int_0^1 f'\phi' + t\phi + f\phi$$

since f is a minimizer of J on $\mathcal{A}$. Now by integrating by parts

$$0 = \int_0^1 (-f'' + t + f)\phi + [f'\phi]_0^1 = 0 + f'(1)\phi(1) - 0 = f'(1)\phi(1).$$

Consequently $f'(1) = 0$ and thus $B\mathrm{ch}(1) = 1$, giving $f(t) = \frac{\mathrm{sh}(t)}{\mathrm{ch}(1)} - t$. Now to compute the actual minimum, it is easier to do an integration by parts

$$J(f) = \frac{1}{2}\int_0^1 (-f'' + f + t)f + [f'f]_0^1 + \int_0^1 \frac{tf(t)}{2}dt = \int_0^1 \frac{tf(t)}{2}dt = \dots.$$

Exercise 2. Consider the problem

$$\min\left\{\int_0^T e^{-t}\left(u'(t)^2 + 5u(t)^2\right) dt \quad : \quad u \in C^1([0,T]),\ u(0) = 1\right\}.$$

Prove that it admits a minimizer, that it is unique, find it, compute the value of the minimum, and the limit of the minimizer (in which sense ?) and of the minimal value as $T \to +\infty$.

Exercise 3. Consider the problem

$$\min\{J(u) := \int_0^1 \left[\frac{1}{2}u'(t)^2 + u(t)f(t)\right] dt\,;\; u \in W^{1,2}([0,1])\}.$$

Find a necessary and sufficient condition on f so that this problem admits a solution.

Solution. We are going to show that it admits a minimizer if and only if f has zero mean. If its mean is positive, say, take u equal to a constant C arbitrarily negative, giving $J(u) = C\int f \xrightarrow{C\to-\infty} -\infty$ and the infimum is $-\infty$. Now suppose $\int f = 0$. Note that the functional is well defined since $W^{1,2} \subset L^\infty$ hence the term $\int fu$ has a meaning. Now, take a minimizing sequence u_n in $H^1([0,1])$. One may add a constant to u_n without changing the value of the functional since f a zero mean, thus we may assume $\int u_n = 0$ for all n. Using Hölder's inequality first, then Poincaré-Wirtinger's inequality, one gets

$$\left|\int_0^1 fu\right| \le \|f\|_{L^2}\|u\|_{L^2} \le C\|f\|_{L^2}\|u'\|_{L^2}$$

so that $J(u) \ge C\|u'\|_{L^2}^2 - C'\|u'\|_{L^2}$. Thus u_n' is bounded in L^2 which implies that u_n is bounded in H^1. We can extract a converging subsequence $u_n \rightharpoonup u$ in H^1. J is lsc for weak H^1 since it is convex and continuous on H^1 thus u is a minimizer.

Exercise 4. Let $L : \mathbb{R} \to \mathbb{R}$ be a strictly convex C^1 function, and consider

$$\min\{\int_0^1 L(u'(t))dt\,;\; u \in C^1([0,1]), u(0) = a, u(1) = b\}.$$

Prove that the solution is $u(t) = (1-t)a + tb$, whatever is L. What happens if L is not C^1 ? and if L is convex but not strictly convex ?

Exercise 5. Prove that we have

$$\inf\{\int_0^1 t|u'(t)|^2 dt\,;\; u \in C^1([0,1]), u(0) = 1, u(1) = 0\} = 0.$$

Is the infimum above attained ?

What about, instead

$$\inf\{\int_0^1 \sqrt{t}|u'(t)|^2 dt\,;\; u \in C^1([0,1]), u(0) = 1, u(1) = 0\} \quad ?$$

Exercise 6. Consider a minimization problem of the form

$$\min\{F(u) := \int_0^1 L(t, u(t), u'(t))dt\,;\; u \in W^{1,1}([0,1]), u(0) = a,\, u(1) = b\},$$

where $L \in C^2([0,1] \times \mathbb{R} \times \mathbb{R})$. We denote as usual by (t, x, v) the variables of L. Suppose that $\bar{u}$ is a solution to the above problem. Prove that we have

$$\frac{\partial^2 L}{\partial v^2}(t, \bar{u}(t), \bar{u}'(t)) \ge 0 \, a.e.$$

Solution. Take a competitor of the form $\bar{u} + \epsilon\phi$ here $\phi \in W_0^{1,1}$ and write

$$\frac{d^2}{d\epsilon^2}_{|\epsilon=0} F(\bar{u} + \epsilon\phi) \ge 0.$$

Let us compute the directional derivatives :

$$\frac{d}{d\epsilon}F(\bar{u}+\epsilon\phi) = \frac{d}{d\epsilon}\int_0^1 L(t,\bar{u}+\epsilon\phi,\bar{u}'+\epsilon\phi')$$
$$= \int_0^1 L_u(t,\bar{u}+\epsilon\phi,\bar{u}'+\epsilon\phi')\phi + L_v(t,\bar{u}+\epsilon\phi,\bar{u}'+\epsilon\phi')\phi'$$

, and

$$0 \leq \frac{d^2}{d\epsilon^2}_{|\epsilon=0}F(\bar{u}+\epsilon\phi) = \int_0^1 L_{uu}\phi^2 + 2L_{uv}\phi\phi' + L_{vv}\phi'^2.$$

For $a, b \in [0,1]$ fxed, set ϕ_n supported in $[a,b]$ and of slope $|\phi_n'| = 1$ in that interval, such that $|\phi_n| \leq 1/n$. Passing to the limit, the terms in $\phi_n'\phi_n$ and ϕ_n^2 vanish and one gets

$$0 \leq \lim_n \int_0^1 (L_{uu}\phi_n^2 + 2L_{uv}\phi_n\phi_n' + L_{vv}\phi_n'^2) = \int_a^b L_{vv}(t,\bar{u},\bar{u}').$$

Since this holds for all a, b, one gets the result.

Exercise 7. Given $f \in C^2(\mathbb{R})$, consider the problem

$$\min\{F(u) := \int_0^1 \Big[(u'(t)^2-1)^2 + f(u(t))\Big]\,dt\,;\ u \in C^1([0,1]), u(0)=a,\ u(1)=b\}.$$

Prove that the problem does not admit any solution if $|b-a| \leq \frac{1}{\sqrt{3}}$.

Solution. Suppose it has a solution u and take any test function $\phi \in W^{1,4}$. The function $\epsilon \mapsto F(u+\epsilon\phi)$ is minimal at $\epsilon = 0$ so that

$$0 \leq \frac{d^2}{d\epsilon^2}_{|\epsilon=0}F(u+\epsilon\phi),$$

provided this quantity is well defined. Let us compute the first derivative :

$$\frac{d}{d\epsilon}_{|\epsilon}F(u+\epsilon\phi) = \int_0^1 4\phi'(u'+\epsilon\phi')((u'+\epsilon\phi')^2-1) + f'(u+\epsilon\phi)\phi,$$

and derive the Euler-Lagrange equation

$$4(u'(u'^2-1))' = f'(u).$$

then its second derivative at 0 :

$$\frac{d^2}{d\epsilon^2}_{|0}F(u+\epsilon\phi) = \int_0^1 4\phi'^2(u'^2-1) + 8\phi'^2u'^2 + f''(u)\phi^2 = 12\int_0^1 \left(u'^2 - \frac{1}{3}\right)\phi'^2 + \phi^2 f''(u).$$

Now take a suitable $\phi = \phi_n$: supported in $[\alpha,\beta] \subseteq [0,1]$ of slope $|\phi'| = 1$ and $|\phi| \leq 1/n$. Then pass to the limit $n \to \infty$ to yield

$$0 \leq 12\int_\alpha^\beta \left(u'^2 - \frac{1}{3}\right).$$

This is true for all α, β hence $|u'| \geq 1/3$ a.e.. Notice that we cannot have $|u'| = \frac{1}{\sqrt{3}}$: in that case the Euler-Lagrange would become $-8u''/3 = f'(u)$ and we could deduce that u is $\mathcal{C}^1$ (even $\mathcal{C}^2$) thus $u' = \pm\frac{1}{\sqrt{3}}$ which is not a minimizer since it does not satisfy the same E-L equation.. Consequently $|b-a| > \frac{1}{\sqrt{3}}$. By contraposition one gets the desired result.

Exercise 8. Consider the problem

$$\min\left\{\int_0^1 \Big[|u'(t)|^2 + \arctan(u(t))\Big]\, dt \;:\; u \in C^1([0,1])\right\},$$

and prove that it has no solutions. Prove the existence of a solution if we add the boundary condition $u(0) = 0$, write the optimality conditions and discuss the regularity of the solution.

Solution. If $u \equiv C$ constant, one has $J(u) = \arctan(C)$, thus

$$\inf J \le \inf_{C\in\mathbb{R}} \arctan(C) = -\frac{\pi}{2}$$

but on the other hand for all u,

$$J(u) = \int_0^1 \Big[|u'(t)|^2 + \arctan(u(t))\Big]\, dt \ge \int_0^1 \arctan u(t) dt \ge -\frac{\pi}{2}.$$

Consequently the infimum is $-\frac{\pi}{2}$ and u is a minimizer if and only if we have equalities n the previous inequalities, which implies that u is constant equal to some C satisfying $\arctan(C) = -\frac{\pi}{2}$ which cannot happen.

Now if one imposes $u(0) = 0$, let us set $\mathcal{A} = \{u \in H^1([0,1]) : u(0) = 0\}$ and look for minimizers of J on $\mathcal{A}$. One has

$$J(u) \ge \|u'\|^2_{L^2} - \frac{\pi}{2}$$

hence u' is bounded in L^2, but $u(x) = \int_0^x u'(t)dt$ thus $\|u\|_{L^\infty} \le \|u'\|_{L^2}$ and u is bounded in $H^1([0,1])$.

Exercise 9. Consider the functional $F : H^1([0,T]) \to \mathbb{R}$ defined through

$$F(u) = \int_0^T \Big(u'(t)^2 + \arctan(u(t) - t)\Big)\, dt.$$

Prove that

a) the problem $(P) := \min\{F(u) \;:\; u \in H^1([0,T])\}$ has no solution ;

b) the problem $(P_a) := \min\{F(u) \;:\; u \in H^1([0,T]), u(0) = a\}$ admits a solution for every $a \in \mathbb{R}$;

c) we have $F(-|u|) \le F(u)$;

d) the solution of (P_a) is unique as soon as $a \le 0$;

e) there exists $L_0 < +\infty$ such that for every $T \le L_0$ the solution of (P_a) is unique for every $a \in \mathbb{R}$

f) the minimizers of (P) and (P_a) are C^∞ functions.

Exercise 10. Prove existence and uniqueness of the solution of

$$\min\left\{\int_\Omega \Big(f(x)|u(x)| + |\nabla u(x)|^2\Big)\, dx \,;\; u \in H^1(\Omega),\; \int_\Omega u = 1\right\},$$

when Ω is an open, connected and bounded subset of $\mathbb{R}^n$ and $f \in L^2(\Omega)$, $f \ge 0$ (the sign of f is not important for existence). Where do we use connectedness ? Also prove that, if Ω is not connected (but has a finite number of connected components and we keep the assumption $f \ge 0$), then we have existence but maybe not uniqueness, and that if we withdraw both connectedness and positivity of f, then maybe we don't even have existence.

Solution. Suppose that Ω is connected. Thus by Poincaré-Wirtinger's inequality one has $\|u - c\|_{L^2} \le C\|\nabla u\|_{L^2}$ where $c = 1/|\Omega|$ is the mean of u (note that this inequality is not true for Ω disconnected). Hence

$$\left|\int_\Omega f|u|\right| \le \|f\|_{L^2}\|u\|_{L^2} = \|f\|_{L^2}\|u - c + c\|_{L^2} \le C + C\|\nabla u\|_{L^2},$$

which implies that

$$J(u) \geq \|\nabla u\|_{L^2}^2 - C - C\|\nabla u\|_{L^2}$$

thus if u_n is a minimizing sequence, $J(u_n) \leq C$ for some finite C and ∇u_n is bounded in L^2. Consequently $u-c$ is bounded in L^2, so is u and therefore u is bounded in H^1. One may extract a weakly converging subsequence $u_n \rightharpoonup u$ in H^1 and by compact injection $u_n \to u$ in L^2. The first part in the integrand is continuous in L^2 strong, and the second part is weakly lsc in H^1, which implies that $J(u) \leq \liminf_n J(u_n)$ and u is a minimizer.

Now assume that we have the following decomposition $\Omega = \bigsqcup_{i=1}^n \Omega_i$ into open connected subsets, and that $f \geq 0$. If $u \in H^1$ is admissible, one may build a better competitor $\tilde{u}$ which is positive. Take $v = |u|$. One has $|\nabla v| = |\nabla u|$ and $\int_\Omega v \geq \int_\Omega u = 1$ thus we divide by $||v||_{L^1} \geq 1$, setting $\tilde{u} = v/\bar{v}$. Since f is positive, one can see that $J(\tilde{u}) \leq J(v) = J(u)$. Thus we may take a minimizing sequence u_n which is nonnegative. For any v, we denote $v^i = v_{|\Omega_i}$ and $\bar{v}^i$ its mean : $\bar{v}^i = \frac{\int_{\Omega_i} f}{|\Omega_i|}$. One has by positivity of f

$$J(v) \geq \int_{\Omega_i} v_i f + |\nabla v_i|^2,$$

the first term on the right being bounded from above

$$\left|\int_{\Omega_i} v_i f\right| \leq \|f\|_{L^2}\|v^i - \bar{v}^i\|_{L^2} + \|f\|_{L^2}\bar{v}^i.$$

and we know that $\bar{v}^i \leq \frac{\int_\Omega v}{|\Omega_i|} = \frac{1}{|\Omega_i|}$. Consequently

$$\left|\int_{\Omega_i} v_i f\right| \leq C\|v^i - \bar{v}^i\|_{L^2} + C \leq C\|\nabla v^i\|_{L^2} + C.$$

Consequently the bound $J(u_n) \leq C$ gives a uniform bound on $\|\nabla u_n^i\|_{L^2}$ and thus on $\|u_n^i\|_{L^2}$ since the mean is bounded. Thus we may conclude as before, u_n^i being bounded in $H^1(\Omega_i)$, extracting a converging subsequence for all i and using the lsc of J. We may not have uniqueness however (even for nonnegative u) : just consider $f = 0$ and Ω the disjoint union of 2 balls with unit volume. You may take u to be either $1/2$ on all Ω or 1 on one ball and 0 on the other : they are both minimizers. If you do not have neither positivity nor connectedness, we may not have existence. Take this time f to be equal to $+1$ on the first ball and -2 on the second one. Choose u_α to be equal to a constant α on the first ball and $1-\alpha$ on the other, so that it has integral 1. $J(u_\alpha) = \alpha - 2|1-\alpha| = -\alpha + 2$ for α large enough, and $J(u_\alpha)$ may be arbitrarily small.

Exercise 11. Fully solve

$$\min\left\{\int_Q \left(|\nabla u(x,y)|^2 + u(x,y)^2\right)\,dx\,dy \;:\; u \in C^1(Q),\ u = \phi \text{ sur } \partial Q\right\},$$

where $Q = [-1,1]^2 \subset \mathbb{R}^2$ and $\phi : \partial Q \to \mathbb{R}$ is given by

$$\phi(x,y) = \begin{cases} 0 & \text{si } x = -1,\ y \in [-1,1] \\ 2(e^y + e^{-y}) & \text{if } x = 1,\ y \in [-1,1] \\ (x+1)(e + e^{-1}) & \text{if } x \in [-1,1],\ y = \pm 1. \end{cases}$$

Find the minimizer and the value of the minimum. Writing the Euler-Lagrange equation is not compulsory, but could help.

Exercise 12. Show that for every function $f : \mathbb{R} \to \mathbb{R}_+$ l.s.c. there exists a sequence of functions $f_k : \mathbb{R} \to \mathbb{R}_+$, each $k-$Lipschitz, such that for every $x \in \mathbb{R}$ the sequence $(f_k(x))_k$ increasingly converges to $f(x)$.

Use this fact and the theorems we saw in class to prove semicontinuity, wrt to weak convergence in $H^1(\Omega)$, of the functional

$$J(u) = \int_\Omega f(u(x))|\nabla u(x)|^p \, dx,$$

where $p \geq 1$ and $f : \mathbb{R} \to \mathbb{R}_+$ is l.s.c.

Solution. Set $f_k(x) = \inf_y f(y) + kd(x,y)$ for $k \geq 1$. Clearly, the sequence (f_k) is increasing and $f_k \leq f$ (take $y = x$ in the infimum). For x fixed, let us show that $f_k(x) \to f(x)$. For all k, there exists some y_k such that $f(y_k) + kd(x, y_k) - \frac{1}{k} \leq f_k(x) \leq f(x)$. By positivity of f, $kd(x, y_k) \leq \frac{1}{k} - f(x) \leq 1$, thus $d(x, y_k) \to 0$ i.e. $y_k \to x$. Passing to the $\liminf_k$ in the inequality $f(y_k) - \frac{1}{k} \leq f_k(x)$ one gets

$$\lim_k f_k(x) \geq \liminf_k f(y_k) - \frac{1}{k} = \liminf_k f(y_k) \geq f(x)$$

by lsc of f. The converse inequality is clear since $f_k \leq f$.

Take $f_k \uparrow f$ as above, set

$$J_k(u) = \int_\Omega f_k(u(x))|\nabla u(x)|^p dx,$$

and $L_k(x, u, v) = f(u)|v|^p$. The functional J_k is lsc weak in H^1 since L_k is measurable in x (it does not depend on it !), continuous in u and convex in v. Now by the monotone convergence theorem $J_k = \lim_k \uparrow J$ and thus J is lsc in H^1 weak as a supremum of lsc functions.

Exercise 13. Find the Poincaré constant of the interval $(-A, A)$, i.e. the smallest constant C such that

$$\int_{-A}^{A} u^2(x)dx \leq C \int_{-A}^{A} (u')^2(x)dx$$

for every function in $H_0^1((-A, A))$.

What is the largest value of A such that $H_0^1((-A, A)) \ni u \mapsto \int_{-A}^{A} [(u')^2(x) - u^2(x)]dx$ is a convex functional ? What about strict convexity ?

Exercise 14. Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and $\phi : \partial\Omega \to \mathbb{R}$ be Lipschitz continuous. Prove that there exists at least a function $\bar{u}$ which is Lipschitz on $\mathbb{R}^n$ and such that $\bar{u} = \phi$ on $\partial\Omega$.

Consider the problem

$$\min \left\{ \int_\Omega \left(|\nabla u|^2 - \varepsilon_0 u^2 \right) \, dx \ : \ u \in H^1(\Omega), \ u - \bar{u} \in H_0^1(\Omega) \right\},$$

where the condition $u - \bar{u} \in H_0^1(\Omega)$ is a way of saying $u = \phi$ on $\partial\Omega$.

Prove that, at least for small $\varepsilon_0 > 0$ the above problem admits a solution, and give an example with large ε_0 where the solution does not exist. Also prove that, for small $\varepsilon_0 > 0$, the solution is unique. What does the smallness of ε_0 depend on ? Write the PDE satisfied by the minimizer.

Exercise 15. Let Ω be an open connected subset in $\mathbb{R}^d$, $a \in L^\infty(\Omega)$ be a function with $a \geq a_0$ where $a_0 > 0$ is a positive constant, and $b \in L^2(\Omega)$ be another function, which is not identically zero. Prove that the following minimization problem admits a solution

$$\min \left\{ \frac{\int_\Omega a|\nabla u|^2 dx}{|\int_\Omega bu \, dx|^2} \ : \ u \in H_0^1(\Omega) \ : \ \int_\Omega bu \, dx \neq 0 \right\},$$

and write the PDE that such a solution satisfies. Finally, compute the value of the above minimum in the case $\Omega = B(0,1) \subset \mathbb{R}^2$, $a(x) = 1$ and $b(x) = |x|$.

Exercise 16. If $f : \mathbb{R}^n \to \mathbb{R}$ is given by $f(x) = |x| \log |x|$, compute f^* and f^{**}.

Solution. Notice that $x \mapsto |x| \log|x|$ is not convex. Indeed $\phi : r \mapsto r \log r$ is convex on $\mathbb{R}_+$ but it is not increasing : it decreases till e^{-1} then increases.

Computation of $f^\star$. One has

$$\begin{aligned} f^\star(y) &= \sup_x x \cdot y - |x| \log|x| \\ &= \sup_{\lambda>0} \lambda r - \lambda \log(\lambda), \end{aligned}$$

where $r = |y|$, since the expression in the sup is greater if y is taken in the same direction as x. Set $g(\lambda) = r - \lambda \log \lambda$. One has $g'(\lambda) = r - \log \lambda - 1$ hence g is maximal at its critical point λ such that $g'(\lambda) = 0 \Leftrightarrow \lambda = e^{r-1}$, which is positive. Consequently

$$f^\star(y) = g(e^{|y|-1}) = e^{|y|-1}.$$

Computation of $f^{\star\star}$. One could use the fact that ϕ (as a function from $\mathbb{R}$) is convex, decreasing from 0 to e^{-1} then increasing, and say that $\phi^{\star\star}$ should be is convex lsc envelope, hence

$$\phi^{\star\star}(x) = \begin{cases} -e^{-1} & \text{if } |x| \le e^{-1} \\ |x| \log|x| & \text{otherwise.} \end{cases}$$

The invariance under rotation allows us to say $f^{\star\star}$ has the same expression.

But let us do it by direct calculation :

$$\begin{aligned} f^{\star\star}(x) &= \sup_y y \cdot x e^{|y|-1} \\ &= \sup_{\lambda>0} \lambda r - e^{\lambda-1} \end{aligned}$$

where $r = |x|$. Set $h(\lambda) = \lambda r - e^{\lambda-1}$ and find it maximal value on $\mathbb{R}_+^\star$. One has $h'(\lambda) = r - e^{\lambda-1}$ and $h'(\lambda) = 0 \Leftrightarrow r = e^{\lambda-1} \Leftrightarrow \lambda = 1 + \log(r)$, so that h increases up to $1 + \log(r)$ then decreases, but this quantity may be negative.

If $|x| = r \le e^{-1}$ then $f^{\star\star} = h(0) = -e^{-1}$. On the other hand if $|x| = r > e^{-1}$ then $f^{\star\star}(x) = h(1 + \log(r)) = |x| \log|x|$. Hence we get

$$f^{\star\star}(x) = \begin{cases} -e^{-1} & \text{if } |x| \le e^{-1} \\ |x| \log|x| & \text{otherwise.} \end{cases}$$

Exercise 17. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Prove that f is strictly convex if and only if f^* is C^1 and that f is $C^{1,1}$ if and only if f^* is elliptic (meaning that there exists $c > 0$ such that $f(x) - c|x|^2$ is convex).

Solution. We recall the following facts about finite-valued functions.

- Convex functions are locally Lipschitz.
- Convex functions are C^1 if and only if they are differentiable at each point, which is in turn equivalent to the subdifferntial being a singleton.
- $p \in \partial f(x) \Leftrightarrow f^*(p) + f(x) = x \cdot p \Leftrightarrow x \in \partial f^*(p)$.
- f is convex if and only if $\partial f(x) \neq \emptyset$ for every x (i.e., for every x there is p s.t. $f(y) \ge f(x) + p \cdot (y - x)$ for all y).
- f is strictly convex if and only if the inequality $f(y) \ge f(x) + p \cdot (y - x)$ above is trict for all $y \neq x$.
- f is elliptic if and only if in the inequality $f(y) \ge f(x) + p \cdot (y - x)$ above we can add a quadratic rest, i.e. $f(y) \ge f(x) + p \cdot (y - x) + c|y - x|^2$. More precisely, this last fact is equivalent to $f(x) - c|x|^2$ being convex.

The last three fact are all proven in the same way : take x_0, x_1 and $x_t = (1 - t)x_0 + tx_1$ and use the tangent (with vector p_t) at point x_t.

Now, to prove $f \in C^1 \Rightarrow f^*$ strictly convex we use the fact that the subdifferential of f are singleton, i.e. for every x there is only one p such that $p \in \partial f(x)$. Now, suppose that f^* is not strictly convex. Then there is a tangent $f^*(p) + x \cdot (p' - p)$ (with $x \in \partial f^*(p)$) which touches f^*, i.e. $f^*(p') \geq f^*(p) + x \cdot (p' - p)$ for all p' but $f^*(p_0) = f^*(p) + x \cdot (p_0 - p)$ for some $p_0 \neq p$. But then, $p' \mapsto f^*(p') - f^*(p) - x \cdot (p' - p)$ is minimal at $p' = p_0$ and we get $x \in \partial f^*(p_0)$, which is a contradiction as x would belong to the subdifferential of two different points.

The opposite implication, f^* strictly convex $\Rightarrow f \in C^1$ is easy : from $f(x) = \sup_p p \cdot x - f^*(p)$ and the strict convexity of f^* we see that there is at most one unique vector p such that $f(x) = p \cdot x - f^*(p)$, i.e. such that $p \in \partial f(x)$.

We pass now to $f \in C^{1,1} \Rightarrow f^*$ elliptic. Set $L = \text{Lip}(\nabla f)$. We will prove that we have the estimate $f^*(p') \geq f^*(p) + x \cdot (p' - p) + \frac{1}{2L}|p' - p|^2$ for $x \in \partial f^*(p)$. To do this, take $\varepsilon, \delta > 0$ and consider

$$\min_{p'} f^*(p') - f^*(p) - x \cdot (p' - p) - \frac{1}{2(L + \varepsilon)}|p' - p|^2 + \frac{\delta}{4}|p' - p|^4.$$

This minimum exists because $f^*(p') - f^*(p) - x \cdot (p' - p) \geq 0$, and the last term grows more than the prevous one. Let us call p_0 a minimum point. At this point we have

$$x_0 \in \partial f^*(p_0), \quad \text{where } x_0 = x + \left(\frac{1}{L + \varepsilon} - A\right)(p_0 - p), \;\; A = \delta|p_0 - p|^2.$$

Note that, by optimality of p_0, comparing to $p' = p$, we have $f^*(p_0) - f^*(p) - x \cdot (p_0 - p) - \frac{1}{2(L+\varepsilon)}|p_0 - p|^2 + \frac{\delta}{4}|p' - p|^4 \leq 0$, which implies $\frac{1}{2(L+\varepsilon)}|p_0 - p|^2 \geq \frac{\delta}{4}|p' - p|^4$, i.e. $A \leq \frac{2}{L+\varepsilon}$. In particular, $|\frac{1}{L+\varepsilon} - A| \leq \frac{1}{L+\varepsilon}$ and

$$|x_0 - x| \leq \frac{1}{L + \varepsilon}|p_0 - p|.$$

Yet, from $x_0 \in \partial f^*(p_0)$ and $x \in \partial f^*(p)$ we get $p_0 \in \partial f(x_0)$ and $p \in \partial f(x)$, and, from the Lipschitz condition on ∇f, we also have $|p_0 - p| \leq L|x_0 - x|$. This is a contradiction unless $p_0 = p$. Hence, the minimum problem above is solved y $p' = p_0$, which means

$$f^*(p') - f^*(p) - x \cdot (p' - p) - \frac{1}{2(L + \varepsilon)}|p' - p|^2 + \frac{\delta}{4}|p' - p|^4 \geq 0 \;\text{ for all } p', \varepsilon > 0, \delta > 0.$$

By letting $\varepsilon, \delta \to 0$ we get the desired estimate and f^* is elliptic.

For the converse implication, consider two points x_0, x_1 and $p_i \in \partial f(x_i)$. This means that p_i maximizes $x_i \cdot p - f^*(p)$, hence

$$f^*(p_0) - x_0 \cdot p_0 \leq f^*(p_1) - x_0 \cdot p_1.$$

Using the fact that f^* is supposed to be elliptic, this inequality can be strengthened, and we can get

$$f^*(p_0) - x_0 \cdot p_0 + \frac{C}{2}|p_0 - p_1|^2 \leq f^*(p_1) - x_0 \cdot p_1.$$

Using again the ellipticity of f^* we also have $f^*(p_0) \geq f^*(p_1) + x_1 \cdot (p_0 - p_1) + \frac{C}{2}|p_1 - p_0|^2$, hence

$$f^*(p_1) - x_0 \cdot p_1 + \frac{C}{2}|p_0 - p_1|^2 \geq f^*(p_1) + x_1 \cdot (p_0 - p_1) + C|p_1 - p_0|^2 - x_0 \cdot p_0,$$

which gives

$$C|p_1 - p_0|^2 \leq (x_0 - x_1) \cdot (p_0 - p_1) \leq |x_0 - x_1||p_0 - p_1|,$$

hence $|p_0 - p_1| \leq C^{-1}|x_0 - x_1|$, which means that ∇f is C^{-1}Lipschitz.

Exercise 18. Given a bounded, smooth and connected domain $\Omega \subset \mathbb{R}^d$, and $f \in L^2(\Omega)$, set $X(\Omega) = \{v \in L^2(\Omega;\mathbb{R}^d) : \nabla \cdot v \in L^2(\Omega)\}$ and consider the minimization problems

$$\begin{aligned}(P) \quad &:= \quad \min\left\{F(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + f(x)u\right) dx \ : \ u \in H^1(\Omega)\right\} \\ (D) \quad &:= \quad \min\left\{G(v) := \int_\Omega \left(\frac{1}{2}|v|^2 + \frac{1}{2}|\nabla \cdot v - f|^2\right) dx \ : \ v \in X(\Omega)\right\},\end{aligned}$$

a) Prove that (P) admits a unique solution ;

b) Prove $\min(P) + \inf(D) \geq 0$;

c) Prove that there exist $v \in X(\Omega)$ and $u \in H^1(\Omega)$ such that $F(u) + G(v) = 0$;

d) Deduce that $\min(D)$ is attained and $\min(P) + \inf(D) = 0$;

e) Justify by a formal inf-sup exchange the duality $\min F(u) = \sup -G(v)$;

f) Prove $\min F(u) = \sup -G(v)$ via a duality proof based on convex analysis.

Exercise 19. Consider the problem

$$\min\left\{\int_\Omega \frac{1}{2}|v|^2 dx + \langle \bar{u}_0, \pi_0\rangle + \langle \bar{u}_1, \pi_1\rangle \quad : \quad \nabla \cdot v = f + \pi_0 - \pi_1\right\},$$

where the minimization is done on the triplets (v, π_0, π_1) with $v \in L^2(\Omega;\mathbb{R}^d)$, $\pi_i \in (H^1(\Omega))'$ satisfying $\langle \pi_i, \phi\rangle = 0$ for every $\phi \in H_0^1(\Omega)$ and $\langle \pi_i, \phi\rangle \geq 0$ for every $\phi \geq 0$. Here $f \in (H^1(\Omega))'$ and $\bar{u}_i \in H^1(\Omega)$ are given.

Find its dual, distinguishing the case $\bar{u}_0 + \bar{u}_1 \geq 0$ or not.

Solution. First, let us derive « formally » the dual problem by an $\inf - \sup$ interversion. To do that, let us encode the constraints in our minimization problem as (convex) functionals which take value 0 if the constraint is satisfied, $+\infty$ otherwise.

For the constraint $\nabla v = f + \pi_0 - \pi_1$, one has

$$\sup_{u \in H^1} -\langle f + \pi_0 - \pi_1, u\rangle - \int v \cdot \nabla u = \begin{cases} 0 & \text{if } \nabla v = f + \pi_0 - \pi_1 \\ +\infty & \text{otherwise.}\end{cases}$$

The constraint $\pi_i \geq 0$ is encoded by

$$\sup_{\psi_i \in (H^1)+} -\langle \pi_i, \psi_i\rangle$$

(every time we write π_i in a functional we mean a sum for $i = 0, 1$). Finally the constraint $\pi_i \perp H_0^1$ is encoded by

$$\sup_{\phi_i \in H_0^1} -\langle \pi_i, \phi_i\rangle.$$

Consequently

$$\begin{aligned}
&\inf_{v\in L^2,\pi_i\in (H^1)'}\left\{\int_\Omega \frac{|v|^2}{2}+\langle \bar u_0,\pi_0\rangle+\langle \bar u_1,\pi_1\rangle \quad : \nabla v = f+\pi_0-\pi_1, \pi_i\geq 0, \pi_i\perp H^1_0\right\}\\
&=\inf_{v\in L^2,\pi_i\in (H^1)'}\Bigg(\int_\Omega \frac{|v|^2}{2}+\langle \bar u_0,\pi_0\rangle+\langle \bar u_1\pi_1\rangle\\
&\qquad + \sup_{u\in H^1,\phi_i\in H^1_0,\psi_i\in (H^1)+} -\langle f+\pi_0-\pi_1,u\rangle-\int v\cdot\nabla u-\langle \pi_i,\psi_i\rangle-\langle\pi_i,\phi_i\rangle\Bigg)\\
&\geq \sup_{u,\phi_i,\psi_i}\inf_{v,\pi_i}\left(\int_\Omega \frac{|v|^2}{2}+\langle \bar u_0,\pi_0\rangle+\langle \bar u_1\pi_1\rangle-\langle f+\pi_0-\pi_1,u\rangle-\int v\cdot\nabla u-\langle \pi_i,\psi_i\rangle-\langle\pi_i,\phi_i\rangle\right)\\
&=\sup_{u,\phi_i,\psi_i}\left[\inf_v\left(\int_\Omega\frac{|v|^2}{2}-v\cdot\nabla u\right)+\inf_{\pi_0}\langle\pi_0,\bar u_0-u-\psi_0-\phi_0\rangle+\inf_{\pi_1}\langle\pi_1,\bar u_1+u-\psi_1-\phi_1\rangle\right]\\
&=\sup_{u,\phi_i,\psi_i}\left[-\int_\Omega\frac{|\nabla u|^2}{2}+\begin{cases}0 & \text{if } \bar u_0-u-\psi_0-\phi_0=0 \text{ and } \bar u_1+u-\psi_1-\phi_1=0\\ -\infty & \text{otherwise.}\end{cases}\right]\\
&=-\inf_{\substack{u\in H^1:\\ \bar u_0-u\in (H^1)^++H^1_0\\ u+\bar u_1\in (H^1)^++H^1_0}}\int_\Omega\frac{|\nabla u|^2}{2}.
\end{aligned}$$

The constraint $w\in (H^1)^++H^1_0$ is a way of saying that $w\geq 0$ on $\partial\Omega$, hence the dual problam is :

$$-\inf_{u\in H^1}\int_\Omega\frac{|\nabla u|^2}{2} \quad : \quad -\bar u_1\leq u\leq \bar u_0 \text{ on } \partial\Omega.$$

One should assume that $\bar u_0+\bar u_1\geq 0$ on $\partial\Omega$.

Proving that the duality holds, i.e. that we have an equality instead of an inequality, is actually not clear, but you may try to prove it to see which arguments of the lectures work and where problems remain. The idea is to set

$$\mathcal F(p)=\min\left\{\int_\Omega\frac12|v|^2dx+\langle\bar u_0,\pi_0\rangle+\langle\bar u_1,\pi_1\rangle \quad : \quad \nabla v=f+\pi_0-\pi_1+p\right\},$$

prove that $\mathcal F$ is convex lsc, compute $\mathcal F^\star$ and write $\mathcal F(0)=\mathcal F^{\star\star}(0)$.

Exercise 20. Let Ω be the $d-$dimensional flat torus (just to avoid boundary conditions, think at a cube), $p,q>1$ two given exponents, $a>0$ and $f:\Omega\to\mathbb R$ a given Lipschitz continuous function. Consider the following minimization problem

$$\inf\left\{\int_\Omega\left(\frac1p|\nabla u|^p-\frac aq|u|^q+fu\right)dx \ : \ u\in W^{1,p}(\Omega)\cap L^q(\Omega),\ \int_\Omega u=0\right\}.$$

a) Prove that, if $q>p$, the inf is $-\infty$ and the minimization problem has no solution.

b) Prove that, if $q<p$, the infimum is attained.

c) Prove that, if $q=p$, the infimum is attained, provided a is small enough.

d) In the cases where the infimum is attained, write the Euler-Lagrange equation solved by the minimizers.

e) Recall the condition on f which guarantee that solutions of $\Delta_p u=f$, satisfy $(\nabla u)^{p/2}\in H^1$ (remember that, for a vector v, the expression v^α is to be intended as equal to a vector w with $|w|=|v|^\alpha$ and $w\in\mathbb R_+v$).

f) For $p\geq 2$ and $2\leq q\leq p$, prove that the solution $\bar u$ satisfies $(\nabla\bar u)^{p/2}\in H^1$.

Exercise 21. Let $H : \mathbb{R}^n \to \mathbb{R}$ be given by

$$H(v) = \frac{(4|v|+1)^{3/2} - 6|v| - 1}{12}.$$

a) Prove that H is C^1 and strictly convex. Is it $C^{1,1}$? Is it elliptic?

b) Compute H^*. Is it C^1, strictly convex, $C^{1,1}$ and/or elliptic?

c) Consider the problem $\min\{\int H(v) \ : \ \nabla \cdot v = f\}$ (on the d-dimensional torus, for simplicity) and find its dual.

d) Supposing $f \in L^2$, prove that the optimal u in the dual problem is H^2.

e) Under the same assumption, prove that the optimal v in the primal problem belongs to $W^{1,p}$ for every $p < 2$ if $d = 2$, for $p = d/(d-1)$ if $3 \leq d \leq 5$, and for $p = 6/5$ if $d \geq 3$.

Exercise 22. Consider the problem

$$\min\{A(v) := \int_{\mathbb{T}^d} H(v(x))dx \ : \ v \in L^2, \nabla \cdot v = f\}$$

for a function H which is elliptic. Prove that the problem has a solution, provided there exists at least an admissible v with $A(v) < +\infty$. Prove that, if f is an H^1 function with zero mean, then the optimal v is also H^1.

Exercise 23. Given a function $g \in L^2([0, L])$, consider the problem

$$\min\left\{\int_0^L \frac{1}{2}|u(t) - g(t)|^2 dt \ : \ u(0) = u(L) = 0, u \in \mathrm{Lip}([0, L]), |u'| \leq 1 \,\text{a.e.}\right\}.$$

a) Prove that this problem admits a solution.

b) Prove that the solution is unique.

c) Find the optimal solution in the case where g is the constant function $g = 1$ in the terms of the value of L, distinguishing $L > 2$ and $L \leq 2$.

d) Computing the value of

$$\sup\left\{-\int_0^L (u(t)z'(t) + |z(t)|)dt \ : \ z \in H^1([0, L])\right\}$$

find the dual of the previous problem by means of a formal inf-sup exchange.

e) Assuming that the equality inf sup = sup inf in the duality is satisfied, write the necessary and sufficient optimality conditions for the solutions of the primal and dual problem. Check that these conditions are satisfied by the solution found in the case $g = 1$.

f) Prove the the equality inf sup = sup inf *(more difficult)*.

Exercise 24. Given $u_0 \in C^1([0, 1])$ consider the problem

$$\min\left\{\int_0^1 \frac{1}{2}|u - u_0|^2 dx \quad : \quad u' \geq 0\right\},$$

which consists in the projection of u_0 onto the set of monotone increasing functions (where the condition $u' \geq 0$ is intended in the weak sense).

a) Prove that this problem admits a unique solution.

b) Write the dual problem

c) Prove that the solution is actually the following : define U_0 through $U_0' = u_0$, set $U_1 := (U_0)^{**}$ to be the largest convex and l.s.c. function smaller than U_0, take $u = U_1'$.