# Università di Pavia, Laurea Magistrale in Matematica

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## Calcolo delle Variazioni

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## Esercizi

**Exercise 1.** Solve the problem

$$\min\{J(f) := \int_0^1 \left[\frac{1}{2}f'(t)^2 + tf(t) + \frac{1}{2}f(t)^2\right] dt\,;\ f \in \mathcal{A}\}, \quad \text{où} \quad \mathcal{A} := \{f \in C^1([0,1])\ :\ f(0) = 0\}.$$

Find the minimal value of $J$ on $\mathcal{A}$ and the function(s) $f$ which attain it, proving that they are actually minimizers

**Exercise 2.** Consider the problem

$$\min\left\{\int_0^L e^{-t}\left(u'(t)^2 + 5u(t)^2\right) dt \quad : \quad u \in C^1([0,L]),\ u(0) = 1\right\}.$$

Prove that it admits a minimizer, that it is unique, find it, compute the value of the minimum, and the limit of the minimizer (in which sense ?) and of the minimal value as $L \to +\infty$.

**Exercise 3.** Consider the problem

$$\min\{J(u) := \int_0^1 \left[\frac{1}{2}u'(t)^2 + u(t)f(t)\right] dt\,;\ u \in W^{1,2}([0,1])\}.$$

Find a necessary and sufficient condition on $f$ so that this problem admits a solution.

**Exercise 4.** Let $L : \mathbb{R} \to \mathbb{R}$ be a strictly convex $C^1$ function, and consider

$$\min\{\int_0^1 L(u'(t))dt\,;\ u \in C^1([0,1]), u(0) = a, u(1) = b\}.$$

Prove that the solution is $u(t) = (1-t)a + tb$, whatever is $L$. What happens if $L$ is not $C^1$ ? and if $L$ is convex but not strictly convex ?

**Exercise 5.** Prove that we have

$$\inf\{\int_0^1 t|u'(t)|^2 dt\,;\ u \in C^1([0,1]), u(0) = 1, u(1) = 0\} = 0.$$

Is the infimum above attained ?

What about, instead

$$\inf\{\int_0^1 \sqrt{t}|u'(t)|^2 dt\,;\ u \in C^1([0,1]), u(0) = 1, u(1) = 0\} \quad ?$$

**Exercise 6.** Consider a minimization problem of the form

$$\min\{F(u) := \int_0^1 L(t, u(t), u'(t))dt\,;\ u \in W^{1,1}([0,1]), u(0) = a,\ u(1) = b\},$$

where $L \in C^2([0,1] \times \mathbb{R} \times \mathbb{R})$. We denote as usual by $(t, x, v)$ the variables of $L$. Suppose that $\bar{u}$ is a solution to the above problem. Prove that we have

$$\frac{\partial^2 L}{\partial v^2}(t, \bar{u}(t), \bar{u}'(t)) \geq 0\, a.e.$$

**Exercise 7.** Given $f \in C^2(\mathbb{R})$, consider the problem

$$\min\{F(u) := \int_0^1 \Big[(u'(t)^2 - 1)^2 + f(u(t))\Big]\, dt\,;\ u \in C^1([0,1]), u(0) = a,\, u(1) = b\}.$$

Prove that the problem does not admit any solution if $|b - a| \leq \frac{1}{\sqrt{3}}$.

**Exercise 8.** Consider the functional $F : H^1([0, L]) \to \mathbb{R}$ defined through

$$F(u) = \int_0^L \Big(u'(t)^2 + \arctan(u(t) - t)\Big)\, dt.$$

Prove that

a) the problem $(P) := \min\{F(u) \,:\, u \in H^1([0, L])\}$ has no solution ;

b) the problem $(P_a) := \min\{F(u) \,:\, u \in H^1([0, L]), u(0) = a\}$ admits a solution for every $a \in \mathbb{R}$ ;

c) we have $F(-|u|) \leq F(u)$ ;

d) the solution of $(P_a)$ is unique as soon as $a \leq 0$ ;

e) there exists $L_0 < +\infty$ such that for every $L \leq L_0$ the solution of $(P_a)$ is unique for every $a \in \mathbb{R}$

f) the minimizers of $(P)$ and $(P_a)$ are $C^\infty$ functions.

**Exercise 9.** Prove existence and uniqueness of the solution of

$$\min\left\{\int_\Omega \Big(f(x)|u(x)| + |\nabla u(x)|^2\Big)\, dx\,;\, u \in H^1(\Omega),\, \int_\Omega u = 1\right\},$$

when $\Omega$ is an open, connected and bounded subset of $\mathbb{R}^n$ and $f \in L^2(\Omega)$, $f \geq 0$ (the sign of $f$ is not important for existence). Where do we use connectedness ? Also prove that, if $\Omega$ is not connected (but has a finite number of connected components and we keep the assumption $f \geq 0$), then we have existence but maybe not uniqueness, and that if we withdraw both connectedness and positivity of $f$, then maybe we don't even have existence.

**Exercise 10.** Fully solve

$$\min\left\{\int_Q \Big(|\nabla u(x,y)|^2 + u(x,y)^2\Big)\, dx\, dy \ :\ u \in C^1(Q),\ u = \phi \text{ sur } \partial Q\right\},$$

where $Q = [-1, 1]^2 \subset \mathbb{R}^2$ and $\phi : \partial Q \to \mathbb{R}$ is given by

$$\phi(x,y) = \begin{cases} 0 & \text{si } x = -1,\, y \in [-1,1] \\ 2(e^y + e^{-y}) & \text{if } x = 1,\, y \in [-1,1] \\ (x+1)(e + e^{-1}) & \text{if } x \in [-1,1],\, y = \pm 1. \end{cases}$$

Find the minimizer and the value of the minimum. Writing the Euler-Lagrange equation is not compulsory, but could help.

**Exercise 11.** Show that for every function $f : \mathbb{R} \to \mathbb{R}_+$ l.s.c. there exists a sequence of functions $f_k : \mathbb{R} \to \mathbb{R}_+$, each $k-$Lipschitz, such that for every $x \in \mathbb{R}$ the sequence $(f_k(x))_k$ increasingly converges to $f(x)$.

Use this fact and the theorems we saw in class to prove semicontinuity, wrt to weak convergence in $H^1(\Omega)$, of the functional

$$J(u) = \int_\Omega f(u(x))|\nabla u(x)|^p\, dx,$$

where $p \geq 1$ and $f : \mathbb{R} \to \mathbb{R}_+$ is l.s.c.

**Exercise 12.** Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and $\phi : \partial\Omega \to \mathbb{R}$ be Lipschitz continuous. Prove that there exists at least a function $\bar{u}$ which is Lipschitz on $\mathbb{R}^n$ and such that $\bar{u} = \phi$ on $\partial\Omega$.

Consider the problem

$$\min\left\{\int_\Omega \left(|\nabla u|^2 - \varepsilon_0 u^2\right) dx \;:\; u \in H^1(\Omega),\; u - \bar{u} \in H^1_0(\Omega)\right\},$$

where the condition $u - \bar{u} \in H^1_0(\Omega)$ is a way of saying $u = \phi$ on $\partial\Omega$.

Prove that, at least for small $\varepsilon_0 > 0$ the above problem admits a solution, and give an example with large $\varepsilon_0$ where the solution does not exist. Also prove that, for small $\varepsilon_0 > 0$, the solution is unique. What does the smallness of $\varepsilon_0$ depend on ? Write the PDE satisfied by the minimizer, and discuss the regularity of the solution.

**Exercise 13.** Let $\Omega$ be an open connected subset in $\mathbb{R}^d$, $a \in L^\infty(\Omega)$ be a function with $a \geq a_0$ where $a_0 > 0$ is a positive constant, and $b \in L^2(\Omega)$ be another function, which is not identically zero. Prove that the following minimization problem admits a solution

$$\min\left\{\frac{\int_\Omega a|\nabla u|^2 dx}{|\int_\Omega bu\, dx|^2} \;:\; u \in H^1_0(\Omega) \;:\; \int_\Omega bu\, dx \neq 0\right\},$$

and write the PDE that such a solution satisfies. Finally, compute the value of the above minimum in the case $\Omega = B(0,1) \subset \mathbb{R}^2$, $a(x) = 1$ and $b(x) = |x|$.

**Exercise 14.** If $f : \mathbb{R}^n \to \mathbb{R}$ is given by $f(x) = |x|\log|x|$, compute $f^*$ and $f^{**}$.

**Exercise 15.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Prove that $f$ is strictly convex if and only if $f^*$ is $C^1$ and that $f$ is $C^{1,1}$ if and only if $f^*$ is elliptic (meaning that there exists $c > 0$ such that $f(x) - c|x|^2$ is convex).

**Exercise 16.** Given a bounded, smooth and connected domain $\Omega \subset \mathbb{R}^d$, and $f \in L^2(\Omega)$, set $X(\Omega) = \{v \in L^2(\Omega;\mathbb{R}^d) \,:\, \nabla\cdot v \in L^2(\Omega)\}$ and consider the minimization problems

$$\begin{aligned}
(P) \quad &:= \quad \min\left\{F(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + f(x)u\right) dx \;:\; u \in H^1(\Omega)\right\}\\
(D) \quad &:= \quad \min\left\{G(v) := \int_\Omega \left(\frac{1}{2}|v|^2 + \frac{1}{2}|\nabla\cdot v - f|^2\right) dx \;:\; v \in X(\Omega)\right\},
\end{aligned}$$

a) Prove that $(P)$ admits a unique solution ;
b) Prove $\min(P) + \inf(D) \geq 0$ ;
c) Prove that there exist $v \in X(\Omega)$ and $u \in H^1(\Omega)$ such that $F(u) + G(v) = 0$ ;
d) Deduce that $\min(D)$ is attained and $\min(P) + \inf(D) = 0$ ;
e) Justify by a formal inf-sup exchange the duality $\min F(u) = \sup -G(v)$ ;
f) Prove that the solution of $(D)$ belongs indeed to $H^1(\Omega;\mathbb{R}^d)$.

**Exercise 17.** Let $\Omega$ be the $d-$dimensional flat torus (just to avoid boundary conditions, think at a cube), $p, q > 1$ two given exponents, $a > 0$ and $f : \Omega \to \mathbb{R}$ a given Lipschitz continuous function. Consider the following minimization problem

$$\inf\left\{\int_\Omega \left(\frac{1}{p}|\nabla u|^p - \frac{a}{q}|u|^q + fu\right) dx \;:\; u \in W^{1,p}(\Omega) \cap L^q(\Omega)\right\}.$$

a) Prove that, if $q > p$, the inf is $-\infty$ and the minimization problem has no solution.
b) Prove that, if $q < p$, the infimum is attained.
c) Prove that, if $q = p$, the infimum is attained, provided $a$ is small enough.

d) In the cases where the infimum is attained, write the Euler-Lagrange equation solved by the minimizers.

e) Recall the condition on $f$ which guarantee that solutions of $\Delta_p u = f$, satisfy $(\nabla u)^{p/2} \in H^1$ (remember that, for a vector $v$, the expression $v^\alpha$ is to be intended as equal to a vector $w$ with $|w| = |v|^\alpha$ and $w \in \mathbb{R}_+ v$).

f) For $p \geq 2$ and $2 \leq q \leq p$, prove that the solution $\bar{u}$ satisfies $(\nabla \bar{u})^{p/2} \in H^1$.

**Exercise 18.** Let $H : \mathbb{R}^n \to \mathbb{R}$ be given by

$$H(v) = \frac{(4|v|+1)^{3/2} - 6|v| - 1}{12}.$$

a) Prove that $H$ is $C^1$ and strictly convex. Is it $C^{1,1}$? Is it elliptic ?

b) Compute $H^*$. Is it $C^1$, strictly convex, $C^{1,1}$ and/or elliptic ?

c) Consider the problem $\min\{\int H(v) \ : \ \nabla \cdot v = f\}$ (on the $d$-dimensional torus, for simplicity) and find its dual.

d) Supposing $f \in L^2$, prove that the optimal $u$ in the dual problem is $H^2$.

e) Under the same assumption, prove that the optimal $v$ in the primal problem belongs to $W^{1,p}$ for every $p < 2$ if $d = 2$, for $p = d/(d-1)$ if $3 \leq d \leq 5$, and for $p = 6/5$ if $d \geq 3$.

**Exercise 19.** Consider the problem

$$\min\{A(v) := \int_{\mathbb{T}^d} H(v(x))dx \ : \ v \in L^2, \nabla \cdot v = f\}$$

for a function $H$ which is elliptic. Prove that the problem has a solution, provided there exists at least an admissible $v$ with $A(v) < +\infty$. Prove that, if $f$ is an $H^1$ function with zero mean, then the optimal $v$ is also $H^1$.

**Exercise 20.** Consider

$$\min\left\{\int_{\mathbb{T}^d} e^{c(x)}\left[\frac{|\nabla u(x)|^2}{2} + \frac{u^2(x)}{2} + f(x)u(x)\right]dx, \ u \in H^1(\mathbb{T}^d)\right\}$$

where $c, f : \mathbb{T}^d \to \mathbb{R}$ are given $C^\infty$ functions.

a) Prove that the problem admits a unique solution.

b) Write the Euler-Lagrange equation of the problem.

c) Prove, using this PDE, that the solution is a $C^\infty$ function.

**Exercise 21.** Let $\Omega$ be an open and bounded subset of $\mathbb{R}^d$, and $p > 1$. Consider the following minimization problem

$$\min\left\{\int_\Omega \left(\frac{1}{p}|\nabla u(x)|^p - \sqrt{1 + \frac{1}{p}|u(x)|^p}\right) dx \ : \ u \in W_0^{1,p}(\Omega)\right\}.$$

a) Prove that it admits a solution.

b) Prove that there exists $\Omega$ such that 0 is not a solution, and hence that the solution is not always unique.

c) Write the PDE solved by any solution.

d) In the case $p = 2$, prove that any solution is $C^\infty$ in the interior of $\Omega$.

**Exercise 22.** Given a function $g \in L^2([0, L])$, consider the problem

$$\min\left\{\int_0^L \frac{1}{2}|u(t) - g(t)|^2 dt \ : \ u(0) = u(L) = 0, u \in \mathrm{Lip}([0, L]), |u'| \leq 1 \text{ a.e.}\right\}.$$

a) Prove that this problem admits a solution.

b) Prove that the solution is unique.

c) Find the optimal solution in the case where $g$ is the constant function $g = 1$ in the terms of the value of $L$, distinguishing $L > 2$ and $L \leq 2$.

d) Computing the value of

$$\sup\left\{-\int_0^L (u(t)z'(t) + |z(t)|)dt \ : \ z \in H^1([0, L])\right\}$$

find the dual of the previous problem by means of a formal inf-sup exchange.

e) Assuming that the equality $\inf\sup = \sup\inf$ in the duality is satisfied, write the necessary and sufficient optimality conditions for the solutions of the primal and dual problem. Check that these conditions are satisfied by the solution found in the case $g = 1$.

f) Prove the the equality $\inf\sup = \sup\inf$ *(more difficult)*.

**Exercise 23.** Given $u_0 \in C^1([0, 1])$ consider the problem

$$\min\left\{\int_0^1 \frac{1}{2}|u - u_0|^2 dx \quad : \quad u' \geq 0\right\},$$

which consists in the projection of $u_0$ onto the set of monotone increasing functions (where the condition $u' \geq 0$ is intended in the weak sense).

a) Prove that this problem admits a unique solution.

b) Write the dual problem

c) Prove that the solution is actually the following : define $U_0$ through $U_0' = u_0$, set $U_1 := (U_0)^{**}$ to be the largest convex and l.s.c. function smaller than $U_0$, take $u = U_1'$.

**Exercise 24.** Given $u \in W^{1,p}_{loc}(\mathbb{R}^d)$, suppose that we have

$$\int_{B(x_0,r)} |\nabla u|^p dx \leq Cr^d f(r),$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as $f(r) = -\log r$ for $r \leq e^{-1}$ and $f(r) = 1$ for $r \geq e^{-1}$. Prove that $u$ is continuous and that we have

$$|u(x) - u(y)| \leq C|x - y|f(|x - y|)$$

(possibly for a different constant $C$).

**Exercise 25.** Let $\Omega$ be an open connected and smooth subset of $\mathbb{R}^d$ such that $\lambda_1(\Omega) > 1$ (remember $\lambda_1(\Omega) := \inf_{v\in H^1_0(\Omega)\setminus\{0\}} ||\nabla v||^2_{L^2}/||v||^2_{L^2}$) and fix $\bar{u} \in H^1(\Omega)$. Prove that the following minimization problem admits a unique solution

$$\min\left\{\int \left(\frac{1}{2}|\nabla u|^2 + \sin(u)\right) dx \ : \ u - \bar{u} \in H^1_0(\Omega)\right\}$$

and prove that $u$ is a solution of the above problem if and only if it solves

$$\begin{cases} \Delta u = \cos(u) & \text{weakly in } \Omega, \\ u = \bar{u} & \text{on } \partial\Omega. \end{cases}$$

Also prove that the optimal $u$ is a $C^\infty$ function inside $\Omega$.

**Exercise 26.** Let $\Omega = B(0,1) \subset \mathbb{R}^d$ be the $d$-dimensional ball, with $d > 1$, $g : \partial\Omega \to \mathbb{R}$ a given Lipschitz function and $f \in L^\infty(\Omega)$. Consider the problem

$$\inf \left\{ J(u) := \int_\Omega \Big( |x||\nabla u(x)|^2 + f(x)u(x) \Big)\, dx \;:\; u = g \text{ on } \partial\Omega \right\}.$$

Find a suitable functional space where $J$ is well-defined (valued in $\mathbb{R}\cup\{+\infty\}$) as well as the boundary condition, and where the minimization problem has a solution. Can we choose $W^{1,p}$ ? $H^1$ ?

Also say whether this solution is unique, and write the Euler-Lagrange equation of the problem. Find the solution when $g$ and $f$ are constant.

**Exercise 27.** Let $\Omega = \mathbb{T}^d$ be the $d$-dimensional flat torus (with $d \geq 2$), and $f \in C^1(\Omega)$ be a given function. Consider the following minimization problem

$$\min \left\{ \int_\Omega \frac{1}{2}(|\nabla u| - 1)_+^2 + \frac{1}{2}(|u| - 1)_+^2 + fu \;:\; u \in W^{1,2}(\Omega) \right\}.$$

a) Prove that it admits a solution.

b) Find an example of $f$ such that the solution is not unique.

c) Write the PDE solved by any solution.

d) Prove that for any solution $u$ we have $(|\nabla u| - 1)_+ \in W^{1,2}(\Omega)$ and $u \in W^{1,2^*}$, where $2^* = \frac{2d}{d-2}$ is the exponent of the Sobolev embedding $W^{1,2} \subset L^{2^*}$ in dimension $d$ (for $d = 2$, prove $u \in W^{1,p}$ for every $p < \infty$).

e) For $d < 4$, prove $u \in C^{0,\alpha}$, for an exponent $\alpha$ to be found ; for $d = 4$, prove $u \in L^p$ for every $p < \infty$ ; for $d > 4$, prove $u \in L^{2^{**}}$ where $2^{**} = \frac{2d}{d-4}$.

f) For $d = 4$ and $d = 5$, prove $u \in C^{0,\alpha}$, for an exponent $\alpha$ to be found.

g) Find the dual of this problem and prove the equality between the primal and dual optimal values.

**Exercise 28.** For given $f \in L^1(\Omega)$ with $\int_\Omega f(x)dx = 0$ and $p > d$ consider the functions $u_p$ which solve

$$\min \left\{ \frac{1}{p} \int |\nabla u|^p dx + \int fu \;:\; u \in W^{1,p}(\Omega) \right\}.$$

Prove that the sequence $u_p$ is compact in $C^0(\Omega)$. What can we say about the possible limits as $p \to \infty$ ? (in particular, do they minimize something ?).

**Exercise 29.** Let $A \subset \mathbb{R}^2$ be a bounded measurable set. Let $\pi(A) \subset \mathbb{R}$ be defined via $\pi(A) = \{x \in \mathbb{R} : \mathcal{L}^1(\{y \in \mathbb{R} : (x,y) \in A\} > 0\}$. Prove that we have $Per(A) \geq 2\mathcal{L}^1(\pi(A))$. Appy this result to the isoperimetric problem in $\mathbb{R}^2$, proving the existence of a solution to

$$\min \left\{ Per(A) \;:\; A \subset \mathbb{R}^2,\, A \text{ bounded } \mathcal{L}^2(A) = 1 \right\}.$$

**Exercise 30.** Let $\Omega \subset \mathbb{R}^d$ be the unit cube $\Omega = (0,1)^d$. Prove that for any measurable set $A \subset \Omega$ there exists unique a solution to the problem

$$\min \left\{ \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx + \int_A u(x)dx - \int_{\Omega\setminus A} u(x)dx \;:\; u \in H_0^1(\Omega) \right\}.$$

Let us call such a solution $u_A$. Write the Euler-Lagrange equation satisfied by $u_A$. Prove that if $A_n \to A$ (in the sense that the Lebesgue measure of the symmetric difference $|A_n \setminus A| + |A \setminus A_n|$ tends to 0) then $u_{A_n} \to u_A$ in the strong $H_0^1$ sense.

Prove that the problem

$$\min\left\{\int_\Omega |u_A(x)|^2 dx + \operatorname{Per}(A) \ : \ A \subset \Omega\right\}$$

admits a solution. Removing the perimeter penalization, prove that we have

$$\inf\left\{\int_\Omega |u_A(x)|^2 dx \ : \ A \subset \Omega\right\} = 0$$

and that the infimum is not attained.

**Exercise 31.** Given a continuous function $L : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, positive and convex in the second variable, and a bounded open domain $\Omega \subset \mathbb{R}^n$, prove that the following minimization problem admits a solution

$$\min\left\{\operatorname{Per}(A) + \int_\Omega (|\nabla u|^p + |u|^p) + \int_A L(u, \nabla u) \ : \ u \in W^{1,p}(\Omega), A \subset \Omega, \ |A| = \frac{|\Omega|}{2}\right\},$$

where $\operatorname{Per}(A)$ stands for the perimeter - in the BV theory - of $A$, and the minimization is performed over $u$ and $A$.

**Exercise 32.** Let $\Omega = B(0,1) \subset \mathbb{R}^N$ be the unit ball in dimension $N$. Consider the following variational problem :

$$\min \ \left\{\int_\Omega |\nabla u|^2 + \int_\Omega (u-1)^2 + Per(A) \ : \ u \in H^1_0(\Omega), A \subset \Omega \text{ of finite perimeter, such that } u = 0 \text{ a.e. on } A^c\right\}.$$

where $Per(A)$ stands for the perimeter of $A$ in the BV sense.

a) Prove that the problem admits a solution $(u, A)$.

b) Prove that for every solution $(u, A)$ we have $0 \leq u \leq 1$.

c) Prove that the problem admits at least a radial solution (i.e. $A = B(0, r)$ and $u = u^*$).

d) Find or characterize the radial solution.

**Exercise 33.** Given an open bounded set $\Omega \subset \mathbb{R}^d$ with volume $|\Omega| > 2\omega_d$ ($\omega_d$ denoting the volume of the unit ball in $\mathbb{R}^d$) and a function $f \in L^1(\Omega)$, prove that the following problem has a solution

$$\min\left\{Per(A) + Per(B) + \int_{A\cap B} f(x)dx \ : \ A, B \subset \Omega, \ |A| = |B| = \omega_d\right\}.$$

$Per(A)$ and $Per(B)$ stand for the perimeters in the BV sense, i.e. the total variation of the corresponding indicator functions (considered as BV functions on the whole $\mathbb{R}^d$, not on $\Omega$).

Also compute the minimal value above when $\Omega$ is the smallest convex domain containing $B(e, 1)$ and $B(-e, 1)$, where $e$ is an arbitrary unit vector $e \in \mathbb{R}^d$ with $|e| = 1$, and $f = 1$ on $\Omega$.

**Exercise 34.** In the disk $B(0,1) \subset \mathbb{R}^2$ we need to place a disk $B(x_0, r)$ centered at $x_0 \in B(0,1)$ and of radius $r \leq 1 - |x_0|$ so that we minimize

$$J(x_0, r) := \frac{1}{r} + \frac{1}{\sqrt{1 - |x_0|^2}} + \int_{B(0,1)} g(x, u_{x_0,r}(x))dx,$$

where $g : B(0,1) \times \mathbb{R} \to \mathbb{R}$ is a given bounded continuous function and $u_{x_0,r}$ denotes the solution of

$$\begin{cases} \Delta u = 1 & \text{dans } B(0,1) \setminus \overline{B(x_0, r)}, \\ u = 0 & \text{sur } \partial B(0,1) \cup \partial B(x_0, r), \end{cases}$$

that we extend to 0 on $B(x_0, r)$. Prove that there exists a solution.

**Exercise 35.** Let us define the following functionals on $X = L^2([-1,1])$

$$F_n(u) := \begin{cases} \frac{1}{2n}\int_{-1}^1 |u'(t)|^2 dt + \frac{1}{2}\int_{-1}^1 |u(t)-t|^2 dt & \text{if } u \in H_0^1([-1,1]), \\ +\infty & \text{otherwise}\,; \end{cases}$$

together with

$$H(u) = \begin{cases} \frac{1}{2}\int_{-1}^1 |u(t)-t|^2 dt & \text{if } u \in H_0^1([-1,1]), \\ +\infty & \text{otherwise}\,; \end{cases}$$

and $F(u) := \frac{1}{2}\int_{-1}^1 |u(t)-t|^2 dt$ for every $u \in X$.

a) Prove that, for each $n$, the functional $F_n$ is l.s.c. for the $L^2$ (strong) convergence ;
b) Prove that also $F$ is l.s.c. for the same convergence, but not $H$ ;
c) Find the minimizer $u_n$ of $F_n$ over $X$ ;
d) Find the limit as $n \to \infty$ of $u_n$. Is it a strong $L^2$ limit ? is it a uniform limit ? a pointwise a.e. limit ?
e) Find the $\Gamma$-limit of $F_n$ (which, without surprise, is one of the functionals $F$ or $H$), proving $\Gamma$-convergence ;
f) Does the functional $H$ admit a minimizer in $X$ ?

**Exercise 36.** Consider the functions $a_n : [0,1] \to$ given by $a_n(x) = a(nx)$ where $a = 2\sum_{k\in\mathbb{Z}} \mathbb{1}_{[2k,2k+1]} + \sum_{k\in\mathbb{Z}} \mathbb{1}_{[2k-1,2k]}$. Given $f \in L^1([0,1])$ with $\int_0^1 f(t)dt = 0$ and $p \in ]1,+\infty[$, compute

$$\lim_{n\to\infty} \min\left\{ \int_0^1 \left( \frac{1}{p} a_n |u'(t)|^p dt + f(t)u(t) \right) dt \;:\; u \in W^{1,p}([0,1]) \right\}.$$

**Exercise 37.** Let $\Omega \subset \mathbb{R}^d$ be the unit square $\Omega = (0,1)^2$ and $\Gamma = \{0\} \times (0,1) \subset \partial\Omega$. Consider the functions $a_n : \Omega \to \mathbb{R}$ defined via

$$a_n(x,y) = \begin{cases} A_0 & \text{if } x \in [\frac{2k}{2n}, \frac{2k+1}{2n}) \text{ for } k \in \mathbb{Z} \\ A_1 & \text{if } x \in [\frac{2k+1}{2n}, \frac{2k+2}{2n}) \text{ for } k \in \mathbb{Z}, \end{cases}$$

where $0 < A_0 < A_1$ are two given values. On the space $L^2(\Omega)$ (endowed with the strong $L^2$ convergence ; every time that we write $\to$ here below we mean this kind of convergence), consider the sequence of functionals

$$F_n(u) = \begin{cases} \int_\Omega a_n |\nabla u|^2 & \text{if } u \in X, \\ +\infty & \text{if not,} \end{cases}$$

where $X \subset L^2(\Omega)$ is the space of functions $u \in H^1(\Omega)$ satisfying $u = 0$ on $\Gamma$ (which can be defined via the condition $u\eta \in H_0^1(\Omega)$ for every cut-off function $\eta \in \mathbb{C}^\infty(\mathbb{R}^2)$ with $\operatorname{spt}(\eta) \cap (\partial\Omega \setminus \Gamma) = \emptyset$). The goal is to find the $\Gamma$-limit of the sequence $(F_n)_n$. Set

$$A_* := \left( \frac{\frac{1}{A_0} + \frac{1}{A_1}}{2} \right)^{-1} \quad \text{and} \quad A_\diamond := \frac{A_0 + A_1}{2}.$$

a) Given a sequence $(u_n)_n$ with $F_n(u_n) \le C$, prove $u \in X$ and $\liminf_n \int_\Omega a_n |\partial_x u_n|^2 \ge A_* \int_\Omega |\partial_x u|^2$.
b) For the same sequence, also prove $\liminf_n \int_\Omega a_n |\partial_y u_n|^2 \ge A_\diamond \int_\Omega |\partial_y u|^2$.
c) For any $u \in X$, find a sequence $u_n$ such that $u_n \to u$, $a_n \partial_x u_n \to A_* \partial_x u$ and $\partial_y u_n \to \partial_y u$.
d) Conclude by finding the $\Gamma$-limit of $F_n$. Is it of the form $F(u) = \int a|\nabla u|^2$ for $u \in X$ ?

**Exercise 38.** Consider the following sequence of minimization problems, for $n \geq 0$,

$$\min \left\{ \int_0^1 \left( \frac{|u'(t)|^2}{\frac{3}{2} + \sin^2(nt)} + \left( \frac{3}{2} + \sin^2(nt) \right) u(t) \right) dt \ : \ u \in H^1([0,1]), u(0) = 0 \right\},$$

calling $u_n$ their unique minimizers and $m_n$ their minimal values.

Prove that we have $u_n(t) \to t^2 - 2t$ uniformly, and $m_n \to -2/3$.

**Exercise 39.** Let $u_n, v_n$ be two sequences of functions belonging to $H^1([0,1])$. Suppose

$$u_n \rightharpoonup u_0, \quad v_n \rightharpoonup v_0, \quad E_n(u_n, v_n, [0,1]) \leq C$$

where $E_n$ is the energy defined for every interval $J \subset [0,1]$ through

$$E_n(u,v,J) := \int_J \left( v(t)|u'(t)|^2 + \frac{1}{2n}|v'(t)|^2 + \frac{n}{2}|1 - v(t)|^2 \right) dt,$$

$C \in \mathbb{R}$ is a given constant, the weak convergence of $u_n$ and $v_n$ occurr in $L^2([0,1])$, and $u_0$ is a function which is piecewise $C^1$ on $[0,1]$ (i.e. there exists a partition $0 = t_0 < t_1 < \cdots < t_N = 1$ such that $u_0 \in C^1(]t_i, t_{i+1}[)$, and the limits of $u_0$ exist finite at $t = t_i^\pm$ but $u_0(t_i^-) \neq u_0(t_i^+)$ for $i = 1, \ldots, N-1$).

Denote by $\mathcal{J}$ the family of all the intervals $J$ compactly contained in one of the open intervals $]t_i, t_{i+1}[$. Also suppose, for simplicity, that $u_n \rightharpoonup u_0$ in $H^1(J)$ for every interval $J \in \mathcal{J}$.

Prove that we necessarily have

a) $v = 1$ a.e. and $v_n \to v$ strongly in $L^2$.
b) $\liminf_{n\to\infty} E_n(u_n, v_n, J) \geq \int_J |u_0'(t)|^2 dt$ for every interval $J \in \mathcal{J}$.
c) $\liminf_{n\to\infty} E_n(u_n, v_n, J) \geq 1$ for every interval $J$ containing one of the points $t_i$.
d) $C \geq \liminf_{n\to\infty} E_n(u_n, v_n, [0,1]) \geq \int_0^1 |u_0'(t)|^2 dt + (N-1)$.

**Exercise 40.** Let $u_\varepsilon$ be solutions of the minimization problems $P_\varepsilon$ given by

$$P_\varepsilon \ := \ \min \left\{ \int_0^\pi \left( \frac{\varepsilon}{2}|u'(t)|^2 + \frac{1}{2\varepsilon} \sin^2(u(t)) + 10^3|u(t) - t| \right) dt \ : \ u \in H^1([0,\pi]) \right\}.$$

Prove that $u_\varepsilon$ converges strongly (the whole sequence, not up to subsequences!!) in $L^1$ to a function $u_0$ as $\varepsilon \to 0$, find this function, and prove that the convergence is actually strong in all the $L^p$ spaces with $p < \infty$. Is it a strong $L^\infty$ convergence?