Deterministic, stochastic and strategical dynamics under density constraints

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1. **Deterministic Dynamics** - Micro and macro models for crowd motions with constraints
   - Disks with no overlapping
   - Density $\rho \leq 1$
   - The continuity equation
2. **Deterministic Dynamics** - Solution by optimal transport
   - A splitting method
   - Few words about optimal transport and Wasserstein distance
   - Recovering the PDE
3. **Strategical Dynamics** - MFG with density penalization or constraints
   - Classical MFG with density penalization
   - How to replace penalizations by constraints
   - Questions
4. **Stochastic Dynamics** - Let's introduce diffusion
   - The PDEs for MFG and crow motion
   - Three possible different schemes

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Deterministic - models

Micro and macro models with constraints

A general model

A particle population needs to move, and each particle - if alone - would follow its own velocity u (depend on time, position...) Yet, particles are rigid disks that cannot overlap, hence, the actual velocity v will not be u if u is too concentrating. Let us suppose $v = P_{adm(q)}(u)$, where q is the particle configuration, $adm(q)$ the set of velocities that do not induce overlapping, $P_{adm(q)}$ the projection on this set.

If every particle is a disk with radius R, located at q_i, we have

$$q \in K := \{q = (q_i)_i \in \Omega^N \ : \ |q_i - q_j| \geq 2R\}$$

$$adm(q) = \{v = (v_i)_i \ : \ (v_i - v_j) \cdot (q_i - q_j) \geq 0 \ \forall (i,j) \ : \ |q_i - q_j| = 2R\}$$

and we solve $q'(t) = P_{adm(q(t))} u(t)$ (with $q(0)$ given).

B. Maury, J. Venel, Handling of contacts in crowd motion simulations, *Traffic and Granular Flow*, 2007.

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Continuous formulation

- The particles population will be described by a density $\rho \in \mathcal{P}(\Omega)$,
- the constraint by $K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$,
- $u : \Omega \to \mathbb{R}^d$ will be a vector field, possibly depending on time or ρ,
- $adm(\rho) = \{v : \Omega \to \mathbb{R}^d \;:\; \nabla \cdot v \geq 0 \text{ on } \{\rho = 1\}\}$,
- P is the projection in $L^2(dx)$ or (which is the same) in $L^2(\rho)$,
- we'll solve $\partial_t \rho_t + \nabla \cdot \left(\rho_t \left(P_{adm(\rho_t)} u_t\right)\right) = 0$.

Difficulty : $v = P_{adm(\rho_t)} u_t$ is not regular, neither depends regularly on ρ.

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Pressures and duality

The set $adm(\rho)$ may be better described by duality :

$$adm(\rho) = \{v \in L^2(\rho) \; : \; \int v \cdot \nabla p \leq 0 \quad \forall p \; : \; p \geq 0, \; p(1-\rho) = 0\}.$$

In this way we can characterize $v = P_{adm(\rho)}(u)$ through

$$u = v + \nabla p, \quad v \in adm(\rho), \quad \int v \cdot \nabla p = 0,$$

$$p \in press(\rho) := \{p \in H^1(\Omega), \; p \geq 0, \; p(1-\rho) = 0\}$$

This function p plays the role of the pressure affecting the movement.

$$\partial_t \rho_t + \nabla \cdot (\rho_t(u_t - \nabla p_t)) = 0$$

$$\rho_t \in K, \; p_t \in press(\rho_t)$$

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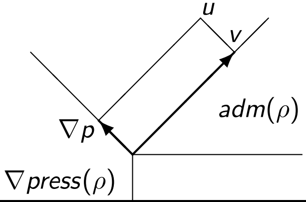

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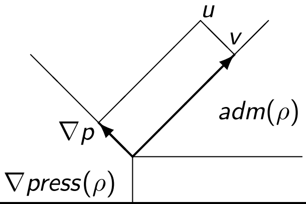

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Deterministic - solutions

Using optimal transport and Wasserstein distance

A splitting scheme for the PDE

Fix a time step $\tau > 0$. We look for a sequence $(\rho_n^\tau)_n$ where ρ_n^τ stands for ρ at time $n\tau$. We first define

$$\tilde{\rho}_{n+1}^\tau = (id + \tau u_{n\tau})_\# \rho_n^\tau \ ; \quad \rho_{n+1}^\tau = P_K(\tilde{\rho}_{n+1}^\tau)$$

where the projection P_K is in the sense of the Wasserstein distance, induced by optimal transport.

The key point is actually using the W_2 projection (instead of L^2 or other projections). It corresponds to the L^2 projection of velocity fields.

B. Maury, A. Roudneff-Chupin and F. Santambrogio, A macroscopic crowd motion model of gradient flow type, *Math. Mod. Meth. Appl. Sci.*

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Optimal transport and Wasserstein distances

If two probabilities $\mu, \nu \in \mathcal{P}(\Omega)$ are given on a compact domain, the Monge-Kantorovitch problem reads

$$\begin{aligned} W_2^2(\mu,\nu) &= \inf\left\{ \int |x - T(x)|^2 d\mu \ : \ T:\Omega\to\Omega,\ T_\#\mu=\nu \right\} \\ &= \inf\left\{ \int |x-y|^2 d\gamma \ : \ \gamma\in\mathcal{P}(\Omega^2),\ (\pi_x)_\#\gamma=\mu, (\pi_y)_\#\gamma=\nu \right\} \\ &= 2\sup\left\{ \int \phi\, d\mu + \int \psi\, d\nu \ : \ \phi(x)+\psi(y) \le \frac{1}{2}|x-y|^2 \right\}. \end{aligned}$$

Under suitable assumptions, there exist an optimal transport T and an optimal function ϕ, called Kantorovich potential, which is Lipschitz continuous. They are linked by $T(x) = x - \nabla\phi(x)$.

Moreover, $W_2(\mu,\nu)$, the square root of the minimal value, is a distance on $\mathcal{P}(\Omega)$ which metrizes the weak-* convergence of probabilities (on compact domains).

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Projections and pressures

Fix a measure $\nu \in \mathcal{P}(\Omega)$ and solve

$$\min \quad \frac{1}{2}W_2^2(\rho,\nu) \ : \ \rho \in K \quad = \quad \min_{\rho \le 1} \ \sup_{\phi,\psi} \int \phi \, d\rho + \int \psi \, d\nu.$$

By duality and inf-sup exchange arguments, the optimal ρ must also solve

$$\min \int \phi \, d\rho \ : \ \rho \le 1,$$

where ϕ is the Kantorovich potential in the transport from ρ to ν. This implies

$$\exists t \ : \ \rho = \begin{cases} 1 & \text{on } \phi < t, \\ 0 & \text{on } \phi > t, \\ \in [0,1] & \text{on } \phi = t \end{cases} \Rightarrow p := (t-\phi)_+ \ge 0, \ \ p(1-\rho) = 0.$$

Hence, $p \in press(\rho)$ and, passing to gradients, we have

$$\rho - \text{a.e.} \quad \nabla p = -\nabla\phi = T(x) - x.$$

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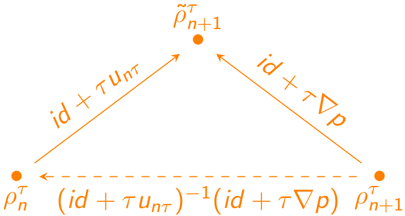
Getting back to the PDE

$T(x) = x + \nabla p(x)$ is the optimal transport from ρ^τ_{n+1} to $\tilde\rho^\tau_{n+1}$. Notice

$$||\nabla p||_{L^2(\rho^\tau_{n+1})} = W_2(\rho^\tau_{n+1}, \tilde\rho^\tau_{n+1}) \le W_2(\rho^\tau_n, \tilde\rho^\tau_{n+1}) \le \tau ||u_{n\tau}||_{L^2(\rho^\tau_n)}.$$

This suggest to scale the pressure (we call it now τp) and get the following situation



Notice that $(id + \tau u_{n\tau})^{-1}(id + \tau\nabla p) = id - \tau(u_{(n+1)\tau} - \nabla p) + o(\tau)$ provided u is regular enough. This allows to get, in the limit $\tau \to 0$, the vector field $v_t = P_{adm(\rho_t)}[u_t]$ and get a solution of the PDE.

Strategical dynamics

Mean Field Games with density penalizations or constraints

MFG with density penalization- 1

In a population of agents everybody follows controlled trajectories

$$y'(t) = f(t, y(t), \alpha(t)), \quad t \in [0, T].$$

For every t, the goal of each agent is to maximize

$$-\int_t^T \left(\frac{|\alpha(s)|^2}{2} + g(\rho_s(y(s))) \right) ds + \Phi(y(T)),$$

where g is a given increasing function of the density ρ_s at time s. The agent hence tries to avoid overcrowded regions.
Let φ be the value function for this problem : it satisfies

$$\partial_t \varphi(t,x) + H(t, x, \nabla\varphi(t,x)) = 0, \quad \varphi(T,x) = \Phi(x) \quad : \quad y(t) = x$$

for a Hamiltonian function H, depending on f and $g(\rho_t)$. The optimal $\alpha(t)$, and hence the evolution of ρ_t, depends on $\nabla\varphi(t,x)$.

MFG with density penalization- 1

In a population of agents everybody follows controlled trajectories

$$y'(t) = f(t, y(t), \alpha(t)), \quad t \in [0, T].$$

For every t, the goal of each agent is to maximize

$$-\int_t^T \left(\frac{|\alpha(s)|^2}{2} + g(\rho_s(y(s))) \right) ds + \Phi(y(T)),$$

where g is a given increasing function of the density ρ_s at time s. The agent hence tries to avoid overcrowded regions.
Let φ be the value function for this problem : it satisfies

$$\partial_t \varphi(t, x) + H(t, x, \nabla \varphi(t, x)) = 0, \quad \varphi(T, x) = \Phi(x) \quad : \quad y(t) = x$$

for a Hamiltonian function H, depending on f and $g(\rho_t)$. The optimal $\alpha(t)$, and hence the evolution of ρ_t, depends on $\nabla \varphi(t, x)$.

MFG with density penalization- 2

The evolution follows a coupled system : φ solves HJB with ρ, which on turn evolves according to $\partial_t \rho + \nabla \cdot (\rho v) = 0$, where $v(t,x)$ depends on $\nabla\varphi(t,x)$.

Typical example : if $f(t,x,\alpha) = \alpha$ then we have

$$\begin{cases} \partial_t \varphi + \frac{|\nabla\varphi|^2}{2} - g(\rho) = 0, \\ \partial_t \rho + \nabla \cdot (\rho \nabla \varphi) = 0, \\ \varphi(T,x) = \Phi(x), \quad \rho(0,x) = \rho_0(x). \end{cases}$$

Stochastic case : we can also insert random effects $dY = f(t,Y,\alpha)dt + dB$, which lets $\Delta\varphi$ and $\Delta\rho$ appear :

$$\partial_t \varphi + \Delta\varphi + \frac{|\nabla\varphi|^2}{2} - g(\rho) = 0 \ : \quad \partial_t \rho - \Delta\rho + \nabla \cdot (\rho \nabla \varphi) = 0.$$

J.-M. Lasry, P.-L. Lions, Mean-Field Games, *Japan. J. Math.* 2007
P. Cardaliaguet, lecture notes, www.ceremade.dauphine.fr/~cardalia/

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MFG with density constraints - 1

How to define a mean field game if we want to replace the penalization $+g(\rho)$ with the constraint $\rho \leq 1$?

Naif idea : when $(\rho_s)_s$ is given, every agent minimizes his own cost paying attention to the constraint $\rho_s(y(s)) \leq 1$. But if ρ already satisfies $\rho \leq 1$, one extra agent will not violate the constraint (non-atomic game). Hence the constraint becomes empty.

Good idea : use the pressure. The unnown is now the pair $(\overline{\rho}, \overline{\alpha})$. We want

$$\partial_t \overline{\rho}_t + \nabla \cdot \left(\overline{\rho}_t \left(P_{adm(\rho_t)}[\overline{\alpha}_t] \right) \right) = 0.$$

The projection of $\overline{\alpha}_t$ onto $adm(\overline{\rho}_t)$ rises a pressure p_t. Every agent tries to solve

$$\max \quad -\int_t^T \frac{|\alpha(s)|^2}{2} ds + \Phi(y(T)), \ : \ y'(s) = \alpha(s) - \nabla p_s(y(s)), \ y(t) = x.$$

A configuration $(\overline{\rho}, \overline{\alpha})$ will be an equilibrium if the original effort field $\overline{\alpha}$ is equal to the optimal one in this problem and if the original densities $\overline{\rho}_t$ are equal to those realized at time t by these optimal trajectories.

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JMFG with density constraints - 2

By computing the Hamiltonian of this problem and the optimal α we get the equations satisfied by the equilibrium, i.e.

$$\begin{cases} \partial_t\varphi + \frac{|\nabla\varphi|^2}{2} - \nabla\varphi\cdot\nabla p = 0, \\ \partial_t\rho + \nabla\cdot(\rho(\nabla\varphi - \nabla p)) = 0, \\ p \geq 0,\ p(1-\rho) = 0, \\ \varphi(T,x) = \Phi(x), \quad \rho(0,x) = \rho_0(x). \end{cases}$$

Unfortunately, no result (existence, uniqueness. . .) is available for the moment.
A PhD thesis (A. Mészáros, Orsay) is ongoing on these questions.

F. Santambrogio, A Modest Proposal for MFG with Density Constraints, *Net. Het. Media*, 2012.

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Stochastic Dynamics

Some perspectives and methods

Which PDE for diffusion and density constraints ?

If we want to model a population which diffuse but is also subject to $\rho \leq 1$, should we apply the projection operator on the drift only or should it also interact with the Laplacian term ?

Good definitions (and existence/uniqueness results) for this notion without strategical issues are a necessary starting point in order to attack later the 2^{nd} order MFG system with density constraints (which could turn out to be simpler than the 1^{st} order one, because of higher regularity).

Fortunately, the Laplacian is consistent with the constraint. Moreover $\Delta\rho = \nabla \cdot \left(\rho \frac{\nabla \rho}{\rho}\right)$ and $\frac{\nabla \rho}{\rho} = 0$ on $\{\rho = 1\}$. Hence the equation is

$$0 = \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (P_{adm(\rho_t)}[u_t]\rho_t) = \partial_t \rho_t + \nabla \cdot \left(P_{adm(\rho_t)}\left[u_t - \frac{\nabla \rho_t}{\rho_t}\right]\rho_t\right)$$

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Different catching-up methods

Take $\tau > 0$ and a density ρ_n^τ. How to choose ρ_{n+1}^τ ?

The cleanest way Take a r.v. $X \sim \rho_n^\tau$ and build $(id + \tau u_{n\tau}) \circ X + B_\tau$ with B a Brownian motion independent of X. Define $\tilde{\rho}_{n+1}^\tau$ as the law of this new r.v., which is given by

$$\tilde{\rho}_{n+1}^\tau = ((id + \tau u_{n\tau})_\# \rho_n^\tau) * \eta_{\sqrt{\tau}},$$

where η_r is a standard Gaussian of size r. Then set $\rho_{n+1}^\tau = P_K(\tilde{\rho}_{n+1}^\tau)$.

A similar one Take the solution ρ of the Fokker-Planck equation

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Estimates as $\tau \to 0$ on the two previous methods are difficult, since they require estimates on the distance $W_2(\rho_n^\tau, \tilde\rho_{n+1}^\tau)$, which roughly corresponds to estimate on the Heat equation between time 0 and time τ. They are available, but under additional regularity assumptions on the initial datum (namely, it should be at least BV).

Hence here is a less meaningful but easier to handle way of defining the next step.

The most efficient way First build $\tilde\rho_{n+1}^\tau := (id + \tau u_{n\tau})_\# \rho_n^\tau$. Then set

$$\rho_{n+1}^\tau = \text{argmin} \int \rho \ln \rho + \frac{1}{2\tau} W_2^2(\rho, \tilde\rho_{n+1}^\tau) \,:\, \rho \in K.$$

This problem is strictly convex and admits a unique solution. This recalls (and actually is taken from) the theory of gradient flows in the Wasserstein space.

L. Ambrosio, N. Gigli, G. Savaré *Gradient Flows*, Birkäuser, 2005,
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Where the PDE comes from

The same variational tricks show that the optimality conditions of $\min_{\rho\in K}\int\rho\ln\rho+\frac{1}{2\tau}W_2^2(\rho,\nu)$ are

$$\exists t\ :\ \rho=\begin{cases}1 & \text{on } \left(\ln\rho+\frac{\phi}{\tau}\right)<t,\\ 0 & \text{on } \left(\ln\rho+\frac{\phi}{\tau}\right)>t,\\ \in[0,1] & \text{on } \left(\ln\rho+\frac{\phi}{\tau}\right)=t.\end{cases}$$

We then define $p=(t-\ln\rho-\frac{\phi}{\tau})_+$ and we get $p\in press(\rho)$. Moreover, $\rho-$a.e. $\nabla p=-\frac{\nabla\rho}{\rho}-\frac{\nabla\phi}{\tau}$, which shows that we have

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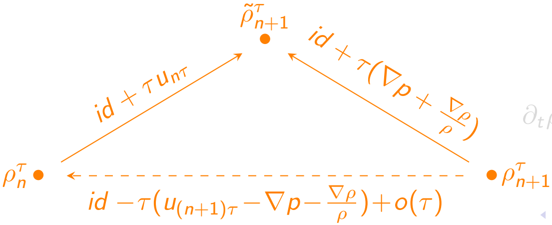

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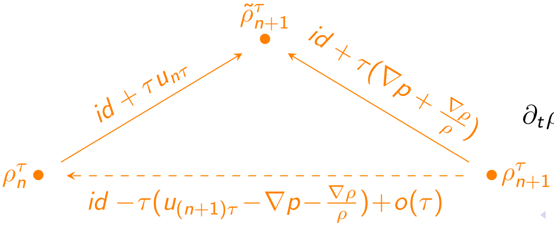

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The End

Thanks for your attention