Phase-field approximation of branched transport problems

Filippo Santambrogio

Laboratoire de Mathématiques d'Orsay, Université Paris-Sud
www.math.u-psud.fr/∼santambr/

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1. Branched Transport : discrete and continuous models
2. Phase-field approximation of singular energies
3. A Modica-Mortola approximation in branched transport theory
4. Applications to numerics

Branched Transport

Discrete and continuous formulations,
transport distances

Branched networks in a discrete framework

Take some points x_i, y_j in $\Omega \subset \mathbb{R}^d$. Inject a mass a_i at x_i and absorb b_j at y_j. Consider weighted oriented graphs $G = (e_h, \hat{e}_h, \theta_h)_h$ (e_h are the edges, $\hat{e}_h$ their orientations, θ_h the weights), satisfying Kirchhoff's law : at each node

incoming + injected mass = outcoming + absorbed mass

For $0 \leq \alpha < 1$, among these graphs we minimize the energy

$$E^\alpha(G) := \sum_h \theta_h^\alpha \mathcal{H}^1(e_h).$$

The inequality $(m_1 + m_2)^\alpha < m_1^\alpha + m_2^\alpha$ makes a branching behavior optimal.

Particular cases : $\alpha = 1$ Monge optimal transport (no joint-trasportation incentive is present) ; $\alpha = 0$: Steiner's minimal connection.

E. N. Gilbert, Minimum cost communication networks, *Bell System Tech. J.*, 1967.
E. N. Gilbert and H. O. Pollak, Steiner minimal trees. *SIAM J. Appl. Math.*, 1968.

From the discrete to the continuous framework

With every G we can associate a vector measure representing the flow

$$u_G := \sum_h \theta_h \hat{e}_h \mathcal{H}^1_{|e_h}.$$

Kirchhoff's law is satisfied if and only if $\nabla \cdot u_G = f^+ - f^-$, where $f^+ = \sum_{i=1}^m a_i \delta_{x_i}$ and $f^- = \sum_{j=1}^n b_j \delta_{y_j}$.

For general $f^+, f^- \in \mathcal{P}(\Omega)$, Q. Xia proposed to extend E^α by relaxation

$$M^\alpha(u) = \inf \left\{ \liminf_n E^\alpha(G_n) \ : \ G_n \text{ finite graph}, \ u_{G_n} \to u \right\},$$

and to minimize M^α under the constraint $\nabla \cdot u = f^+ - f^-$. We also have

$$M^\alpha(u) = \begin{cases} \int_M \theta^\alpha d\mathcal{H}^1 & \text{if } u = U(M, \theta, \xi), \\ +\infty & \text{otherwise.} \end{cases}$$

where $U(M, \theta, \xi)$ is the rectifiable vector measure $u = \theta\xi \cdot \mathcal{H}^1_{|M}$ ($\theta : M \to \mathbb{R}^+$ is a real multiplicity and $\xi : M \to \mathbb{R}^d$, $|\xi| = 1$ an orientation of M).

Q. Xia, Optimal Paths related to Transport Problems. *Comm. Cont. Math.*, 2003.
F. Maddalena, S. Solimini, J.M. Morel, A variational model of irrigation patterns. *Int. Free Bound.*, 2003.

Branched transport distances

The cost is not proportional to the "mass" θ but to θ^α ; small masses are penalized and singular measures are easier to reach.

On a bounded domain Ω, if $\alpha = 1$ we can always connect with finite Monge cost any pair of probabilities, but here it is the case only for α close to 1. Set

$$d_\alpha(f^+, f^-) := \min\{M^\alpha(u) \,:\, \nabla \cdot u = f^+ - f^-\}.$$

If $\alpha > 1 - \frac{1}{d}$, then $d_\alpha < +\infty$ for any $f^+, f^- \in \mathcal{P}(\Omega)$ and d_α is **a distance over $\mathcal{P}(\Omega)$ metrizing weak topology**. Sharp comparison results with the Wasserstein distances W_p also exist :

$$W_{1/\alpha} \le d_\alpha \le W_1^\beta, \qquad \text{for } \beta = d(\alpha - (1 - \frac{1}{d})).$$

If $\alpha \le 1 - \frac{1}{d}$, only "low dimensional" measures are reachable by branched transport (the best ones being atomic measures, the worst Lebesgue).

J.-M. Morel, F. S., Comparison of distances between measures, *Appl. Math. Lett.*, 2007.
F. Maddalena, S. Solimini, Transport distances and irrigation models, *J. Conv. An*, 2009.

Elliptic approximations

$\Gamma-$convergence for singular energies

Preliminaries : $\Gamma-$convergence

On a metric space X let $F_n : X \to \mathbb{R} \cup \{+\infty\}$ be a sequence of functions. We define the two lower-semicontinuous functions F^- and F^+ (called $\Gamma - \liminf$ and $\Gamma - \limsup$ F^+ of this sequence, respectively) by

$$F^-(x) := \inf\{\liminf_{n\to\infty} F_n(x_n) : x_n \to x\}, F^+(x) := \inf\{\limsup_{n\to\infty} F_n(x_n) : x_n \to x\}.$$

If $F^- = F^+ = F$ coincide, then we say $F_n \xrightarrow{\Gamma} F$.

Among the properties of $\Gamma-$convergence we have the following :

- if there exists a compact set $K \subset X$ such that $\inf_X F_n = \inf_K F_n$ for any n, then F attains its minimum and $\inf F_n \to \min F$;
- if $(x_n)_n$ is a sequence of minimizers for F_n admitting a subsequence converging to x, then x minimizes F
- if F_n is a sequence $\Gamma-$converging to F, then $F_n + G$ will $\Gamma-$converge to $F + G$ for any continuous function $G : X \to \mathbb{R} \cup \{+\infty\}$.

E. De Giorgi, T. Franzoni. Su un tipo di convergenza variazionale. *Atti Lincei*, 1975.
A. Braides. *Γ-convergence for beginners*. 2002

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Elliptic approximation of the perimeter functional

Theorem (Modica-Mortola)

Define the functional F_ε on $L^1(\Omega)$ through

$$F_\varepsilon(u) = \begin{cases} \frac{1}{\varepsilon}\int W(u(x))dx + \varepsilon \int |\nabla u(x)|^2 dx & \text{if } u \in H^1(\Omega); \\ +\infty & \text{otherwise.} \end{cases}$$

Then, if $W(0) = W(1) = 0$ and $W(t) > 0$ for any $t \neq 0, 1$, we have $F_\varepsilon \xrightarrow{\Gamma} F$, where F is given by

$$F(u) = \begin{cases} c\,Per(S) & \text{if } u = I_S \text{ and } S \text{ is a finite perimeter set;} \\ +\infty & \text{otherwise,} \end{cases}$$

and the constant c is given by $c = 2\int_0^1 \sqrt{W(t)}dt$.

L. Modica, S. Mortola, Un esempio di Γ-convergenza. *Boll. UMI*, 1977.

Other approximations of singular energies

Vector framework : Ginzburg-Landau approximation (Bethuel-Brezis-Helein)

$$\min \frac{1}{\varepsilon}\int (1-|u|)^2 + \varepsilon \int |\nabla u|^2, \quad u \in H^1(\Omega;\mathbb{R}^d).$$

Gradient framework : Aviles-Giga, Ambrosio-DeLellis-Mantegazza (Modica-Mortola results for higher order energies)

$$\min \frac{1}{\varepsilon}\int F(\nabla u) + \varepsilon \int |D^2 u|^2.$$

Mumford-Shah : Ambrosio-Tortorelli

$$\min_{u,v} \int_\Omega (v^2+\sqrt{\epsilon})|\nabla u|^2 + \alpha \int_\Omega \frac{(1-v)^2}{4\varepsilon} + \varepsilon|\nabla v|^2 + \beta \int_\Omega (u-g)^2.$$

Atomic energies on the line : Bouchitté-Dubs-Seppecher

$$\min \frac{1}{\varepsilon}\int W(u) + \varepsilon \int |u'|^2, \quad \text{with } \lim_{t\to\infty} \frac{W(t)}{t} = 0.$$

Modica-Mortola for Branched Transport

$\Gamma-$convergence results

Ideas, conjectures and goals

It would be natural to approximate the minimization of M^α with some minimization problems defined on **regular** vector fields u (instead of singular measures) having a "true" divergence. What about

$$\min \frac{1}{\varepsilon}\int |u|^\alpha + \varepsilon \int |\nabla u|^2, \quad u \in H^1(\Omega;\mathbb{R}^d),\ \nabla \cdot u = f \quad ?$$

Two goals :

- **(theory)** make a bridge with the theory of elliptic approximation for singular energies
- **(applications)** produce an efficient numerical method for finding optimal branched structures.

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Heuristics for the exponents

$\frac{1}{\varepsilon}\int|u|^\alpha+\varepsilon\int|\nabla u|^2$ is not the correct choice. We consider more generally

$$M_\varepsilon^\alpha(u)=\varepsilon^{\gamma_1}\int|u|^p+\varepsilon^{\gamma_2}\int|\nabla u|^2.$$

Consider a measure $U(S,\theta,\xi)$, concentrated on a segment S with constant multiplicity θ, and approximate it with a smooth u_A on a strip of width A :

$$M_\varepsilon^\alpha\approx\varepsilon^{\gamma_1}A^{d-1}\left(\frac{\theta}{A^{d-1}}\right)^p\mathcal{H}^1(S)+\varepsilon^{\gamma_2}A^{d-1}\left(\frac{\theta}{A^d}\right)^2\mathcal{H}^1(S).$$

Minimizing over possible widths A gives the optimal values

$$A\approx\varepsilon^{\frac{\gamma_2-\gamma_1}{2d-p(d-1)}}\theta^{\frac{2-p}{2d-p(d-1)}};\quad M_\varepsilon^\alpha\approx\varepsilon^{\gamma_2-(\gamma_2-\gamma_1)\frac{d+1}{2d-p(d-1)}}\theta^{2-(2-p)\frac{d+1}{2d-p(d-1)}}\mathcal{H}^1(S).$$

The correct choice for approximating M^α is

$$p=\frac{2-2d+2\alpha d}{3-d+\alpha(d-1)};\quad\frac{\gamma_1}{\gamma_2}=\frac{(d-1)(\alpha-1)}{3-d+\alpha(d-1)}<0.$$

We get $p\in]0,1[$ as soon as $\alpha\in]1-\frac{1}{d},1[$.

A Γ−convergence theorem

Let $\mathcal{M}(\Omega)$ be the space of finite vector measures on Ω with values in $\mathbb{R}^d$ and such that their divergence is a finite scalar measure. On this space we consider the weak convergence of both u and $\nabla \cdot u$. We stick to the case $d = 2$ and define

$$M_\varepsilon^\alpha(u) = \begin{cases} \varepsilon^{\alpha-1} \int_\Omega |u(x)|^p dx + \varepsilon^{\alpha+1} \int_\Omega |\nabla u(x)|^2 dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

for $p = \frac{4\alpha-2}{\alpha+1}$ (using the exponent we found before).

Theorem

Suppose $d = 2$ and $\alpha \in]\frac{1}{2}, 1[$: then we have Γ−convergence of the functionals M_ε^α to cM^α, with respect to the convergence of $\mathcal{M}(\Omega)$, as $\varepsilon \to 0$. Here c is a finite and positive constant, given by $c = \alpha^{-1} \left(4c_0\alpha/(1-\alpha)\right)^{1-\alpha}$, where $c_0 = \int_0^1 \sqrt{t^p - t}\,dt$.

F.S. A Modica-Mortola approximation for branched transport, *CRAS*, 2010.

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What lacks in the theorem

- We should prove compactness of the minimizers sequence u_ε, if we want to deduce $u_\varepsilon \to u_{opt}$.
- We only adressed $\Gamma-$convergence of the energies, but ignored the divergence constraint.
- It is only stated for dimension 2.
- It only works for $\alpha > 1 - \frac{1}{d}$

Some questions and answers – I

Compactness of u_ε : is it possible to prove an L^1 bound on the minimizers ? $M_\varepsilon^\alpha(u_\varepsilon) \le C$ is not sufficient (as in the limit problem, bounded energy configurations don't have mass bounds, but optimal configurations do have, due to no-cycles conditions), but the minimizers should have extra properties (no cycles). Also, it is possible to artifically add a constraint $\int |u| \le C$ which does not affect the limit, but that's not satisfactory. OPEN

Divergence constraint : can we find a sequence $f_\varepsilon \rightharpoonup f$ and add the constraint $\nabla \cdot u = f_\varepsilon$ in the approximating problems ? (notice that we need $f_\varepsilon \in L^2$). We have to adapt the divergence of the vector field u_ε that we build in the previous $\Gamma-$convergence proof. This can be done thanks to the following estimate : define d_α^ε as d_α, but with the approximated energy M_ε^α instead of M^α ; then

$$d_\alpha^\varepsilon(f^+, f^-) \le \omega\left(W_1(f^+, f^-)^\beta + \varepsilon^{\gamma_2}||f||_{L^2}^2\right) \qquad (\omega(t) \approx t + t^\alpha).$$

This allows to control the cost for modifying the divergence and allows to add the divergence constraints into the functional and the Γ-convergence result. SOLVED

A. Monteil Uniform estimates for a Modica-Mortola type approximation of branched transportation, *ESAIM COCV*, 2017

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Some questions and answers – II

Exponents $\alpha \leq 1 - \frac{1}{d}$: the main issue is $p < 0$. But this can be fixed

Theorem

Suppose $d = 2$ and $\alpha \in]0, \frac{1}{2}[$; let $B \in C^0(\mathbb{R}_+)$ such that $B(0) = 0$, $B(t) > 0$ for $t > 0$, $\lim_{t\to\infty} \frac{B(t)}{t^p} = 1$, $B'(0) > 0$, $p = \frac{4\alpha-2}{\alpha+1} \in]-2, 0[$. Define M_ε^B through

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The higher-dimensional case : the problem was the proof, not suitable to codimension > 1. But it can be done differently, yet only for $\alpha > 1 - \frac{1}{d}$.

Both : PARTIALLY SOLVED

E. Oudet, F. S., A Modica-Mortola approximation for branched transport and applications, *Arch. Rati. Mech. An.*, 2011.

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Numerics

Finding “good” local minima

Idea of the numerical method

The exact identification of global optimal networks in the combinatorial context is NP hard (with respect to the number of sources and targets). The method we propose here is, on the contrary, purely continuous : it requires to find a vector field on the whole Ω and is not influenced by this number. The main difficulties are related both to the approximation of singular and irregular functions and to the strongly non-convex cost functional.

Idea : (by E. Oudet, who already used this approach for other problems admitting $\Gamma-$convergence approximations) observe that for $\varepsilon \gg 1$ the functional M^α_ε is "almost" convex. Hence we perform a (projected) gradient descent on M^α_ε for ε large. Then, decreasing the value of ε step by step, we start a new descent for $M^\alpha_{\varepsilon_{k+1}}$ starting from the u_{ε_k} found at the previous step.

Projection : Once fixed a suitable f_ε, L^2 approximation of f, we need to solve at every step $\min\left\{\int \frac{1}{2}|u-u_0|^2 \,:\, \nabla\cdot u = f_\varepsilon\right\}$. By duality, this becomes

$$\max\left\{-\int \frac{1}{2}|\nabla\varphi|^2 - \varphi(f_\varepsilon - \nabla\cdot u_0)\right\}$$

and just requires to solve a Laplacian.

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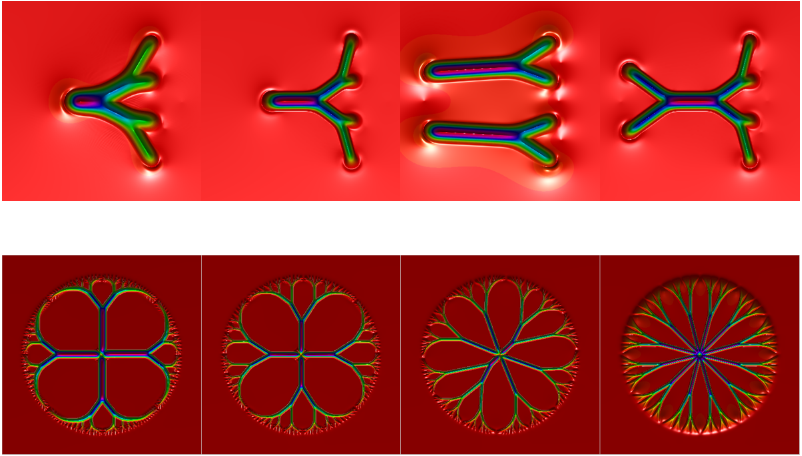
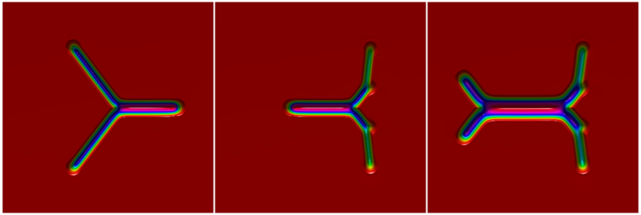
Thick optimal networks



(Numerical computations by E. Oudet, from our joint paper)

The Steiner limit

We can also attack **Steiner problem**. If some points $x_0, \ldots, x_n$ are given, take as a source measure $f^+ = \delta_{x_0}$ and as a destination $f^- = \sum_{i=1}^n \frac{1}{n}\delta_{x_i}$. This imposes connectedness of the networks. Then use M_ε^B and $\alpha \to 0$.



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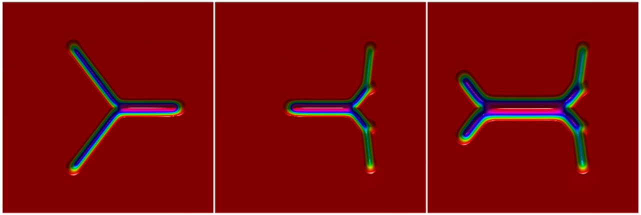

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A variant : the optimal shape to be irrigated

Fix $\alpha > 1 - 1/d$. What is the **best shape to be irrigated**, for the branched transport cost, from δ_0 ?

$$\min \{ d_\alpha(I_A, \delta_0) \,:\, |A| = 1 \} .$$

Equivalently, solve

$$\min \{ d_\alpha(f^+, \delta_0) \,:\, f^+ \in \mathcal{P}(\Omega),\, f^+ \le 1 \} .$$

Note that, for $\alpha = 1$, the solution is the ball of unit volume. What about $\alpha < 1$? How to adapt the numerics when f^+ is not fixed ? Let f_ε^- be a suitable approximation of δ_0, and solve

$$\min \{ M_\varepsilon^\alpha(u) \,:\, 0 \le \nabla \cdot u + f_\varepsilon^- \le 1 \} .$$

The main difference is in the projection. We need to solve

$$\min \left\{ \int \frac{1}{2} |u - u_0|^2 \,:\, 0 \le \nabla \cdot u + f_\varepsilon^- \le 1 \right\}$$

which becomes the non-smooth problem

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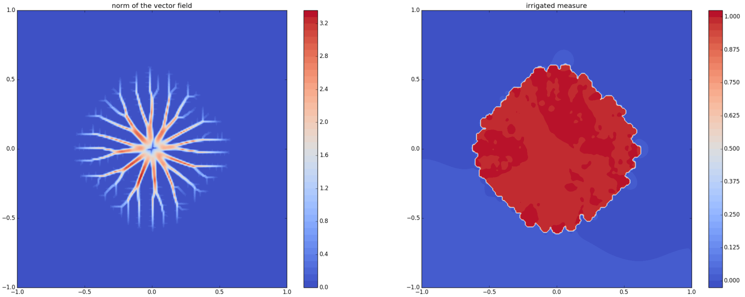
A fractal shape



New computations are in progress. This one was obtained by solving the non-smooth optimization problem in the projection step by a FISTA method (with very small gradient step).

Numerical computations done by P. Pegon. A collaboration with E. Oudet is in progress.

. . . the end. . .

thanks for your attention.