# A fractal shape optimization problem in branched transport

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#### Branched Transport

- Discrete and continuous models
- Landscape function
- Fractals???
- The optimal shape to be irrigated
- Approximation via a phase-field approach
- Output State Numerics for the optimal shape

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#### Branched Transport

# Discrete and continuous formulations,

river basins, fractals

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## Branched networks in a discrete framework

Take some points  $x_i, y_j$  in  $\Omega \subset \mathbb{R}^d$ . Inject a mass  $a_i$  at  $x_i$  and absorb  $b_j$  at  $y_j$ . Consider weighted oriented graphs  $G = (e_h, \hat{e}_h, \theta_h)_h$  ( $e_h$  are the edges,  $\hat{e}_h$  their orientations,  $\theta_h$  the weights), satisfying Kirchhoff's law: at each node

incoming + injected mass = outcoming + absorbed mass

For 0  $\leq \alpha <$  1, among these graphs we minimize the energy

$$E^{\alpha}(G) := \sum_{h} \theta^{\alpha}_{h} \mathcal{H}^{1}(e_{h}).$$

The inequality  $(m_1 + m_2)^{\alpha} < m_1^{\alpha} + m_2^{\alpha}$  makes a branching behavior optimal. **Particular cases:**  $\alpha = 1$  Monge optimal transport (no joint-trasportation incentive is present);  $\alpha = 0$ : Steiner's minimal connection.

E. N. GILBERT, Minimum cost communication networks, *Bell System Tech. J.*, 1967. E. N. GILBERT AND H. O. POLLAK, Steiner minimal trees. *SIAM J. Appl. Math.*, 1968.

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#### From the discrete to the continuous framework

With every G we can associate a vector measure representing the flow

$$u_G := \sum \theta_h \hat{e}_h \mathcal{H}^1_{|e_h}.$$

Kirchhoff's law is satisfied if and only<sup>h</sup>if  $\nabla \cdot u_G = \mu - \nu$ , where  $\mu = \sum_{i=1}^m a_i \delta_{x_i}$ and  $\nu = \sum_{j=1}^n b_j \delta_{y_j}$ . For general  $\mu, \nu \in \mathcal{P}(\Omega)$ , Q. Xia proposed to extend  $E^{\alpha}$  by relaxation  $M^{\alpha}(u) = \inf \left\{ \liminf_n E^{\alpha}(G_n) : G_n \text{ finite graph, } u_{G_n} \to u \right\},$ 

and to minimize  $M^{lpha}$  under the constraint  $abla \cdot u = \mu - \nu$ . We also have

$$M^{lpha}(u) = egin{cases} \int_{M} heta^{lpha} d\mathcal{H}^1 & ext{ if } u = U(M, heta,\xi), \ +\infty & ext{ otherwise.} \end{cases}$$

where  $U(M, \theta, \xi)$  is the rectifiable vector measure  $u = \theta \xi \cdot \mathcal{H}^1_{|M}$  ( $\theta : M \to \mathbb{R}^+$  is a real multiplicity and  $\xi : M \to \mathbb{R}^d$ ,  $|\xi| = 1$  an orientation of M).

 Q. XIA, Optimal Paths related to Transport Problems. Comm. Cont. Math., 2003.
 F. MADDALENA, S. SOLIMINI, J.M. MOREL, A variational model of irrigation patterns. Int. Free Bound., 2003.

M. COLOMBO, A. DE ROSA, A. MARCHESE, S. STUVARD On the lower semicontinuous envelope of functionals defined on polyhedral chains, preprint.  $\langle \Box \rangle \langle \Box \rangle \langle$ 

#### Branched transport distances

The cost is not proportional to the "mass"  $\theta$  but to  $\theta^{\alpha}$ ; small masses are penalized and singular measures are easier to reach.

On a bounded domain  $\Omega$ , if  $\alpha = 1$  we can always connect with finite Monge cost any pair of probabilities, but here it is the case only for  $\alpha$  close to 1. Set

$$d_{\alpha}(\mu,\nu) := \min\{M^{\alpha}(u) : \nabla \cdot u = \mu - \nu\}.$$

If  $\alpha > 1 - \frac{1}{d}$ , then  $d_{\alpha} < +\infty$  for any  $\mu, \nu \in \mathcal{P}(\Omega)$  and  $d_{\alpha}$  is a distance over  $\mathcal{P}(\Omega)$  metrizing weak topology. Sharp comparison results with the Wasserstein distances  $W_p$  also exist:

$$W_{1/lpha} \leq {\it d}_{lpha} \leq W_1^eta, \quad ext{ for } eta = {\it d}(lpha - (1 - rac{1}{{\it d}})).$$

If  $\alpha \leq 1 - \frac{1}{d}$ , only "low dimensional" measures are reachable by branched transport (the best ones being atomic measures, the worst Lebesgue).

J.-M. MOREL, F. S., Comparison of distances between measures, *Appl. Math. Lett.*, 2007. F. MADDALENA, S. SOLIMINI, Transport distances and irrigation models, *J. Conv. An*, 2009.

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#### The landscape function associated with stable river basins

In the study of river basins, given a rain input  $\mu$  and a single outlet  $\nu = \delta_0$ , geophysicists look for a pair (N, z): N is the drainage network and  $z : \operatorname{spt}(\mu) \to \mathbb{R}$  is the elevation of each point. Such a pair is stable iff two facts occurr: water flows along the steepest descent of z ( $\nabla z$  is parallel to the newtork and opposite to its direction) and a slope-discharge relation of the form

$$|\nabla z| = \theta^{\alpha - 1}$$

is satisfied ( $\theta(x)$  is the amount of water passing through x, and  $\alpha \approx 1/2$  is a fixed exponent). In a discrete netork framework, such a pair can be found by minimizing an  $E^{\alpha}$  energy, using the graph G as a network, and defining

$$z(x) = \sum_{h \in E_x} \theta_h^{\alpha - 1} \mathcal{H}^1(e_h),$$

where  $E_x$  is the set of edges in the unique path connecting 0 to x (the optimal G is acyclic)... We can do the same for the continuous case.

R. RIGON, A. RINALDO, I. RODRIGUEZ-ITURBE, E. IJJASZ-VASQUEZ, R. L. BRAS Optimal channel networks: a framework for the study of river basin morphology, *Water Res. Res.*, 1993.

## The landscape function associated with branched transport

Given an optimal *u*, we define (warning: proper definitions are needed!)

$$z(x) = \int_0^1 heta^{lpha - 1}(\omega(t)) |\omega'(t)| dt$$

where  $\omega: [0,1] \to M$  is any curve with  $\omega(0) = 0, \omega(1) = x$  and  $\omega'$  oriented as  $-\xi$  a.e. We have

- z is well-defined (independent of  $\omega$ ) and l.s.c.
- $z(x) \ge z(x_0) \theta(x_0)^{\alpha-1}\xi(x_0) \cdot (x x_0) + o(|x x_0|)$  (slope-discharge condition, and direction of the slope)
- if  $\Omega$  has an interior cone condition and  $\mu \ge c > 0$  on  $\Omega$ , then  $z \in C^{0,\beta}(\Omega)$ (then generalized to lower-dimensional measures  $\mu$ )

• 
$$d_{\alpha}(\mu, \delta_0) = \int z d\mu$$
 and for  $\mu_{\varepsilon} = (1 - \varepsilon)\mu + \varepsilon \tilde{\mu}$   
 $d_{\alpha}(\mu_{\varepsilon}, \delta_0) \leq d_{\alpha}(\mu, \delta_0) + \alpha \varepsilon \int z d(\tilde{\mu} - \mu)$ 

(the first variation of  $d_{\alpha}$  is  $\alpha z$ ).

F. S., Optimal Channel Networks, Landscape Function and Branched Transport, *Int. Free Bound.*, 2007.

A. BRANCOLINI, S. SOLIMINI, On the Hölder Regularity of the Landscape Function  $_{E}$  IFB, 2011,  $_{\circ,\circ}$ 

## Just an idea of the optimal networks

From one source point  $\nu = \delta_0$  to the uniform measure  $\mu = \frac{1}{2\pi} \mathcal{H}^1_{|\mathbb{S}^1}$ , for different values of  $\alpha$ .



(numerics by E. Oudet; I'll explain later where the simulations come from)

A very general reference for branched transport:

M. BERNOT, V. CASELLES, J.M. MOREL, Optimal Transportation Networks, Models and Theory, LNM 2009

# Fractals

The solutions of the branched transport problem that we can guess, simulate or compute show some sort of *fractal behavior*, but, in which sense?

- the optimal network if necessarily one-dimensional, and has no fractal dimension
- yet, some scaling laws on its branching behavior exist: Brancolini and Solimini proved that from every branch *L* of lenght  $\ell$  and for every  $\varepsilon < \ell$ , the number of branches stemming from *L* with length between  $\varepsilon$  and  $2\varepsilon$  are  $O(\ell/\varepsilon)$ , with some universal bounds.
- the frontiers between adjacent **irrigation basins** (water divide lines) are a good candidate to be of fractal dimension (conjecture by J.-M. Morel)...
- or other sets related to branched transport and to the landscape z...

I. RODRIGUEZ-ITURBE ET AL. Fractal structures as least energy dissipation patterns: the case of river networks, *Geophys. Res. Lett.*, 1992.

I. RODRIGUEZ-ITURBE, A. RINALDO Fractal River Basins, 1997.

A. BRANCOLINI, S. SOLIMINI., Fractal regularity results on optimal irrigation patterns, J. Math. Pures. Appl., 2014.

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A. BRANCOLINI, S. SOLIMINI., Fractal regularity results on optimal irrigation patterns, *J. Math. Pures. Appl.*, 2014. Theory: Branched Transport, landscape, shape optimization Approximation

#### Fractal divide lines



Picture from https://de.wikipedia.org/wiki/Wasserscheide ((C)creative commons)

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Theory: Branched Transport, landscape, shape optimization Approximation Numerics

# A shape optimization problem

Fix an exponent  $\alpha > 1 - 1/d$ . What is the **best shape to be irrigated**, for the branched transport cost, from  $\delta_0$ ?

$$\min\left\{d_{\alpha}(I_{\mathcal{A}},\delta_{0})\ :\ |\mathcal{A}|=1\right\}.$$

Equivalently, solve

$$\min\left\{d_{\alpha}(\mu,\delta_{0})\,:\,\mu\in\mathcal{P}(\Omega),\,\mu\leq1\right\}.$$

Note that, for  $\alpha = 1$ , the solution is the ball of unit volume. What about  $\alpha < 1$ ? The conjecture is that the boundary  $\partial A$  of the optimal set A is indeed of fractal dimension.

work in progress with P. Pegon (Orsay) and Q. Xia (UC Davis)

## What we can prove on the optimal set

- The optimal  $\mu$  exists and is indeed of the form  $\mu = I_A$ .
- The set A is of the form  $A = \{z \leq C\}$ . If we call  $m_{\alpha}$  the minimal value, we have  $C = m_{\alpha}(1 + \frac{1}{\alpha d})$ .
- The function z is  $C^{0,\beta}$  globally on A (and equal to  $+\infty$  outside A).
- A is a closed set with negligible boundary.
- The point 0 belongs to the interior of A (as well as all points x with z(x) < C).
- The boundary  $\partial A$  is of dimension at most  $d \beta$ .

... but we are far from proving that the dimension of  $\partial A$  is  $d-\beta$  (or even that A is not a smooth set, or even not a ball!).

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# Elliptic approximation

A phase-field approach

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# Ideas, conjectures and goals

It would be natural to approximate the minimization of  $M^{\alpha}$  with some minimization problems defined on **regular** vector fields u (instead of singular measures) having a true divergence.

What about

$$\min\left\{\frac{1}{\varepsilon}\int|u|^{\alpha}+\varepsilon\int|\nabla u|^{2},\quad u\in H^{1}(\Omega;\mathbb{R}^{d}),\ \nabla\cdot u=f\right\} \quad ?$$

Two goals:

- (theory) make a bridge with the theory of elliptic approximation for singular energies (Modica-Mortola, Allen-Cahn, Ambrosio-Tortorelli...)
- (applications) produce an efficient numerical method for finding optimal branched structures.

We expect u to be close to 0 far from the network and  $|u| \approx \infty$  close to M.

#### Heuristics for the exponents

Consider a measure  $U(S, \theta, \xi)$ , concentrated on a segment S with constant multiplicity  $\theta$ , and approximate it with a smooth  $u_A$  on a strip of width A around S. Then

$$M_{\varepsilon}^{\alpha} \approx \varepsilon^{\gamma_1} A^{d-1} \left(\frac{\theta}{A^{d-1}}\right)^{\rho} + \varepsilon^{\gamma_2} A^{d-1} \left(\frac{\theta}{A^d}\right)^2$$

Minimizing over possible widths A gives the optimal values

$$A \approx \varepsilon^{\frac{\gamma_2 - \gamma_1}{2d - \rho(d - 1)}} \theta^{\frac{2 - \rho}{2d - \rho(d - 1)}}; \quad M_{\varepsilon}^{\alpha} \approx \varepsilon^{\gamma_2 - (\gamma_2 - \gamma_1)\frac{d + 1}{2d - \rho(d - 1)}} \theta^{2 - (2 - \rho)\frac{d + 1}{2d - \rho(d - 1)}}.$$

The correct choice for approximating  $M^{\alpha}$  is

$$p = \frac{2 - 2d + 2\alpha d}{3 - d + \alpha(d - 1)}; \quad \frac{\gamma_1}{\gamma_2} = \frac{(d - 1)(\alpha - 1)}{3 - d + \alpha(d - 1)} < 0.$$
  
Notice  $p \in ]0, 1[$  as soon as  $\alpha \in ]1 - \frac{1}{d}, 1[$ .

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# A $\Gamma$ -convergence theorem

Let  $\mathcal{M}(\Omega)$  be the space of finite vector measures on  $\Omega$  with values in  $\mathbb{R}^d$  whose divergence is a finite scalar measure, endowed with the weak convergence of both u and  $\nabla \cdot u$ . We stick to the case d = 2 and  $\alpha \in ]\frac{1}{2}, 1[$ . We define  $M_{\varepsilon}^{\alpha}(u) = \begin{cases} \varepsilon^{\alpha-1} \int_{\Omega} |u(x)|^p dx + \varepsilon^{\alpha+1} \int_{\Omega} |\nabla u(x)|^2 dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$ for  $p = \frac{4\alpha-2}{\alpha+1}$  (using the exponent we found before).

#### Theorem

The functionals  $M_{\varepsilon}^{\alpha}$   $\Gamma$ -converge to  $cM^{\alpha}$ , with respect to the convergence of  $\mathcal{M}(\Omega)$ , as  $\varepsilon \to 0$  ( $c = c(\alpha)$  is a finite and positive constant). For suitable smooting  $\mu_{\varepsilon} \rightharpoonup \mu$ ,  $\nu_{\varepsilon} \rightharpoonup \nu$ , the limits of  $\operatorname{argmin}\{M_{\varepsilon}^{\alpha}(u): \nabla \cdot u = \mu_{\varepsilon} - \nu_{\varepsilon}\}$  are minimizers for  $M^{\alpha}(u)$  under the constraint  $\nabla \cdot u = \mu - \nu$ .

 $\rm F.S.$  A Modica-Mortola approximation for branched transport, CRAS, 2010  $\rm A.$  MONTEIL Uniform estimates for a Modica-Mortola type approximation of branched transportation, ESAIM COCV, 2017.

#### Numerics

# Finding "good" local minima: networks and shapes

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# Idea of the numerical method

The exact identification of global optimal networks in the combinatorial context is NP hard (with respect to the number of sources and targets). The method based on the elliptic approximation is, on the contrary, purely continuous : it requires to find a vector field on the whole  $\Omega$  and is not influenced by this number. The main difficulties are related both to the approximation of singular and irregular functions and to the strongly non-convex cost functional.

**Idea :** (by E. Oudet, who already used this approach for other problems admitting  $\Gamma$ -convergence approximations) observe that for  $\varepsilon \gg 1$  the functional  $M_{\varepsilon}^{\alpha}$  is close from being convex. Hence we perform a gradient descent on  $M_{\varepsilon}^{\alpha}$  for  $\varepsilon$  large. Then, decreasing the value of  $\varepsilon$  step by step, we start a gradient descent for  $M_{\varepsilon_{k+1}}^{\alpha}$  starting from the  $u_{\varepsilon_k}$  found at the previous step.

**Warning:** No guarantee of convergence to a true minimizer, we only expect to select a "good" local minimum.

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# Thick optimal networks

Run for each  $\varepsilon$  a projected gradient algorithm (projecting on the constraint  $\nabla \cdot u = f_{\varepsilon}$ ), and reduce  $\varepsilon$  step by step:



E. OUDET AND F. SANTAMBROGIO, A Modica-Mortola approximation for branched transport and applications, *Arch. Rati. Mech. An.*, 2011.

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## Looking for optimal shapes

How to adapt this approach when  $\mu$  is not fixed? Let  $\nu_\varepsilon$  be a suitable approximation of  $\delta_0,$  and solve

$$\min\left\{M^{\alpha}_{\varepsilon}(u) \,:\, 0 \leq \nabla \cdot u + \nu_{\varepsilon} \leq 1\right\}.$$

The main difference concerns the projection step. For fixed  $f = \mu_{\varepsilon} - \nu_{\varepsilon}$  we needed to solve

$$\min\left\{\int \frac{1}{2}|u-u_0|^2 : \nabla \cdot u = f\right\}$$

which, by duality became

$$\max\left\{-\int \frac{1}{2}|\nabla \varphi|^2 - \varphi(f - \nabla \cdot u_0)\right\}$$

and just required to solve a Laplacian. Now, we must instead solve

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Theory: Branched Transport, landscape, shape optimization Approximation Numerics

# A fractal shape



New computations are in progress. This one was obtained by solving the nonsmooth optimization problem in the projection step by a FISTA method (with very small gradient step).

Numerical computations done by P. Pegon. A collaboration with E. Qudet is in progress

... the end...

thanks for your attention.

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