# “Exact” semi-discrete approaches to moment measures, affine spheres, gradient flows and variational MFG

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Optimal Transportation and Applications
Pisa, November 13, 2018

Maud, two weeks old today

# Outline

1. Moment measures
   - Optimal transport formulation for moment measures
   - Optimization among convex functions
   - Semi-discrete optimization and discrete Brascamp-Lieb inequality
2. Variant: $q$-moment measures
   - Optimal transport formulation
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   - Applications to variational Mean Field Games
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# Moment Measures

**Definition** A probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ is said to be the moment measure of a convex function $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ if $\mu = (\nabla\psi)_\#(e^{-\psi}\mathrm{d}x)$.

**Applications** Kähler geometry (given a convex function $\psi$ one can construct a Kahler manifold $X_\psi$ whose Ricci tensor is half of the metric tensor iff the moment measure of $\psi$ is uniform on a convex set), Stein kernels (writing $\mu$ as a moment measure of $\psi$ allows useful change-of-variables)...

**Natural questions** given $\mu$, can we find $\psi$? is it unique? which measures $\mu$ are moment measurs of suitable functions $\psi$? to which class of convex function should we restrict?

Note that if $\psi$ is nice and the log-concave density $e^{-\psi}$ is integrable, then $\mu$ is *non-degenerate* (spt $\mu$ is full-dimensional), and $m_1(\mu) := \int |y|\mathrm{d}\mu(y) < \infty$; moreover, $\int y\mathrm{d}\mu(y) = \int \nabla\psi(x)e^{-\psi(x)}\mathrm{d}x = -\int \nabla(e^{-\psi}) = 0$.


D. Cordero-Erausquin, B. Klartag, Moment Measures, *J. Funct. An.* 2015

R. Berman, B. Berndtsson, Real Monge-Ampere equations and Kähler-Ricci solitons on toric log Fano varieties, *Ann. Fac. Sci. Toulouse*, 2013

S. K. Donaldson, Kahler geometry on toric manifolds, and some other manifolds with large symmetry, 2008.

M. Fathi, Stein kernels and moment maps, *Ann. Prob.*, to appear.

# An optimal transport interpretation

Cordero-Erausquin and Klartag proposed a variational formulation to find $\psi$ given $\mu$. This can also be done using optimal transport.

We look for a log-concave density $\rho$ such that the Brenier map $\nabla\psi$ from $\rho$ to $\mu$ satisfies $\psi + \log\rho = 0$. This is an optimality condition for

$$\min_{\rho} \quad \mathcal{E}(\rho) + \mathcal{T}(\rho, \mu)$$

where $\mathcal{E}(\rho) = \int \rho \log \rho \, \mathrm{d}x$ and $\mathcal{T}$ is the *maximal correlation* transport cost

$$\begin{aligned}
\mathcal{T}(\rho,\mu) &= \max\{\mathbb{E}[X \cdot Y] \, : \, X \sim \rho, Y \sim \mu\} \\
&= \max\left\{\int x \cdot y \, \mathrm{d}\gamma(x,y) \, : \, \gamma \text{ transp. plan from } \rho \text{ to } \mu\right\} \\
&= \min\left\{\int \psi \mathrm{d}\rho + \int \psi^* \mathrm{d}\mu \, : \, \psi \text{ convex}\right\}
\end{aligned}$$

F. Santambrogio, Dealing with moment measures via entropy and optimal transport, *J. Funct. An.* 2016

# Bounded and unbounded domains, continuity of $\psi$

Fix a convex compact set $\Omega \subset \mathbb{R}^d$, and solve $\min_{\rho \in \mathcal{P}(\Omega)} \mathcal{E}(\rho) + \mathcal{T}(\rho, \mu)$. A solution exists, and $\mu$ is a moment measure... with no condition on $\mu$!

Yet, we have $\rho > 0$ on $\Omega$, and $\psi$ is bounded on $\Omega$, $\psi = +\infty$ on $\mathbb{R}^d \setminus \Omega$. The discontinuity of $\psi$ is the issue (note that if $\operatorname{spt} \mu$ is compact then $\psi$ is expected to be Lipschitz).

**Definition** $\psi$ is said to be *essentially continuous* if for $\mathcal{H}^{d-1}$-a.e. point $x \in \partial\{\psi < +\infty\}$ we have $\lim_{x' \to x} \psi(x') = +\infty$.

**Result** if $\rho$ solves $\min_{\rho \in \mathcal{P}_1(\mathbb{R}^d)} \mathcal{E}(\rho) + \mathcal{T}(\rho, \mu)$ then $\rho = e^{-\psi}$, $\nabla\psi$ is the Brenier map from $\rho$ to $\mu$, and $\psi$ is essentially continuous (i.e. $\rho \to 0$ a.e. on the boundary of $\operatorname{spt} \rho$).

# Bounds on $\mathcal{E}$ and $\mathcal{T}$, existence

Using $t\log t + e^{s-1} \geq st$ we get

$$\mathcal{E}(\rho) \geq \int \rho(x)h(x)\,\mathrm{d}x - \int e^{h(x)-1}\mathrm{d}x$$

for any function $h$ such that $e^{h-1} \in L^1(\mathbb{R}^d)$ and $h \in L^1(\rho)$. Using $h = -\sqrt{|x|}$ we get

$$\mathcal{E}(\rho) \geq -\int \rho(x)\sqrt{|x|}\mathrm{d}x - \int e^{-\sqrt{|x|}-1}\mathrm{d}x \geq -C - \sqrt{m_1(\rho)}.$$

For $\mathcal{T}$, if we suppose that $\mu$ is centered ($\int y\mathrm{d}\mu(y) = 0$) and non-degenerate, then there exists $c > 0$ such that

$$\mathcal{T}(\rho,\mu) \geq cm_1(\rho) \quad \text{for all } \rho \text{ with } \int x\mathrm{d}\rho(x) = 0.$$

Minimizing sequences $(\rho_n)_n$ can be supposed to be also centered if $\mu$ is. Then we can apply the above bounds, get tightness for minizing sequences, and prove existence of a minimizer.

# Geodesic convexity, uniqueness and characterization

Is the optimal $\rho$ unique? is the condition "$\rho = e^{-\psi}$, with $\mu$ moment measure of $\psi$" equivalent to being optimal?
This requires some convexity, but $\rho \mapsto \mathcal{T}(\rho, \mu)$ is concave!

Informally (we are not supposing $\rho, \mu \in \mathcal{P}_2(\mathbb{R}^d)$) we have

$$\mathcal{T}(\rho, \mu) = \int \frac{1}{2}|x|^2 \mathrm{d}\rho(x) + \int \frac{1}{2}|y|^2 \mathrm{d}\mu(y) - \frac{1}{2} W_2^2(\rho, \mu),$$

and we know that $\rho \mapsto \int \frac{1}{2}|x|^2 \mathrm{d}\rho(x)$ is 1-geodesically convex in $W_2$, while $\rho \mapsto \frac{1}{2} W_2^2(\rho, \mu)$ is 1-geodesically concave.
And $\mathcal{E}$ is geodesically convex thanks to McCann's condition.

We can then prove

**Uniqueness and characterization** If $\mu$ is centered and not supported on a hyperplane, the solution of $\min_{\rho \in \mathcal{P}_1(\mathbb{R}^d)} \mathcal{E}(\rho) + \mathcal{T}(\rho, \mu)$ exists and is unique up to translations. Moreover, being a solution of this optimization problem is equivalent to being of the form $\rho = e^{-\psi}$, with $\mu$ moment measure of $\psi$.

R. J. McCann, A convexity principle for interacting gases, *Adv. Math.* 1997

# Optimization among convex functions

We are looking at a min-min problem

$$\min\left\{\mathcal{E}(\rho)+\int\psi(x)\rho(x)\,\mathrm{d}x+\int\psi^*(y)\,\mathrm{d}\mu(y)\;:\;\rho\in\mathcal{P}_1(\mathbb{R}^d),\psi\text{ convex}\right\}.$$

If we consider $\psi$ as a secondary variable, for every $\rho$ we take $\min_\psi$, we obtain $\mathcal{T}(\rho,\mu)$. If instead we first compute $\min_\rho$, we have to compute

$$\inf\left\{\int\rho(x)\log\rho(x)\,\mathrm{d}x+\int\psi(x)\rho(x)\,\mathrm{d}x\;:\;\rho\geq 0,\int\rho(x)\,\mathrm{d}x=1\right\}.$$

Setting $c=\int e^{-\psi(x)}\mathrm{d}x$, the above inf equals $-\log c$. Hence we need to solve

$$\min\left\{\int\psi^*\,\mathrm{d}\mu-\log\left(\int e^{-\psi}\mathrm{d}x\right)\;:\;\psi\text{ convex}\right\}.$$

The variational approach by Cordero-Erausquin and Klartag considers the same problem in terms of $u=\psi^*$: the first term becomes linear and the second, magically, convex (thanks to Prékopa-Leindler inequality).

# Semi-discrete approach to the JKO scheme

In the JKO scheme for gradient flows in $W_2$ we solve problems of the form

$$\min_\rho \quad \frac{1}{2} W_2^2(\rho,\mu) + \mathcal{F}(\rho),$$

where $\mathcal{F}(\rho) = \int f(\rho(x))\mathrm{d}x$. This can be formulated as

$$\min_{u \text{ convex} : \nabla u \in \Omega} \quad \frac{1}{2}\int_\Omega |\nabla u(x) - x|^2 \mathrm{d}\mu(x) + \mathcal{F}((\nabla u)_{\#}\mu).$$

Suppose that $\mu = \sum_j a_j \delta_{x_j}$ is discrete and look for a convex function on its support $S := \{x_j\}_j$. Require that its *subdifferential*

$$\partial u(x) := \{p \in \mathbb{R}^d \ : \ u(x) + p \cdot (y - x) \le u(y) \text{ for all } y \in S\}$$

is non-empty. Any element of $\partial u(x)$ can play the role of $\nabla u(x)$, but, how to define $F$? Spreading the mass $a_j$ uniformly on $\partial u(x_j)$ one obtains a functional (convex if $f$ satisfies McCann's condition)

$$u \mapsto \sum_j f\left(\frac{a_j}{|\partial u(x_j) \cap \Omega|}\right) |\partial u(x_j) \cap \Omega|,$$

which depends on the areas of the subdifferential cells.

J.-D. Benamou, G. Carlier, Q. Mérigot, É. Oudet Discretization of functionals involving the Monge-Ampère operator, *Num. Math.*, 2016.

# Voronoi and Laguerre cells, and semi-discrete OT

Given some points $(x_j)_j$, their Voronoi cells $V_j$ are defined by

$$V_j := \left\{x \in \Omega \,:\, \frac{1}{2}|x - x_j|^2 \le \frac{1}{2}|x - x_{j'}|^2 \text{ for all } j'\right\}.$$

These cells $V_j$ are convex polyhedra. In optimal transport a variant of these cells is useful: given values $\phi_j$, we look for the *Laguerre cells*

$$W_j := \left\{x \in \Omega \,:\, \frac{1}{2}|x - x_j|^2 - \phi_j \le \frac{1}{2}|x - x_{j'}|^2 - \phi_{j'} \text{ for all } j'\right\}.$$

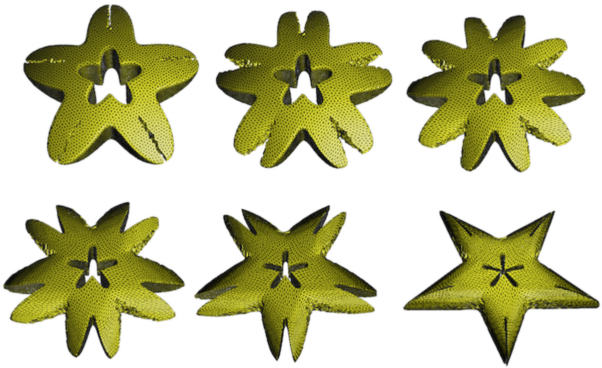
Note that the Laguerre cells corresponding to $\phi_j := -u(x_j) + \frac{1}{2}|x_j|^2$ are nothing but the subdifferentials of $u$.
If $\rho$ is a density on $\Omega$ and $\mu = \sum_j a_j \delta_{x_j}$, then finding an optimal transport from $\rho$ to $\mu$ is equivalent to finding $\phi_j$ such that $\rho(W_j) = a_j$ for every $j$.

This amounts to solving the dual Kantorovich problemin terms of the finite-dimensional variable $\phi$:

$$\max_{\phi} \sum_j a_j \phi_j + \int_\Omega \phi^c \mathrm{d}\rho := \sum_j \left(a_j \phi_j + \int_{W_j} \left(\frac{1}{2}|x - x_j|^2 - \phi_j\right) \mathrm{d}\rho(x)\right).$$

# Algorithmic geometry and semi-discrete numerics for OT

Algorithmic geometry is now able to deal with Laguerre cells of many ($10^9$) points in 2D and 3D, compute their measures, and differentiate this in terms of the values of $\psi_j$.



$W_2$ geodesic between two different stars in 3D, computed via semi-discrete optimal transport ($10^6$ Dirac masses to a $10^6$ simplex triangulation).

Q. Mérigot, A multiscale approach to optimal transport, *Comput. Graph. Forum*, 2011.
B. Lévy A numerical algorithm for $L^2$ semi-discrete optimal transport in 3D, *M2AN*, 2015.

# A better semi-discrete approach for moment measures

The natural counterpart of the BCMO scheme to solve the moment measure optimization problem would be discretize $\mu$ as $\sum_j a_j\delta_{x_j}$ and minimize

$$(u,P)\mapsto\sum_j x_jP_j+a_j\log\left(\frac{a_j}{|\partial u(x_j)\cap\Omega|}\right),\quad P_j\in\partial u(x_j).$$

**Main difficulty:** we need to use $\Omega=\mathbb{R}^d$, and the method does not work. Instead, it is possible to solve

$$\min\int u\,\mathrm{d}\mu-\log\left(\int e^{-u^*}\mathrm{d}x\right):=\sum_j a_ju_j-\log\left(\sum_j\int_{\partial u(x_j)}e^{-x\cdot x_j+u_j}\mathrm{d}x\right)$$

Let us set $F(u)=\sum_j\int_{\partial u(x_j)}e^{-x\cdot x_j+u_j}\,\mathrm{d}x$ and $G=-\log F$.
We will solve by convex optimization methods $\min <a,u> +G(u)$, or, equivalently, $\min <a,u>:\ F(u)=1$, via Newton's method.

# Derivatives and Hessian

We can compute the derivatives of $F$ and $G$.

$$\partial_j F = \int_{\partial u(x_j)} e^{-u^*} \mathrm{d}x; \quad \text{for } i \neq j \quad \partial_{ij} F = \int_{\Gamma_{ij}} e^{-u^*} \frac{1}{|x_i - x_j|} \mathrm{d}\mathcal{H}^{d-1},$$

where $\Gamma_{ij} = \partial u(x_j) \cap \partial u(x_i)$ (note $i \sim j$ if $\Gamma_{ij} \neq \emptyset$), and

$$\partial_{jj} F = \int_{\partial u(x_j)} e^{-u^*} \mathrm{d}x - \sum_{i: i \sim j, i \neq j} \int_{\Gamma_{ij}} e^{-u^*} \frac{1}{|x_i - x_j|} \mathrm{d}\mathcal{H}^{d-1}.$$

Set $\lambda_j := \int_{\partial u(x_j)} e^{-u^*} \mathrm{d}x$ and $\gamma_{ij} := \int_{\Gamma_{ij}} e^{-u^*} \frac{1}{|x_i - x_j|} \mathrm{d}\mathcal{H}^{d-1}$. Note $F = \sum_i \lambda_i$.

$$\partial_{ij} G = -\frac{\partial_{ij} F}{F} + \frac{\partial_i F \partial_j F}{F^2}.$$

Take $v \in \mathbb{R}^N$ and compute (tedious computations + manipulations)

$$\sum_{ij} (\partial_{ij} G) v_i v_j = \frac{1}{F} \sum_{ij: i \sim j} \gamma_{ij} |v_i - v_j|^2 - \frac{1}{F} \sum_j \lambda_j |v_j - \bar{v}|^2,$$

where $\bar{v} = \sum_i \lambda_i v_i / F$.

# Discrete Brascamp-Lieb inequality

For the convergence of the Newton method, we would like uniform convexity of $G$, at least in the orthogonal space to the trivial invariance direction (where $u_j$ is an affine function of $x_j$).

**Brascamp-Lieb inequality**: for $\psi$ convex and $\int ve^{-\psi} = 0$, we have

$$\int |v|^2 e^{-\psi} \le \int (D^2\psi)^{-1}(\nabla v, \nabla v)e^{-\psi}$$

with equality if and only if $v = \hat{w} \cdot \nabla\psi$ for a fixed $\hat{w}$.

**Discrete counterpart:** take $v = v_j$ on $\partial u(x_j)$ and $\psi = u^*$, and suppose $\sum_j \lambda_j v_j = 0$. Then

$$\sum_j \lambda_j |v_j|^2 \le \sum_{ij:i\sim j} \int_{\Gamma_{ij}} e^{-\psi} \frac{1}{|x_i - x_j|} |v_i - v_j|^2 = \sum_{ij:i\sim j} \gamma_{ij} |v_i - v_j|^2,$$

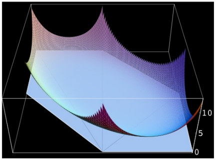
with equality in the same case, i.e. when $v_j$ is a linear function of $u_j$. Applied to $v - \overline{v}$, this gives the desired lower bound on $D^2G$.

H. J. Brascamp, E. H. Lieb, On extensions of the Brunn-Minkowski and Prékopa- Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, *J. Funct. Anal.*, 1976.

Work in progress with B. Klartag and Q. Mérigot

# Numerics for moment measures

Few numerical studies exist for moment measures, in particular when $\mu$ is the uniform measure on a hexagon or a regular polygon in 2D (including work by Donaldson and other geometers).




The techniques coming from semi-discrete OT make this challenge almost trivial: we discretize $\mu$ as an atomic measure, keeping it centered, and non-supported on a hyperplane. Then we apply a damped Newton algorithm to the minimization of $G$.

Convergence along Newton iterations are guaranteed by uniform convexity and we can consider that the solution is "exact". Convergence as the discetization of $\mu$ becomes finer can be proven by Γ-convergence.

C. Doran, M. Headrick, C. P. Herzog, J. Kantor, T. Wiseman Numerical Kaehler-Einstein metric on the third del Pezzo, *Comm. Math. Phys.* 2008.
R. S. Bunch, S. K. Donaldson Numerical approximations to extremal metrics on toric surfaces, *Handbook Geom. An.*, 2008.
J. Kitagawa, Q. Mérigot, B. Thibert Convergence of a Newton algorithm for semi-discrete optimal transport *JEMS*, to appear.

**Variant** Given $q > 0$, a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ is said to be the $q$–moment measure of a convex function $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ if we have $\mu = (\nabla\psi)_\#(\frac{1}{\psi(x)^{d+q}}\mathrm{d}x)$.

The condition $q > 0$ guarantees integrability of $\psi^{-(d+q)}$. We consider now

$$\min_{\rho} \quad \mathcal{E}_\alpha(\rho) + \mathcal{T}(\rho,\mu)$$

where $\mathcal{E}_\alpha(\rho) = -\int \frac{1}{\alpha}(\rho^{ac})^\alpha \mathrm{d}x$, $\alpha = 1 - \frac{1}{d+q} \in (0,1)$, and $\rho^{ac}$ is the density of the absolutely continuous part of $\rho$.
$\mathcal{E}_\alpha$ is known to be l.s.c. for the weak-* convergence as soon as $\alpha \in (0,1)$. It is geodesically convex for $\alpha > 1 - 1/d$ (i.e. $q > 0$). For lower bounds use

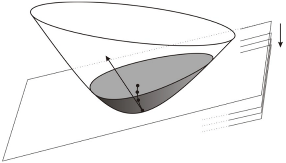
$$-\frac{1}{\alpha}t^\alpha + \left(\frac{1}{\alpha} - 1\right)(s_-)^{-\frac{\alpha}{1-\alpha}} \geq ts.$$

This provides $E_\alpha(\rho) \geq -C - Cm_1(\rho)^\delta$ but $\delta > d/(d+q-1)$ and we need $q > 1$ for tightness and existence.

**Result** If $q > 1$ and $\mu$ is centered and non-degenerate there exists unique (up to translations) an optimal $\rho$, which is a.c. and bounded from above, and is of the form $\rho = \psi^{-(d+q)}$, where $\nabla\psi$ is the Brenier map from $\rho$ to $\mu$.

# Connections with affine spheres

**Affine normals** Given a smooth strictly convex body $\Omega$ and $x_0 \in \partial\Omega$, take $\Pi_0$ the tangent plane to $\partial\Omega$ at $x_0$, $(\Pi_t)_t$ a family of parallel plans to $\Pi_0$, $\Omega_t := \Omega \cap \Pi_t$ and $z(t)$ the barycenter of $\Omega_t$. The tangent direction at $t = 0$ to the curve $t \mapsto z(t)$ is called affine normal to $\Omega$ at $x_0$.



**Affine spheres** An affine sphere is a convex body where the affine normal vectors at each point of the boundary point towards a same common point. If $\partial\Omega$ is parameterized as a graph of a function, being an affine sphere is a 3rd order PDE on this function.

**PDE characterization** Take a convex function $\psi > 0$ on $\mathbb{R}^d$ and define $n : \mathbb{R}_+ \times \mathbb{R}^d$ via $n(1, x) = \psi(x)$ and then extended by 1-homogeneity. Take the set $B := \{n \leq 1\}$ and let $K$ be the intersection $\partial B \cap (\{0\} \times \mathbb{R}^d)$. Then $B$ is an affine hemisphere centered at the origin if and only if $\psi$ solves $\psi^{d+2} \det(D^2\psi) = c$, i.e. if the 2-moment measure of $\psi$ is the uniform measure on the convex set $K^o$.

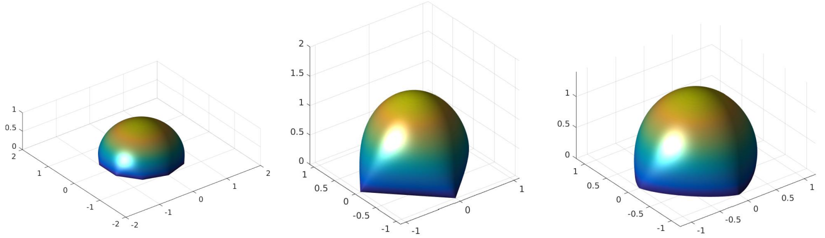
B. Klartag Affine hemispheres of elliptic type, *Algebra i Analiz*, 2017

# Semidiscrete optimization and numerical results

With the same discretization, we now need now to solve

$$\min < a, u > - \frac{1}{d+q-1} \int \frac{1}{(u^*)^{d+q-1}} \;:\; \int \frac{1}{(u^*)^{d+q}} = 1.$$

The estimates for the uniform convexity are now based on a discrete version of the *dimensional Brascamp-Lieb* inequality.
The Newton approach is also efficient and allows for fast computations.



Few affine hemispheres in $\mathbb{R}^3$ based on different convex sets in $\mathbb{R}^2$.

V. H. Nguyen Dimensional variance inequalities of Brascamp-Lieb type and a local approach to dimensional Prékopa's theorem. *J. Funct. An.* 2014

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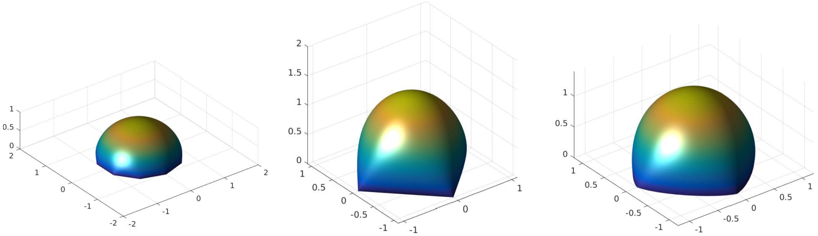

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V. H. NGUYEN Dimensional variance inequalities of Brascamp-Lieb type and a local approach to dimensional Prékopa's theorem. *J. Funct. An.* 2014

# Moreau-Yosida regularization of internal energies

**Definition** On a Hilbert space $X$, given $H : X \to \mathbb{R} \cup \{+\infty\}$, its Moreau-Yosida regularization $H_\varepsilon$ is defined as $H_\varepsilon(x) := \inf_y \frac{|x-y|^2}{2\varepsilon} + H(y)$. It is always semi-concave, and convex if $H$ is convex. If $H$ is l.s.c., we have $H_\varepsilon \to H$ as $\varepsilon \to 0$, with $\Gamma$-convergence.

In the Wasserstein space, we could take $\mathcal{F} : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ and define

$$\mathcal{F}_\varepsilon(\mu) := \min_\rho \frac{W_2^2(\mu,\rho)}{2\varepsilon} + \mathcal{F}(\rho).$$

Suppose $\mu = \sum_j a_j \delta_{x_j}$ and $\mathcal{F}(\rho) = \int f(\rho(x))\mathrm{d}x$. Using the transport cost $c(x,y) = |x-y|^2/2$, the dual of the above optimization problem is

$$\max_\phi \int \frac{\phi}{2\varepsilon}\mathrm{d}\mu - \int f^*\left(-\frac{\phi^c(x)}{2\varepsilon}\right)\mathrm{d}x.$$

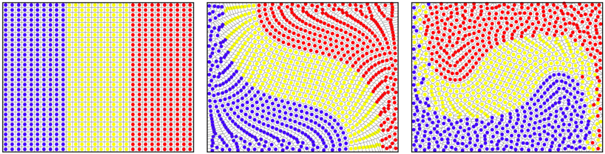
Setting $\frac{|x|^2}{2} - \phi(x) = u(x)$ and $\frac{|y|^2}{2} - \phi^c(y) = u^*(y)$, the above problem is very similar to the previous ones (only, $\phi^c$ is not affine on each cell, but quadratic). Algorithmic geometry can easily find the optimal $\rho$ given $\mu$, and differentiate $\mathcal{F}_\varepsilon$ in terms of the positions and masses of the atoms of $\mu$.

# Incompressible Euler equation

Euler equations for incompressible fluids $\partial_t v + (v \cdot \nabla) v = \nabla p, \nabla \cdot v = 0$ are known to represent geodesics among measure-preserving diffeomorphism. In the Brenier's variational interpretation, they can be obtained via

$$\min \int_C K(\omega) \mathrm{d}Q(\omega) \;:\; (e_t)_\# Q = \mathrm{d}x, (e_0, e_1)_\# Q = \overline{\gamma} \in \mathcal{P}(\Omega \times \Omega),$$

where $C$ is the space of curves $\omega : [0,1] \to \Omega$ and $K(\omega) := \int |\omega'(t)|^2/2 \mathrm{d}t$. The condition $(e_t)_\# Q = \mathrm{d}x$ can be replaced, as a variational approximation, by adding $\mathcal{F}_\varepsilon((e_t)_\# Q)$ where $\mathcal{F}(\rho) = 0$ if $\rho = \mathrm{d}x$, $+\infty$ if not.




Y. Brenier The least action principle and the related concept of generalized flows for incompressible perfect fluids, *J. Amer. Math. Soc.* 1989.

Q. Mérigot, J.-M. Mirebeau Minimal geodesics along volume preserving maps, through semi-discrete optimal transport *SINUM* 2016.

T. Gallouët, Q. Mérigot A Lagrangian scheme à la Brenier for the incompressible Euler equations *Foundation of Computational Mathematics*, to appear

# Variational MFG with local congestion costs - 1

In a population of agents everybody chooses its own trajectory, solving

$$\min \int_0^T \left(\frac{|x'(t)|^2}{2} + g(\rho_t(x(t)))\right) \mathrm{d}t + \Psi(x(T)),$$

with given initial point $x(0)$; suppose that $g$ is increasing, i.e. agents try to avoid overcrowded regions. Yet, $\rho$ depends on their overall choices, and hence we look for a Nash equilibrum. If $\varphi$ is their value function we have

$$\begin{cases} -\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} = g(\rho), \\ \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0, \\ \varphi(T,x) = \Psi(x), \quad \rho(0,x) = \rho_0(x). \end{cases}$$


J.-M. Lasry, P.-L. Lions, Jeux à champ moyen. (I & II) *C. R. Math. Acad. Sci. Paris,* 2006, Mean-Field Games, *Japan. J. Math.* 2007.

M. Huang, R.P. Malhamé, P.E. Caines, Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle, *Comm. Info. Syst.* 2006

P.-L. Lions, courses at Collège de France, 2006/12, videos available on the web

P. Cardaliaguet, lecture notes, www.ceremade.dauphine.fr/~cardalia/

# Variational MFG with local congestion costs - 2

It happens that an equilibrium can be found by minimizing the (global) energy

$$\mathcal{A}(\rho, v) := \int_0^T \int_\Omega \left( \frac{1}{2} \rho_t |v_t|^2 + f(\rho_t) \right) + \int_\Omega \Psi \rho_T$$

where $f' = g$ under the constraint $\partial_t \rho + \nabla \cdot (\rho v) = 0$.
This can also be written in Lagrangian terms as

$$\min \left\{ \int_C K \mathrm{d}Q + \int_0^T \mathcal{F}((e_t)_\# Q) + \int_\Omega \Psi \mathrm{d}(e_T)_\# Q, \; Q \in \mathcal{P}(C), (e_0)_\# Q = \rho_0 \right\}.$$

or, for curves valued in the Wasserstein space, as

$$\min \left\{ \int_0^T \left( \frac{1}{2} |\rho'|(t)^2 + \mathcal{F}(\rho(t)) \right) \mathrm{d}t + \int_\Omega \Psi \mathrm{d}\rho_T \; : \; \rho(0) = \rho_0 \right\}.$$

P. Cardaliaguet, P.J. Graber. Mean field games systems of first order. *ESAIM: COCV*, 2015.
J.-D. Benamou, G. Carlier Augmented Lagrangian methods for transport optimization, Mean-Field Games and degenerate PDEs, *JOTA*, 2015.
J.-D. Benamou, G. Carlier, F. Santambrogio, Variational Mean Field Games, *Active Particles I*, 2016

# Lagrangian approximation of variational MFG

We can consider atomic measures on $C$, i.e. $Q = \sum_j a_j \delta_{\omega_j}$ where, for simplicity, $a_j = 1/N$ (but different choices could model *major players*).
Yet $(e_t)_\# Q = \sum_j a_j \delta_{\omega_j(t)}$ is atomic and $\mathcal{F} = +\infty$.

**Idea:** replacing $\mathcal{F}$ with $\mathcal{F}_\varepsilon$ the problem is perfectly posed on atomic measures, and Γ-converges (non-trivial details) to the problem with $\mathcal{F}$.
The corresponding optimization problem is non-convex, but provides Lagrangian trajectories for a MFG with an approximate congestion cost.

Computations are easy for $f(s) = s \log s$ (i.e. $g(s) = \log s$), where densities are Gaussians on each cells, and for the particular case $f(s) = 0$ for $s \in [0, 1]$, $= +\infty$ if not (density-costrained MFG: here densities are uniform on cells, and vanish outside, as in partial transport problems).

P. Cardaliaguet, A. Mészáros, F. Santambrogio, First order Mean Field Games with density constraints: Pressure equals Price, *SIAM J. Contr. Opt.,* 2016
Work in progress with Q. Mérigot and C. Sarrazin; Master and PhD thesis of C. Sarrazin.

# Lagrangian approximation of gradient flows with diffusion

The same idea (replacing $\mathcal{F}$ with $\mathcal{F}_\varepsilon$ ) can be used in gradient flows

$$\partial_t\rho - \nabla\cdot\left(\rho\nabla\left(\frac{\delta\mathcal{F}}{\delta\rho}\right)\right) - \nabla\cdot(\rho\nabla V) = 0,$$

in particular for local functionals $\mathcal{F}(\rho) = \int f(\rho(x))\mathrm{d}x$ which give raise to diffusion and would not be well-defined on atomic measures (note that we added a potential energy $\int V\mathrm{d}\rho$).

For instance, the Fokker-Planck equation $\partial_t\rho - \Delta\rho - \nabla\cdot(\rho\nabla V)$ uses $\mathcal{F} = \mathcal{E}$ and lets Gaussian appear in the Moreau-Yosida regularization.

The idea is looking for $\mu(t) = \sum_j a_j\delta_{\omega_j(t)}$, with

$$\omega_j'(t) = -\nabla V(\omega_j(t)) - \nabla_j\mathcal{F}_\varepsilon(\mu(t)).$$

Convergence proofs cannot only be based on Γ-convergence and are not trivial (or require extra assumptions), as usual when dealing with gradient flows (a control on the slope is needed).

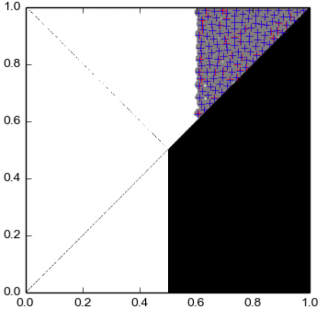
E. Sandier, S. Serfaty Gamma-convergence of gradient flows with applications to Ginzburg-Landau, *Comm. PDE*, 2004

# Numerical approximation of crowd motion

Using $\mathcal{F}(\rho) = 0$ if $\rho \leq 1$, $+\infty$ otherwise we obtain a crowd motion model of gradient-flow type.

$$\begin{cases} \partial_t \rho - \nabla \cdot (\rho(\nabla V + \nabla p)) \\ p \geq 0, \rho \leq 1, p(1-\rho) = 0. \end{cases}$$

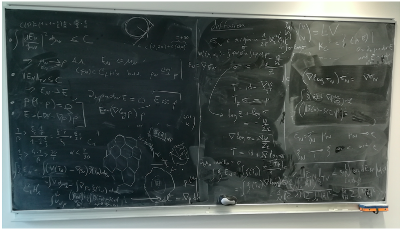
The Moreau-Yosida computation for the constraint part consists in partial transport. The gradient $\nabla_j \mathcal{F}_\varepsilon$ w.r.t. to the positions of the atoms equals $\frac{x_j - \beta_j}{\varepsilon}$, where $\beta_j$ is the barycenter of the cell corresponding to $x_j$.

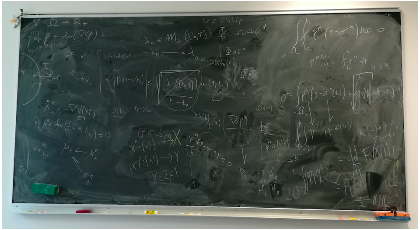




B. Maury, A. Roudneff-Chupin and F. Santambrogio, A macroscopic crowd motion model of gradient flow type, *Math. Mod. Meth. Appl. Sci.*, 2010


Work in progress with Q. Mérigot and F. Stra





## *The End*

Thanks for your attention

Few images have been stolen from various sources on the internet. I thank their authors.