Modeling and PDE questions in continuous Wardrop equilibria

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PDE models for multi-agent phenomena

Outline

1. Discrete models on networks
2. Continuous models on a traffic domain
3. Short-term problem: numerics
4. Long-term problem: very degenerate elliptic PDEs, weak flows, and Augmented Lagrangian
5. Capacity constraints: pressure and price

Network models

The idea

Suppose two very different roads connect two cities: a straight-line highway and a longer country path.
If everybody chooses the former, it will become congested and less performant than the latter. Hence everybody will change his mind and take the other one. And it will be even worse !

Is there an equilibrium?

The objects

- A finite graph with edges e, a set of sources S and destinations D,
- the set $C(s,d) = \{\omega \text{ from } s \text{ to } d\}$ of possible paths from s to d,
- a demand input $\overline{\gamma}(s,d)$ denoting the quantity of commuters going from $s \in S$ to $d \in D$,
- an unknown repartition strategy (to be looked for) $q = (q_\omega)_\omega$ such that $\sum_{\omega \in C(s,d)} q_\omega = \overline{\gamma}(s,d)$,
- a consequent traffic intensity (depending on q) $i_q = (i_q(e))_e$ given by $i_q(e) = \sum_{e \in \omega} q_\omega$,
- an increasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(i_q(e))$ represents the congestioned cost (per unit length) of the edge e,
- the cost for each path ω, given by $c_q(\omega) = \sum_{e \in \omega} g(i_q(e)) \,\text{length}(e)$,
- the induced distance on sources-destination pairs: $\mathrm{d}_q(s,d) := \min\{c_q(\omega) \,:\, \omega \in C(s,d)\}$.

Wardrop equilibria

The global strategy q represents the overall distribution of choices of commuters' paths. Imposing a **Nash equilibrium** condition (no single commuter wants to change his mind, provided all the others keep the same strategy) gives the following condition:

$$\omega \in C(s,d),\ q_\omega > 0 \Rightarrow c_q(\omega) = \mathrm{d}_q(s,d)\ (= \min\{c_q(\tilde\omega)\ :\ \tilde\omega \in C(s,d)\}).$$

This condition is well-known among economists and engineers as Wardrop equilibrium.

The existence of at least an equilibrium comes from the following variational principle.

J. G. Wardrop, Some theoretical aspects of road traffic research, *Proc. Inst. Civ. Eng.*, 1952.

Variational principle

Optimizing an overall congestion cost means minimizing a quantity

$$\sum_e H(i_q(e)) \,\text{length}(e)$$

($H : \mathbb{R}^+ \to \mathbb{R}^+$ being an increasing function) among all possible strategies q (i.e. under the constraint $\gamma_q = \overline{\gamma}$, where $\gamma_q(s,d) := \sum_{\omega \in C(s,d)} q_\omega$).

Optimality conditions: **if q is optimal for a cost function H, then it is a Wardrop equilibrium** for $g = H'$. Hence, to get a Wardrop equilibrium it is sufficient to solve a convex optimization problem (where H will be the primitive of g).

This problem **does not** amount to minimizing the total cost of all commuters! indeed, the total cost is obtained by using $H(t) = tg(t)$ (only when $g(t) = t^p$ the two problems are equivalent). Hence, there is a cost of anarchy.

M. Beckmann, C. McGuire, C. Winsten *Studies in the Economics of Transportation*, 1956.

Short- and Long-term problems

Instead of fixing $\overline{\gamma}$, we can admit, in the minimization, all strategies q compatible with any $\gamma \in \Gamma$, where

$$\Gamma = \left\{ \gamma \,:\, \sum_d \gamma(s,d) = \mu(s),\, \sum_s \gamma(s,d) = \nu(d) \right\},$$

$\mu(s)$ being the quantity of commuters starting from the source s and $\nu(d)$ the number of those arriving at d.

In a **short-term problem** it is more natural to consider γ as given. On the contrary, in a **long-term problem** we can consider that μ and ν (typically residential and working areas) are more stable over time, and leave γ as an unknown.

When we minimize under the constraint $\gamma_q \in \Gamma$, we obtain as optimality conditions

- q is a Wardrop equlibrium for $g = H'$,
- γ_q solves $\min_{\gamma\in\Gamma} \sum_{s,d} \mathrm{d}_q(s,d)\gamma(s,d)$, which is an optimal transport problem from μ to ν for the cost given by d_q.

Continuous models

Continuous formulation with measures

In a domain $\Omega \subset \mathbb{R}^n$ the demand is represented by measures $\gamma \in \mathcal{P}(\Omega\times\Omega)$. We are given a set $\Gamma \subset \mathcal{P}(\Omega\times\Omega)$, the set of admissible demand couplings: either $\Gamma = \{\overline{\gamma}\}$ (short-term) or

$$\Gamma = \Pi(\mu, \nu) = \{\gamma \in \mathcal{P}(\Omega \times \Omega) \; : \; (\pi_X)_\# \gamma = \mu, \; (\pi_Y)_\# \gamma = \nu\}$$

(long term). Let us also set

$$C = \{\text{Lipschitz paths } \omega : [0,1] \to \Omega\}$$
$$C(s,d) = \{\omega \in C : \omega(0) = s,, \omega(1) = d\}.$$

We look for a probability $Q \in \mathcal{P}(C)$ such that $(\pi_{0,1})_\# Q \in \Gamma$ ($\pi_t : C \to \Omega$ is given by $\pi_t(\omega) = \omega(t)$ and $\pi_{t,s}(\omega) = (\omega(t), \omega(s))$): it can be expressed as $Q = Q^{s,d} \otimes \gamma$ with $Q^{s,d} \in \mathcal{P}(C(s,d))$ and $\gamma \in \Gamma$.
We want to define a traffic intensity $i_Q \in \mathcal{M}^+(\Omega)$ such that

$$i_Q(A) = \text{“how much ” the movement takes place in } A \ldots$$

Traffic intensity and overall congestion

For $\phi \in C^0(\Omega)$ and $\omega \in C$ set

$$L_\phi(\omega) = \int_0^1 \phi(\omega(t))|\omega'(t)|dt.$$

Define i_Q by

$$< i_Q, \phi >= \int_C L_\phi(\omega)dQ(\omega).$$

Optimization: we minimize the convex functional

$$F(i_Q) = \begin{cases} \int H(i_q(x))dx & \text{if } i_q \ll \mathcal{L}^n, \\ +\infty & \text{otherwise} \end{cases}$$

among all admissible strategies Q, H being a convex, increasing and superlinear function. Typically $H(t) = t^p$ or $H(t) \approx t^p$.

Optimality conditions

Let $\overline{Q}$ be a minimizer and set $\overline{\kappa} = H'(i_{\overline{Q}})$. For $\kappa \geq 0$ set

$$\mathrm{d}_\kappa(s, d) = \inf_{\omega \in C(s,d)} L_\kappa(\omega)$$

(it is the conformal Riemannian distance induced by κ; we will call it **weighted distance**).

- $\overline{\gamma}$ minimizes $\int \mathrm{d}_{\overline{\kappa}} d\gamma$ among $\gamma \in \Gamma$,
- $\overline{Q}$–a.e. $L_{\overline{\kappa}}(\omega) = \mathrm{d}_{\overline{\kappa}}(\omega(0), \omega(1))$.

Hence $\overline{\gamma}$ solves a Kantorovich **transport problem** and **almost any path is geodesic** for a cost κ depending on Q (i.e. Wardrop equilibrium with respect to H').

G. Carlier, C. Jimenez and F. Santambrogio, Optimal transportation with traffic congestion and Wardrop equilibria, *SIAM J. Control Optim.* 2008.

Why a continuous model?

Where to use a continuous model instead of a network one?
In crowd and **pedestrian motion**, for instance.
Or, as a **large scale limit** of car vehicle traffic models (if we avoid to simulate a very dense network traffic problem, we avoid the *curse of dimensionality* and wan have a rough idea of the most congested areas).

These problemas share the same flavor of **Mean Field Games**, with the only difference that the model is not really dynamic (time is only fictitious, and traffic intensity is considered as a function of x, not of (t, x)).

J.-M. Lasry, P.-L. Lions, Mean-Field Games, *Japan. J. Math.* 2007
P.-L. Lions, courses at Collège de France, 2006/12, videos available online
P. Cardaliaguet, lecture notes, www.ceremade.dauphine.fr/~cardalia/

. . . for the variational case, also see

J.-D. Benamou, G. Carlier, F. Santambrogio, Variational Mean Field Games, 2016

Some other remarks

- If $H(t) = t^p$, $p > 1$, then $g(0) = H'(0) = 0$: moving on a road with no traffic would cost zero (infinite speed, no fuel comsumption...). One should rather take $g(0) > 0$, for instance $H(t) = t + \frac{1}{p}t^p$ with $p > 1$.
- If we want $\min F(i_Q) < +\infty$, we need to find Q with $i_Q \in L^p$.
  - if $\Gamma = \Pi(\mu, \nu)$, we can use L^p estimates on the **transport density** which work for $\mu, \nu \in L^p$ (but the sharp assumption is $\mu - \nu \in W^{-1,p}$)
  - if $\Gamma = \{\overline{\gamma}\}$ and $\mu = \nu = \mathcal{L}^n$ Brenier's construction for Incompressible Euler gives $i_Q \in L^\infty$ (for general μ, ν, compose with diffeomorphisms);
  - if $\Gamma = \{\overline{\gamma}\}$ and μ and ν are discrete one can construct by hand a Q such that $i_Q(x) \approx |x - x_i|^{-1}$ near the atoms x_i, which implies $i_Q \in L^p$ for $p < n$.
- In general, $\bar{\kappa}$ is just measurable ($L^{p'}$) and $L_{\bar{\kappa}}(\omega)$ not well-defined. Solutions: work hard to **define $L_{\bar{\kappa}}$ and $\mathrm{d}_{\bar{\kappa}}$** (use C^0 bounds on d_κ when $\kappa \in L^{n+\varepsilon}$, or other ideas from Incompressible Euler in order to choose a special representative $\hat{\kappa}$), or prove smoothness (via PDEs).

L. De Pascale, L. C. Evans, A. Pratelli, Integral estimates for transport densities, *BLMS*, 2004
Y. Brenier, Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations, *Comm. Pure Appl. Math.*, 1999.
L. Ambrosio, A. Figalli, Geodesics in the space of measure-preserving maps and plans, *Arch. Rati. Mech. Anal.*, 2009.

The short-term problem

Short term – duality and numerics

$$(P) = \min_Q \int_\Omega H(i_Q); \quad (D) = \max_{\kappa \geq 0} - \int H^*(\kappa) + \int \mathrm{d}_\kappa d\overline{\gamma}.$$

$$\text{Duality:} \quad (P) = (D).$$

In the network case, the dual problem is usually used instead of the primal one for numerical purposes (simpler constraints, smaller dimension...).

We do the same in the continuous case. Then, from an optimal κ one can retrieve the corresponding traffic density by $H'(i_Q) = \kappa$.
This dual problem involves computing geodesic distances according to κ, i.e. viscosity solutions of the Eikonal equation $|\nabla \mathcal{U}| = \kappa$. We will solve

$$(DD) = \quad \min J(\kappa) = \sum_i H^*(\kappa_i) - \sum_{j,k} \overline{\gamma}(j,k) \mathcal{U}_{x_j;\kappa}(x_k)$$

where $\mathcal{U}_{x;\kappa}(y)$ is a discretized solution of $|\nabla U| = \kappa$ with $\mathcal{U}(x) = 0$, computed at y and we will compute $\mathcal{U}$ via the FMM algorithm.

F. Benmansour, G. Carlier, G. Peyré, F. Santambrogio, Numerical Approximation of Continuous Traffic Congestion Equilibria, *Net. Het. Media*, 2009.

Fast marching methods for weighted distances

Take $\kappa \geq 0$ defined on a square grid of size h. Set $\mathcal{U}(x_0) = 0$. Look for

$$(D_x\mathcal{U})^2_{i,j} + (D_y\mathcal{U})^2_{i,j} = h^2(\kappa_{i,j})^2,$$

where we denote

$$(D_x\mathcal{U})_{i,j} := \max\{(\mathcal{U}_{i,j} - \mathcal{U}_{i-1,j}), (\mathcal{U}_{i,j} - \mathcal{U}_{i+1,j}), 0\}/h,$$
$$(D_y\mathcal{U})_{i,j} := \max\{(\mathcal{U}_{i,j} - \mathcal{U}_{i,j-1}), (\mathcal{U}_{i,j} - \mathcal{U}_{i,j+1}), 0\}/h.$$

The **Fast Marching Method (FMM)** is a numerical method introduced by Sethian for efficiently solving this system (whose solution converges to $c_\kappa(\cdot, x_0)$, which is itself a solution of $|\nabla\mathcal{U}| = \kappa$). The numerical complexity of the FMM is $O(N\log(N))$ operations for a grid with N points.

E. Rouy, A. Tourin A viscosity solution approach to shape from shading. *SINUM* 1992.

J. A. Sethian *Level Set Methods and Fast Marching Methods.*, 1999.

Subgradients

The problem (DD) is convex ($\kappa \mapsto \mathcal{U}_{x,\kappa}(y)$ is concave) and we can solve it by a gradient method. We use a variant of the FMM which also computes, besides the value of $\mathcal{U}$, its derivatives w.r.t. perturbations in κ (essentially based on **automatic differentiation**) and costs $O(N^2 \log(N))$.

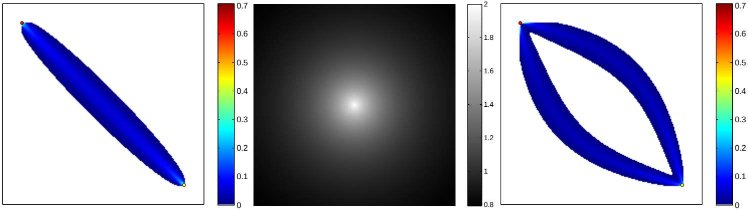

Figure: Examples of the subgradient computation. On the left, an element of $\partial^-_\kappa \mathcal{U}(\kappa)$ when κ is a constant metric; in the middle, a non constant (gaussian) metric κ ; on the right, an element of $\partial^-_\kappa \mathcal{U}(\kappa)$ for this κ.

F. Benmansour, G. Carlier, G. Peyré, F. Santambrogio, Derivatives with Respect to Metrics and Applications: Subgradient Marching Algorithm, *Num. Math., 2010.*

Traffic equilibria - 1

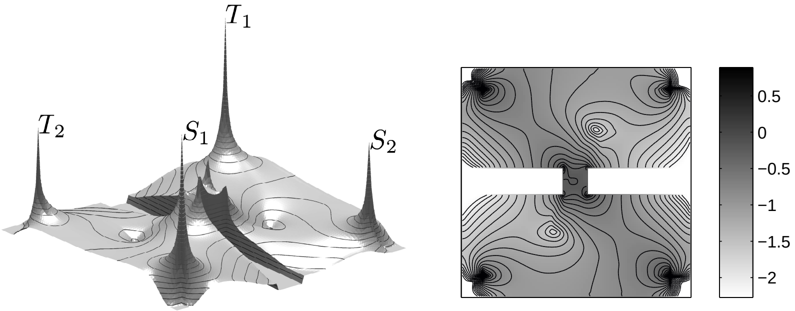

Figure: Two sources and two targets, with a river and a bridge on a symmetric configuration and an asymmetric traffic weights.

Traffic equilibria- 2

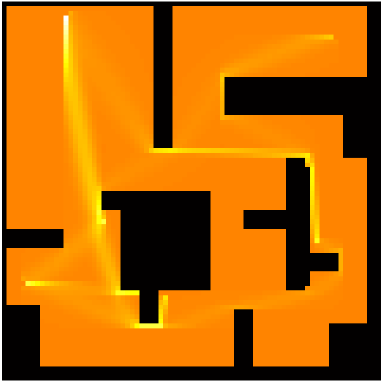

Figure: Running of the subgradient algorithm

Long-term problem: prescribed divergence, very degenerate PDEs, Augmented Lagrangian methods

Minimal flow problem

Let's define sort of a vector traffic density: let $v_Q \in \mathcal{M}^n(\Omega)$ be given by

$$< v_Q, \vec{\phi} >= \int_C \left(\int_0^1 \vec{\phi}(\omega(t)) \cdot \omega'(t)\, dt \right) dQ(\omega)$$

for all $\vec{\phi} \in C^0(\Omega; \mathbb{R}^n)$.
We can check $\nabla \cdot v_Q = (\pi_0)_\# Q - (\pi_1)_\# Q = \mu - \nu$. Moreover, $|v_Q| \leq i_Q$.

Question: can we replace i_Q with $|v_Q|$? can we minimize among all $v \in \mathcal{M}^n(\Omega)$ with $\nabla \cdot v = \mu - \nu$?

Take $\mathcal{H}(z) = H(|z|)$: let us consider

$$(VP): \qquad \min \int \mathcal{H}(v) \ : \ \nabla \cdot v = \mu - \nu.$$

This problem (only for $\Gamma = \Pi(\mu, \nu)$) is a congested variant of the Beckmann's formulation for the linear Monge problem: $\min \int |v| \ : \ \nabla \cdot v = \mu - \nu$.

M. Beckmann, A continuous model of transportation, *Econometrica*, 1952.

Equivalence

Obviously we have $(VP) \le (P)$. Take a minimizer $\overline{v}$ for (VP) and build a Q such that $i_Q \le |\overline{v}|$. This is possible, but we need to follow, in the spirit of Dacorogna-Moser's techniques, the integral curves of a vector field built from $\overline{v}$:

$$\begin{cases} \omega_x'(t) = \frac{\overline{v}(\omega_x(t)),}{\mu_t(\omega_x(t))} \\ \omega_x(0) = x. \end{cases} \qquad \mu_t(y) = (1-t)\mu(y) + t\nu(y),$$

Everything would be nice if $\overline{v}/\mu_t$ was Lipschitz. Anyway, by approximation, it is possible to obtain the following Smirnov-like result: for every v with $\nabla \cdot v = \mu - \nu$ there exists Q with $(\pi_0)_\# Q = \mu, (\pi_1)_\# Q = \nu$ and $i_Q \le |v|$.

B. Dacorogna, J. Moser, On a partial differential equation involving the Jacobian determinant. *Ann. Inst. H. Poincaré Anal. Non Lin.*, 1990

S. K. Smirnov, Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows, *Algebra i Analiz*, 1993.

F. Santambrogio, A Dacorogna-Moser approach to flow decomposition and minimal flow problems, *Proc. SMAI 2013*.

Need for regularity

Anyway, if we want a uniquely defined flow of curves Q, it would be desirable to have better regularity of $\overline{v}$. Less than Lipschitz could be enough, if weuse the **DiPerna-Lions theory**. We will suppose $\mu, \nu \geq c > 0$ and μ, ν Lipschitz; we need $\overline{v} \in W^{1,1} \cap L^\infty$.
$\overline{v}$ satisfies $\nabla\mathcal{H}(\overline{v}) = \nabla u$ (divergence-free perturbations).

$$\overline{v} = \nabla\mathcal{H}^*(\nabla u); \quad \nabla \cdot \nabla\mathcal{H}^*(\nabla u) = \mu - \nu.$$

If $H(t) = t^2$: **standard elliptic regularity**
If $H(t) = t^p$: $p'-$**Laplacian**
And what if $H(t) = t + \frac{1}{p}t^p$? in this case we have a very degenerate elliptic equation, since $\mathcal{H}^*(z) = 0$ for all $z \in B_1$.

R. J. DiPerna, P. L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Inv. Math.* 1989.

Regularity: here we are!

Regularity for a very degenerate elliptic equation:

$$\nabla \cdot G(\nabla u) = f$$

with $G(z) = (|z| - 1)_+^{p'-1} \frac{z}{|z|}$ and $f \in W^{1,p} : \Rightarrow G(\nabla u)^{p/2} \in H^1$.
For the proof: translation methods for the $p-$Laplacian, (similar to results about stress regularity by Carstensen-Müller, Fonseca-Fusco-Marcellini...).
For global estimates, higher regularity of $\partial\Omega$ is required ($C^{3,1}$).

Also:

$$\nabla \cdot G(\nabla u) = f$$

with $G = \nabla\mathcal{H}$, $D^2\mathcal{H}(z) \geq cI_n$ outside $B_2(0)$, $f \in L^{n+\varepsilon} \Rightarrow \nabla u \in L^\infty$.
The proof is based on Moser's iteration, and only large values of the gradient really matter. Once $\nabla u \in L^\infty$, the H^1 result is true also for $f \in BV$.

L. Brasco, G. Carlier, F. Santambrogio, Congested traffic dynamics, weak flows and very degenerate elliptic equations, *J. Math. Pures et Appl.*, 2010.

The H^1 regularity can also be obtained via duality methods:

F. Santambrogio, Regularity via duality in calculus of variations and degenerate elliptic PDEs, preprint

Even more: continuity!

Consider

$$\nabla \cdot G(\nabla u) = f$$

with $G = \nabla \mathcal{H}$, $\mathcal{H} \in C^2(\mathbb{R}^n)$, $D^2\mathcal{H}(z) \geq c_\delta I_n$ outside $B_{1+\delta}$, $f \in L^{2+\varepsilon}$.

Then, in dimension $n = 2$: $G(\nabla u) \in W^{1,2} \cap L^\infty \Rightarrow g(\nabla u) \in C^0$ for all $g \in C^0(\mathbb{R}^2)$ with $g = 0$ sur B_1. In particular $G(\nabla u) \in C^0$.

F. Santambrogio, V. Vespri, Continuity in two dimensions for a very degenerate elliptic equation, *Nonlinear Analysis*, 2010.

Later, this result has been (much) improved by Colombo and Figalli, who proved C^0 regularity in any dimension without apriori summability assumptions on $G(\nabla u)$, by proving $C^{1,\alpha}$ results on u outside $\{|\nabla u| < 1 + \delta\}$ (and generalizing to functions H degenerate on more general convex sets).

M. Colombo, A. Figalli, Regularity results for very degenerate elliptic equations, *J. Math. Pures Appl.* 2012.

Augmented Lagrangian

The formulation with prescribed divergence allows for easy duality results and for a numerical treatment in terms of Saddle Points: to solve

$$\min \int H(v) \;:\; \nabla \cdot v = f$$

we look for a saddle point of

$$L(v,(w,\phi)) := \int v \cdot (\nabla\phi + w) - H^*(w) + f\phi.$$

The saddle points are the same as those of

$$\tilde{L}(v,(w,\phi)) := \int v \cdot (\nabla\phi + w) - H^*(w) + f\phi - \frac{\tau}{2}|\nabla\phi + w|^2.$$

$\tilde{L}$ (the so-called *Augmented Lagrangian*) can be treated via alternate maximization in ϕ (solving $\tau\Delta\phi = \nabla \cdot v - f - \tau\nabla \cdot w$), in w (a strictly convex pointwise problem) and then a gradient update on v ($v \mapsto v - \tau(\nabla\phi + w)$), according to what is often called the **ALG2** method.

M. Fortin, R. Glowinski *Augmented Lagrangian methods*, 1983.
J.-D. Benamou, Y. Brenier A computational fluid mechanics solution to the Monge- Kantorovich mass transfer problem, *Numer. Math.*, 2000.

Congested flows via Augmented Lagrangian

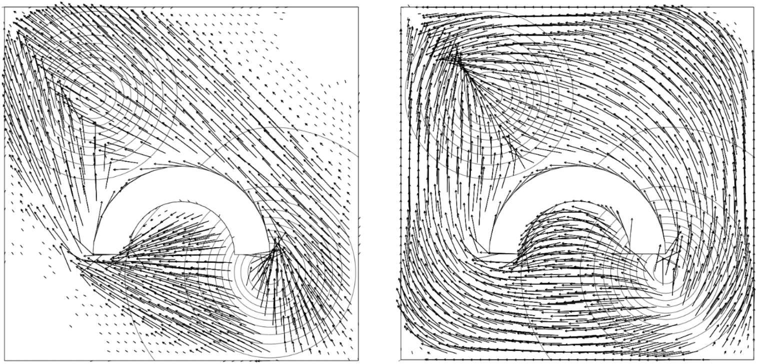

Figure: Congested flows with an obstacle, for $H(t) = t + \frac{1}{p}t^p$, for different values of p.

J.-D. Benamou, G. Carlier Augmented Lagrangian methods for transport optimization, Mean-Field Games and degenerate PDEs, *JOTA*, 2015

A variant: capacity constraints

Continuous Wardrop equilibria with capacity constraints

How to define an equilibrium if instead of penalizing traffic congestion via $g(i_Q)$ we want to impose a constraint $i_Q \leq 1$?
Naive idea: when i_Q is given, every agent minimizes his own cost paying attention to the constraint $i_Q \leq 1$. But if i_Q already satisfies $i_Q \leq 1$, one extra agent will not violate the constraint (it's a *non-atomic game*). Hence the constraint becomes empty.

Let us start, instead, from the variational formuation

$$\min\left\{\int i_Q\,dx \ : \ i_Q \leq 1\right\}$$

or

$$\min\left\{\int |v(x)|\,dx \ : \ \nabla\cdot v = \rho_0 - \rho_1,\ |v| \leq 1\right\}.$$

The constraint can be dualized, thus obtaining

$$\min_Q \sup_{p\geq 0} \int i_Q\,dx + \int p\, i_Q\,dx - \int p\,dx.$$

This introduces an extra term $p \geq 0$ in the optimality condition.

Pressure and Price

Open Problem Given $\overline{\gamma} \in \mathcal{P}(\Omega \times \Omega)$ (or - which is easier - given $\mu, \nu \in \mathcal{P}(\Omega)$), find $Q \in \mathcal{P}(C)$ and p smooth enough such that

- $(e_0, e_1)_\# Q = \overline{\gamma}$ (or $(e_0)_\# Q = \mu, (e_1)_\# Q = \nu$)
- $p \geq 0, \; i_Q \leq 1, \; p(1 - i_Q) = 0$
- Q-a.e. curve γ is geodesic for the distance d_κ with $\kappa = 1 + p$

The new term p plays the role of a *pressure* associated with the incompressibility constraint $i_Q \leq 1$, but is also a *price* to be paid to pass through saturated regions where $i_Q = 1$. One can see that $p = (\nabla u - 1)_+$ where u solves (in the long-term case)

$$\min \int (\nabla u - 1)_+ dx + \int u \, d(\mu - \nu).$$

Difficulty: the length L_κ is not well-defined, and here p will only be L^1 (or, even, worse, a positive measure). The Ambrosio-Figalli theory required some assumptions on the maximal function of κ, hence some regularity must be prove. The situation is similar to MFG with density constraints (but is not yet solved).

P. Cardaliaguet, A. Mészáros, F. Santambrogio, First order Mean Field Games with density constraints: Pressure equals Price, *SICON*, 2016

The End

Thanks for your attention