Microscopic and macroscopic modeling of passive and active crowds

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Outline

Part I

Passive Crowds: spontaneous velocity and density constraints

Non-overlapping disks

A particle population needs to move, and each particle - if alone - would follow its own velocity u (depending on time, position... a typical case is: $u = -\nabla D$, where $D(x) = \mathrm{dist}(x, \Gamma)$).

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If q is the particle configuration, we define $adm(q)$ the set of velocities that do not induce overlapping: if every particle is a disk with radius R, located at q_i, we have

$$q \in K := \{q = (q_i)_i \in \Omega^N \ : \ |q_i - q_j| \geq 2R\}$$

$$adm(q) = \left\{v = (v_i)_i \ : \ (v_i - v_j) \cdot (q_i - q_j) \geq 0 \ \forall (i,j) \ : \ |q_i - q_j| = 2R\right\}$$

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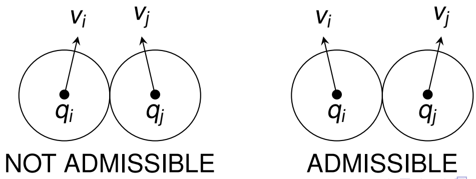

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$$\tilde{q}^{\tau}_{n+1} = q^{\tau}_{n} + \tau u(n\tau), \quad q^{\tau}_{n+1} = P_K[\tilde{q}^{\tau}_{n+1}]$$

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(for a small time step $\tau > 0$) and is the same as the differential inclusion

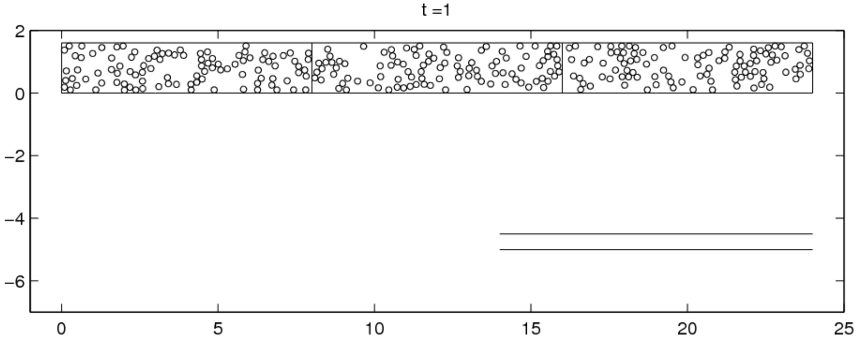
$$q'(t) \in u(t) - N_K(q(t))$$

where N_K is the normal cone to the set K

$$N_K(q_0) = \{v \ : \ q_1 \in K \Rightarrow v \cdot (q_1 - q_0) \leq o(|q_1 - q_0|)\}.$$

It is important here that K, even if not convex in Ω^N, is as at least *prox-regular* (the projection on K is well defined on a neighborhood of K).
B. Maury, J. Venel, Handling of contacts in crowd motion simulations, *Traffic and Granular Flow*, 2007.

From the RER to the escalator



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A Continuous formulation

The continuity equation. The PDE $\partial_t\rho_t + \nabla\cdot(\rho_t v_t) = 0$, classical in fluid mechanics, describes the evolution of a density $\rho(t,x)$ composed of particles following the velocity field $v(t,x)$ (i.e. each particle follows a trajectory of $x'(t) = v(t,x(t))$). It is obtained just by imposing the local conservation of mass.

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Difficulty : $v_t = P_{adm(\rho_t)}[u_t]$ is not regular (remember that the equation $x'(t) = v_t(x(t))$ is well-posed for $v_t \in \text{Lip}$), neither depends regularly on ρ. Extra tools than the standard PDE methods are needed. . .

Pressures and duality

The set $adm(\rho)$ may be better described by duality :

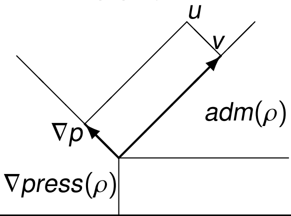
$$adm(\rho) = \left\{ v \in L^2(\rho) \ : \ \int v \cdot \nabla p \leq 0 \quad \forall p \ : \ p \geq 0, \, p(1-\rho) = 0 \right\}.$$

In this way we can characterize $v = P_{adm(\rho)}[u]$ through

$$u = v + \nabla p, \quad v \in adm(\rho), \quad \int v \cdot \nabla p = 0,$$

$$p \in press(\rho) := \{p \in H^1(\Omega), \, p \geq 0, \, p(1-\rho) = 0\}$$

This function p plays the role of the pressure affecting the movement.



$$\partial_t \rho_t + \nabla \cdot \big(\rho_t (u_t - \nabla p_t) \big) = 0$$

$$\rho_t \in K, \ p_t \in press(\rho_t)$$

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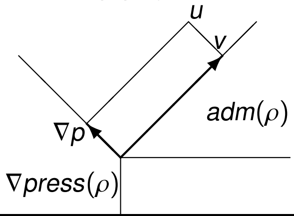

$$\partial_t \rho_t + \nabla \cdot \big(\rho_t (u_t - \nabla p_t)\big) = 0$$

$$\rho_t \leq 1,\ p_t \geq 0,\ p_t(1-\rho_t) = 0$$

A catching-up scheme for the PDE

Fix a time step $\tau > 0$. We look for a sequence $(\rho_n^\tau)_n$ where ρ_n^τ stands for ρ at time $n\tau$. We first define

$$\tilde{\rho}_{n+1}^\tau = (id + \tau u_{n\tau})_{\#}\rho_n^\tau \; ; \quad \rho_{n+1}^\tau = P_K(\tilde{\rho}_{n+1}^\tau)$$

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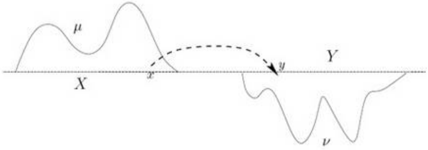
The key point is actually using the W_2 projection (instead of L^2 or other projections). It corresponds to the L^2 projection of velocity fields.

B. Maury, A. Roudneff-Chupin and F. Santambrogio, A macroscopic crowd motion model of gradient flow type, *Math. Mod. Meth. Appl. Sci.*

B. Maury, A. Roudneff-Chupin, F. Santambrogio and J. Venel, Handling congestion in crowd motion modeling *Net. Het. Media*

Optimal transport and Wasserstein distances - 1



If two distributions of mass $\mu, \nu \in \mathcal{P}(\Omega)$ are given on a compact domain, the Monge-Kantorovich problem reads:

$$W_2^2(\mu,\nu) = \inf\left\{ \int |x - T(x)|^2 d\mu \ : \ T : \Omega \to \Omega, \ T_{\#}\mu = \nu \right\}$$

$$= \inf\left\{ \int |x - y|^2 d\gamma \ : \ \gamma \in \mathcal{P}(\Omega^2), \ (\pi_x)_{\#}\gamma = \mu, (\pi_y)_{\#}\gamma = \nu \right\}$$

Optimal transport and Wasserstein distances - 1

Push-forward of a measure. Given $\mu \in \mathcal{P}(X)$ and $T : X \mapsto Y$, we define a new distribution $T_\# \mu$ of mass on Y, called *push-forward* (or image measure) of μ through T as the one that we obtain if we "tell" every particle of μ to move from x to $T(x)$. Mathematically, $\int f \, d(T_\# \mu) = \int f \circ T \, d\mu$ for every f.

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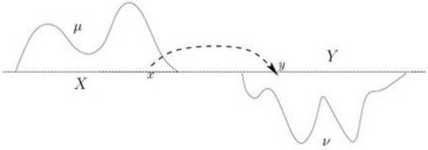

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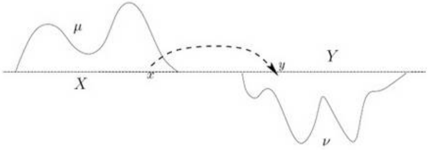

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Under suitable assumptions, there exist an optimal transport $T = \nabla u$, which is the gradient of a convex function (Brenier's theorem).

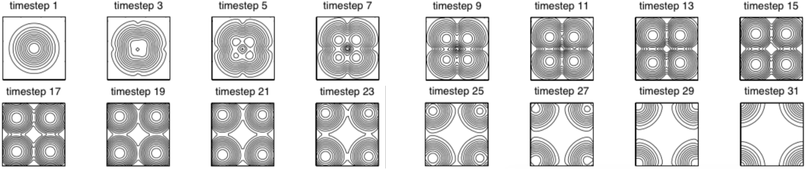
$W_2(\mu,\nu)$, the square root of the minimal value, is a distance on $\mathcal{P}(\Omega)$ which metrizes the weak-* convergence of probabilities (on compact domains).

G. Monge, Mémoire sur la théorie des déblais et des remblais, 1781

L. Kantorovich, On the transfer of masses, 1942.

Y. Brenier, Décomposition polaire et réarrangement monotone des champs de vecteurs, *CRAS*, 1987.

Optimal transport and Wasserstein distances - 2



There is also a dynamical formulation

$$W_2^2(\mu,\nu) = \inf\left\{ \int_0^1 \int \rho_t |v_t|^2 \, dxdt \ : \ \partial_t \rho + \nabla \cdot (\rho v) = 0, \ \rho_0 = \mu, \rho_1 = \nu \right\}$$

$$= \inf\left\{ \int_0^1 \int \frac{|w_t|^2}{\rho_t} \, dxdt \ : \ \partial_t \rho + \nabla \cdot w = 0, \quad \rho_0 = \mu, \rho_1 = \nu \right\}$$

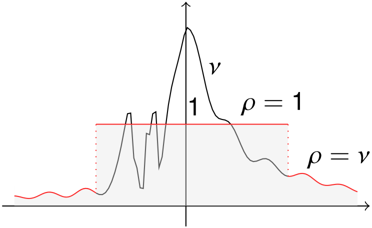
This kinetic energy minimization is the so-called *Benamou-Brenier* formulation, which amounts to a convex optimization problem, solvable by Augmented Lagrangian methods.

J.-D. Benamou, Y. Brenier A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, *Numer. Math.*, 2000.

Projections and pressures

Fix a measure $\nu \in \mathcal{P}(\Omega)$ and solve

$$\min \quad \left\{ W_2^2(\rho, \nu) \ : \ \rho \in K \quad \right\} = \min\left\{ \int |T(x) - x|^2 \, d\nu \ : \ T_\# \nu \leq 1 \right\}.$$



Let S be the optimal transport from ρ to ν.
If $\rho(x_0) < 1$, then $S(x_0) = x_0$.
More generally, $S(x) = x + \nabla p(x)$, with $p \geq 0$ and $p(1 - \rho) = 0$ (hence, p is a pressure).

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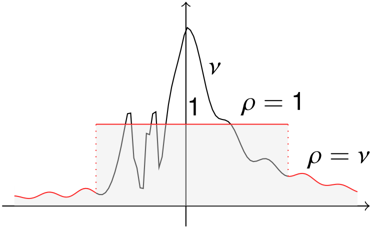

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To prove this optimality condition, consider $T_\varepsilon := (id + \varepsilon v) \circ T$, with $\nabla \cdot v > 0$ on $A = \{\rho = 1\}$. Then $(T_\varepsilon)_{\#}\nu \le 1$ and the optimality provides

$$\frac{d}{d\varepsilon} \int |T_\varepsilon(x) - x|^2 d\nu \left(= \int |y + \varepsilon v(y) - S(y)|^2 d\rho \right)_{|\varepsilon=0} = \int v(y) \cdot (y - S(y))\rho(y) \ge 0.$$

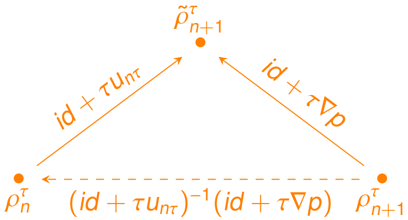
Using $S(y) = \nabla u(y) = y + \nabla p(y)$, formally, on A this gives $\int_A (\nabla \cdot v) p \ge 0$, hence $p \ge 0$ on A.

Getting back to the PDE

$S(x) = x + \nabla p(x)$ is the optimal transport from ρ^τ_{n+1} to $\tilde\rho^\tau_{n+1}$. Notice

$$||\nabla p||_{L^2(\rho^\tau_{n+1})} = W_2(\rho^\tau_{n+1}, \tilde\rho^\tau_{n+1}) \leq W_2(\rho^\tau_n, \tilde\rho^\tau_{n+1}) \leq \tau ||u_{n\tau}||_{L^2(\rho^\tau_n)}.$$

Let us scale the pressure (we call it τp) and get the following situation

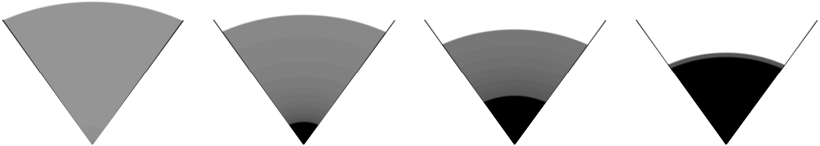


Notice that $(id + \tau u_{n\tau})^{-1}(id + \tau\nabla p) = id - \tau(u_{(n+1)\tau} - \nabla p) + o(\tau)$ provided u is regular enough. This allows to get, in the limit $\tau \to 0$, the vector field $v_t = P_{adm(\rho_t)}[u_t]$ and get a solution of the PDE.

An explicit example

$u(x) = -\nabla \text{dist}(x, \Gamma)$, where the exit Γ is the bottom of the cone.

Closed door



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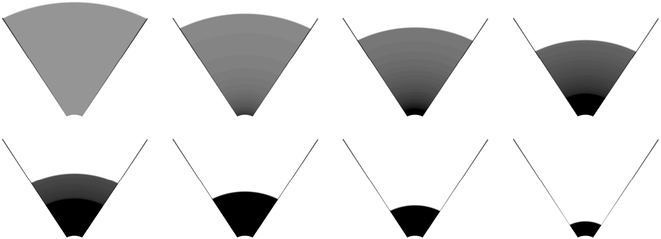

Micro vs Macro

5 obstacles, micro — 5 obstacles, macro

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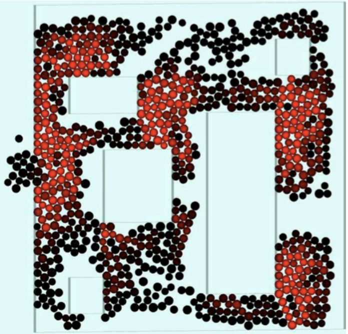

Part II

Active Crowds: game theory and control problems

What are MFG?

The theory of Mean Field Games has been introduced by Lasry and Lions to describe the evolution of a population, where each agent has to choose the strategy (i.e., a path) which best fits his preferences, but is affected by the others through a global *mean field*.
It is a differential game, with a continuum of players, all indistinguishable and all negligible, and we look for a *Nash equilibrium*, which can be translated into a system of PDEs.

What are MFG?

Nash equilibrium. In a finite game, each player i chooses a strategy $s_i \in S_i$ and gets a payoff $f_i(s_i, s_{-i})$, where $s_{-i} = (s_j)_{j \neq i}$. A configuration of choices is an equilibrium is for every i s_i optimizes $f_i(\cdot, s_{-i})$.

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J.-M. Lasry, P.-L. Lions, Mean-Field Games, *Japan. J. Math.* 2007
P.-L. Lions, courses at Collège de France, 2006/12, videos available at
http://www.college-de-france.fr/site/pierre-louis-lions/_course.htm
P. Cardaliaguet, lecture notes, www.ceremade.dauphine.fr/~cardalia/

Limit of finite games

The goal behind the theory is to study the limit as $N \to \infty$ of games of N player, each one choosing a trajectory $x_i(t)$ and optimizing a quantity

$$\int_0^T \left(\frac{|x_i'(t)|^2}{2} + g_i(x_1(t), \dots, x_N(t)) \right) dt + \Psi_i(x_i(T)).$$

In particular, we are interested in the case where g_i penalizes points close to too many other players x_j, $j \neq i$.

We will suppose that g_i only depends on the position x_i and on the distribution of the other player, and that all players have the same preferences. And **we will not study the discrete case** and pass to the limit, but **directly study the continuous case.**

MFG with density penalization- 1

In a population of agents everybody chooses its own trajectory, solving

$$\min \int_0^T \left(\frac{|x'(t)|^2}{2} + g(\rho_t(x(t))) \right) dt + \Psi(x(T)),$$

with given initial point $x(0)$; here g is a given increasing function of the density ρ_t at time t (we take $g(0) = 0$ and $g \geq 0$). The agent hence tries to avoid overcrowded regions.

Input: the evolution of the density ρ_t.

A crucial tool is the value function φ for this problem, defined as

$$\varphi(t_0, x_0) := \min \left\{ \int_{t_0}^T \left(\frac{|x'(t)|^2}{2} + g(\rho_t(x(t))) \right) dt + \Psi(x(T)),\ x(t_0) = x_0 \right\}.$$

MFG with density penalization- 2

Optimal control theory tells us that φ solves

$$(HJ) \qquad -\partial_t\varphi(t,x)+\frac{1}{2}|\nabla\varphi(t,x)|^2=g(\rho_t(x)),\quad \varphi(T,x)=\Psi(x).$$

Moreover, the optimal trajectories $x(t)$ follow $x'(t)=-\nabla\varphi(t,x(t))$.

Hence, given the initial ρ_0, we can find the density at time t by solving

$$(CE) \qquad \partial_t\rho-\nabla\cdot(\rho\nabla\varphi)=0,$$

which give as **Output:** the evolution of the density ρ_t.
We have an equilibrium if **Input = Output**.
This requires to solve a coupled system (HJ)+(CE):

$$\begin{cases} -\partial_t\varphi+\frac{|\nabla\varphi|^2}{2}=g(\rho),\\ \partial_t\rho-\nabla\cdot(\rho\nabla\varphi)=0,\\ \varphi(T,x)=\Psi(x),\quad \rho(0,x)=\rho_0(x).\end{cases}$$

Stochastic case : we can also insert random effects $dX=\alpha dt+dB$, obtaining $-\partial_t\varphi-\Delta\varphi+\frac{|\nabla\varphi|^2}{2}-g(\rho)=0\ :\quad \partial_t\rho-\Delta\rho-\nabla\cdot(\rho\nabla\varphi)=0.$

Variational principle

It happens that an equilibrium is found by minimizing the (global) energy

$$\mathcal{A}(\rho, v) := \int_0^T \int_\Omega \left(\frac{1}{2}\rho_t |v_t|^2 + G(\rho_t) \right) + \int_\Omega \Psi \rho_T$$

among pairs (ρ, v) such that $\partial_t \rho + \nabla \cdot (\rho v) = 0$, with given ρ_0, where G is the anti-derivative of g, i.e. $G' = g$ (in particular, G is convex).

Important: this problem is convex in the variables $(\rho, w := \rho v)$ and it recalls Benamou-Brenier formulation for optimal transport.
This formulation can be used to do numerics!!

Warning: this is not the total cost for all the agents, as we put $G(\rho)$ instead of $\rho g(\rho)$. The equilibrium minimizes an overall energy (it's a *potential game*), but not the total cost: there is a *price of anarchy*, and interesting questions about *regulation* and tolls.

J.-D. Benamou, G. Carlier Augmented Lagrangian methods for transport optimization, Mean-Field Games and degenerate PDEs, preprint.

Measures on possible trajectories

The same variational problem can also be written in the following way: let $\mathcal{C} = W^{1,2}([0,T];\Omega)$ be the space of curves valued in Ω and $e_t : \mathcal{C} \to \Omega$ the evaluation map, $e_t(\gamma) = \gamma(t)$. Solve

$$\min\left\{\int_{\mathcal{C}} \mathcal{K} dQ + \int_0^T \mathcal{G}((e_t)_\# Q) + \int_\Omega \Psi d(e_T)_\# Q,\ Q \in \mathcal{P}(\mathcal{C}), (e_0)_\# Q = \rho_0\right\},$$

where $\mathcal{K} : \mathcal{C} \to \mathbb{R}$ and $\mathcal{G} : \mathcal{P}(\Omega) \to \overline{\mathbb{R}}$ are given by $K(\gamma) = \frac{1}{2}\int_0^T |\gamma'|^2$ and $\mathcal{G}(\rho) = \int G(\rho(x))dx$.

Existence: by semicontinuity in the space $\mathcal{P}(\mathcal{C})$.

Optimality conditions: take $\overline{Q}$ optimal, $\widetilde{Q}$ another competitor, and $Q_\varepsilon = (1-\varepsilon)\overline{Q} + \varepsilon\widetilde{Q}$. Setting $\rho_t = (e_t)_\# \overline{Q}$ and $h(t,x) = g(\rho_t(x))$, differentiating w.r.t. ε gives

$$J_h(\widetilde{Q}) \geq J_h(\overline{Q}),$$

where J_h is the linear functional

$$J_h(Q) = \int \mathcal{K} dQ + \int_0^T\int_\Omega h(t,x)\, d(e_t)_\# Q + \int_\Omega \Psi\, d(e_T)_\# Q.$$

Back to an equilibrium

Look at J_h. If everything goes well (ignoring regularity issues) we can also write $\int_0^T\int_\Omega h(t,x)\,d(e_t)_\# Q = \int_{\mathcal{C}} dQ \int_0^T h(t,\gamma(t))dt$ and hence we get that

$$Q \mapsto \int_{\mathcal{C}} dQ(\gamma)\left(\mathcal{K}(\gamma) + \int_0^T h(t,\gamma(t))dt + \Psi(\gamma(T))\right)$$

is minimal for $Q = \overline{Q}$. Hence $\overline{Q}$ is concentrated on curves minimizing $\mathcal{K}(\gamma) + \int_0^T h(t,\gamma(t))dt + \Psi(\gamma(T))$. This means **Input=Output**.

Back to an equilibrium

Continuous Nash equilibria. In a non-atomic game with indistinguishable players, given a set of strategy S consider measures $Q \in \mathcal{P}(S)$; each Q induces a payoff $f_Q : S \to \mathbb{R}$ and we look for Q such that Q-a.e. s optimizes f_Q.

Look at J_h. If everything goes well (ignoring regularity issues) we can also write $\int\int h(t,x)\, d(e_t)_\# Q = \int dQ \int_0^T h(t,\gamma(t))dt$ and hence we get that

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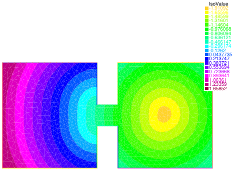
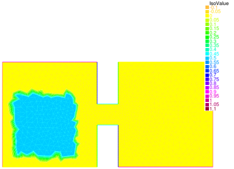
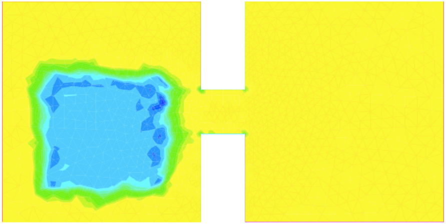
Indeed, $\overline{Q}$-almost all the curves γ minimize the above functional, hence they MUST satisfy $\gamma'(t) = -\nabla\varphi(t,\gamma(t))$. But then $\rho_t = (e_t)_\#\overline{Q}$ evolves according to (CE), and conversely φ solves (HJ) with $h = g(\rho_t)\dots$

An example - simulations by convex optimization







Left top: final potential Ψ,
left bottom: initial density ρ_0,
right: evolution

J.-D. Benamou, G. Carlier, F. Santambrogio, Variational Mean Field Games (to appear a book chapter in Active Particles Volume 1, Theory, Methods, and Applications)

MFG with density constraints - 1

How to define a mean field game if we want to replace the penalization $+g(\rho)$ with the constraint $\rho \leq 1$?

MFG with density constraints - 1

How to define a mean field game if we want to replace the penalization $+g(\rho)$ with the constraint $\rho \leq 1$?

Naïve idea: when $(\rho_t)_t$ is given, every agent minimizes his own cost paying attention to the constraint $\rho_t(x(t)) \leq 1$. But if ρ already satisfies $\rho \leq 1$, one extra agent will not violate the constraint (it'a *non-atomic game*). Hence the constraint becomes empty.

Instead, let's look at the variational problem

$$\min\left\{\int_0^T\int_\Omega \frac{1}{2}\rho_t|v_t|^2 + \int_\Omega \Psi\rho_T \;:\; \rho \leq 1\right\}.$$

This problem is also obtained as the limit $m \to \infty$ of $g_m(\rho) = \rho^m$. Indeed in this case we have $G_m(\rho) = \frac{1}{m+1}\rho^{m+1}$ and

$$\lim_{m\to\infty} \frac{1}{m+1}\rho^{m+1} = \begin{cases} 0 & \text{if } \rho \leq 1, \\ +\infty & \text{if } \rho > 1 \end{cases}.$$

F. Santambrogio, A Modest Proposal for MFG with Density Constraints, *NHM*, 2012.

MFG with density constraints - 2

The system with density penalization was

$$\begin{cases} -\partial_t \varphi + \frac{|\nabla \varphi|^2}{2} = \rho^m, \\ \partial_t \rho - \nabla \cdot (\rho \nabla \varphi) = 0, \\ \varphi(T, x) = \Psi(x), \quad \rho(0, x) = \rho_0(x). \end{cases}$$

MFG with density constraints - 2

The system with density constraints is

$$\begin{cases} -\partial_t \varphi + \frac{|\nabla \varphi|^2}{2} = p, \\ \partial_t \rho - \nabla \cdot (\rho \nabla \varphi) = 0, \\ \varphi(T, x) = \Psi(x), \quad \rho(0, x) = \rho_0(x), \\ \end{cases}$$

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since $\rho^m \geq 0$ and $\rho^m(1-\rho) \to 0$ if $\rho \leq 1$.

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From this system, we come back to a control problem: each agent solves
$\min \int_0^T \left(\frac{|x'(t)|^2}{2} + p(t,x(t))\right)dt + \Psi(x(T))$.

Here p is a **pressure** arising from the incompressibility constraint $\rho \leq 1$ but finally acts as a **price** (only acting on saturated regions). Of course, in order to give a meaning to the above problem we need a bit of regularity,

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P. Cardaliaguet, A. Mészáros, F. Santambrogio, First order Mean Field Games with density constraints: Pressure equals Price, preprint

The End

Thanks for your attention