ERRATUM FOR SECTION 5.4.3

There was a mistake in this section: it is not true that if u is a solution then $h(u)$ is a subsolution, differently from the case $F = 0$. Yet, this is not a problem, as one can directly prove that the properties that were stated for positive subsolutions are actually ture for solutions, paying attention to some signs. Here below we correct this, and also simplify some arguments. In red are the main differences.

Section 5.4.3

For completeness, we want to study what happens for solutions of $\nabla\cdot(A(x)\nabla u) = \nabla\cdot F$.

We will first start with the case of solutions $u \in H_0^1(\Omega)$ (i.e. with homogeneous Dirichlet boundary conditions) and define the notion of solution in this same case. **Definition 5.22** A function u is said to be a solution of the Dirichlet problem $\nabla\cdot(A(x)\nabla u) = \nabla\cdot F$ in Ω if it belongs to $H_0^1(\Omega)$ and $\int_\Omega A(x)\nabla u\cdot\nabla\varphi = \int_\Omega F\cdot\nabla\varphi$ for every function $\varphi \in H_0^1(\Omega)$.

We have the following result.

Proposition 5.23

(1) If u is a solution and $h : \mathbb{R} \to \mathbb{R}_+$ is a positive bounded function, then we have

$$\Lambda_{min}^2 \int_\Omega h(u)|\nabla u|^2 \,\mathrm{d}x \le \int_\Omega |F|^2 h(u)\,\mathrm{d}x.$$

(2) If u is a solution and $F \in L^d(\Omega)$ then $u \in L^p(\Omega)$ for all $p < \infty$.

(3) If u is a solution and $F \in L^d(\Omega)$ then $|u|^p \in H_0^1(\Omega)$ for all $p \in [1, \infty)$.

(4) If u is a solution then for every $p \in [0, \infty)$ we have

$$\Lambda_{min}^2 \int_\Omega |u|^p|\nabla u|^2 \,\mathrm{d}x \le \int_\Omega |F|^2|u|^p \,\mathrm{d}x. \tag{5.6}$$

(5) If u is a solution and $F \in L^q(\Omega)$ for some $q > d$, then $u \in L^\infty(\Omega)$ and $||u||_{L^\infty}$ is bounded by a constant only depending on $d, q, \Lambda_{min}, \Omega$ and $||F||_{L^q}$.

Proof. (1) Let us use $\varphi = g(u)$, with $g(0) = 0$ and $g' = h$. Since g is Lipschitz continuous, we have $g(u)$ is in H_0^1. We then obtain

$$\int_\Omega A(x)\nabla u\cdot\nabla u h(u)\,\mathrm{d}x \le \int_\Omega F\cdot\nabla u h(u)\,\mathrm{d}x.$$

This implies

$$\Lambda_{min}\int_\Omega h(u)|\nabla u|^2\mathrm{d}x \le \int_\Omega |F||\nabla u|h(u)\mathrm{d}x \le \left(\int_\Omega |F|^2 h(u)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^2 h(u)\mathrm{d}x\right)^{\frac{1}{2}}$$

which provides the claim.

(2) If $d = 2$ then we have $u \in L^p(\Omega)$ for all $p < \infty$ because this is true for every H^1 function. If $d > 2$, then we have $|F|^2 \in L^{d/2}$ and $(\frac{d}{2})' = \frac{d}{d-2} = \frac{2^*}{2}$. Assume now $u \in L^p$ for a certain p and take $h(s) = |s|^m$ with $m2^*/2 = p$. This function cannot be used in the estimate that we just proved since it is not globally bounded but can be approximated by a sequence of bounded functions h_n such that $h_n \leq h$ (take, for instance, $h_n(s) = \min\{h(s), n\}$) for which we have

$$\Lambda_{min}^2 \int_\Omega h_n(u)|\nabla u|^2 \, dx \leq \int_\Omega |F|^2 h_n(u) \, dx \leq \int_\Omega |F|^2 h(u) \, dx < \infty,$$

where the finiteness of the last integral comes from $|F|^2 \in L^{d/2}$ and $h(u) = |u|^m \in L^{(d/2)'}$. Passing to the limit this implies

$$\int_\Omega |u|^m |\nabla u|^2 \, dx < \infty \quad \Rightarrow \quad |u|^{\frac{m}{2}+1} \in H^1 \quad \Rightarrow \quad |u|^{\frac{m}{2}+1} \in L^{2^*},$$

which means $u \in L^{p+2^*}$. This proves that the integrability of u can be improved by induction and $u \in L^p(\Omega)$ for all $p < \infty$.

(3) Now that we know $u \in L^p(\Omega)$ for all $p < \infty$ the same argument as before provides $u^{\frac{m}{2}+1} \in H^1$ for all $m = 2p/2^*$. Note that we can also use $p < 1$ and even $p = 0$.

(4) Given $p \in [0, \infty[$ we define $h(s) = |s|^p$. This allows us to obtain the desired estimate as in the first point of the proof.

(5) Let us start from the case $|\Omega| = 1$. In this case the L^m norm of u is increasing in m. Set $m_0 := \inf\{m \geq 1 \, : \, ||u||_{L^m} > 1\}$. If $m_0 = +\infty$ then $||u||_{L^m} \leq 1$ for all m and $||u||_{L^\infty} \leq 1$. In this case there is nothing to prove. If $m_0 > 1$ then $||u||_{L^m} \leq 1$ for all $m < m_0$ and, by Fatou's lemma, we also have $||u||_{L^{m_0}} \leq 1$. If $m_0 = 1$ we then have $||u||_{L^{m_0}} = ||u||_{L^1} \leq C||u||_{L^2} \leq C||\nabla u||_{L^2}$ and we know from Lemma 5.9 that u is bounded in H^1 in terms of $||F||_{L^2}$, and hence of $||F||_{L^q}$. In any case, we obtain that $||u||_{L^{m_0}}$ is bounded in terms of $||F||_{L^q}$, and for $m > m_0$ we have $||u||_{L^m} \geq 1$, an inequality which we will use to simplify the computations.

Using (5.6) we have

$$\frac{4}{(p+2)^2} \int_\Omega |\nabla(|u|^{\frac{p}{2}+1})|^2 \, dx \leq C \int_\Omega |F|^2 |u|^p \, dx \leq C \left(\int_\Omega |u|^{\frac{pq}{q-2}} \, dx \right)^{\frac{q-2}{q}},$$

where the constant C depends on Λ_{min} and $||F||_{L^q(\Omega)}$. Let us fix an exponent β such that $\beta < \frac{2^*}{2}$ but $\beta > \frac{q}{q-2}$. This is possible because $q > d$. We then use the Sobolev injection of H_0^1 into $L^{2\beta}$ and obtain

$$\left(\int_\Omega |u|^{(p+2)\beta} \right)^{\frac{1}{\beta}} \leq C(p+2)^2 \left(\int_\Omega |u|^{\frac{pq}{q-2}} \right)^{\frac{q-2}{q}}.$$

We now raise to the power $1/(p+2)$ and assume $pq/(q-2) \geq m_0$. We then have

$$||u||_{L^{p\beta}} \leq ||u||_{L^{(p+2)\beta}} \leq C(p+2)^{\frac{2}{p+2}} ||u||_{L^{\frac{pq}{q-2}}}^{\frac{p}{p+2}} \leq C(p+2)^{\frac{2}{p+2}} ||u||_{L^{\frac{pq}{q-2}}},$$

where we used $||u||_{L^{\frac{pq}{q-2}}} \geq 1$. Setting $r = \beta(q-2)/q > 1$ and $m_k = m_0 r^k$ we then have

$$||u||_{L^{m_{k+1}}} \leq C(p_k+2)^{\frac{2}{p_k+2}} ||u||_{L^{m_k}},$$

where $p_k = m_k \frac{q-2}{q}$. Passing to the logarithms and using the exponential behavior of p_k as in the proof of Theorem 5.18 we then obtain

$$||u||_{L^{m_k}} \leq C||u||_{L^{m_0}} \leq C,$$

where all constants depend on $d, q, \Lambda_{min}, \Omega$ and $||F||_{L^q}$.

For the case where $|\Omega| \neq 1$, we simply perform a scaling: if u is a subsolution in Ω with the matrix field $A(x)$ and the datum $F(x)$ then we define u_R, A_R and F_R as $u_R(x) = u(Rx)$, $A_R(x) = A(Rx)$ and $F_R(x) = F(Rx)$, and we obtain $\nabla \cdot (A_R \nabla u_R) \geq R\nabla \cdot F_R$ in $\frac{1}{R}\Omega$. Indeed, for every non-negative test function we have $\int_\Omega A(x)\nabla u \cdot \nabla\varphi = -\int_\Omega F \cdot \nabla\varphi$ and the change of variable $x = Ry$ gives

$$\int_{\frac{1}{R}\Omega} A_R(y)\nabla u(Ry) \cdot \nabla\varphi(Ry)\,\mathrm{d}y = -\int_\Omega F_R(y) \cdot \nabla\varphi(Ry)\,\mathrm{d}y,$$

which gives the desired condition using $\nabla u(Ry) = \frac{1}{R}\nabla u_R(y)$ and $\nabla\varphi(Ry) = \frac{1}{R}\nabla\varphi_R(y)$. Choosing R such that $|\frac{1}{R}\Omega| = 1$ provides the desired estimate on $||u_R||_{L^\infty} = ||u||_{L^\infty}$.

□

We can now obtain a precise result on solutions.

Theorem 5.24

If $u \in H_0^1(\Omega)$ is a solution of $\nabla \cdot (A(x)\nabla u) = \nabla \cdot F$ in Ω with $F \in L^q(\Omega)$ for some $q > d$, then $u \in L^\infty(\Omega)$ and $||u||_{L^\infty} \leq C(d, q, \Lambda_{min}, \Omega)||F||_{L^q}$. Moreover, if $\Omega = B(x_0, R)$ we have $C(d, q, \Lambda_{min}, \Omega) = R^{1-\frac{d}{q}} C(d, q, \Lambda_{min})$.

Proof. We already proved that $||u||_{L^\infty}$ can be bounded in terms of $||F||_{L^q}$, without being precise about the exact dependence of the bound on this norm.

We now need to investigate more precisely the dependence of the bound. The map $F \mapsto u$ is well-defined (thanks to an easy variant of Lemma 5.9 from L^2 to H^1 but we proved that it maps L^q into L^∞ and that it is bounded on the unit ball of L^q. It is linear, so we necessarily have $||u||_{L^\infty} \leq C||F||_{L^q}$. The constant C depends here on all the data, including the domain. When $\Omega = B_R := B(x_0, R)$ we perform the same scaling as in the last part of the proof of Proposition 5.23, and we obtain

$$\begin{aligned} ||u||_{L^\infty} = ||u_R||_{L^\infty} &\leq C||RF_R||_{L^q(B_1)} \\ &= CR\left(\int_{B_1} |F(Ry)|^q\,\mathrm{d}y\right)^{1/q} = CR^{1-\frac{d}{q}}\left(\int_{B_R} |F(y)|^q\,\mathrm{d}y\right)^{1/q}. \end{aligned}$$

□

The rest of the section can be kept unchanged