Local H^2 regularity

This document aims at clarifying the proof of the local H^2 regularity for the equation $\nabla\cdot(A\nabla u) = f$, which allows to differentiate the equation and get a PDE for u'.

Notations:

Given a function g defined over Ω and valued in $\mathbb{R}$ or $\mathbb{R}^N$ we define its incremental ratio $\tau_h[g]$ through

$$\tau_h[g](x) := \frac{g(x+h\vec{e}) - g(x)}{h}.$$

Here $\vec{e}$ is a fixed unit vector. We omit the dependence of τ_h on $\vec{e}$ to ease the notation. Warning: $\tau_h[g]$ is only defined for $x \in \Omega\cap(\Omega - h\vec{e})$, i.e. far from a part of the boundary of Ω. We will only use $\tau_h[g]$ when it is multiplied times a cut-off function which guarantees that its value near the boundary does not matter.

Properties:

Characterization of the H^1 functions: it is well known that, whenever $K \subset \Omega$, $d(K, \partial\Omega) \geq h$ and $g \in H^1(\Omega)$, we have $||\tau_h[g]||_{L^2(K)} \leq ||\nabla g||_{L^2(\Omega)}$ (in order to prove it, just write the incremental ration as an integral of the derivative, use Jensen and change the order of the integrals). Also, it is well known that, whenever $K \subset \Omega$ with $d(K, \partial\Omega) > 0$, if $g \in L^2(\Omega)$ is such that $||\tau_h[g]||_{L^2(K)} \leq C$ for all $h < d(K, \partial\Omega)$ and all $\vec{e} \in \mathbb{S}^{N-1}$, then $g \in H^1(K)$ and $||\nabla g||_{L^2(K)} \leq C$.

Change-of-variables – Discrete Integration buy Parts (DIP): for any g and ϕ we have

$$\int g\tau_{-h}[\phi] = -\int \tau_h[g]\phi$$

(this comes from a simple change of variable $y = x - h\vec{e}$).

Commutation with the gradient: for any $g \in H^1(\Omega)$ we have $\nabla\big(\tau_h[g]\big) = \tau_h[\nabla g]$.

Product: it holds $\tau_h[fg] = \tilde{f}\tau_h[g] + g\tau_h[f]$, where $\tilde{f}(x) = f(x+h\vec{e})$.

Now, suppose $u \in H^1(\Omega)$ is a solution of $\nabla\cdot(A\nabla u) = f$, where $f \in L^2(\Omega)$ and $A : \Omega \to \mathcal{M}^{N\times N}$ is a function valued in the symmetric positive-definite matrices with $\lambda I \leq A(x) \leq \Lambda I$ for all x, and we suppose A to be Lischitz continuous.

Theorem 0.1. *Under the above assumptions $u \in H^2_{loc}(\Omega)$.*

Proof. Let us a fix a cut-off function $\eta \in C^\infty_c(\Omega)$ with $\eta = 1$ on a given open set $K \subset \Omega$ with $d(K, \partial\Omega) > 0$ and $\eta = 0$ on the points x such that $d(x, \partial\Omega) \leq \frac{1}{2}d(K, \partial\Omega)$. Let us also fix $0 < h < \frac{1}{2}d(K, \partial\Omega)$.

We want to give a bound on $||\tau_h[\nabla u]\eta||_{L^2(\Omega)}$, which implies an L^2 bound on $\tau_h[\nabla u]$ on K and hence $\nabla u \in H^1(K)$ and $u \in H^2(K)$. This means $u \in H^2_{loc}(\Omega)$.

From the equation we have $\int A\nabla u \cdot \nabla \phi = -\int f\phi$ for any $\phi \in C_c^\infty(\Omega)$. If we use $\tau_{-h}[\phi]$ as a test function we get, for any $\phi \in C_c^\infty(\Omega)$ (but also for any $\phi \in H_0^1(\Omega)$, by density)

$$\int \tau_h\big[A\nabla u\big] \cdot \nabla \phi = -\int \tau_h[f]\phi.$$

Let us take $\phi = \tau_h[u]\eta^2$, which is meaningful and admissible thanks to the presence of the cut-off function. Computing $\nabla\phi = 2\eta\tau_h[u]\nabla\eta + \tau_h[\nabla u]\eta^2$ we get

$$\int \tau_h\big[A\nabla u\big] \cdot \tau_h[\nabla u]\,\eta^2 = -\int \tau_h[f]\tau_h[u]\eta^2 - 2\int \tau_h\big[A\nabla u\big] \cdot \nabla\eta\,\tau_h[u]\eta.$$

We develop $\tau_h\big[A\nabla u\big] = \tilde{A}\,\tau_h[\nabla u] + \nabla u\,\tau_h[A]$, where $\tilde{A}(x) = A(x + h\vec{e})$.

Hence we get

$$\int \tilde{A}\,\tau_h[\nabla u] \cdot \tau_h[\nabla u]\,\eta^2 = -\int \tau_h[f]\tau_h[u]\eta^2 - 2\int \tau_h\big[A\nabla u\big] \cdot \nabla\eta\,\tau_h[u]\eta - \int \nabla u\,\tau_h[A] \cdot \tau_h[\nabla u]\eta^2.$$

Let us set

$$\begin{aligned} X &:= ||\tau_h[\nabla u]\,\eta||_{L^2}, \\ Y_1 &:= -\int \tau_h[f]\tau_h[u]\eta^2, \\ Y_2 &:= \int \tau_h\big[A\nabla u\big] \cdot \nabla\eta\,\tau_h[u]\eta, \\ Y_3 &:= \int \nabla u\,\tau_h[A] \cdot \tau_h[\nabla u]\eta^2. \end{aligned}$$

The quantity X is the term that we want to estimate. From $\tilde{A} \geq \lambda I$, we have $\lambda X^2 \leq Y_2 - 2Y_2 + Y_3$. Our goal is to estimate the terms Y_i with terms which are independent of h, or with terms which are linear in X. In this way, having a square at the left-hand side and linear terms at the right-hand side, we get a bound on $||\tau_h[\nabla u]\,\eta||_{L^2}$.

We start from Y_1. We first perform a DIP, so that we get

$$Y_1 = -\int \tau_h[f]\tau_h[u]\eta^2 = \int f\,\tau_h\big[\tau_h[u]\eta^2\big] \leq ||f||_{L^2}||\tau_h\big[\tau_h[u]\eta^2\big]||_{L^2}.$$

Now, we estimate $||\tau_h\big[\tau_h[u]\eta^2\big]||_{L^2}$ with $||\nabla\big(\tau_h[u]\eta^2\big)||_{L^2}$ and we compute the gradient:

$$||\nabla\big(\tau_h[u]\,\eta^2\big)||_{L^2} \leq ||(\nabla\tau_h[u])\eta^2||_{L^2} + ||\tau_h[u]\,\nabla\eta^2||_{L^2} \leq ||(\nabla\tau_h[u])\eta||_{L^2} + C||\tau_h[u]||_{L^2} \leq X + C||\nabla u||_{L^2},$$

where we used the inequality $\eta^2 \leq \eta$, $|\nabla\eta^2| \leq C$ (where the constant depends on K and Ω and degenerates if K approaches $\partial\Omega$) and $||\tau_h[u]||_{L^2} \leq ||\nabla u||_{L^2}$. Hence, Y_1 is bounded by something linear in X.

Let's pass to the second term. We have

$$Y_2 = \int \tau_h\big[A\nabla u\big] \cdot \nabla\eta\, \tau_h[u]\eta = \int \tilde{A}\, \tau_h\big[\nabla u\big] \cdot \nabla\eta\, \tau_h[u]\eta + \int \tau_h[A]\nabla u \cdot \nabla\eta\, \tau_h[u]\eta.$$

In this sum, the first term may be estimated by using $|\tilde{A}| \le C$ and $|\nabla\eta^2| \le C$ by a quantity of the form $C||(\nabla\tau_h[u])\eta||_{L^2}||\tau_h[u]||_{L^2} \le ||\nabla u||_{L^2}X$. For the second we use $|\tau_h[A]| \le C$ (since A is Lipschitz) and $|\eta\nabla\eta| \le C$ and we bound it with $||\tau_h[u]||_{L^2}||\nabla u||_{L^2} \le ||\nabla u||^2_{L^2}$. Hence, the first term is linear in X and the last is bounded: we have $|Y_2| \le ||\nabla u||_{L^2}X + ||\nabla u||^2_{L^2}$.

Finally, we look at Y_3, for which we write

$$Y_3 = \int \nabla u\, \tau_h[A] \cdot \tau_h[\nabla u]\eta^2 \le C||\eta\nabla u||_{L^2}X,$$

by using again $|\tau_h[A]| \le C$. We have $Y_3 \le C||\nabla u||_{L^2}X$.

Globally we got

$$\lambda X^2 \le ||f||_{L^2}X + CX||\nabla u||_{L^2} + C||\nabla u||^2_{L^2}.$$

This implies $X \le C(||f||_{L^2} + ||\nabla u||_{L^2})$, with a constant C depending on the cut-off function η (i.e. on $d(K, \partial\Omega)$), on λ, Λ and on the Lipschitz constant of A. □