Variational Mean Field Games

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Outline

1. What are MFG? Where do they come from?
2. A coupled system of PDEs
3. Different variational problems
4. Coming back to an equilibrium
5. Some words on regularity
6. Density constraints instead of penalization

What are MFG?

The theory of Mean Field Games (introduced by Lasry and Lions, and at the same time by Huang, Malhamé and Caines) describes the evolution of a population, where each agent has to choose the strategy (i.e., a path) which best fits his preferences, but is affected by the others through a global *mean field* effect.

It is a differential game, with a continuum of players, all indistinguishable and all negligible. It is a typical congestion game (agents try to avoid the regions with high concentrations) and we look for a *Nash equilibrium*, which can be translated into a system of PDEs.

J.-M. Lasry, P.-L. Lions, Jeux à champ moyen. (I & II) *C. R. Math. Acad. Sci. Paris,* 2006 + Mean-Field Games, *Japan. J. Math.* 2007

M. Huang, R.P. Malhamé, P.E. Caines, Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle, *Comm. Info. Syst.* 2006

P.-L. Lions, courses at Collège de France, 2006/12, videos available on the web

P. Cardaliaguet, lecture notes, `www.ceremade.dauphine.fr/~cardalia/`

Limit of finite games

The goal behind the theory is to study the limit as $N \to \infty$ of games of N player, each one choosing a trajectory $x_i(t)$ and optimizing a quantity

$$\int_0^T \left(\frac{|x_i'(t)|^2}{2} + g_i(x_1(t), \dots, x_N(t)) \right) dt + \Psi_i(x_i(T)).$$

In particular, we are interested in the case where g_i penalizes points close to too many other players x_j, $j \neq i$.
Note that we consider here deterministic mean field games (no stochastic effects in the trajectories $x_i(t)$).

We will suppose that g_i only depends on the position x_i and on the distribution of the other players, and that all players have the same preferences. And **we will not study the discrete case** and pass to the limit, but **directly study the continuous case**.

MFG with density penalization- 1

In a population of agents everybody chooses its own trajectory, solving

$$\min \int_0^T \left(\frac{|x'(t)|^2}{2} + g(x, \rho_t(x(t))) \right) dt + \Psi(x(T)),$$

with given initial point $x(0)$; here $g(x, \cdot)$ is a given increasing function of the density ρ_t at time t. The agent hence tries to avoid overcrowded regions.

Input: the evolution of the density ρ_t.

A crucial tool is the value function φ for this problem, defined as

$$\varphi(t_0, x_0) := \min \left\{ \int_{t_0}^T \left(\frac{|x'(t)|^2}{2} + g(x, \rho_t(x(t))) \right) dt + \Psi(x(T)),\ x(t_0) = x_0 \right\}.$$

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MFG with density penalization- 2

Optimal control theory tells us that φ solves

$$(HJ) \qquad -\partial_t\varphi(t,x) + \frac{1}{2}|\nabla\varphi(t,x)|^2 = g(x,\rho_t(x)), \quad \varphi(T,x) = \Psi(x).$$

Moreover, the optimal trajectories $x(t)$ follow $x'(t) = -\nabla\varphi(t,x(t))$.

Hence, given the initial ρ_0, we can find the density at time t by solving

$$(CE) \qquad \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0,$$

which give as **Output:** the evolution of the density ρ_t.
We have an equilibrium if **Input = Output**.
This requires to solve a coupled system (HJ)+(CE):

$$\begin{cases} -\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} = g(x,\rho), \\ \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0, \\ \varphi(T,x) = \Psi(x), \quad \rho(0,x) = \rho_0(x). \end{cases}$$

Stochastic case : we can also insert random effects $dX = \alpha dt + dB$, obtaining $-\partial_t\varphi - \Delta\varphi + \frac{|\nabla\varphi|^2}{2} - g(x,\rho) = 0\ ; \quad \partial_t\rho - \Delta\rho - \nabla\cdot(\rho\nabla\varphi) = 0.$

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Variational principle

It happens that an equilibrium is found by minimizing the (global) energy

$$\mathcal{A}(\rho, v) := \int_0^T \int_\Omega \left(\frac{1}{2} \rho_t |v_t|^2 + G(x, \rho_t) \right) + \int_\Omega \Psi \rho_T$$

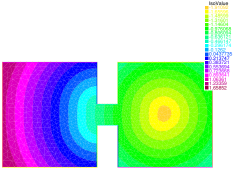
among pairs (ρ, v) such that $\partial_t \rho + \nabla \cdot (\rho v) = 0$, with given ρ_0, where $G(x, \cdot)$ is the anti-derivative of $g(x, \cdot)$, i.e. $G(x, \cdot)' = g(x, \cdot)$ (in particular, $G(x, \cdot)$ is convex).

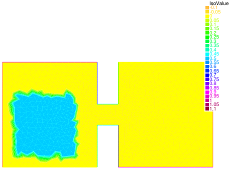
Warning: as it often happens in congestion games, this is not the total cost for all the agents, as we put $G(x, \rho)$ instead of $\rho g(x, \rho)$.The equilibrium minimizes an overall energy (it's a *potential game*), but not the total cost: there is a *price of anarchy*.

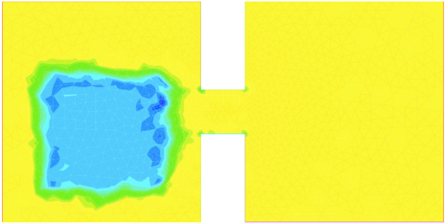
Important: this problem is convex in the variables $(\rho, w := \rho v)$ and it recalls Benamou-Brenier formulation for optimal transport.
This formulation can be used to do numerics!!

J.-D. Benamou, Y. Brenier A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, *Numer. Math.*, 2000.
J.-D. Benamou, G. Carlier Augmented Lagrangian methods for transport optimization, Mean-Field Games and degenerate PDEs, *JOTA*, 2015.

An example - simulations by convex optimization







Left top: final potential Ψ,
left bottom: initial density ρ_0,
right: evolution

J.-D. Benamou, G. Carlier, F. Santambrogio, Variational Mean Field Games, *Active Particles Vol. I,* 2016

Duality

As all convex minimization problem, $\min \mathcal{A}$ admits a dual problem, obtained from

$$\min_{\rho,v} \mathcal{A}(\rho, v) + \sup_{\phi} \int_0^T \int_\Omega (\rho \partial_t \phi + \nabla\phi \cdot \rho v) + \int_\Omega \phi_0 \rho_0 - \int_\Omega \phi_T \rho_T,$$

interchanging inf and sup. We get

$$\sup \left\{ -\mathcal{B}(\phi, p) := \int_\Omega \phi_0 \rho_0 - \int_0^T \int_\Omega G^*(x, p) \ : \ \phi_T \le \Psi, \ -\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = p \right\},$$

where G^* is the Legendre transform of G (w.r.t. p), i.e. $G^*(x, p) = \sup_q \ pq - G(x, q)$.

For optimal (ρ, v, ϕ, p) we have (ρ-a.e.) $v = -\nabla\phi$, $p = g(x, \rho)$ and $\phi_T = \Psi$ i.e., a solution to the MFG system (up to some technicalities).

Measures on the space of trajectories

The same variational problem can also be written in the following way: let $\mathcal{C} = H^1([0,T];\Omega)$ be the space of curves valued in Ω and $e_t : \mathcal{C} \to \Omega$ the evaluation map, $e_t(\gamma) = \gamma(t)$. Solve

$$\min\left\{ \int_{\mathcal{C}} K dQ + \int_0^T \mathcal{G}((e_t)_\# Q) + \int_\Omega \Psi d(e_T)_\# Q,\ Q \in \mathcal{P}(\mathcal{C}), (e_0)_\# Q = \rho_0 \right\},$$

where $K : \mathcal{C} \to \mathbb{R}$ and $\mathcal{G} : \mathcal{P}(\Omega) \to \overline{\mathbb{R}}$ are given by $K(\gamma) = \frac{1}{2}\int_0^T |\gamma'|^2$ and $\mathcal{G}(\rho) = \int G(x, \rho(x))dx$. ($\#$ denotes image measure, or push-forward).

Existence: by semicontinuity in the space $\mathcal{P}(\mathcal{C})$.

Optimality conditions: take $\overline{Q}$ optimal, $\widetilde{Q}$ another competitor, and $Q_\varepsilon = (1-\varepsilon)\overline{Q} + \varepsilon\widetilde{Q}$. Setting $\rho_t = (e_t)_\# \overline{Q}$ and $h(t,x) = g(x, \rho_t(x))$, differentiating w.r.t. ε gives

$$J_h(\widetilde{Q}) \geq J_h(\overline{Q}),$$

where J_h is the linear functional

$$J_h(Q) = \int K dQ + \int_0^T \int_\Omega h(t,x)(e_t)_\# Q + \int_\Omega \Psi d(e_T)_\# Q.$$

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Back to an equilibrium

Look at J_h. It is well-defined for $h \geq 0$ measurable.
But if $h \in C^0$ we can also write $\int_0^T \int_\Omega h(t,x)(e_t)_\# Q = \int_C dQ \int_0^T h(t,\gamma(t))dt$ and hence we get that

$$Q \mapsto \int_C dQ(\gamma)\left(K(\gamma) + \int_0^T h(t,\gamma(t))dt + \Psi(\gamma(T))\right)$$

is minimal for $Q = \overline{Q}$. Hence $\overline{Q}$ is concentrated on curves minimizing $K(\gamma) + \int_0^T h(t,\gamma(t))dt + \Psi(\gamma(T))$. This means **Input=Output**.

A rigorous proof can also be done even for $h \notin C^0$ but one has to choose a precise representative. Techniques from incompressible fluid mechanics **(incompressible Euler** *à la* **Brenier)** allow to handle the case $g(x,\rho) \approx \rho^q$, $h \in L^{q'}$, $q, q' > 1$ using $\hat{h}(x) := \limsup_{r\to 0} \fint_{B(x,r)} h(t,y)dy$ (maximal function needed to justify some convergences, which requires h to be better than L^1).

L. Ambrosio, A. Figalli, On the regularity of the pressure field of Brenier's weak solutions to incompressible Euler equations, *Calc. Var. PDE*, 2008.

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Trajectories on the space of measures

The very same variational problem can also be written in a third way. Use the space of probabilities $\mathbb{W}_2(\Omega)$ endowed with the Wasserstein distance W_2 (enduced by optimal transport) and look for a curve $(\rho(t))_{t\in[0,T]}$ solving

$$\min\left\{\int_0^T\left(\frac{1}{2}|\rho'|(t)^2+\mathcal{G}(\rho(t))\right)dt+\int_\Omega \Psi d\rho_T\ \rho(0)=\rho_0\right\},$$

(here $|\rho'|(t):=\lim_{s\to t}\frac{W_2(\rho(s),\rho(t))}{|s-t|}$ is the metric derivative of the curve ρ).
Existence is also easy by semicontinuity and by Ascoli-Arzelà applied in the space of curves from $[0,T]$ to the compact metric space $\mathbb{W}_2(\Omega)$. Useful estimates and optimality conditions can be obtained via time-discretization: fix $\tau=T/N>0$ and look for a sequence $\rho_0,\rho_1,\dots,\rho_N$ solving

$$\min\left\{\sum_{k=1}^N\left(\frac{W_2^2(\rho_k,\rho_{k-1})^2}{2\tau}+\tau\mathcal{G}(\rho_k)\right)+\int_\Omega \Psi d\rho_N\right\}.$$

For those who know OT and gradient flows: each ρ_k minimizes a functional involving $W_2^2(\rho,\rho_{k\pm1})$, in a very similar way as for the JKO scheme (but with two distance terms); techniques from this field (for instance the *flow interchange* method) can provide estimates on ρ (ongoing work with H. Lavenant).

The problem of regularity

Obtaining classical solutions to the MFG system is a hard question. One possible strategy, suggested by P-L Lions, is to reduce everything to a (non-linear and degenerate) elliptic equation in φ. For instance, if $g(x,\rho) = \rho$, we can replace ρ with $-\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2$ and obtain

$$\partial_{tt}\varphi + \frac{1}{2}\Delta_4\varphi - 2\partial_t\nabla\varphi \cdot \nabla\varphi - \partial_t\varphi\Delta\varphi = 0.$$

This PDE is degenerate elliptic and corresponds to the minimization of $\iint(\partial_t\varphi - \frac{1}{2}|\nabla\varphi|^2)^2$ (with suitable boundary conditions; actually, this is just the dual problem).

It is easier when $g(x,\rho) = \log\rho$, which reduces degeneracy

$$\Delta_{t,x}\varphi + \nabla\varphi \cdot D^2\varphi \cdot \nabla\varphi - 2\partial_t\nabla\varphi \cdot \nabla\varphi = 0$$

(it is actually non-degenerate as soon as $|\nabla\varphi|$ is bounded). This corresponds to $\min \iint e^{(-\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2)}$.

Yet, let us see **a different technique, based on duality** (originating again from Brenier's works on incompressible Euler).

Using duality

Take arbitrary (ρ, v) and (ϕ, p) admissible in the primal and dual problem. Compute

$$\mathcal{A}(\rho, v) + \mathcal{B}(\phi, p)$$
$$= \int_\Omega (\Psi - \phi_T)\rho_T + \int_0^T \int_\Omega (G(x,\rho) + G^*(x,p) - p\rho) + \frac{1}{2}\int_0^T \int_\Omega \rho |v + \nabla\phi|^2.$$

Notice $(G(x,\rho) + G^*(x,p) - p\rho) \geq \frac{\lambda}{2}|\rho - g^{-1}(x,p)|^2$ where $\lambda = \inf g'$. Suppose $\lambda > 0$.

We know $\min \mathcal{A} + \min \mathcal{B} = 0$. Take $(\rho, v), (\phi, p)$ optimal.
We get

$$\begin{aligned} \rho &= g^{-1}(x,p) & \\ \Psi &= \phi_T & \text{on } \{\rho_T > 0\} \\ v &= -\nabla\phi & \text{on } \{\rho > 0\}, \end{aligned}$$

i.e. (again) a solution of the MFG system, in a suitable sense.

H^1 regularity from duality

Suppose for simplicity $\Omega = \mathbb{T}^d$ to be the flat torus. We go on from

$$\mathcal{A}(\rho, v) + \mathcal{B}(\phi, p) \geq c \int_0^T \int_\Omega |\rho - g^{-1}(x, p)|^2.$$

Again, take $(\rho, v), (\phi, p)$ optimal. Take (ρ^δ, v^δ) translation of (ρ, v) (i.e. $\rho^\delta(t, x) = \rho(t, x + \delta)$, up to some cut-off functions to correct at $t = 0$ and $t = T$).
From the fact that $\delta \mapsto \mathcal{A}(\rho^\delta, v^\delta)$ is smooth and minimal for $\delta = 0$, we can prove $\mathcal{A}(\rho^\delta, v^\delta) \leq \mathcal{A}(\rho, v) + C|\delta^2|$. We get

$$\int_0^T \int_\Omega |\rho^\delta - \rho|^2 = \int_0^T \int_\Omega |\rho^\delta - g^{-1}(x, p)|^2 \leq \mathcal{A}(\rho^\delta, v^\delta) + \mathcal{B}(\phi, p) \leq C|\delta|^2,$$

which means $\rho \in L^2_{loc}((0, T); H^1(\Omega))$. We can also adapt to time translation and obtain $\rho \in H^1_{loc(t,x)}$. We can also get $\iint \rho |D^2 \phi|^2 < \infty$.

A. Prosinski, F. Santambrogio, Global-in-time regularity via duality for congestion-penalized Mean Field Games, *Stochastics*, 2017.

MFG with density constraints - 1

How to define a mean field game if we want to replace the penalization $+g(x,\rho)$ with the constraint $\rho \le 1$? (we can keep a penalization depending on x, say $+f(x)$)

Naïve idea: when $(\rho_t)_t$ is given, every agent minimizes his own cost paying attention to the constraint $\rho_t(x(t)) \le 1$. But if ρ already satisfies $\rho \le 1$, one extra agent will not violate the constraint (it's a *non-atomic game*). Hence the constraint becomes empty.

Instead, let's look at the variational problem

$$\min\left\{\int_0^T\int_\Omega\left(\frac{1}{2}\rho_t|v_t|^2+\rho_t f(x)\right)+\int_\Omega \Psi\rho_T\ :\ \rho\le 1\right\}.$$

It means $G(x,\rho)=f(x)\rho$ for $\rho\in[0,1]$ and $+\infty$ otherwise. There is a dual

$$\sup\left\{\int_\Omega \phi_0\rho_0-\int_0^T\int_\Omega (p-f)_+\ :\ \phi_T\le\Psi,\ -\partial_t\phi+\frac{1}{2}|\nabla\phi|^2=p\right\}.$$

This problem is also obtained as the limit $m\to\infty$ of $g(\rho)=f(x)+\rho^m$. Indeed the functional $\frac{1}{m+1}\int\rho^{m+1}$ Γ-converges to the constraint $\rho\le 1$.

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MFG with density constraints - 2

The system we get is

$$\begin{cases} -\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} = p = f + \tilde{p}, \\ \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0, \\ \tilde{p} \geq 0,\ \rho \leq 1,\ \tilde{p}(1-\rho) = 0, \\ \varphi(T,x) = \Psi(x), \quad \rho(0,x) = \rho_0(x) \end{cases}$$

(where $\tilde{p} = p - f$). Agents solve min $\int_0^T \left(\frac{|x'(t)|^2}{2} + p(t,x(t))\right)dt + \Psi(x(T))$.

Here $\tilde{p}$ is a **pressure** arising from the incompressibility constraint $\rho \leq 1$ but finally acts as a **price**. In order to give a meaning to the above problem we need a bit of regularity. The same kind of duality argument, as in the works by Brenier and Ambrosio-Figalli, allows to get

$$\tilde{p} \in L^2_{loc}((0,T); BV(\Omega)).$$

P. Cardaliaguet, A. Mészáros, F. Santambrogio, First order Mean Field Games with density constraints: Pressure equals Price, *SIAM J. Contr. Opt.*, 2016

Y. Brenier, Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations, *Comm. Pure Appl. Math.*, 1999.

The End

Thanks for your attention