

Sharp bounds on the half-space two-point function for high-dimensional Bernoulli percolation

Romain Panis^{1,a}, Bruno Schapira^{1,b}

¹Université Claude Bernard Lyon 1, Institut Camille Jordan, UMR 5208, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France;, ^apanis@math.univ-lyon1.fr; ^bschapira@math.univ-lyon1.fr

Abstract. We consider Bernoulli percolation on \mathbb{Z}^d with $d > 6$. We prove an up-to-constant estimate for the critical two-point function restricted to a half-space. This completes previous results of Chatterjee and Hanson (Commun. Pure Appl. Math., 2021), and Chatterjee, Hanson, and Sosoe (Commun. Math. Phys., 2023), and solves a question asked by Hutchcroft, Michta, and Slade (Ann. Probab., 2023).

Résumé. Nous considérons la percolation de Bernoulli sur \mathbb{Z}^d avec $d > 6$. Nous prouvons des bornes optimales à constante multiplicative près pour la fonction à deux points restreinte à un demi-espace, au point critique. Cela complète des résultats antérieurs de Chatterjee et Hanson (Commun. Pure Appl. Math., 2021), et Chatterjee, Hanson, et Sosoe (Commun. Math. Phys., 2023), et résout une question posée par Hutchcroft, Michta, et Slade (Ann. Probab., 2023)

MSC2020 subject classifications: Primary 60K35

1. Introduction

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with vertex set \mathcal{V} and edge set \mathcal{E} . If $x, y \in \mathcal{V}$, we write $x \sim y$ to say that $\{x, y\} \in \mathcal{E}$. We consider Bernoulli percolation on \mathcal{G} . Given $p \in [0, 1]$, we construct a random subgraph of \mathcal{G} by independently keeping (resp. deleting) each edge of \mathcal{E} with probability p (resp. $1 - p$). The associated measure is denoted by \mathbb{P}_p . We focus on the following examples of graphs \mathcal{G} : for $d \geq 2$,

- (i) Nearest-neighbour model: $\mathcal{V} = \mathbb{Z}^d$ and $\mathcal{E} = \{\{x, y\} : \|x - y\|_1 = 1\}$ where $\|\cdot\|_1$ is the ℓ^1 norm on \mathbb{R}^d ;
- (ii) Spread-out model with *spread* parameter $L \geq 1$: $\mathcal{V} = \mathbb{Z}^d$ and $\mathcal{E} = \{\{x, y\} : \|x - y\|_1 \leq L\}$.

It is well-known (see for instance [Gri99]) that the model undergoes a non-trivial phase transition as the parameter p varies: one has $p_c \in (0, 1)$ with

$$(1) \quad p_c := \inf\{p \in [0, 1] : \mathbb{P}_p[0 \leftrightarrow \infty] > 0\},$$

where $\{0 \leftrightarrow \infty\}$ is the event that the origin lies in an infinite connected component.

In this paper, we study *high-dimensional* percolation, meaning that we work in dimensions $d > 6$. This corresponds to the (conjectured) *mean-field* regime of the model. We refer to [Gri99, Sla06, Pan24, Hut25] and references therein for more information on the particular role played by the dimension $d = 6$ in percolation theory. We investigate properties of the critical measure $\mathbb{P} = \mathbb{P}_{p_c}$. A fundamental quantity in its analysis is the so-called (restricted) *two-point function*, which is defined as follows: if $A \subset \mathbb{Z}^d$ and $x, y \in \mathbb{Z}^d$,

$$(2) \quad \tau_A(x, y) := \mathbb{P}[x \overset{A}{\leftrightarrow} y],$$

where $\{x \overset{A}{\leftrightarrow} y\}$ is the event that there exists an *open path* (i.e. a path made of edges that were kept) fully contained in A which connects x and y . When $A = \mathbb{Z}^d$, we drop it from the above notation.

The starting point in the study of high-dimensional Bernoulli percolation is the following estimate on the critical two-point function: for every $x, y \in \mathbb{Z}^d$,

$$(*) \quad \tau(x, y) \asymp \frac{1}{1 + |x - y|^{d-2}},$$

where \asymp means that the ratio of the two quantities is bounded away from 0 and infinity by two constants which only depend on d (and potentially the spread parameter L), and where $|\cdot|$ denotes the ℓ^∞ norm on \mathbb{R}^d . The *lace expansion* approach developed by Brydges and Spencer [BS85] (see [Sla06] for a review) has been successfully implemented to derive a more precise version of $(*)$ for nearest-neighbour percolation in dimensions $d > 10$ [HS90, Har08, FvdH17], and sufficiently spread-out percolation (i.e. $L \gg 1$) in dimensions $d > 6$ [HHS03]. An alternative proof of $(*)$ in the latter setting has recently been obtained in [DCP25a].

We are interested in the behaviour of the critical two-point function restricted to the half-space $\mathbb{H} := \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d : x_1 \geq 0\}$. In this setting, the main difficulty comes from the lack of full translation invariance. Nevertheless, several partial results have been obtained. The first set of results goes back to [CH20].

Proposition 1.1 ([CH20]). *Let $d > 6$ and assume that $(*)$ holds. Then, for every $K \geq 1$, there exist $c, C > 0$ such that the following holds:*

(a) *For every $x, y \in \mathbb{H}$ which satisfy $|x - y| \leq K \min(x_1, y_1)$,*

$$(3) \quad \frac{c}{1 + |x - y|^{d-2}} \leq \tau_{\mathbb{H}}(x, y) \leq \frac{C}{1 + |x - y|^{d-2}}.$$

(b) *For every $x, y \in \mathbb{H}$ which satisfy $x_1 = 0$ and $|x - y| \leq Ky_1$,*

$$(4) \quad \frac{c}{1 + |x - y|^{d-1}} \leq \tau_{\mathbb{H}}(x, y) \leq \frac{C}{1 + |x - y|^{d-1}}.$$

(c) *For every $x, y \in \mathbb{H}$ which satisfy $x_1 = y_1 = 0$,*

$$(5) \quad \frac{c}{1 + |x - y|^d} \leq \tau_{\mathbb{H}}(x, y) \leq \frac{C}{1 + |x - y|^d}.$$

This result identifies three different regimes of decay for $\tau_{\mathbb{H}}(x, y)$. However, Proposition 1.1 does not give any information on how the two-point function interpolates between these regimes. Partial steps in this direction were taken in the subsequent work [CHS23]. Below, we let $\mathbf{e}_1 = (1, 0, \dots, 0)$.

Proposition 1.2 ([CHS23]). *Let $d > 6$ and assume that $(*)$ holds. Then, there exist $c, C > 0$ such that the following holds:*

(a) *For every $m \geq 1$, and every $x \in \mathbb{H}$,*

$$(6) \quad \tau_{\mathbb{H}}(x, m\mathbf{e}_1) \leq C \frac{1 + m}{1 + |x - m\mathbf{e}_1|^{d-1}}.$$

(b) *For every $m \geq 1$, and every $x \in \mathbb{H}$, if $x_1 \geq \frac{1}{2}|x|$ and $|x| \geq 4m$, then*

$$(7) \quad \tau_{\mathbb{H}}(x, m\mathbf{e}_1) \geq c \frac{1 + m}{1 + |x - m\mathbf{e}_1|^{d-1}}.$$

Remark 1.3. *Some of the estimates stated in Propositions 1.1 and 1.2 have recently been derived in the context of spread-out percolation [DCP25a] (and also in the context of the weakly-self-avoiding walk model [DCP25b]) using very different methods.*

The estimates of Proposition 1.2 are inefficient in the situation where both x and y lie near the boundary of \mathbb{H} . By analogy with Green function estimates (see e.g [LL10]), Hutchcroft, Michta, and Slade [HMS23, Remark 3.4] conjectured a behaviour for $\tau_{\mathbb{H}}(x, y)$ in the regime where $\max(x_1, y_1) \leq |x - y|$. Our main result is a proof of their conjecture. It provides a sharp (up-to-constant) estimate on $\tau_{\mathbb{H}}(x, y)$ for every $x, y \in \mathbb{H}$. We will need the following notation: if $x, y \in \mathbb{H}$, we let $r_{x,y} := \min(x_1, |x - y|)$.

Theorem 1.4. *Let $d > 6$ and assume that $(*)$ holds. Then, there exist $c, C > 0$ such that, for every $x, y \in \mathbb{H}$,*

$$(8) \quad c \frac{(1 + r_{x,y}) \cdot (1 + r_{y,x})}{1 + |x - y|^d} \leq \tau_{\mathbb{H}}(x, y) \leq C \frac{(1 + r_{x,y}) \cdot (1 + r_{y,x})}{1 + |x - y|^d}.$$

Remark 1.5. *It is interesting to compare our result with Proposition 1.2. The latter result can be rephrased as follows: if $d > 6$ and $(*)$ holds, then there exist $c, C > 0$ such that, for every $x, y \in \mathbb{H}$,*

$$(9) \quad \tau_{\mathbb{H}}(x, y) \leq C \frac{1 + \min(r_{x,y}, r_{y,x})}{1 + |x - y|^{d-1}},$$

and, assuming (for instance) that $x_1 \geq \frac{1}{2}|x|$ and $|x| \geq 4|y|$,

$$(10) \quad \tau_{\mathbb{H}}(x, y) \geq c \frac{1 + y_1}{1 + |x - y|^{d-1}}.$$

Therefore, Theorem 1.4 corresponds to Proposition 1.2 in the regime where $r_{x,y} \asymp |x - y|$.

As an immediate corollary of Theorem 1.4, we obtain an alternative (short and easy) proof of [HMS23, Proposition 3.1] (which motivated [HMS23, Remark 3.4]). For every $n \in \mathbb{Z}$, let $\mathbb{H}_n := \mathbb{H} - n\mathbf{e}_1$.

Corollary 1.6. *Let $d > 6$ and assume that $(*)$ holds. Then, there exists $C > 0$ such that, for every $n \geq 0$,*

$$(11) \quad \varphi_{p_c}(\mathbb{H}_n) := p_c \sum_{\substack{x \in \mathbb{H}_n \\ y \notin \mathbb{H}_n \\ x \sim y}} \tau_{\mathbb{H}_n}(0, x) \leq C.$$

Remark 1.7. (i) *This result was also derived in [DCP25a] in the context of spread-out percolation, and in [DCP25b] in the context of the weakly self-avoiding walk model.*

(ii) *Corollary 1.6 implies the uniform boundedness of the expected number of critical pioneers of half-spaces.*

Proof of Corollary 1.6. By translation invariance, one has

$$(12) \quad \varphi_{p_c}(\mathbb{H}_n) = p_c \sum_{\substack{x \in \mathbb{H}_n \\ y \notin \mathbb{H}_n \\ x \sim y}} \tau_{\mathbb{H}}(0, x).$$

By Theorem 1.4, there exists $C_1 > 0$ such that, for every $x \in \mathbb{H}$

$$(13) \quad \tau_{\mathbb{H}}(0, x) \leq C_1 \frac{1 + \min(x_1, |x|)}{1 + |x|^d}.$$

Plugging (13) in (12) concludes the proof. \square

Finally, let us mention that the upper bound in Theorem 1.4 is also useful in the recent [ASS25] (see Lemma 4.3 there).

Notations. We let $\|\cdot\|$ (resp. $|\cdot|$) denote the standard Euclidean norm (resp. the ℓ^∞ norm) on \mathbb{R}^d . If $f, g > 0$, we write $f \lesssim g$ (or $g \gtrsim f$) if there exists $C > 0$, which only depends on d (and potentially the spread parameter L) such that $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$, we write $f \asymp g$.

Given $A, B, C \subset \mathbb{Z}^d$, we write $\{A \overset{C}{\longleftrightarrow} B\}$ for the event that there exists an open path in C connecting A and B , and we omit the superscript C , when $C = \mathbb{Z}^d$.

We now introduce various ‘‘half-space notations’’. Observe that these notations are slightly different from the standard ones. Given a subset $A \subset \mathbb{Z}^d$, we define the inner boundary of A in \mathbb{H} ,

$$(14) \quad \partial A := \{z \in A : \exists z' \sim z \text{ with } z' \in \mathbb{H} \cap A^c\},$$

where we recall that $z' \sim z$ means that z and z' are neighbors in the graph \mathcal{G} under consideration. For $z \in \mathbb{H}$, and $r \geq 0$, we denote the box of radius r centered at z in \mathbb{H} as

$$(15) \quad B_r(z) = \{y \in \mathbb{H} : |y - z| \leq r\},$$

and just write B_r when z is the origin.

The van den Berg–Kesten inequality. If E and F are two percolation events, we write $E \circ F$ for the event of *disjoint* occurrence of E and F , that is, the event that there exist two disjoint sets \mathcal{I} and \mathcal{J} of edges such that the configuration restricted to \mathcal{I} (resp. \mathcal{J}) is sufficient to decide that E (resp. F) occurs. The van den Berg–Kesten (BK) inequality (see [Gri99, Section 2.3]) states that for two *increasing* events (i.e. events that are stable under the action of opening edges) E and F , one has

$$(BK) \quad \mathbb{P}[E \circ F] \leq \mathbb{P}[E]\mathbb{P}[F].$$

2. Proof of Theorem 1.4

In the rest of the paper, we work either in the nearest-neighbour or in the spread-out setting (with spread parameter $L \geq 1$). Additionally, we assume that $d > 6$ and that $(*)$ holds. The proof of Theorem 1.4 is based on Propositions 1.1 and 1.2 and on two new ingredients: Propositions 2.1 and 2.2. We state these results here and prove them in later sections.

We observe the following consequence of the BK inequality: for every $x, y \in \mathbb{H}$, letting $n = \lfloor |x - y|/3 \rfloor$ and assuming that $n \geq 1$ (resp. $n \geq L$ in the spread-out case), one has

$$(16) \quad \tau_{\mathbb{H}}(x, y) \leq \sum_{u \in \partial B_n(x)} \sum_{v \in \partial B_n(y)} \tau_{B_n(x)}(x, u) \cdot \tau_{\mathbb{H}}(u, v) \cdot \tau_{B_n(y)}(v, y).$$

Indeed, exploring an open self-avoiding path γ from x to y , and decomposing it according to the last vertex u visited by γ before exiting $B_n(x)$, and the first vertex v such that the restriction of γ to the portion between v and y lies in $B_n(y)$ gives

$$(17) \quad \{x \overset{\mathbb{H}}{\longleftrightarrow} y\} \subset \bigcup_{u \in \partial B_n(x)} \bigcup_{v \in \partial B_n(y)} \{x \overset{B_n(x)}{\longleftrightarrow} u\} \circ \{u \overset{\mathbb{H}}{\longleftrightarrow} v\} \circ \{v \overset{B_n(y)}{\longleftrightarrow} y\}.$$

See Figure 1 for an illustration. Using a union bound and (BK) gives (16).

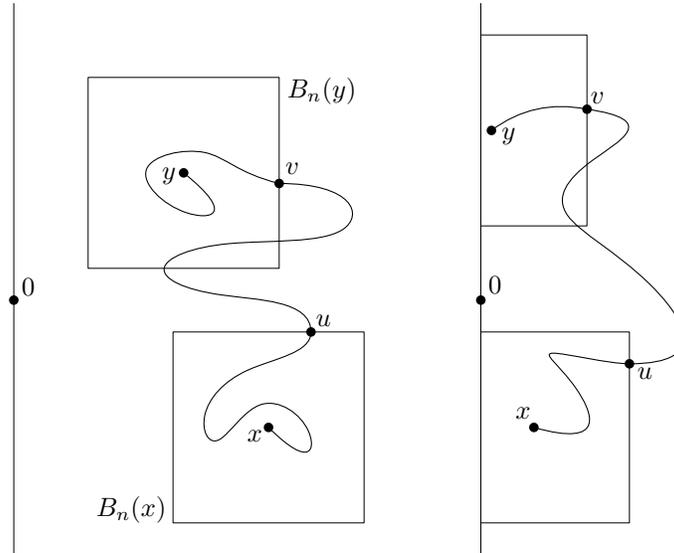


FIG 1. An illustration of the decomposition used to obtain (16) (in the nearest-neighbour case). The black bold path represents an open self-avoiding path from x to y . Depending on the values of x_1 and y_1 , the boxes $B_n(x)$ and $B_n(y)$ may “touch” the boundary of \mathbb{H} . The reversed inequality of Proposition 2.1 decomposes paths from x to y similarly, except that there is an additional restriction to vertices u and v satisfying $u_1, v_1 \geq \varepsilon n$.

Our first result provides a reversed inequality (see also Remark 2.4 below) up to some small multiplicative constant. Given $\varepsilon \in (0, 1)$, $r \geq 0$ and $x \in \mathbb{H}$, we write

$$(18) \quad \partial B_r^\varepsilon(x) = \{u \in \partial B_r(x) : u_1 \geq \varepsilon r\}.$$

Proposition 2.1. *For every $\varepsilon \in (0, 1/2)$, there exist $c, n_0 > 0$, such that for every $x, y \in \mathbb{H}$, letting $n = \lfloor |x - y|/3 \rfloor$ and assuming that $n \geq n_0$, one has*

$$(19) \quad \tau_{\mathbb{H}}(x, y) \geq c \sum_{u \in \partial B_n^{\varepsilon}(x)} \sum_{v \in \partial B_n^{\varepsilon}(y)} \tau_{B_n(x)}(x, u) \cdot \tau_{\mathbb{H}}(u, v) \cdot \tau_{B_n(y)}(v, y).$$

The second important new ingredient is the following estimate.

Proposition 2.2. *For every $n \geq 1$ and every $x \in \mathbb{H}$,*

$$(20) \quad \sum_{u \in \partial B_n(x)} \tau_{B_n(x)}(x, u) \lesssim \frac{1 + \min(x_1, n)}{n}.$$

Furthermore, there exists $\varepsilon \in (0, 1/2)$, such that, for every $n \geq 1$ and every $x \in \mathbb{H}$,

$$(21) \quad \sum_{u \in \partial B_n^{\varepsilon}(x)} \tau_{B_n(x)}(x, u) \gtrsim \frac{1 + \min(x_1, n)}{n}.$$

Remark 2.3. (1) *The case $x_1 = 0$ of (20) was derived in [CH20, Lemma 26]. It is also known (see [HS14, Theorem 1.5]) that for every $n \geq 1$ and every $x \in \mathbb{H}$,*

$$(22) \quad \sum_{u \in \partial B_n(x)} \tau_{B_n(x)}(x, u) \lesssim 1.$$

(2) *A weak version of (20) was derived in [DCP25a] in the context of (sufficiently) spread-out percolation. There, the authors obtained (see [DCP25a, Lemma 3.5]) the existence of $c_0 = c_0(d) > 0$ such that for every $n \geq 1$ and every $x \in \mathbb{H}$,*

$$(23) \quad \sum_{u \in \partial B_n(x)} \tau_{B_n(x)}(x, u) \lesssim \left(\frac{1 + \min(x_1, n)}{n} \right)^{c_0}.$$

We postpone the proofs of these two propositions and give a short proof of our main result.

Proof of Theorem 1.4. We begin with the upper bound. By (*), one has, for every $u, v \in \mathbb{H}$,

$$(24) \quad \tau_{\mathbb{H}}(u, v) \leq \tau(u, v) \lesssim \frac{1}{1 + |u - v|^{d-2}}.$$

Together with (16) and (20), this yields the existence of $C_1 > 0$ such that, for every $x, y \in \mathbb{H}$ with $n = \lfloor |x - y|/3 \rfloor \geq 1$ (resp. $n \geq L$ in the spread-out case),

$$(25) \quad \begin{aligned} \tau_{\mathbb{H}}(x, y) &\lesssim \frac{1}{n^{d-2}} \left(\sum_{u \in \partial B_n(x)} \tau_{B_n(x)}(x, u) \right) \cdot \left(\sum_{v \in \partial B_n(y)} \tau_{B_n(y)}(y, v) \right) \\ &\lesssim \frac{(1 + r_{x,y}) \cdot (1 + r_{y,x})}{n^d} \leq C_1 \frac{(1 + r_{x,y}) \cdot (1 + r_{y,x})}{1 + |x - y|^d}, \end{aligned}$$

where (recall that $r_{x,y} = \min(x_1, |x - y|)$) we used that $\min(x_1, n) \lesssim r_{x,y}$ and $\min(y_1, n) \lesssim r_{y,x}$. In the situation where $n = 0$ (resp. $n < L$ in the spread-out case), a similar bound holds trivially (to the cost of potentially increasing C_1), since $\max(r_{x,y}, r_{y,x}) \leq \max(1, L)$.

The lower bound follows similarly by combining Proposition 2.1 together with (4) and (21). Indeed, let $\varepsilon \in (0, 1/2)$ be given by Proposition 2.2 and $n_0 = n_0(\varepsilon)$ be given by Proposition 2.1, and observe that for every $x, y \in \mathbb{H}$ with

$$n = \lfloor |x - y|/3 \rfloor \geq n_0,$$

$$\begin{aligned}
(26) \quad \tau_{\mathbb{H}}(x, y) &\stackrel{(19)}{\gtrsim} \sum_{u \in \partial B_n^\varepsilon(x)} \sum_{v \in \partial B_n^\varepsilon(y)} \tau_{B_n(x)}(x, u) \cdot \tau_{\mathbb{H}}(u, v) \cdot \tau_{B_n(y)}(v, y) \\
&\gtrsim \min_{\substack{u \in \partial B_n^\varepsilon(x) \\ v \in \partial B_n^\varepsilon(y)}} \tau_{\mathbb{H}}(u, v) \cdot \left(\sum_{u \in \partial B_n^\varepsilon(x)} \tau_{B_n(x)}(x, u) \right) \cdot \left(\sum_{v \in \partial B_n^\varepsilon(y)} \tau_{B_n(y)}(y, v) \right) \\
&\stackrel{(21)}{\gtrsim} \min_{\substack{u \in \partial B_n^\varepsilon(x) \\ v \in \partial B_n^\varepsilon(y)}} \tau_{\mathbb{H}}(u, v) \cdot \frac{(1 + r_{x,y}) \cdot (1 + r_{y,x})}{n^2},
\end{aligned}$$

where we used that $\min(x_1, n) \gtrsim r_{x,y}$ and $\min(y_1, n) \gtrsim r_{y,x}$. Observe that if $u \in \partial B_n^\varepsilon(x)$ and $v \in \partial B_n^\varepsilon(y)$, then $\frac{n}{3} \leq |u - v| \leq \frac{10}{\varepsilon} \min(u_1, v_1)$. Using (3) with $K = 10/\varepsilon$ gives $c_1 = c_1(\varepsilon)$ such that,

$$(27) \quad \min_{\substack{u \in \partial B_n^\varepsilon(x) \\ v \in \partial B_n^\varepsilon(y)}} \tau_{\mathbb{H}}(u, v) \geq \frac{c_1}{n^{d-2}}.$$

Plugging (27) in (26) gives $c_2 > 0$ such that, for every $x, y \in \mathbb{H}$ such that $n \geq n_0$,

$$(28) \quad \tau_{\mathbb{H}}(x, y) \gtrsim \frac{(1 + r_{x,y}) \cdot (1 + r_{y,x})}{n^d} \geq c_2 \frac{(1 + r_{x,y}) \cdot (1 + r_{y,x})}{1 + |x - y|^d}.$$

Again, the bound in the case $n \leq n_0$ follows straightforwardly (to the cost of potentially decreasing c_2). This concludes the proof. \square

Remark 2.4. Retrospectively, by combining (*), Theorem 1.4, and (20), we can deduce that there exists some constant $c > 0$ such that, for every $x, y \in \mathbb{H}$ with $n = \lfloor |x - y|/3 \rfloor \geq 1$,

$$(29) \quad \tau_{\mathbb{H}}(x, y) \geq c \sum_{u \in \partial B_n(x)} \sum_{v \in \partial B_n(y)} \tau_{B_n(x)}(x, u) \cdot \tau_{\mathbb{H}}(u, v) \cdot \tau_{B_n(y)}(v, y),$$

which strengthens the result of Proposition 2.1. Let us provide a short proof of (29). For u, v as above, (*) gives that $\tau_{\mathbb{H}}(u, v) \lesssim \frac{1}{|x - y|^{d-2}} = \frac{1}{n^{d-2}}$, and (20) gives that

$$(30) \quad \sum_{u \in \partial B_n(x)} \tau_{B_n(x)}(x, u) \lesssim \frac{1 + \min(x_1, n)}{n}, \quad \sum_{v \in \partial B_n(y)} \tau_{B_n(y)}(y, v) \lesssim \frac{1 + \min(y_1, n)}{n}.$$

Combining these observations, we get that the sum on the right-hand side of (29) is bounded by a quantity of order $\frac{(1 + \min(x_1, n))(1 + \min(y_1, n))}{n^d}$, which is itself bounded by (a multiple of) $\tau_{\mathbb{H}}(x, y)$ by Theorem 1.4.

3. Proof of Proposition 2.2

We now turn to the proof of Proposition 2.2.

Proof of (20). Let $n \geq 1$. The case $x_1 = 0$ (resp. $x_1 \leq L - 1$ in the spread-out case) was derived¹ in [CH20, Lemma 26]. We now consider a general point $x \in \mathbb{H}$. Decomposing an open self-avoiding path γ from x to u according to the earliest point $v \in \partial B_{n/4}(u)$ (along γ) such that the portion of γ between v and u lies in $B_{n/4}(u)$ and using (BK), we obtain for any $u \in \partial B_n(x)$,

$$(31) \quad \mathbb{P}[x \xrightarrow{B_n(x)} u] \leq \sum_{v \in \partial B_{n/4}(u) \cap B_n(x)} \mathbb{P}[x \xrightarrow{B_n(x)} v] \cdot \mathbb{P}[v \xrightarrow{B_{n/4}(u) \cap B_n(x)} u].$$

¹For full disclosure, [CH20] only treats the case $x_1 = 0$. However, it is easy to extend their result to the case $x_1 \leq L - 1$ by using the Fortuin–Kasteleyn–Ginibre (FKG) inequality (see [Gri99, Chapter 2.2]).

Write $x = (x_1, x_\perp)$, with $x_\perp \in \mathbb{Z}^{d-1}$. For every $v = (v_1, v_\perp) \in \partial B_{n/4}(u) \cap B_n(x)$, letting $\tilde{x} = x - (0, v_\perp)$ and $\tilde{v} = v - (0, v_\perp)$, (6) gives that

$$(32) \quad \mathbb{P}[x \xrightarrow{B_n(x)} v] = \mathbb{P}[\tilde{x} \xrightarrow{B_n(\tilde{x})} v_1 \mathbf{e}_1] = \mathbb{P}[\tilde{v} \xrightarrow{B_n(x_1 \mathbf{e}_1)} x_1 \mathbf{e}_1] \lesssim \frac{\min(1 + v_1, 1 + x_1)}{1 + |x - v|^{d-1}} \lesssim \frac{1 + \min(x_1, n)}{n^{d-1}},$$

where we used translation invariance in the first two equalities. Plugging (32) in (31) and using (20) for $x_1 = 0$ (resp. $x_1 \leq L - 1$), we deduce that, for every $x \in \mathbb{H}$ and $u \in \partial B_n(x)$,

$$(33) \quad \mathbb{P}[x \xrightarrow{B_n(x)} u] \lesssim \frac{1 + \min(x_1, n)}{n^{d-1}} \cdot \left(\sum_{v \in \partial B_{n/4}(u) \cap B_n(x)} \mathbb{P}[v \xrightarrow{B_{n/4}(u) \cap B_n(x)} u] \right) \lesssim \frac{1 + \min(x_1, n)}{n^d}.$$

Summing over $u \in \partial B_n(x)$ concludes the proof. \square

Proof of (21). Let $\varepsilon \in (0, 1/2)$ to be fixed. Let $n \geq 1$ and $x \in \mathbb{H}$. Without loss of generality, we may assume that $x = (x_1, 0, \dots, 0)$. We first assume that $n \geq \frac{1}{2}x_1$. Recall that $\mathbf{e}_1 = (1, 0, \dots, 0)$. On the one hand, by² (3) and (7), one has

$$(34) \quad \tau_{\mathbb{H}}(x, x + 2n\mathbf{e}_1) \gtrsim \frac{1 + \min(x_1, n)}{n^{d-1}}.$$

On the other hand, decomposing an open self-avoiding path from x to $x + 2n\mathbf{e}_1$ according to the first point in $\partial B_n(x)$ it visits and using (BK) gives

$$(35) \quad \tau_{\mathbb{H}}(x, x + 2n\mathbf{e}_1) \lesssim \max_{u \in \partial B_n^\varepsilon(x)} \tau_{\mathbb{H}}(u, x + 2n\mathbf{e}_1) \cdot \left(\sum_{u \in \partial B_n^\varepsilon(x)} \tau_{B_n(x)}(x, u) \right) \\ + \max_{u \in \partial B_n(x) \setminus \partial B_n^\varepsilon(x)} \tau_{\mathbb{H}}(u, x + 2n\mathbf{e}_1) \cdot \left(\sum_{u \in \partial B_n(x) \setminus \partial B_n^\varepsilon(x)} \tau_{B_n(x)}(x, u) \right).$$

Using (*) and (6) give that, for n large enough (in terms of ε),

$$(36) \quad \max_{u \in \partial B_n^\varepsilon(x)} \tau_{\mathbb{H}}(u, x + 2n\mathbf{e}_1) \lesssim \frac{1}{n^{d-2}}, \quad \max_{u \in \partial B_n(x) \setminus \partial B_n^\varepsilon(x)} \tau_{\mathbb{H}}(u, x + 2n\mathbf{e}_1) \lesssim \frac{\varepsilon n}{n^{d-1}}.$$

Combining (34), (35), and (36) gives, for n large enough,

$$(37) \quad \frac{1 + \min(x_1, n)}{n^{d-1}} \lesssim \frac{1}{n^{d-2}} \cdot \left(\sum_{u \in \partial B_n^\varepsilon(x)} \tau_{B_n(x)}(x, u) \right) + \varepsilon \cdot \frac{1 + \min(x_1, n)}{n^{d-1}},$$

where we used (20) to get $\sum_{u \in \partial B_n(x) \setminus \partial B_n^\varepsilon(x)} \tau_{B_n(x)}(x, u) \lesssim \frac{1 + \min(x_1, n)}{n}$. Choosing $\varepsilon \in (0, 1/2)$ small enough concludes the proof in the case $n \geq \frac{1}{2}x_1$ and n large enough (in terms of ε). The remaining values of n can be handled by adjusting the value of the constant in (21).

It remains to treat the case $n < \frac{1}{2}x_1$. Observe that for this choice, one has $B_n(x) = \{u \in \mathbb{Z}^d : |u - x| \leq n\}$ (that is, the \mathbb{Z}^d -box of radius n around x is fully included in \mathbb{H}). Thus, we may use [DCT16] (which gives that $\varphi_{p_c}(B_n(x)) := \sum_{\substack{u \in \partial B_n(x) \\ v \notin B_n(x) \\ u \sim v}} \tau_{B_n(x)}(x, u) p_c \geq 1$) to conclude that, for any $\varepsilon \in (0, 1/2)$,

$$(38) \quad \sum_{u \in \partial B_n^\varepsilon(x)} \tau_{B_n(x)}(x, u) \geq \frac{1}{2} \sum_{u \in \partial B_n(x)} \tau_{B_n(x)}(x, u) \gtrsim 1.$$

This concludes the proof. \square

²Let us give more details. If $\frac{1}{2}x_1 \leq n \leq 2x_1$, then (3) (for a proper choice of K) gives (34). If $n \geq 2x_1$, we may apply (7) (since $x_1 + 2n \geq 4x_1$) to get (34).

4. Proof of Proposition 2.1

The proof of Proposition 2.1 is technically more involved. We will need a number of preliminary results and notations.

We rely on the notion of *regular points* initially introduced in [KN11], and recently revisited in [ASS25]. In particular, we take advantage of the geometric and convenient definition of regular points chosen in [ASS25]. Most of the arguments below are adaptations of intermediate results that already appeared in this paper. It is worth mentioning that our setting is often simpler. For instance, our Lemma 4.3 requires an averaged estimate on the expected number of regular points, while [ASS25, Section 6] derives finer and stronger estimates (quite similarly to what is done in [KN11]). Our main contribution is to use these regular points to derive “reversed Simon–Lieb type inequalities”.

In Section 4.1, we present a general lower bound for the probability that two sets are connected, which we express in terms of their $(d-4)$ -capacity. In Section 4.2, we recall all the necessary definitions to introduce the theory of regularity of [KN11], taking here the viewpoint of [ASS25]. Finally, we give the proof of Proposition 2.1 in Section 4.3. Recall that we have assumed that $d > 6$ and that (*) holds.

4.1. A general lower bound for the connection probability of two finite sets

We present here a general lower bound for the probability that two finite subsets of \mathbb{H} are connected by an open path in terms of the product of their $(d-4)$ -capacity. A non-restricted version of this result has already appeared in [ASS25]. We adapt their argument to the setting of restricted percolation on \mathbb{H} . Since the proof in [ASS25] is only sketched, we provide here a more detailed argument for the reader’s convenience.

Recall that $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^d . Given a non-empty finite subset $A \subset \mathbb{Z}^d$, we define its $(d-4)$ -capacity as

$$(39) \quad \text{Cap}_{d-4}(A) := \left(\inf \left\{ \sum_{a,b \in A} \mu(a)\mu(b)(1 + \|a-b\|)^{4-d} : \mu \text{ probability measure on } A \right\} \right)^{-1}.$$

Given two finite sets $A, B \subset \mathbb{Z}^d$, we let $d(A, B) := \min_{a \in A, b \in B} \|a-b\|$, and $\text{diam}(A) := \max_{a, a' \in A} \|a-a'\|$.

Lemma 4.1. *For every $c_1 > 0$, there exists $c > 0$ such that the following holds. For every finite $A, B \subseteq \mathbb{H}$ such that $d(A, B) \geq c_1 \cdot \max(\text{diam}(A), \text{diam}(B))$, one has*

$$(40) \quad \mathbb{P}[A \overset{\mathbb{H}}{\longleftrightarrow} B] \geq c \cdot \frac{(\min_{a \in A, b \in B} \tau_{\mathbb{H}}(a, b))^2}{d(A, B)^{2-d}} \cdot \text{Cap}_{d-4}(A) \cdot \text{Cap}_{d-4}(B).$$

Proof. As for Lemma 8.1 in [ASS25], the proof is based on a second moment method. More precisely, given two probability measures μ and ν supported respectively on A and B , consider the random variable

$$(41) \quad X = \sum_{a \in A} \sum_{b \in B} \mu(a)\nu(b) \cdot \mathbf{1}\{a \overset{\mathbb{H}}{\longleftrightarrow} b\}.$$

Using Cauchy–Schwarz’s inequality, we get

$$(42) \quad \mathbb{P}[A \overset{\mathbb{H}}{\longleftrightarrow} B] \geq \mathbb{P}[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$

The first moment $\mathbb{E}[X]$ can be easily lower bounded: using that μ and ν are probability measures, we obtain that

$$(43) \quad \mathbb{E}[X] \geq \min_{a \in A, b \in B} \tau_{\mathbb{H}}(a, b).$$

We now upper bound $\mathbb{E}[X^2]$. Write

$$(44) \quad \mathbb{E}[X^2] = \sum_{\substack{a, a' \in A \\ b, b' \in B}} \mu(a)\mu(a')\nu(b)\nu(b')\mathbb{P}[a \overset{\mathbb{H}}{\longleftrightarrow} b, a' \overset{\mathbb{H}}{\longleftrightarrow} b'].$$

Exploring an open self-avoiding path from a to b , and then one from a' to b' , and using (BK), we obtain that for every $a, a' \in A$ and $b, b' \in B$,

$$(45) \quad \mathbb{P}[a \overset{\mathbb{H}}{\leftrightarrow} b, a' \overset{\mathbb{H}}{\leftrightarrow} b'] \leq \tau_{\mathbb{H}}(a, b)\tau_{\mathbb{H}}(a', b') + \sum_{w, w' \in \mathbb{H}} \tau_{\mathbb{H}}(a, w)\tau_{\mathbb{H}}(a', w)\tau_{\mathbb{H}}(w, w')\tau_{\mathbb{H}}(w', b)\tau_{\mathbb{H}}(w', b') \\ + \sum_{w, w' \in \mathbb{H}} \tau_{\mathbb{H}}(a, w)\tau_{\mathbb{H}}(w, b')\tau_{\mathbb{H}}(w, w')\tau_{\mathbb{H}}(w', a')\tau_{\mathbb{H}}(w', b).$$

See Figure 2 for an illustration.

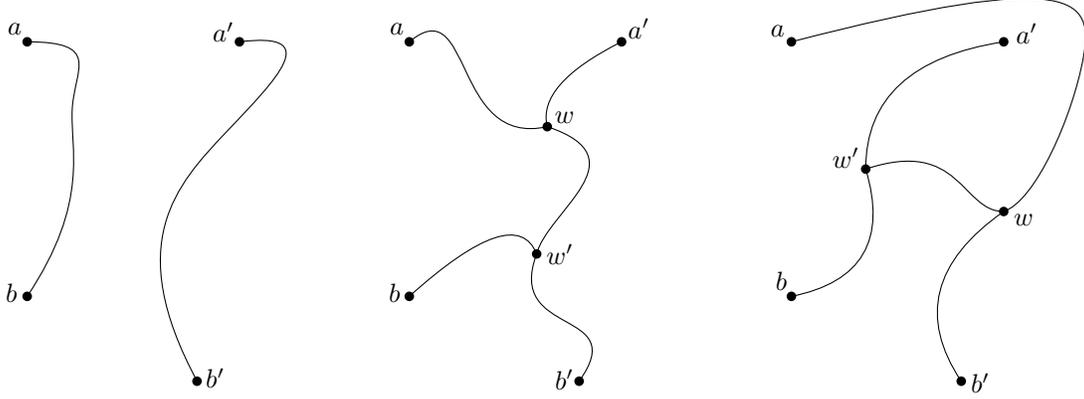


FIG 2. An illustration of the diagrams underlying the proof of (45). The black bold paths are open self-avoiding paths. If the event $\{a \overset{\mathbb{H}}{\leftrightarrow} b, a' \overset{\mathbb{H}}{\leftrightarrow} b'\}$ occurs, then one of the situations must occur (for some $w, w' \in \mathbb{H}$). Each diagram corresponds to a term on the right-hand side of (45).

We begin with the analysis of the first sum on the right-hand side of (45). Letting $r = d(A, B)/2$ and using (*), we find that

$$(46) \quad \Sigma(a, a', b, b') := \sum_{w, w' \in \mathbb{H}} \tau_{\mathbb{H}}(a, w)\tau_{\mathbb{H}}(a', w)\tau_{\mathbb{H}}(w, w')\tau_{\mathbb{H}}(w', b)\tau_{\mathbb{H}}(w', b') \\ \lesssim \frac{1}{r^{d-2}} \left(\sum_{w \in \mathbb{H}} \tau_{\mathbb{H}}(a, w)\tau_{\mathbb{H}}(a', w) \right) \cdot \left(\sum_{w' \in \mathbb{H}} \tau_{\mathbb{H}}(w', b)\tau_{\mathbb{H}}(w', b') \right) \\ + \sum_{\substack{w, w' \in \mathbb{H} \\ \|w - w'\| \leq r}} \tau_{\mathbb{H}}(a, w)\tau_{\mathbb{H}}(a', w)\tau_{\mathbb{H}}(w, w')\tau_{\mathbb{H}}(w', b)\tau_{\mathbb{H}}(w', b').$$

Using (*) and a classical estimate (see for instance [DCP25a, Proposition B.1]),

$$(47) \quad \sum_{w \in \mathbb{H}} \tau_{\mathbb{H}}(a, w)\tau_{\mathbb{H}}(a', w) \leq \sum_{w \in \mathbb{Z}^d} \tau(a, w)\tau(w, a') \lesssim (1 + \|a - a'\|)^{4-d},$$

and likewise,

$$(48) \quad \sum_{w' \in \mathbb{H}} \tau_{\mathbb{H}}(w', b)\tau_{\mathbb{H}}(w', b') \lesssim (1 + \|b - b'\|)^{4-d}.$$

We now look at the second sum on the second line of (46). If w is at distance at least $r/2$ from A , then,

$$(49) \quad \sum_{\substack{w' \\ w: d(w, A) \geq r/2, \|w - w'\| \leq r}} \tau_{\mathbb{H}}(a, w)\tau_{\mathbb{H}}(a', w)\tau_{\mathbb{H}}(w, w')\tau_{\mathbb{H}}(w', b)\tau_{\mathbb{H}}(w', b') \\ \lesssim \frac{1}{r^{2d-4}} \sum_{w'} \tau_{\mathbb{H}}(w', b)\tau_{\mathbb{H}}(w', b') \sum_{w: \|w - w'\| \leq r} \tau(w, w') \lesssim \frac{1}{r^{2d-6}} (1 + \|b - b'\|)^{4-d},$$

where in the first inequality we used (*), and in the second one we used (48) and (*) one more time to get $\sum_{w: \|w-w'\| \leq r} \tau(w, w') \lesssim r^2$. Similarly,

$$(50) \quad \sum_{\substack{w \\ w': d(w', B) \geq r/2, \|w-w'\| \leq r}} \tau_{\mathbb{H}}(a, w) \tau_{\mathbb{H}}(a', w) \tau_{\mathbb{H}}(w, w') \tau_{\mathbb{H}}(w', b) \tau_{\mathbb{H}}(w', b') \lesssim \frac{1}{r^{2d-6}} (1 + \|a - a'\|)^{4-d}.$$

By definition of r , it is impossible to have $d(w, A) < r/2$, $d(w', B) < r/2$, and $\|w - w'\|$ simultaneously. Hence,

$$(51) \quad \sum_{\substack{w, w' \in \mathbb{H} \\ \|w-w'\| \leq r}} \tau_{\mathbb{H}}(a, w) \tau_{\mathbb{H}}(a', w) \tau_{\mathbb{H}}(w, w') \tau_{\mathbb{H}}(w', b) \tau_{\mathbb{H}}(w', b') \lesssim \frac{1}{r^{2d-6}} \left((1 + \|a - a'\|)^{4-d} + (1 + \|b - b'\|)^{4-d} \right).$$

Therefore, if we define for a probability measure ρ , its energy as

$$(52) \quad \mathcal{E}_{d-4}(\rho) := \sum_{u, v \in \mathbb{Z}^d} \rho(u) \rho(v) (1 + \|u - v\|)^{4-d},$$

one obtains from the previously displayed equations that

$$(53) \quad \sum_{\substack{a, a' \in A \\ b, b' \in B}} \mu(a) \mu(a') \nu(b) \nu(b') \Sigma(a, a', b, b') \lesssim \frac{1}{r^{d-2}} \mathcal{E}_{d-4}(\mu) \mathcal{E}_{d-4}(\nu) + \frac{1}{r^{2d-6}} \left(\mathcal{E}_{d-4}(\mu) + \mathcal{E}_{d-4}(\nu) \right) \\ \lesssim \frac{1}{r^{d-2}} \cdot \mathcal{E}_{d-4}(\mu) \mathcal{E}_{d-4}(\nu),$$

where in the second inequality, we used that, since (by hypothesis) $r \gtrsim \max(\text{diam}(A), \text{diam}(B))$, one has $\mathcal{E}_{d-4}(\rho) \gtrsim \frac{1}{r^{d-4}}$, for any probability measure ρ supported on A or B .

Similar computations allow to treat the last sum in the right-hand side of (45). More precisely, using repeatedly (*), we can see that it is upper bounded up to some multiplicative constant by

$$(54) \quad \frac{1}{r^{d-2}} \left(\sum_{w \in \mathbb{H}} \tau_{\mathbb{H}}(a, w) \tau_{\mathbb{H}}(w, b') \right) \cdot \left(\sum_{w' \in \mathbb{H}} \tau_{\mathbb{H}}(w', a') \tau_{\mathbb{H}}(w', b) \right) + \sum_{\substack{w, w' \in \mathbb{H} \\ \|w-w'\| \leq r}} \tau_{\mathbb{H}}(a, w) \tau_{\mathbb{H}}(w, b') \tau_{\mathbb{H}}(w, w') \tau_{\mathbb{H}}(w', a') \tau_{\mathbb{H}}(w', b) \\ \lesssim \frac{1}{r^{d-2}} \left\{ \frac{1}{r^{2(d-4)}} + \frac{1}{r^{d-2}} \sum_{\substack{w, w' \in \mathbb{H} \\ \|w-w'\| \leq r}} \left(\tau_{\mathbb{H}}(a, w) \tau_{\mathbb{H}}(w, w') \tau_{\mathbb{H}}(w', a') + \tau_{\mathbb{H}}(w, b') \tau_{\mathbb{H}}(w, w') \tau_{\mathbb{H}}(w', b) \right) \right\} \\ \lesssim \frac{1}{r^{d-2}} \left\{ \mathcal{E}_{d-4}(\mu) \mathcal{E}_{d-4}(\nu) + \frac{1}{r^{d-2}} \left(\frac{1}{1 + \|a - a'\|^{d-6}} + \frac{1}{1 + \|b - b'\|^{d-6}} \right) \right\} \\ \lesssim \frac{1}{r^{d-2}} \left\{ \mathcal{E}_{d-4}(\mu) \mathcal{E}_{d-4}(\nu) + \frac{\mathcal{E}_{d-4}(\nu)}{1 + \|a - a'\|^{d-4}} + \frac{\mathcal{E}_{d-4}(\mu)}{1 + \|b - b'\|^{d-4}} \right\},$$

where in the second inequality, we used (*) to argue that

$$(55) \quad \sum_{\substack{w, w' \in \mathbb{H} \\ \|w-w'\| \leq r}} \tau_{\mathbb{H}}(a, w) \tau_{\mathbb{H}}(w, w') \tau_{\mathbb{H}}(w', a') \leq \sum_{w, w' \in \mathbb{Z}^d} \tau(a, w) \tau(w, w') \tau(w', a') \lesssim \frac{1}{1 + \|a - a'\|^{d-6}}.$$

Summing (54) over $a, a' \in A$, and $b, b' \in B$ against $\mu(a) \mu(a') \nu(b) \nu(b')$, we obtain the same upper bound as in (53), and conclude that

$$(56) \quad \mathbb{E}[X^2] \lesssim \frac{1}{r^{d-2}} \cdot \mathcal{E}_{d-4}(\mu) \mathcal{E}_{d-4}(\nu).$$

Optimising over the choices of μ and ν and combining the result with (42) and (43) concludes the proof. \square

4.2. Regular points, line good points, extended cluster

We present here the basis of a technique first introduced in [KN11] to derive the one-arm exponent in high-dimensional critical percolation. It is based on a notion of regularity. Here, we will adapt to our setting the definition of regular points from [ASS25] which is stated in purely geometric terms.

We write $\mathcal{C}(x; A) := \{z \in A : x \xleftrightarrow{A} z\}$ for the cluster of a point x restricted to a set A . Fix $n \geq 1$ and $x \in \mathbb{H}$, and for $z \in \partial B_n(x)$, and $s > 0$, consider the event

$$(57) \quad \mathcal{T}_s(z) := \{|\mathcal{C}(z; B_n(x)) \cap B_s(z)| \leq s^4(\log s)^7\} \cap \{|\mathcal{C}(z; B_n(x)) \cap B_s(z) \cap \partial B_n(x)| \leq s^2(\log s)^7\}.$$

Definition 4.2 (K -regular points). *Given $K > 0$, we call $z \in \partial B_n(x)$ a K -regular point, if the events $\mathcal{T}_s(z)$ hold for all $s \geq K$.*

Let $\varepsilon \in [0, 1/2)$, We denote by $X_n^{\varepsilon, K\text{-reg}}(x)$ the number of points on $\partial B_n^\varepsilon(x)$ (recall (18)), which are K -regular and connected to x in $B_n(x)$. Also, denote by $X_n^\varepsilon(x)$ the number of points on $\partial B_n^\varepsilon(x)$ which are connected to x in $B_n(x)$ (the so-called pioneers). It turns out that most of the pioneers are regular, and consequently one can show the following lemma.

Lemma 4.3. *There exist $K_0 \geq 1$ and $n_0 \geq 1$ such that the following holds. For every $K \geq K_0$, every $n \geq n_0$, every $\varepsilon \in [0, 1/2)$, and every $x \in \mathbb{H}$,*

$$(58) \quad \mathbb{E}[X_n^{\varepsilon, K\text{-reg}}(x)] \geq \frac{1}{2} \cdot \mathbb{E}[X_n^\varepsilon(x)].$$

We defer the proof of this lemma to Section 4.4 and introduce now the notion of K -line good points. For this, one first needs to consider a maximal subset of the set of K -regular points of $\partial B_n^\varepsilon(x)$ which has the property that all its points are at distance at least $2K$ one from each other. Denote by $\mathcal{X}_n^{\varepsilon, K\text{-reg}}(x)$ one such maximal subset chosen uniformly at random. Then, if $z \in \mathcal{X}_n^{\varepsilon, K\text{-reg}}(x)$, we consider a line segment of length K emanating from z , outside $B_n(x)$ and orthogonal to its boundary (choose one arbitrarily if there are many). Call z' the endpoint of this line segment. We say that z' is a **K -line good point** if all the edges on the line segment between z and z' are open. More generally, for any $z \in \mathcal{X}_n^{\varepsilon, K\text{-reg}}(x)$, we denote by L_z the maximal open segment emanating from z orthogonally to $B_n(x)$, of length at most K .

We next define the **extended cluster** of x in $B_n(x)$, which we denote by $\mathcal{C}_n^e(x)$, as the cluster of x in $B_n(x)$ together with all the line segments L_z for $z \in \mathcal{X}_n^{\varepsilon, K\text{-reg}}(x)$.

We say that a set A is **K -admissible** for the pair (x, n) , if $\mathbb{P}[\mathcal{C}_n^e(x) = A] > 0$, and for such admissible set we denote by $\partial_* A$ its set of points which are at distance exactly K from $B_n(x)$. Hence by definition $\partial_* \mathcal{C}_n^e(x)$ is the set of K -line good points.

One interest of the notion of regularity, which has been noticed and used extensively in [ASS25], is that in any dimension $d > 6$, admissible sets have a $(d-4)$ -capacity which is comparable to their cardinality. Indeed, the following lemma was observed in [ASS25, Claim 6.1].

Lemma 4.4. *There exists a constant $c > 0$, such that for every $n, K \geq 1$, every $\varepsilon \in [0, 1/2)$, every $x \in \mathbb{H}$, and every K -admissible set A for the pair (x, n) , one has*

$$(59) \quad \text{Cap}_{d-4}(\partial_* A) \geq c|\partial_* A|.$$

Proof. For the reader's convenience, we include a short proof. By taking μ to be the uniform measure on $\partial_* A$ in the definition of the $(d-4)$ -capacity, one gets

$$(60) \quad \text{Cap}_{d-4}(\partial_* A) \geq \frac{|\partial_* A|^2}{\sum_{a, a' \in \partial_* A} (1 + \|a - a'\|)^{4-d}}.$$

Let ω be a percolation configuration realising $\{\mathcal{C}_n^e(x) = A\}$ (note that it exists since A is K -admissible for the pair (x, n)). By definition, to every fixed $a \in \partial_* A$ corresponds a unique $z_a \in \partial B_n(x)$ such that ω realises the events $\{z_a \xleftrightarrow{B_n(x)} x\}$ and $\mathcal{T}_s(z_a)$ for every $s \geq K$. Thus, one has that, for every $a \in \partial_* A$, and every $s \geq K$,

$$(61) \quad |\partial_* A \cap B_s(a)| \lesssim |\partial_* A \cap B_s(z_a)| \lesssim |\mathcal{C}(z_a; B_n(x)) \cap B_s(z_a) \cap \partial B_n(x)| \leq s^2(\log s)^7,$$

where the implicit constants do not depend on x, n, K, ε , and A . As a consequence, for every fixed $a \in \partial_* A$, we find that

$$(62) \quad \sum_{a' \in \partial_* A} (1 + \|a - a'\|)^{4-d} \lesssim 1 + \sum_{i \geq \log_2(K)} \frac{|\partial_* A \cap (B_{2^{i+1}}(a) \setminus B_{2^i}(a))|}{2^{i(d-4)}} \lesssim 1 + \sum_{i \geq \log_2(K)} \frac{2^{2i}(\log 2^i)^7}{2^{i(d-4)}} \lesssim 1,$$

where in the first inequality we used that (by definition) $(\partial_* A \setminus \{a\}) \cap B_K(a) = \emptyset$, in the second inequality we used (61), and where (again) the implicit constants do not depend on x, n, K, ε , and A . This concludes the proof. \square

4.3. Conclusion

We now have all the necessary material to prove our desired result.

Proof of Proposition 2.1. Let $\varepsilon \in (0, 1/2)$. Let $x, y \in \mathbb{H}$ and set $n = \lfloor |x - y|/3 \rfloor$. Fix K_0, n_0 as in Lemma 4.3. Let $K \geq K_0$ to be chosen large enough and assume that $n \geq n_0$.

We first observe that

$$(63) \quad \tau_{\mathbb{H}}(x, y) \geq \sum_{A, B} \mathbb{P}[\mathcal{C}_n^e(x) = A, \mathcal{C}_n^e(y) = B, \partial_* A \xrightarrow{\text{off}(A \cup B)} \partial_* B],$$

where $\{\partial_* A \xrightarrow{\text{off}(A \cup B)} \partial_* B\}$ is the event that $\partial_* A$ is connected to $\partial_* B$ by an open path in \mathbb{H} that avoids $A \cup B$, except at its end points. Note next that the two events $\{\mathcal{C}_n^e(x) = A, \mathcal{C}_n^e(y) = B\}$ and $\{\partial_* A \xrightarrow{\text{off}(A \cup B)} \partial_* B\}$ depend on different sets of edges, and are thus independent. Hence, we get

$$(64) \quad \tau_{\mathbb{H}}(x, y) \geq \sum_{A, B} \mathbb{P}[\mathcal{C}_n^e(x) = A, \mathcal{C}_n^e(y) = B] \cdot \mathbb{P}[\partial_* A \xrightarrow{\text{off}(A \cup B)} \partial_* B].$$

Given A, B which are K -admissible respectively for (x, n) and (y, n) , we define

$$(65) \quad C = A \cup B \cup \left(\bigcup_{a \in \partial_* A} B_K(a) \right) \cup \left(\bigcup_{b \in \partial_* B} B_K(b) \right).$$

Now, we fix an arbitrary ordering of the elements of $\partial_* A$ and $\partial_* B$. On the event $\{\partial_* A \xrightarrow{\text{off}(A \cup B)} \partial_* B\}$, we denote by Y_1 the first element $a \in \partial_* A$ for this ordering such that $\partial B_K(a)$ is connected to $\cup_{b \in \partial_* B} B_K(b)$ by an open path that avoids C , and let Y_2 be the first element $b \in \partial_* B$ such that $\partial B_K(Y_1)$ is connected to $\partial B_K(b)$ by an open path that avoids C . Finally, we let

$$(66) \quad H = (A \cup B) \setminus \left(\bigcup_{z \in \partial_* A \cup \partial_* B} L_z \right),$$

where for any $a \in \partial_* A$ we let L_a be the line segment of length K between a and $B_n(x)$, and similarly for $b \in \partial_* B$. Letting $c(K) > 0$ be the probability that all edges are open in a box of size K , we find

$$(67) \quad \begin{aligned} \mathbb{P}[\partial_* A \xrightarrow{\text{off}(A \cup B)} \partial_* B] &\geq \sum_{a \in \partial_* A} \sum_{b \in \partial_* B} \mathbb{P}[Y_1 = a, Y_2 = b, \text{all edges in } B_K(a) \cup B_K(b) \text{ are open}] \\ &= c(K)^2 \cdot \mathbb{P}\left[\left(\cup_{a \in \partial_* A} B_K(a)\right) \xrightarrow{\text{off } C} \left(\cup_{b \in \partial_* B} B_K(b)\right)\right] \\ &= c(K)^2 \cdot \mathbb{P}\left[\left(\cup_{a \in \partial_* A} B_K(a)\right) \xrightarrow{\text{off } H} \left(\cup_{b \in \partial_* B} B_K(b)\right)\right] \\ &\geq c(K)^2 \cdot \mathbb{P}[\partial_* A \xrightarrow{\text{off } H} \partial_* B]. \end{aligned}$$

We then write

$$(68) \quad \mathbb{P}[\partial_* A \xrightarrow{\text{off } H} \partial_* B] = \mathbb{P}[\partial_* A \leftrightarrow \partial_* B] - \mathbb{P}[\partial_* A \xrightarrow{\text{via } H} \partial_* B],$$

where $\{\partial_* A \xrightarrow{\text{via } H} \partial_* B\}$ denotes the event that $\partial_* A$ and $\partial_* B$ are connected by an open path, and all open paths that connect them intersect H . We claim that for any constant $\delta > 0$, one can find $K \geq K_0$ large enough, so that for all admissible sets A and B ,

$$(69) \quad \mathbb{P}[\partial_* A \xrightarrow{\text{via } H} \partial_* B] \leq \frac{\delta}{n^{d-2}} \cdot |\partial_* A| \cdot |\partial_* B|.$$

To see this, we note that by a union bound, it suffices to show that for K sufficiently large, for every $a \in \partial_* A$ and $b \in \partial_* B$, one has

$$(70) \quad \mathbb{P}[a \xrightarrow{\text{via } H} b] \leq \frac{\delta}{n^{d-2}}.$$

Now, decomposing an open self-avoiding path connecting a and b through H according to the first point in H it visits, and using (BK) and (*), we deduce that

$$(71) \quad \begin{aligned} \mathbb{P}[a \xrightarrow{\text{via } H} b] &\leq \sum_{u \in H} \tau_{\mathbb{H}}(a, u) \tau_{\mathbb{H}}(u, b) \lesssim \frac{1}{n^{d-2}} \left(\sum_{u \in H \cap A} \tau_{\mathbb{H}}(a, u) + \sum_{u \in H \cap B} \tau_{\mathbb{H}}(u, b) \right) \\ &\lesssim \frac{1}{n^{d-2}} \sum_{i \geq \log_2(K)} \frac{2^{4i} (\log 2^i)^7}{2^{i(d-2)}} \lesssim \frac{1}{n^{d-2} \sqrt{K}}, \end{aligned}$$

where we used that

$$(72) \quad \sum_{u \in H \cap A} \tau_{\mathbb{H}}(a, u) \lesssim \sum_{i \geq \log_2(K)} \frac{|H \cap A \cap (B_{2^{i+1}}(a) \setminus B_{2^i}(a))|}{2^{i(d-2)}} \lesssim \sum_{i \geq \log_2(K)} \frac{2^{4i} (\log 2^i)^7}{2^{i(d-2)}},$$

which follows by using that $H \cap B_{K-1}(a) = \emptyset$, and by proving—similarly to (61)—that $|H \cap A \cap B_s(a)| \lesssim s^4 (\log s)^7$ for every $s \geq K$. As a consequence, (71) gives (70)—and hence (69)—by choosing K large enough.

On the other hand, by combining Lemmas 4.1 and 4.4, we obtain that for every K -admissible sets A and B ,

$$(73) \quad \mathbb{P}[\partial_* A \longleftrightarrow \partial_* B] \gtrsim \left(\min_{\substack{a \in \partial_* A \\ b \in \partial_* B}} \tau_{\mathbb{H}}(a, b) \right)^2 \cdot n^{d-2} \cdot |\partial_* A| \cdot |\partial_* B|.$$

If $a \in \partial_* A$ (resp. $b \in \partial_* B$), then $a_1 \geq \varepsilon n$ (resp. $b_1 \geq \varepsilon n$). As a result, for $(a, b) \in \partial_* A \times \partial_* B$, one has $n \lesssim |a - b| \lesssim \min(a_1, b_1)$ (where the implicit constants depend on ε). This is where we crucially require that $\varepsilon > 0$. We can therefore use (3) to conclude that

$$(74) \quad \left(\min_{\substack{a \in \partial_* A \\ b \in \partial_* B}} \tau_{\mathbb{H}}(a, b) \right)^2 \gtrsim (n^{2-d})^2.$$

Altogether, this shows that, with the notation of Lemma 4.3,

$$(75) \quad \begin{aligned} \tau_{\mathbb{H}}(x, y) &\gtrsim \frac{1}{n^{d-2}} \sum_{A, B} \mathbb{P}[\mathcal{C}_n^e(x) = A, \mathcal{C}_n^e(y) = B] \cdot |\partial_* A| \cdot |\partial_* B| \\ &= \frac{1}{n^{d-2}} \cdot \mathbb{E}[|\partial_* \mathcal{C}_n^e(x)|] \cdot \mathbb{E}[|\partial_* \mathcal{C}_n^e(y)|] \\ &\gtrsim \frac{1}{n^{d-2}} \cdot \mathbb{E}[X_n^{\varepsilon, K-\text{reg}}(x)] \cdot \mathbb{E}[X_n^{\varepsilon, K-\text{reg}}(y)] \\ &\gtrsim \frac{1}{n^{d-2}} \mathbb{E}[X_n^\varepsilon(x)] \cdot \mathbb{E}[X_n^\varepsilon(y)], \end{aligned}$$

where we used Lemma 4.3 in the last inequality. Using (*) in (75) yields the existence of $c = c(\varepsilon) > 0$ such that, for every $x, y \in \mathbb{H}$ satisfying $\lfloor |x - y|/3 \rfloor \geq n_0$,

$$(76) \quad \tau_{\mathbb{H}}(x, y) \geq c \sum_{u \in \partial B_n^\varepsilon(x)} \sum_{v \in \partial B_n^\varepsilon(y)} \tau_{B_n(x)}(x, u) \cdot \tau_{\mathbb{H}}(u, v) \cdot \tau_{B_n(y)}(v, y).$$

This concludes the proof. \square

4.4. Proof of Lemma 4.3

The proof of Lemma 4.3 is very similar to the proofs of Theorem 4 in [KN11] and Proposition 5.7 in [ASS25]. First, one needs to introduce a local density condition. To be more precise, fix $n \geq 1$, $x \in \mathbb{H}$, and for $s > 0$ and $z \in \partial B_n(x)$, consider the event

$$(77) \quad \begin{aligned} \mathcal{T}_s^{\text{loc}}(z) = & \left\{ |\mathcal{C}(y; B_{s^d}(z) \cap B_n(x)) \cap B_s(z)| \leq s^4 (\log s)^4, \forall y \in B_s(z) \right\} \\ & \cap \left\{ |\mathcal{C}(y; B_{s^d}(z) \cap B_n(x)) \cap B_s(z) \cap \partial B_n(x)| \leq s^2 (\log s)^4, \forall y \in B_s(z) \cap \partial B_n(x) \right\} \\ & \cap \left\{ \exists \text{ at most } (\log s)^3 \text{ disjoint paths from } B_s(z) \text{ to } \partial B_{s^d}(z) \text{ in } B_n(x) \right\}. \end{aligned}$$

The interest of this event, when compared to $\mathcal{T}_s(z)$, is that it only depends on the configuration of the percolation inside the box $B_{s^d}(z)$, and is thus a purely local event, while to determine whether $\mathcal{T}_s(z)$ holds or not, one needs a priori to know the configuration in the whole box $B_n(x)$. The drawback is that this event is a priori less likely than $\mathcal{T}_s(z)$, but as Lemma 4.6 below shows, it is still extremely likely, and furthermore, the following simple fact holds by construction (see Claim 4.1 in [KN11] or Claim 5.4 in [ASS25]).

Lemma 4.5. *One has for every $n \geq 1$, every $x \in \mathbb{H}$, every $z \in \partial B_n(x)$, and every $s > 0$,*

$$(78) \quad \mathcal{T}_s^{\text{loc}}(z) \subseteq \mathcal{T}_s(z).$$

As already mentioned, another fact we will use, and which is proved in [ASS25, Claim 5.5], is the following.

Lemma 4.6. *There exists a constant $c > 0$, such that for every $n \geq 1$, every $x \in \mathbb{H}$, every $z \in \partial B_n(x)$, and every $s > 0$,*

$$(79) \quad \mathbb{P}[\mathcal{T}_s^{\text{loc}}(z)] \geq 1 - \exp(-c(\log s)^4).$$

The last fact we shall need is Lemma 1.1 from [KN11], which we state here for the sake of completeness.

Lemma 4.7. *There exist positive constants c and C , such that for every $u \in \mathbb{H}$, every $s > 0$, and every $v \in \partial B_s(u)$,*

$$(80) \quad \tau_{B_s(u)}(u, v) \geq c \exp(-C(\log s)^2).$$

Using the FKG inequality, it follows from Lemma 4.7, that the probability to connect two arbitrary points of $\partial B_s(u)$ (for some $u \in \mathbb{H}$) is at least $c^2 \exp(-2C(\log s)^2)$. We are now in a position to prove Lemma 4.3.

Proof of Lemma 4.3. Fix $x \in \mathbb{H}$ and $\varepsilon \in (0, 1)$. Let $n, K \geq 1$ to be fixed. Let us say that a point $z \in \partial B_n(x)$ is s -locally bad if the event $\mathcal{T}_s^{\text{loc}}(z)$ does not hold, and let us denote by $X_n^{\varepsilon, s\text{-loc-bad}}(x)$ the number of points on $\partial B_n^\varepsilon(x)$ which are s -locally bad and connected to x in $B_n(x)$. Note that, due to Lemma 4.5, one has

$$(81) \quad \mathbb{E}[X_n^{\varepsilon, K\text{-reg}}(x)] \geq \mathbb{E}[X_n^\varepsilon(x)] - \sum_{s \geq K} \mathbb{E}[X_n^{\varepsilon, s\text{-loc-bad}}(x)].$$

We now upper bound each term of the above sum separately. First of all, for s such that $8s^d > n$, we simply use Lemma 4.6 and a union bound over all the points on $\partial B_n(x)$, to get (for some $c_1 = c_1(d) > 0$)

$$(82) \quad \mathbb{E}[X_n^{\varepsilon, s\text{-loc-bad}}(x)] \lesssim n^{d-1} \cdot \exp(-c_1(\log n)^4).$$

Additionally, for s such that s^2 is much larger than n^{d-1} (or larger than the total number of points on $\partial B_n(x)$), the set of s -locally bad points is empty by definition so that $\mathbb{E}[X_n^{\varepsilon, s\text{-loc-bad}}(x)] = 0$. On the other hand one has by Lemma 4.7 (and the remark following it),

$$(83) \quad \mathbb{E}[X_n^\varepsilon(x)] \gtrsim \exp(-C_1(\log n)^2),$$

for some constant $C_1 > 0$. Combining the two previously displayed equations, one can choose n large enough so that

$$(84) \quad \sum_{s: 8s^d > n} \mathbb{E}[X_n^{\varepsilon, s\text{-loc-bad}}(x)] \leq \frac{1}{4} \cdot \mathbb{E}[X_n^\varepsilon(x)].$$

We now fix $s \geq K$ such that $8s^d \leq n$ and consider the set $U = \{u \in \mathbb{Z}^d : u_i \in \{0, s^d\} \forall i = 1, \dots, d\}$. For each $w \in U$, we define

$$(85) \quad \mathcal{B}(w) = \{B_{2s^d}(z) : z \in w + 4s^d \cdot \mathbb{Z}^d\}.$$

Denote by $Q(w)$ the union of all the boxes of this partition which intersect $\partial B_n^\varepsilon(x)$. We explore them using the following algorithm. First, we reveal the whole percolation configuration outside the union of these boxes. Next, if at least one of the boxes of $Q(w)$ is connected to x by an open path, that remains in the explored region, we choose one at random and reveal the configuration inside it, and we continue as long as there still exist an unexplored box of $Q(w)$ which is connected to x via an open path in the explored region. We let $N(w)$ be the number of boxes of $Q(w)$ which have been revealed by this algorithm. Note that during this exploration procedure, each time we reveal the configuration inside a new box, almost surely, conditionally on the configuration outside the box, the probability that it contains a pioneer point is at least $c_2 \exp(-C_2(\log s)^2)$ by Lemma 4.7 and the remark following it, for some constant $c_2, C_2 > 0$. Hence, for any $w \in U$,

$$(86) \quad \mathbb{E}[X_n^\varepsilon(x)] \geq c_2 \exp(-C_2(\log s)^2) \cdot \mathbb{E}[N(w)],$$

and hence also (since $|U| = 2^d$),

$$(87) \quad \mathbb{E}[X_n^\varepsilon(x)] \geq \frac{c_2}{2^d} \exp(-C_2(\log s)^2) \cdot \sum_{w \in U} \mathbb{E}[N(w)].$$

For a box $q \in Q(w)$, call the *interior* of q the set of points in q which are at distance at least s^d from the points which are in $q^c \cap B_n(x)$. Observe that as w varies in U , the union of all the interiors of the boxes $q \in Q(w)$ covers the whole boundary $\partial B_n^\varepsilon(x)$ (recall that we assume $8s^d \leq n$). Note also that for a point z on $\partial B_n^\varepsilon(x)$ which is in the interior of a box $q \in Q(w)$, the event $\mathcal{T}_s^{\text{loc}}(z)$ only depends on the configuration of edges inside q . Since there are at most order s^{d^2} such points in each box $q \in Q(w)$, a union bound and Lemma 4.6 give that for some constants $c_3, C_3 > 0$, and for any s as above,

$$(88) \quad \mathbb{E}[X_n^{\varepsilon, s\text{-loc-bad}}(x)] \leq C_3 s^{d^2} \exp(-c_3(\log s)^4) \sum_{w \in U} \mathbb{E}[N(w)].$$

Hence, using again (87), and taking K large enough ensures that for all $n \geq 1$,

$$(89) \quad \sum_{s: K \leq s \leq (n/8)^{1/d}} \mathbb{E}[X_n^{\varepsilon, s\text{-loc-bad}}(x)] \leq \frac{1}{4} \cdot \mathbb{E}[X_n^\varepsilon(x)].$$

Together with (81) and (84), this concludes the proof of the lemma. \square

Acknowledgments

RP thanks Hugo Duminil-Copin for stimulating discussions at an early stage of this project.

Funding

The first author was supported by the Swiss National Science Foundation through a Postdoc.Mobility grant. The second author was supported by the grant ANR-22-CE40-0012 (project LOCAL).

References

- [ASS25] Amine Asselah, Bruno Schapira, and Perla Sousi. Capacity in high dimensional percolation. *Preprint*, 2025. <https://arxiv.org/pdf/2509.21253>.
- [BS85] David Brydges and Thomas Spencer. Self-avoiding walk in 5 or more dimensions. *Communications in Mathematical Physics*, **97**(1):125–148, 1985.
- [CH20] Shirshendu Chatterjee and Jack Hanson. Restricted percolation critical exponents in high dimensions. *Communications on Pure and Applied Mathematics*, **73**(11):2370–2429, 2020.
- [CHS23] Shirshendu Chatterjee, Jack Hanson, and Philippe Sosoe. Subcritical connectivity and some exact tail exponents in high dimensional percolation. *Communications in Mathematical Physics*, **403**(1):83–153, 2023.

- [DCP25a] Hugo Duminil-Copin and Romain Panis. An alternative approach for the mean-field behaviour of spread-out Bernoulli percolation in dimensions $d > 6$. *Probability Theory and Related Fields*, 2025.
- [DCP25b] Hugo Duminil-Copin and Romain Panis. An alternative approach for the mean-field behaviour of weakly self-avoiding walks in dimensions $d > 4$. *Probability Theory and Related Fields*, 2025.
- [DCT16] Hugo Duminil-Copin and Vincent Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. *Communications in Mathematical Physics*, **343**:725–745, 2016.
- [FvdH17] Robert Fitzner and Remco W. van der Hofstad. Mean-field behavior for nearest-neighbor percolation in $d > 10$. *Electronic Journal of Probability*, **22**:43, 2017.
- [Gri99] Geoffrey Grimmett. *Percolation*, volume **321**. Springer, 1999.
- [Har08] Takashi Hara. Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. *The Annals of Probability*, **36**(2):530–593, 2008.
- [HHS03] Takashi Hara, Remco van der Hofstad, and Gordon Slade. Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. *The Annals of Probability*, **31**(1):349–408, 2003.
- [HMS23] Tom Hutchcroft, Emmanuel Michta, and Gordon Slade. High-dimensional near-critical percolation and the torus plateau. *The Annals of Probability*, **51**(2):580–625, 2023.
- [HS90] Takashi Hara and Gordon Slade. Mean-field critical behaviour for percolation in high dimensions. *Communications in Mathematical Physics*, **128**(2):333–391, 1990.
- [HS14] Remco van der Hofstad and Artem Sapozhnikov. Cycle structure of percolation on high-dimensional tori. *Annales de l'Institut Henri Poincaré: Probabilités et Statistiques*, **50**:999–1027, (2014).
- [Hut25] Tom Hutchcroft. Dimension dependence of critical phenomena in long-range percolation. *Preprint*, 2025. <https://arxiv.org/pdf/2510.03951>.
- [KN11] Gady Kozma and Asaf Nachmias. Arm exponents in high dimensional percolation. *Journal of the American Mathematical Society*, **24**(2):375–409, 2011.
- [LL10] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume **123**. Cambridge University Press, 2010.
- [Pan24] Romain Panis. *Applications of path expansions to statistical mechanics*. PhD thesis, PhD thesis, University of Geneva, 2024.
- [Sla06] Gordon Slade. *The Lace Expansion and Its Applications: Ecole D'Eté de Probabilités de Saint-Flour XXXIV-2004*. Springer, 2006.