Large Deviations for Intersections of Random Walks

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Abstract

We prove a Large Deviations Principle for the number of intersections of two independent infinite-time ranges in dimension five and more, improving upon the moment bounds of Khanin, Mazel, Shlosman and Sinaï [10]. This settles, in the discrete setting, a conjecture of van den Berg, Bolthausen and den Hollander [7], who analyzed this question for the Wiener sausage in finite-time horizon. The proof builds on their result (which was resumed in the discrete setting by Phetpradap [13]), and combines it with a series of tools that were developed in recent works of the authors [2, 3, 6]. Moreover, we show that most of the intersection occurs in a single box where both walks realize an occupation density of order one.

1 Introduction

1.1 Overview and results

In 1921, Pólya [14] presents his recurrence theorem, inspired by some counter-intuitive observation on the large number of intersections two random walkers in a park would make. A hundred years later, the study of intersections of random walks is still active, and produces perplexing problems. This paper is devoted to estimating deviations for the number of sites two infinite trajectories both visit, when dimension is five or larger.

It is known since the work of Erdös and Taylor [9], that the number of intersections of two independent random walk ranges on $\mathbb{Z}^d$ is almost surely infinite if $d \leq 4$, and finite if $d \geq 5$. In 1994, Khanin, Mazel, Shlosman and Sinaï [10] obtain the following bounds in dimension $d \geq 5$: for any $\varepsilon > 0$, and all t large enough,

$$\exp(-t^{1-\frac{2}{d}+\varepsilon}) \leq \mathbb{P}(|\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty| > t) \leq \exp(-t^{1-\frac{2}{d}-\varepsilon}), \tag{1.1}$$

where $\mathscr{R}_\infty$ and $\widetilde{\mathscr{R}}_\infty$ denote two independent ranges, and $|\Lambda|$ is the number of sites of $\Lambda \subset \mathbb{Z}^d$. About ten years later, van den Berg, Bolthausen and den Hollander [7] prove a Large Deviations Principle for the Wiener sausage (the continuous counterpart of the range), in *a finite-time horizon*. Their result was resumed in the

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discrete setting by Phetpradap [13] and reads as follows: for any $b>0$, there exists a positive constant $\mathscr{I}(b)$, such that

$$\lim_{t\to\infty}\frac{1}{t^{1-\frac{2}{d}}}\log\mathbb{P}(|\mathscr{R}_{bt}\cap\widetilde{\mathscr{R}}_{bt}|>t)=-\mathscr{I}(b), \tag{1.2}$$

where $\mathscr{R}_{bt}$ and $\widetilde{\mathscr{R}}_{bt}$ denote the ranges of two independent walks up to time $\lfloor bt\rfloor$. Furthermore, through an analysis of the variational formula of the rate function, the authors of [7] show that $\mathscr{I}(b)$ reaches a plateau and conjecture that the rate function for the infinite-time problem coincides with the value of $\mathscr{I}$ at the plateau. Our first result confirms this conjecture. The ranges of two independent simple random walks is denoted $\{\mathscr{R}_n, n\in\mathbb{N}\cup\{\infty\}\}$ and $\{\widetilde{\mathscr{R}}_n, n\in\mathbb{N}\cup\{\infty\}\}$.

Theorem 1.1. *Assume $d\geq 5$. The following limit exists and is positive:*

$$\mathscr{I}_\infty:=\lim_{t\to\infty}-\frac{1}{t^{1-\frac{2}{d}}}\log\mathbb{P}(|\mathscr{R}_\infty\cap\widetilde{\mathscr{R}}_\infty|>t). \tag{1.3}$$

Moreover, there exists $b_*>0$, such that for all $b>b_*$,

$$\mathscr{I}_\infty=\mathscr{I}(b)=\lim_{t\to\infty}-\frac{1}{t^{1-\frac{2}{d}}}\log\mathbb{P}(|\mathscr{R}_{bt}\cap\widetilde{\mathscr{R}}_{bt}|>t). \tag{1.4}$$

For $\mathscr{I}_\infty$ and b_*, [7] presents variational formulas whose thorough study leads to a rich and precise phenomenology. Namely, that the two walks adopt the same strategy, the so-called Swiss cheese during a time b_*t, in a ball-like region whose volume should be of order t, leaving holes everywhere of size order 1. After time b_*t, the two walks would roam as typical random walks.

Our second result shows that a fraction arbitrarily close to one of the desired number of intersections occurs in a box with volume of order t. To state the result, define $Q(x,r):=[x-r/2,x+r/2)^d$, for $x\in\mathbb{Z}^d$, and $r>0$.

Theorem 1.2. *For any $\varepsilon>0$, there exists a constant $L=L(\varepsilon)>0$, such that*

$$\lim_{t\to\infty}\mathbb{P}(\exists x\in\mathbb{Z}^d\,:\,|\mathscr{R}_\infty\cap\widetilde{\mathscr{R}}_\infty\cap Q(x,Lt^{1/d})|>(1-\varepsilon)t\;|\;|\mathscr{R}_\infty\cap\widetilde{\mathscr{R}}_\infty|>t)=1. \tag{1.5}$$

Our proof provides some bound on $L(\varepsilon)$, which is (stretched) exponential in $1/\varepsilon$. We note that it is expected that L should depend on ε, since the Swiss cheese is delocalized, see [7]. Concerning the (random) site $X(t,\varepsilon)$ realizing the centering of the box appearing in the statement of Theorem 1.2, not much is known. Our proof yields tightness of $X(t,\varepsilon)/t^{1/d}$.

Sznitman in [15] formalized precisely the picture of Swiss cheese using a tilted version of the Random Interlacements, but so far no rigorous link has been established with the large deviations for the volume of the range nor for the intersection of two ranges.

Our techniques are robust enough to consider other natural functionals of two ranges, which do not seem to be tractable by moment methods, as in [10]. In

particular in [5] we consider the functional $\chi_{\mathscr{C}}(\cdot,\cdot)$ defined for finite subsets $A,B\subseteq\mathbb{Z}^d$, by

$$\chi_{\mathscr{C}}(A,B)=\mathrm{cap}(A)+\mathrm{cap}(B)-\mathrm{cap}(A\cup B),$$

where $\mathrm{cap}(A):=\sum_{x\in A}\mathbb{P}_x(\mathscr{R}[1,\infty)\cap A=\varnothing)$, denotes the capacity of A. It turns out that this definition may be extended to infinite subsets. Indeed, one has for any finite $A,B\subseteq\mathbb{Z}^d$, $\chi_{\mathscr{C}}(A,B)\le\chi(A,B)$ with

$$\chi(A,B):=2\sum_{x\in A}\sum_{y\in B}\mathbb{P}_x(\mathscr{R}[1,\infty)\cap A=\varnothing)\cdot G(y-x)\cdot\mathbb{P}_y(\mathscr{R}[1,\infty)\cap B=\varnothing),$$

and it makes sense to consider $\chi(\mathscr{R}_\infty,\widetilde{\mathscr{R}}_\infty)$. In [5], we show using similar arguments as here that in dimension $d\ge 7$, for some positive constants c_1,c_2, and all t large enough,

$$\exp(-c_1t^{1-\frac{2}{d-2}})\le\mathbb{P}(\chi(\mathscr{R}_\infty,\widetilde{\mathscr{R}}_\infty)>t)\le\exp(-c_2t^{1-\frac{2}{d-2}}).$$

These bounds are used in turn to derive a moderate deviations principle for the capacity of the range in the Gaussian regime.

Interestingly, a related object, the mutual intersection local time defined by

$$J_\infty:=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbf{1}\{S_i=\widetilde{S}_j\},$$

has a stretched exponential tail with a different exponent. Indeed, Khanin et al. in [10], also show that for some positive constants c and c', for all t large enough,

$$\exp(-c\sqrt{t})\le\mathbb{P}(J_\infty>t)\le\exp(-c'\sqrt{t}).$$

Chen and Mörters [8] then prove that the limit of $t^{-1/2}\cdot\log\mathbb{P}(J_\infty>t)$ exists and has a nice variational representation. Our proofs allow to consider some intermediate quantity, the time spent by one walk on the range of the other walk, and show that its tail distribution has the same speed of decay as the intersection of two ranges. More precisely, consider two independent walks S and $\widetilde{S}$, and denote by $\widetilde{\ell}_\infty$ the local times associated to $\widetilde{S}$ (see below for a definition).

Proposition 1.3. *There exists two positive constants c_1 and c_2, such that for any $t>0$,*

$$\exp(-c_1t^{1-\frac{2}{d}})\le\mathbb{P}(\widetilde{\ell}_\infty(\mathscr{R}_\infty)>t)\le\exp(-c_2t^{1-\frac{2}{d}}). \tag{1.6}$$

Furthermore, for any $\varepsilon>0$, there exists an integer $N=N(\varepsilon)$, such that

$$\lim_{t\to\infty}\mathbb{P}\left(\exists x_1,\dots,x_N\in\mathbb{Z}^d:\widetilde{\ell}_\infty\left(\mathscr{R}_\infty\cap\bigcup_{i=1}^{N}Q(x_i,t^{1/d})\right)>(1-\varepsilon)t\mid\widetilde{\ell}_\infty(\mathscr{R}_\infty)>t\right)=1. \tag{1.7}$$

Analogous results such as Theorems 1.1 and 1.2 would hold for $\widetilde{\ell}_\infty(\mathscr{R}_\infty)$, conditionally on obtaining first an analogue of (1.2) for $\widetilde{\ell}_{bt}(\mathscr{R}_{bt})$, which is presumably true, but not available at the moment.

Let us remark also that the problem we address here has a flavor of a much studied problem of *random walk in random landscape*, where the random landscape is produced here by another independent walk. Here also, it appears interesting to study a *quenched regime*, where one walk is frozen in a typical realization, whereas the second tries to hit t sites of the first range. This problem is still untouched, and we believe that our techniques will shed some light on it.

1.2 Proof strategy

While the proof in [10] used a moment method and some ingenious computations, our proof is based on more geometric arguments.

There are two parts. In the first one, we show that conditionally on the intersection event, with probability going to one, the whole intersection takes place in a finite number of boxes (as in Proposition 1.3 above). In the second part we use the full power of the LDP (1.2) and the concavity of the speed $t \mapsto t^{1-\frac{2}{d}}$ to show how the *energy gain* in reducing the number of boxes wins over the *entropy loss* associated to reducing the volume where the two walks meet. This leads to Theorem 1.2, which in turns leads to Theorem 1.1.

The first part is itself obtained in three steps. First we reduce the time window to a finite time interval, using that it is unlikely for one walk to intersect the range of the other walk after a time of order $\exp(\beta \cdot t^{1-\frac{2}{d}})$, for some large β. This leaves however a lot of room for the places where the action could take place (since we recall it holds in a box with volume of order t only). In particular decomposing space into boxes and using a union bound type argument would not work, at least not directly. Our main idea to overcome this difficulty is to divide space according to the occupation density of the range, which we do at different space-scales depending on the density we are considering, in a similar fashion as in [3, 5]. Then we use a fundamental tool from [3] which gives a priori bounds on the size of these regions, with the conclusion that it is only in those with high density (of order one) that the intersection occurs. Finally we use another recent result from [6], which bounds the probability to cover a positive fraction of any *fixed* union of distant boxes. When we further impose that these boxes are visited by another independent walk, one can sum over all possible centers of the boxes, and this yields some bound on the number of boxes, with volume of the right order, that are needed to cover the region where the intersection occurs.

For the second part of the proof, we decompose the journeys between a finite number of boxes into excursions either within one box, or joining two boxes. Then some surgery is applied. We cut the excursions between different boxes and replace them by excursions drawn independently with starting points sampled according to the harmonic measure. This allows to compare the probability of the event when the walk realizes the intersection in N different boxes, to the product of the probabilities of realizing (smaller) intersection in each of these boxes, and one can then use (1.2) to bound these probabilities. This is also where the concavity is used,

to quantify how much one box is better than many. The surgery arguments are used again to restore the journeys and yield Theorem 1.1.

1.3 Organization

The paper is organized as follows. In the next section we recall the main notation, and the tools that will be used in the proofs, which for the most part appeared in our previous works [2, 3, 5, 6]. In Section 3 we give a detailed plan of the proofs of ours main results. The latter are then proved in the remaining sections 4–5–6.

2 Notation and main tools

2.1 Notation and basic results

Let $\{S_n\}_{n\geq 0}$ be a simple random walk on $\mathbb{Z}^d$. We denote by $\mathbb{P}_x$ its law starting from x, which we abbreviate as $\mathbb{P}$ when $x=0$. We mainly assume here that $d\geq 5$, yet some results hold for all $d\geq 3$, in which case we shall mention it explicitly. For $n\in\mathbb{N}\cup\{\infty\}$, we write the range of the walk up to time n as $\mathscr{R}_n := \{S_0,\dots,S_n\}$. More generally for $n\leq m$ two (possibly infinite) integers, we consider the range between times n and m, defined as $\mathscr{R}[n,m] := \{S_n,\dots,S_m\}$. For $\Lambda\subseteq\mathbb{Z}^d$, and $n\in\mathbb{N}\cup\{\infty\}$, we define the time spent in Λ as

$$\ell_n(\Lambda) := \sum_{k=0}^{n} \mathbf{1}\{S_k\in\Lambda\},$$

and simply let $\ell_n(z)$ be the time spent on a site $z\in\mathbb{Z}^d$. The **Green's function** is defined by

$$G(x,z) := \sum_{k=0}^{\infty} \mathbb{P}_x(S_k=z) = \mathbb{E}_x[\ell_\infty(z)].$$

By translation invariance one has for any $x,z\in\mathbb{Z}^d$, $G(x,z)=G(0,z-x)=:G(z-x)$. Thus for any $\Lambda\subset\mathbb{Z}^d$, and any $x\in\mathbb{Z}^d$,

$$\mathbb{E}_x[\ell_\infty(\Lambda)] = \sum_{z\in\Lambda} G(x,z) = \sum_{z\in\Lambda-x} G(z) =: G(\Lambda-x).$$

The asymptotics of Green's function are well known (see Theorem 4.3.1 of [12]). If $\|\cdot\|$ denotes the Euclidean norm, there exists $C>0$, such that

$$\forall z\in\mathbb{Z}^d,\qquad G(z)\leq \frac{C}{1+\|z\|^{d-2}}. \tag{2.1}$$

Finally, recall a basic hitting time estimate (Proposition 6.5.1 of [12]). For all $r>0$ and $z\in\mathbb{Z}^d$, with $\|z\|>r$,

$$\mathbb{P}\big(\,|\mathscr{R}_\infty\cap Q(z,r)|>0\,\big)\leq C\Big(\frac{r}{\|z\|}\Big)^{d-2}. \tag{2.2}$$

2.2 Preliminaries

We recall and discuss here a series of known results on which relies our proof. Most of them come from our recent works [2, 3, 5, 6].

In fact the first one is older and shows that the tail distribution of the time spent in a region is controlled simply by its mean value, when starting from the worst point. We recall its short proof for completeness.

Lemma 2.1 ([1])**.** *Let $\Lambda \subseteq \mathbb{Z}^d$ be a (non necessarily finite) subset of $\mathbb{Z}^d$, $d \geq 3$. Then for any $t > 0$,*

$$\mathbb{P}(\ell_\infty(\Lambda) > t) \leq 2\exp\left(-\frac{t \cdot \log 2}{2\sup_{x\in\Lambda} G(\Lambda - x)}\right).$$

Proof. The result simply follows from the fact that by Markov's inequality (and the Markov property), the random variable $\frac{\ell_\infty(\Lambda)}{2\sup_{x\in\Lambda}\mathbb{E}_x[\ell_\infty(\Lambda)]} = \frac{\ell_\infty(\Lambda)}{2\sup_{x\in\Lambda} G(\Lambda - x)}$, is stochastically bounded by a geometric random variable with parameter $1/2$. □

We need also to estimate the expected time spent (or equivalently the sum of the Green's function) on the range of an independent random walk. For this we use several facts. The first one is the following well-known simple lemma.

Lemma 2.2. *There exists $C > 0$, such that for any finite subset $\Lambda \subseteq \mathbb{Z}^d$, $d \geq 3$, one has*

$$G(\Lambda) = \sum_{z\in\Lambda} G(z) \leq C|\Lambda|^{2/d}.$$

Proof. The result follows from the bound (2.1), and observing that the resulting sum is maximized (at least up to a constant) when points of Λ are all contained in a ball of side-length of order $|\Lambda|^{1/d}$. □

Now we decompose the points of the range in several subsets according to the occupation density in some neighborhoods of these points, and use that Green's function is additive in the sense that for any disjoints subsets $\Lambda, \Lambda' \subseteq \mathbb{Z}^d$, $G(\Lambda \cup \Lambda') = G(\Lambda) + G(\Lambda')$. Thus we need to estimate the Green's function of regions with some prescribed density, which is the content of Lemma 2.3 below. Recall that for $r \geq 1$, and $x \in \mathbb{Z}^d$, the cube centered at x of side r is

$$Q(x,r) := [x - r/2, x + r/2)^d,$$

The next result is Lemma 4.3 from [5]. Its proof is similar to the proof of Lemma 2.2.

Lemma 2.3 ([5])**.** *Assume $d \geq 3$. There exists a constant $C > 0$, such that the following holds. For any integer $r \geq 1$, any $\rho > 0$, and any finite subset $\Lambda \subseteq \mathbb{Z}^d$, satisfying*

$$|\Lambda \cap Q(z,r)| \leq \rho \cdot r^d, \quad \text{for all } z \in r\mathbb{Z}^d,$$

one has

$$G(\Lambda \cap Q(0,r)^c) \leq C\rho^{1-\frac{2}{d}}|\Lambda|^{2/d}.$$

We now turn to estimating the number of points in the range of a random walk, around which the walk realizes a certain occupation density. For $n \in \mathbb{N}$, $r \geq 1$, and $\rho > 0$, we define

$$\mathscr{R}_n(r,\rho) = \{x \in \mathscr{R}_n : |\mathscr{R}_n \cap Q(x,r)| > \rho \cdot r^d\}. \tag{2.3}$$

Theorem 2.4 ([3]). *Assume $d \geq 3$. There are positive constants κ, and C_0, such that for any n, r and L positive integers and $\rho > 0$, satisfying*

$$\rho r^{d-2} \geq C_0 \cdot \log n, \tag{2.4}$$

one has

$$\mathbb{P}\big(|\mathscr{R}_n(r,\rho)| > L\big) \leq \exp\big(-\kappa \cdot \rho^{2/d} \cdot L^{1-2/d}\big).$$

A weaker version of this result first appeared in [2], with the stronger condition $\rho r^{d-2} \geq C_0(\frac{L}{\rho r^d})^{2/d} \log n$, and the elimination of the L dependence is fundamental here.

Finally the following result is used to reduce the number of boxes where most of the intersection occurs. For $r \geq 1$, some integer, we denote by $\mathscr{X}_r$ the collection of finite subsets of $\mathbb{Z}^d$, whose points are at distance at least r from each other. For $\mathscr{C} \subseteq \mathbb{Z}^d$, we let $Q_r(\mathscr{C}) := \cup_{x \in \mathscr{C}} Q(x,r)$.

Theorem 2.5 ([6]). *Assume $d \geq 3$. There exist positive constants κ and C, such that for any $\rho > 0$, $r \geq 1$ and $\mathscr{C} \in \mathscr{X}_{4r}$, satisfying*

$$\rho r^{d-2} > C,$$

one has

$$\mathbb{P}\Big(\ell_\infty(Q(x,r)) > \rho r^d,\ \forall x \in \mathscr{C}\Big) \leq C\exp(-\kappa\rho \cdot cap(Q_r(\mathscr{C}))).$$

Remark 2.6. *Using the well-known bound $cap(\Lambda) > c|\Lambda|^{1-2/d}$, for any finite $\Lambda \subseteq \mathbb{Z}^d$, and some universal constant $c > 0$, we have $cap(Q_r(\mathscr{C})) \geq cr^{d-2}|\mathscr{C}|^{1-2/d}$. This latter bound is used later.*

3 Plan of the proof

Recall that we consider two independent walks $\{S_n\}_{n\geq 0}$ and $\{\widetilde{S}_n\}_{n\geq 0}$. All quantities associated to the second walk will be decorated with a tilde. With a slight abuse of notation we still denote by $\mathbb{P}$ the law of the two walks.

The first step is to reduce the problem to a finite time horizon. For this we simply use a first moment bound, and the well-known fact that for any $n \geq 1$ (see [11, Proposition 3.2.3]), for some constant $C > 0$,

$$\mathbb{E}[\widetilde{\ell}_\infty(\mathscr{R}[n,\infty))] = \sum_{z \in \mathbb{Z}^d} G(z) \cdot \mathbb{P}(z \in \mathscr{R}[n,\infty)) \leq Cn^{\frac{4-d}{2}}.$$

Using next Markov's inequality we deduce (see also [9, Lemma 9] for a similar statement),

$$\mathbb{P}(\widetilde{\mathscr{R}}_\infty \cap \mathscr{R}[n,\infty) \neq \varnothing) \leq \mathbb{P}(\widetilde{\ell}_\infty(\mathscr{R}[n,\infty)) \geq 1) \leq \mathbb{E}[\widetilde{\ell}_\infty(\mathscr{R}[n,\infty))] \leq Cn^{\frac{4-d}{2}}. \tag{3.1}$$

Thanks to this inequality, it suffices in fact to consider only the intersection of the two walks up to a time n of order $\exp(\beta t^{1-2/d})$, with β some appropriate constant.

The second step is the following proposition. Recall the definition (2.3).

Proposition 3.1. *For any $\beta \geq 1$, there exist positive constants c and C, such that for any $t > 0$, one has with $n := \exp(\beta t^{1-\frac{2}{d}})$,*

$$\mathbb{P}\left(\sup_{x\in\mathbb{Z}^d} G(\mathscr{R}_n - x) > Ct^{2/d} \right) \leq C\exp(-ct^{1-\frac{2}{d}}). \tag{3.2}$$

Furthermore, for any $\varepsilon > 0$ and $K > 0$, there exists $\rho = \rho(\varepsilon, K, \beta)$, and $A = A(\varepsilon, K, \beta)$, such that for t large enough

$$\mathbb{P}\left(\sup_{x\in\mathbb{Z}^d} G\Big(\mathscr{R}_n \setminus \mathscr{R}_n(At^{1/d}, \rho) - x\Big) > \varepsilon t^{2/d} \right) \leq C\exp(-Kt^{1-\frac{2}{d}}). \tag{3.3}$$

One can moreover choose ρ and A, such that $\frac{\log(1/\rho)}{\log(1/\varepsilon)}$ and $\frac{\log A}{\log(1/\varepsilon)}$ remain bounded as $\varepsilon \to 0$.

The third step is to obtain that most of the intersection, up to an arbitrary fraction $\varepsilon \cdot t$, occurs in a finite number of boxes, say $N(\varepsilon)$, each of radius $L(\varepsilon)\cdot t^{1/d}$, and at distances $L^2(\varepsilon)\cdot t^{1/d}$ from each other. The dependance on ε of $N(\varepsilon)$ and $L(\varepsilon)$ turns out to play a crucial role through the bound (6.2). That leads to Proposition 1.3, as well as its analogue for the mutual intersection, which we state as a separate proposition.

Proposition 3.2. *There exist two positive constants c_1 and c_2, such that for any $t > 0$,*

$$\exp(-c_1 t^{1-\frac{2}{d}}) \leq \mathbb{P}(|\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty| > t) \leq \exp(-c_2 t^{1-\frac{2}{d}}). \tag{3.4}$$

Furthermore, for any $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$, such that if $\mathscr{V} := \widetilde{\mathscr{R}}_\infty \cap \mathscr{R}_\infty$

(3.5)

$$\lim_{t\to\infty} \mathbb{P}\left(\exists x_1, \ldots, x_N \in \mathbb{Z}^d \,:\, |\mathscr{V} \cap \Big(\bigcup_{i=1}^N Q(x_i, t^{1/d}) \Big)| > (1-\varepsilon)t \;|\; |\mathscr{V}| > t \right) = 1.$$

One can choose $N(\varepsilon)$, such that $\frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$ remains bounded as $\varepsilon \to 0$.

The lower bound in (3.4) follows of course from (1.2), but for the sake of completeness, we provide another independent argument based on [2], which makes the proof of Propositions 1.3 and 3.2 independent of [7]. The upper bound in (3.4) on the other hand simply follows from Lemma 2.1, together with (3.1) and (3.2). Now concerning (3.5), note that it would follow as well from (3.1), (3.3) and Lemma 2.1, if we could combine it with Theorem 2.4, since we just need to show that the set $\mathscr{R}_n(At^{1/d}, \rho)$ can be covered by a finite number of cubes. This would be fine

indeed, if we could choose the constants A and ρ given by (3.3) so as to satisfy the condition (2.4).

However, since in fact they may be small, we use instead Theorem 2.5.

The rest of the proof relies on the results of [7, 13], and (1.2). We first reduce the region where most of the intersection occurs, from an arbitrary finite number of boxes to a unique, possibly enlarged, one; in other words we prove Theorem 1.2. This part is based on the concavity of the map $t \mapsto t^{1-2/d}$, which implies that distributing the total intersection t on more than one box increases the cost of the deviations. Note that it is crucial here to know the exact constant in the exponential, which is why we need (1.2). We also use some surgery on the trajectories of the two walks; that is first a decomposition into excursions between the various boxes, and then a cutting/gluing argument to ensure that intersections inside each box occur in time-windows of order t, so to make (1.2) applicable. Finally, the same operation of surgery allows also to deduce Theorem 1.1 from Theorem 1.2 and (1.2).

Now the end of the proof is organized as follows. We first prove Proposition 3.1 in Section 4. We then prove Propositions 1.3 and 3.2 in Section 5, and finally we conclude the proofs of Theorems 1.1 and 1.2 in Section 6.

4 Proof of Proposition 3.1

We first introduce a decomposition of the range into subsets according to the occupation density of their neighborhoods, at different scales and bound the cardinality of each subset using Theorem 2.4. Then we prove (3.2) and (3.3) separately in Subsections 4.2 and 4.3 respectively.

4.1 Multi-scale decomposition of the range

Our approach relies on a simple multi-scale analysis of the occupation densities where space and density are scaled together. More precisely we introduce a sequence of densities $\{\rho_i\}_{i\geq 0}$ and associated space-scales $\{r_i\}_{i\geq 0}$ defined respectively, for any integer $i \geq 0$, by

$$\rho_i := 2^{-i}, \quad \text{and} \quad \rho_i \cdot r_i^{d-2} = C_0 \log n, \tag{4.1}$$

with C_0 the constant appearing in (2.4).

It might be that on small scales, say r_j for $j < i$, the density around some site of the range *remains small*, whereas it overcomes ρ_i at scale r_i. To encapsulate this idea we define for $i \geq 1$ (recall (2.3) and note that by definition $\mathscr{R}_n(r_0, \rho_0)$ is empty),

$$\Lambda_i := \mathscr{R}_n(r_i, \rho_i) \setminus \bigcup_{1\leq j<i} \mathscr{R}_n(r_j, \rho_j), \quad \text{and} \quad \Lambda_i^* = \mathscr{R}_n \setminus \bigcup_{1\leq j<i} \mathscr{R}_n(r_j, \rho_j). \tag{4.2}$$

When dealing with these sets we will use two facts: on one hand for each $i \geq 1$, Λ_i is a subset of $\mathscr{R}_n(r_i, \rho_i)$, and thus Theorem 2.4 will provide some control on its

volume. On the other hand, using that $\Lambda_i^* \subseteq \mathscr{R}_n(r_{i-1},\rho_{i-1})^c$, and by cutting a box into 2^d disjoint sub-boxes of side-length twice smaller, we can see that

$$|\Lambda_i^* \cap Q(z,r_{i-1})| \le 2^d \rho_{i-1} r_{i-1}^d, \quad \text{for all } z \in \mathbb{Z}^d, \text{ and all } i>0. \tag{4.3}$$

Note also that since $\Lambda_i \subseteq \Lambda_i^*$, the same bounds hold for Λ_i.

By Theorem 2.4, we have for some constant $\kappa > 0$, for any $\lambda \ge 1$, and any $i \ge 1$,

$$\mathbb{P}\big(|\Lambda_i| > \lambda\big) \le \exp(-\kappa \rho_i^{2/d} \cdot \lambda^{1-2/d}). \tag{4.4}$$

Note also that since $|\mathscr{R}_n| \le n+1$, the set Λ_i is empty when $\rho_i r_i^d > n+1$, or equivalently when $C_0 r_i^2 \log n > n+1$. In particular, for n large enough,

$$\Lambda_i = \varnothing, \quad \text{for all } i > (d-2)\log_2(n). \tag{4.5}$$

Now for L positive integer, define the good event:

$$\mathscr{E}_L := \left\{ |\Lambda_i| \le \rho_i^{-\frac{2}{d-2}} \cdot Lt, \quad \text{for all } i \ge 1 \right\}.$$

Then (4.4) and (4.5) show that for some constant $C>0$ (and $n = \exp(\beta t^{1-2/d})$),

$$\mathbb{P}(\mathscr{E}_L^c) \le C\log_2(n)\exp(-\kappa(Lt)^{1-\frac{2}{d}}) \le C\exp(-\frac{\kappa}{2}\cdot(Lt)^{1-\frac{2}{d}}). \tag{4.6}$$

4.2 Proof of (3.2)

We claim that for some constant $C>0$,

$$\mathscr{E}_1 = \{|\Lambda_i| \le \frac{t}{\rho_i^{2/(d-2)}},\ \forall i \ge 1\} \subseteq \{\sup_{x\in\mathbb{Z}^d} G(\mathscr{R}_n - x) \le Ct^{2/d}\}. \tag{4.7}$$

By (4.6), this would imply the desired result, so let us start proving (4.7).

Assume that the event $\mathscr{E}_1$ holds, and let us bound $\sup_{x\in\mathbb{Z}^d} G(\mathscr{R}_n - x)$.

We fix some $x \in \mathbb{Z}^d$, and divide space into concentric shells as follows: for integers $k \ge 1$, set

$$\mathscr{S}_k := Q(x,r_k)\backslash Q(x,r_{k-1}),$$

and $\mathscr{S}_0 = Q(x,r_0)$. Then, by additivity

$$G(\mathscr{R}_n - x) = \sum_{k\ge 0} G(\mathscr{S}_k \cap \mathscr{R}_n).$$

By Lemma 2.2 and (2.1), one has on $\mathscr{E}_1$,

$$G(\mathscr{R}_n \cap \mathscr{S}_0) \le G(\mathscr{S}_0) \le Cr_0^2 \le C(\log n)^{\frac{2}{d-2}} \le Ct^{2/d}, \tag{4.8}$$

with C some positive constant, whose value might change from line to line. Furthermore, for any $k \ge 1$, recalling (4.2),

$$G(\mathscr{S}_k \cap \mathscr{R}_n) = \sum_{j=1}^{k} G\big(\mathscr{S}_k \cap \Lambda_j\big) + G(\mathscr{S}_k \cap \Lambda_{k+1}^*).$$

By (2.1) and (4.3), one has for any $k \geq 1$,

$$G(\mathscr{S}_k \cap \Lambda^*_{k+1}) \leq C \cdot \frac{|\mathscr{S}_k \cap \Lambda^*_{k+1}|}{r_{k-1}^{d-2}} \leq C \cdot \frac{\rho_k r_k^d}{r_{k-1}^{d-2}} \leq C \cdot \frac{\log n}{r_k^{d-4}},$$

using also (4.1) for the last inequality. Summing over k gives

$$\sum_{k\geq 1} G(\mathscr{S}_k \cap \Lambda^*_{k+1}) \leq C \frac{\log n}{r_0^{d-4}} \leq C(\log n)^{1-\frac{d-4}{d-2}} \leq C(\log n)^{\frac{2}{d-2}} \leq Ct^{2/d}.$$

On the other hand by Lemma 2.3, for any $j \geq 1$, on $\mathscr{E}_1$,

$$\sum_{k\geq j} G(\mathscr{S}_k \cap \Lambda_j) = G(\Lambda_j \cap Q(x, r_{j-1})^c) \leq C\rho_{j-1}^{1-\frac{2}{d}} |\Lambda_j|^{2/d} \leq C\rho_j^{1-\frac{2}{d}(1+\frac{2}{d-2})} t^{2/d} \leq C\rho_j^{\frac{d-4}{d-2}} t^{2/d}.$$

Summing over $j \geq 1$, gives

$$\sum_{j\geq 1} \sum_{k\geq j} G(\mathscr{S}_k \cap \Lambda_j) \leq Ct^{2/d},$$

which concludes the proof of (4.7), and (3.2).

4.3 Proof of (3.3)

Let us give some ε and K, and then fix L such that $\mathbb{P}(\mathscr{E}_L^c) \leq C\exp(-Kt^{1-2/d})$, which is always possible by (4.6).

Next, for $\delta > 0$, and I integer, define

$$\mathscr{R}_n(I, \delta) := \bigcup_{i\leq I} \mathscr{R}_n(r_i, \delta\rho_i).$$

We claim that one can find $\delta \in (0,1)$ and $I \geq 0$, such that

$$\mathscr{E}_L \subseteq \{ \sup_{x\in\mathbb{Z}^d} G(\mathscr{R}_n \backslash \mathscr{R}_n(I,\delta) - x) \leq \varepsilon t^{2/d} \}. \tag{4.9}$$

This would conclude the proof, since for any fixed I and δ, one can find A and ρ, such that

$$\mathscr{R}_n(I, \delta) \subseteq \mathscr{R}_n(At^{1/d}, \rho).$$

So let us prove (4.9) now. Fix some $x \in \mathbb{Z}^d$, and consider the decomposition of space into concentric shells $(\mathscr{S}_k)_{k\geq 0}$, as in the previous subsection. By Lemma 2.2, one has

$$G((\mathscr{R}_n \backslash \mathscr{R}_n(I,\delta)) \cap \mathscr{S}_0) \leq C\delta^{2/d} r_0^2 \leq C\delta^{2/d} t^{2/d},$$

and for any $1 \leq k \leq I$, by (2.1),

$$G((\mathscr{R}_n \backslash \mathscr{R}_n(I,\delta)) \cap \mathscr{S}_k) \leq \frac{C\delta\rho_k r_k^d}{r_{k-1}^{d-2}} \leq C\delta \frac{\log n}{r_k^{d-4}}.$$

Thus, summing over $k \leq I$, yields

$$\sum_{1\leq k\leq I} G((\mathscr{R}_n \backslash \mathscr{R}_n(I,\delta)) \cap \mathscr{S}_k) \leq C\delta \frac{\log n}{r_0^{d-4}} \leq C\delta t^{2/d}.$$

On the other hand, since for any $\delta \le 1$, $\cup_{i\le I}\Lambda_i \subseteq \mathscr{R}_n(I,\delta)$, one has for any $k > I$,

$$G((\mathscr{R}_n \backslash \mathscr{R}_n(I,\delta)) \cap \mathscr{S}_k) \le \sum_{j=I+1}^{k} G(\Lambda_j \cap \mathscr{S}_k) + G(\Lambda^*_{k+1} \cap \mathscr{S}_k),$$

and the same bounds as in the previous subsection give on $\mathscr{E}_L$,

$$\sum_{k\ge I+1} G((\mathscr{R}_n \backslash \mathscr{R}_n(I,\delta)) \cap \mathscr{S}_k) \le Ct^{2/d} \sum_{j\ge I+1} \rho_j^{\frac{d-4}{d-2}} + C \sum_{k\ge I+1} \frac{\log n}{r_k^{d-4}} \le C\rho_I^{\frac{d-4}{d-2}} t^{2/d}.$$

Altogether, we see that by choosing I large enough, and δ small enough, we get (4.9), concluding the proof of (3.3). Finally the fact that $1/\rho$ and A can be chosen, so that they grow at most polynomially in $1/\varepsilon$ by construction.

5 Proof of Propositions 1.3 and 3.2

5.1 Proof of (1.6) and (3.4).

We start with the lower bounds. Note that it suffices to do it for the intersection of two ranges, that is for (3.4), and for a finite time horizon. For this we use Proposition 4.1 from [2], which entails the following fact:

Proposition 5.1 ([2]). *Assume $d \ge 3$. There are positive constants ρ, κ and C, such that for n large enough, for any subset $\Lambda \subseteq Q(0, n^{1/d})$, with $|\Lambda| > C$, one has*

$$\mathbb{P}(|\mathscr{R}_n \cap \Lambda| > \rho|\Lambda|) \ge \exp(-\kappa \cdot n^{1-2/d}).$$

Note that Proposition 4.1 in [2] is stated for dimension 3 only, but its proof applies mutatis mutandis in higher dimension.

Now for $\alpha = 1/\rho^2$ we force, at a cost given by Proposition 5.1, the range $\widetilde{\mathscr{R}}_{\alpha t}$ to cover a fraction ρ of $Q(0,r)$ with $r = (\alpha t)^{1/d}$, and in turn force $\mathscr{R}_{\alpha t}$ to cover a fraction ρ of $\widetilde{\mathscr{R}}_{\alpha t} \cap Q(0,r)$. Observe that one has the inclusion

$$\begin{aligned}\{|\widetilde{\mathscr{R}}_{\alpha t} \cap Q(0,r)| > \rho r^d\} \cap \{|\mathscr{R}_{\alpha t} \cap \widetilde{\mathscr{R}}_{\alpha t} \cap Q(0,r)| > \rho |\widetilde{\mathscr{R}}_{\alpha t} \cap Q(0,r)|\} \\ \subseteq \{|\mathscr{R}_{\alpha t} \cap \widetilde{\mathscr{R}}_{\alpha t}| > \rho^2 r^d = t\},\end{aligned}$$

which concludes the proof of the lower bounds.

Concerning the upper bounds, as was already mentioned, they simply follow from (3.1), (3.2) (say with $\beta = 1$), together with Lemma 2.1.

5.2 Proof of (1.7) and (3.5)

We first state and prove a corollary of Theorem 2.5 (and Remark 2.6), which might be of general interest. Recall that $\mathscr{X}_r$ is the collection of finite subsets of $\mathbb{Z}^d$, whose points are at distance at least r from each other, and for N positive integer, let $\mathscr{X}_{r,N}$ be the subset of $\mathscr{X}_r$ formed by subsets of cardinality N.

Proposition 5.2. *Let $\{S_n\}_{n\geq 0}$ and $\{\widetilde{S}_n\}_{n\geq 0}$ be two independent simple random walks on $\mathbb{Z}^d$, $d \geq 5$. There exist positive constants κ and C, such that for any integers r and N, and any $\rho > 0$, satisfying*

$$\rho r^{d-2} > CN^{2/d} \log N, \tag{5.1}$$

one has

$$\mathbb{P}\left(\exists \mathscr{C} \in \mathscr{X}_{4r,N} : \ell_\infty(Q(x,r)) > \rho r^d, \widetilde{\mathscr{R}}_\infty \cap Q(x,r) \neq \varnothing, \forall x \in \mathscr{C}\right) \leq Ce^{-\kappa \rho r^{d-2} N^{1-\frac{2}{d}}}. \tag{5.2}$$

Remark 5.3. *An important difference here with the statement of Theorem 2.5 is that the set $\mathscr{C}$ is not fixed, and this is compensated by the fact that we impose another independent walk to visit all the cubes centered at points of $\mathscr{C}$. Also, the condtion (5.1) could be further weakened to $\rho r^{d-2} > C \log N$, by using the proof of Theorem 2.5. However, since the number of boxes, N, will not depend on t, in the sequel, it is useless to improve on (5.1).*

Before proving Proposition 5.2, let us assume Proposition 5.2 for a while, and conclude the proofs of (1.7) and (3.5).

Proof of (1.7) *and* (3.5). First we choose β large enough, so that the probability of the event $\{\widetilde{\ell}_\infty(\mathscr{R}[n,\infty)) \geq 1\}$ is negligible, when we take $n = \exp(\beta t^{1-2/d})$, which is always possible by (3.1) and the lower bound in (3.4).

Next, by Lemma 2.1 and (3.3), it suffices to show that for any fixed $A > 0$ and $\rho \in (0,1)$, the set $\mathscr{R}_n(At^{1/d}, \rho) \cap \widetilde{\mathscr{R}}_\infty$ can be covered by at most N disjoint cubes of side length $At^{1/d}$, for some well-chosen constant $N \in \mathbb{N}$. To see this, we first fix the constant N large enough, such that the bound obtained in (5.2) with $r = At^{1/d}$, and t large enough, is negligible when compared to the lower bound in (3.4). Then we define inductively a sequence of boxes as follows. First if the set $\mathscr{R}_n(At^{1/d}, \rho) \cap \widetilde{\mathscr{R}}_\infty$ is nonempty, pick some point x_1 in it. Then, if the set $\mathscr{R}_n(At^{1/d}, \rho) \cap \widetilde{\mathscr{R}}_\infty \cap Q(x_1, 4At^{1/d})^c$ is empty, stop the procedure. Otherwise pick some x_2 in it, and continue like this until we exhaust all points of $\mathscr{R}_n(At^{1/d}, \rho) \cap \widetilde{\mathscr{R}}_\infty$. Note that the points we define by this procedure $x_1, x_2, \ldots$ are all at distance at least $4At^{1/d}$ one from each other by definition. Furthermore, for each i, one has by definition $|Q(x_i, At^{1/d}) \cap \mathscr{R}_n| \geq \rho A^d t$. Thus by Proposition 5.2, the probability that we end up with more than N cubes is negligible. Finally this means that with (conditional) probability going to 1, as $t \to \infty$, we can cover $\mathscr{R}_n(At^{1/d}, \rho) \cap \widetilde{\mathscr{R}}_\infty$ by at most N cubes of side length $4At^{1/d}$, which concludes the proofs of (1.7) and (3.5) (since each such cube is in turn the union of only a fixed number of cubes of side length $t^{1/d}$). □

Remark 5.4. *In the previous proof, note that if A and $1/\rho$ grow at most polynomially in $1/\varepsilon$, so does N by construction.*

Proof of Proposition 5.2. Note that by replacing r by $2r$, ρ by $\rho/2^d$, and N by $\lceil N/2\rceil$ if necessary, one restricts to subsets $\mathscr{C}$ whose points belong to $2r\mathbb{Z}^d\setminus\{0\}$. Fix now such set $\mathscr{C}\in\mathscr{X}_{4r,N}$, and denote by $x_1,\dots,x_N$ its elements. Note first that for any r and ρ satisfying (5.1), with C large enough, Theorem 2.5 (and Remark 2.6) yield for some constant κ,

$$\mathbb{P}(\ell_\infty(Q(x,r))>\rho r^d,\ \forall x\in\mathscr{C})\le C\exp(-\kappa\rho r^{d-2}N^{1-\frac{2}{d}}). \tag{5.3}$$

On the other hand, by (2.2) one has

$$\mathbb{P}\big(\mathscr{R}_\infty\cap Q(x,r)\neq\varnothing,\ \forall x\in\mathscr{C}\big)\le(Cr^{d-2})^N\cdot G(x_1,\dots,x_N), \tag{5.4}$$

where we denote by $\mathfrak{S}_N$ the set of permutations of $\{1,\dots,N\}$ and

$$G(x_1,\dots,x_N):=\sum_{\sigma\in\mathfrak{S}_N}G(x_{\sigma_1})\prod_{i=1}^{N-1}G\big(x_{\sigma_{i+1}}-x_{\sigma_i}\big).$$

For any $q\in(1,2)$, using Hölder's inequality (with all sums over the x_i running over $2r\mathbb{Z}^d\setminus\{0\}$)

$$\begin{aligned}\sum_{x_1}\cdots\sum_{x_N}G^q(x_1,\dots,x_N)&\le\sum_{x_1}\cdots\sum_{x_N}(N!)^{q-1}\sum_{\sigma\in\mathfrak{S}_N}G^q(x_{\sigma_1})\prod_{i=1}^{N-1}G^q(x_{\sigma_{i+1}}-x_{\sigma_i})\\&\le(N!)^q\Big(\sum_{z\in2r\mathbb{Z}^d\setminus\{0\}}G^q(z)\Big)^N.\end{aligned} \tag{5.5}$$

Now fix some $q\in(\frac{d}{d-2},2)$, and note that by (2.1), one has (with a possibly larger constant C),

$$\sum_{z\in2r\mathbb{Z}^d\setminus\{0\}}G^q(z)\le Cr^{q(2-d)},$$

so that (5.4) and (5.5) give,

$$\sum_{x_1,\dots,x_N\in2r\mathbb{Z}^d\setminus\{0\}}\mathbb{P}^q\big(\mathscr{R}_\infty\cap Q(x,r)\neq\varnothing,\ \forall x\in\mathscr{C}\big)\le C^{2N}\cdot(N!)^q. \tag{5.6}$$

Then (5.3) and (5.6) yield

$$\begin{aligned}&\mathbb{P}\big(\exists\mathscr{C}\in\mathscr{X}_{4r,N}:\ell_\infty(Q(x,r))>\rho r^d,\widetilde{\mathscr{R}}_\infty\cap Q(x,r)\neq\varnothing,\forall x\in\mathscr{C}\big)\\&\quad\le\sum_{\mathscr{C}\in\mathscr{X}_{4r,N}}\mathbb{P}\big(\ell_\infty(Q(x,r))>\rho r^d,\ \forall x\in\mathscr{C}\big)\times\mathbb{P}\big(\mathscr{R}_\infty\cap Q(x,r)\neq\varnothing,\ \forall x\in\mathscr{C}\big)\\&\quad\le\sum_{\mathscr{C}\in\mathscr{X}_{4r,N}}\mathbb{P}^{2-q}\big(\ell_\infty(Q(x,r))>\rho r^d,\ \forall x\in\mathscr{C}\big)\times\mathbb{P}^q\big(\mathscr{R}_\infty\cap Q(x,r)\neq\varnothing,\ \forall x\in\mathscr{C}\big)\\&\quad\le C^{2N}(N!)^q\cdot\exp(-\kappa(2-q)\rho r^{d-2}N^{1-\frac{2}{d}}),\end{aligned}$$

and we conclude the proof using hypothesis (5.1). □

6 Proof of Theorems 1.1 and 1.2

Let us define $\mathscr{I}_\infty := \lim_{b\to\infty} \mathscr{I}(b)$, with $\mathscr{I}(b)$ as in (1.2). Since it is easier to realize a large intersection in infinite time, rather than in any finite time, we already know that

$$\liminf_{t\to\infty} \frac{1}{t^{1-\frac{2}{d}}} \log \mathbb{P}(|\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty| \geq t) \geq -\mathscr{I}_\infty. \tag{6.1}$$

The proofs of Theorems 1.1 and 1.2 are now based on the following result, which uses the following notation. For k, L, and t some positive integers, and $\delta \in (0,1)$, define the event (where the k centers of our cubes are in $\mathbb{Z}^d$).

$$\mathscr{A}(k,L,\delta,t) := \left\{ \exists x_1,\dots,x_k : \begin{array}{ll} \|x_i - x_j\| \geq L^2 t^{1/d} & \forall i \neq j \\ |\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty \cap Q(x_i, Lt^{1/d})| \geq \delta t & \forall 1 \leq i \leq k \\ |\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty \cap (\bigcup_{i=1}^k Q(x_i, Lt^{1/d}))| \geq t & \end{array} \right\}.$$

Proposition 6.1. *There exist $C > 0$ and $L_0 \geq 1$, such that for any $L \geq L_0$, $k \leq L$, and $\delta \in (0,1)$,*

$$\limsup_{t\to\infty} \frac{1}{t^{1-\frac{2}{d}}} \log \mathbb{P}(\mathscr{A}(k,L,\delta,t)) \leq -\mathscr{I}_\infty \left(1 + (1 - \frac{1}{2^{2/d}})[(k-1)\delta]^{1-2/d}\right) + \frac{C \log k}{\log L}. \tag{6.2}$$

Note that Theorem 1.1 follows from Theorem 1.2 and Proposition 6.1, applied with $k = 1$.

Remark 6.2. *The bound (6.2) is the key large deviation estimate behind our localisation result. It quantifies the trade off between the number of boxes, say $k \leq N(\varepsilon)$, and the parameter $L(\varepsilon)$ which fixes the radius of the boxes (in a scale $t^{1/d}$), and the spacing of their centers. The concavity of the rate function gives a cost for having k boxes of the order $k^{1-2/d}$, whereas the entropy gain in having the opportunity to travel through k boxes at distance L^2 (in a scale $t^{1/d}$) is of order $\log(k)/\log(L)$. Thus, (6.2) is efficient if L is an exponential in $1/\varepsilon$, since we know from Remark 5.4 that $N(\varepsilon)$, which bounds the number of boxes, grows like a polynomial in $1/\varepsilon$.*

Before we prove Proposition 6.1, let us see how it allows to prove Theorem 1.2.

Proof of Theorem 1.2. For $N \geq 1$ some integer and $t > 0$, define the event

$$\mathscr{B}_{N,t} := \left\{ \exists x_1,\dots,x_N \in \mathbb{Z}^d : |\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty \cap \left(\bigcup_{i=1}^N Q(x_i, t^{1/d}) \right)| \geq t \right\},$$

and for $L \geq 1$ another integer, set

$$\mathscr{B}_{N,L,t} := \left\{ \exists x_1,\dots,x_N \in \mathbb{Z}^d : \begin{array}{ll} \|x_i - x_j\| \geq L^2 t^{1/d} & \forall i \neq j \\ |\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty \cap \left(\bigcup_{i=1}^N Q(x_i, Lt^{1/d})\right)| \geq t & \end{array} \right\}. \tag{6.3}$$

First, it is a simple (deterministic) observation that for any $N \geq 1$, $L_0 \geq 1$, and $t > 0$, one has

$$\mathscr{B}_{N,t} \subset \bigcup_{k=0}^{N} B_{N,L_k,t}, \quad \text{with} \quad L_k = (2L_{k-1})^2 \quad \forall k \geq 1. \tag{6.4}$$

Indeed, assume $\mathscr{B}_{N,t}$ holds, and consider $x_1, \ldots, x_N$ realizing this event. Let also $I_0 := \{1, \ldots, N\}$. If the $(x_i)_{i \in I_0}$ are all at distance at least $L_0^2 t^{1/d}$ one from each other, we stop and $\mathscr{B}_{N,L_0,t}$ holds. If not, consider the first index i, such that x_i is at distance smaller than $L_0^2 t^{1/d}$ from one of the x_j, with $j < i$, and set $I_1 = I_0 \backslash \{i\}$. Set also $L_1 = (2L_0)^2$, and restart the algorithm with I_1 and L_1 in place of I_0 and L_0 respectively. Since this procedure stops in at most N steps, we deduce (6.4). Note that we may end up with less than N points, but since we do not impose the intersection of the ranges with all cubes to be nonempty, we may always add some distant points at the end of the construction.

Next, let $K > 0$ be some fixed constant. We claim that for any reals $\varepsilon \in (0,1)$, $t > 0$, and any integers $N \leq \varepsilon^{-K}$, $L \geq 1$, one has

$$\mathscr{B}_{N,L,t} \subseteq \bigcup_{k=1}^{N} \mathscr{A}\left(k, L, \frac{\varepsilon^{\frac{d}{d-1}}}{2^{(d-1)K} k}, (1-\varepsilon)t\right). \tag{6.5}$$

To see this, assume that the event $\mathscr{B}_{N,L,t}$ holds, and consider $x_1, \ldots, x_N$ realizing it. Set $k_0 = N$, and $J_0 = \{1, \ldots, N\}$, and then let

$$J_1 := \{i \in J_0 : |\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty \cap Q(x_i, Lt^{1/d})| \geq \frac{\varepsilon}{2k_0}\}.$$

Note that by definition of $\mathscr{B}_{N,L,t}$ and J_1,

$$|\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty \cap (\bigcup_{i \in J_1} Q(x_i, Lt^{1/d}))| \geq (1 - \frac{\varepsilon}{2})t.$$

Thus if $|J_1| \geq \varepsilon^{\frac{1}{d-1}} k_0$, we are done, since in this case $\mathscr{B}_{N,L,t} \subset \mathscr{A}(k_1, L, \frac{\varepsilon^{\frac{d}{d-1}}}{2k_1}, (1 - \frac{\varepsilon}{2})t)$, with $k_1 := |J_1|$. If not, define

$$J_2 := \{i \in J_1 : |\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty \cap Q(x_i, Lt^{1/d})| \geq \frac{\varepsilon}{4k_1}\}.$$

One has by definition,

$$|\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty \cap (\bigcup_{i \in J_2} Q(x_i, Lt^{1/d}))| \geq (1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{4})t.$$

Thus if $|J_2| \geq \varepsilon^{\frac{1}{d-1}} k_1$, we are done as well, and if not we continue defining inductively $(J_i)_{i \geq 1}$ and $(k_i)_{i \geq 1}$ as above, until either $|J_i| \geq \varepsilon^{\frac{1}{d-1}} k_{i-1}$, or $|J_i| = 1$, for some i. Note that in the latter case one has $\mathscr{B}_{N,L,t} \subseteq \mathscr{A}(1, L, 1-\varepsilon, (1-\varepsilon)t)$. Since on the other hand at each step we reduce the cardinality of the set of points by a factor at least $\varepsilon^{1/(d-1)}$, and by hypothesis $N \leq \varepsilon^{-K}$, this algorithm must stop in at most $(d-1)K$ steps, and this proves well (6.5).

Recall next that Proposition 3.2 says that for any ε, there exists some integer $N = N(\varepsilon)$, such that

$$\lim_{t\to\infty} \mathbb{P}(\mathscr{B}_{N,(1-\varepsilon)t} \mid |\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty| \geq t) = 1,$$

and furthermore, by Remark 5.4, that one can find a constant K, such that $N(\varepsilon) \leq \varepsilon^{-K}$, at least for ε small enough. Moreover, the constant K being fixed, Proposition 6.1 and the lower bound (6.1) also show that for any ε small enough, any $L \geq \exp(1/\varepsilon)$, and $2 \leq k \leq N(\varepsilon)$,

$$\lim_{t\to\infty} \mathbb{P}(\mathscr{A}(k, L, \frac{\varepsilon^{\frac{d}{d-1}}}{2^{(d-1)K}k}, (1-\varepsilon)^2 t) \mid |\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty| \geq t) = 0.$$

Thus Theorem 1.2 follows from (6.4) and (6.5), taking $L_0 \geq \exp(1/\varepsilon)$, and noting that for any $L \leq L'$, and $\delta \leq 1$, one has the inclusion $\mathscr{A}(1, L, \delta, t) \subseteq \mathscr{B}_{1,L',t}$. □

It remains now to prove Proposition 6.1. For this we need the following lemma.

Lemma 6.3. *Assume $q \in (0,1]$. For any integer $k \geq 1$, and $t_1, \ldots, t_k$ positive numbers, we have*

$$t_1^q + \cdots + t_k^q \geq \Big(\sum_{i=1}^k t_i\Big)^q + (1 - \frac{1}{2^{1-q}})\Big((k-1)\min_{i\leq k}(t_i)\Big)^q. \tag{6.6}$$

Proof. The proof is by induction. For $k = 2$, assume $t_1 \geq t_2 > 0$. Then (6.6) reduces to seeing that

$$t_1^q + \frac{1}{2^{1-q}} t_2^q \geq (t_1 + t_2)^q.$$

If we set $x = t_2/t_1$, we need to show that for $0 \leq x \leq 1$,

$$1 + \frac{x^q}{2^{1-q}} \geq (1+x)^q.$$

By taking derivatives of the two terms, the problem reduces to checking that $2x < 1 + x$ for $0 \leq x \leq 1$, which is indeed true. The induction follows: set $\alpha = 1 - 1/2^{1-q}$, and write

$$\begin{aligned}
t_k^q + \sum_{i=1}^{k-1} t_i^q &\geq t_k^q + \Big(\sum_{i=1}^{k-1} t_i\Big)^q + \alpha\Big((k-2)\min_{i\leq k-1}(t_i)\Big)^q \\
&\geq \Big(\sum_{i=1}^k t_i\Big)^q + \alpha\Big\{\min(t_k, \sum_{i=1}^{k-1} t_i)^q + \Big((k-2)\min_{i\leq k-1}(t_i)\Big)^q\Big\} \\
&\geq \Big(\sum_{i=1}^k t_i\Big)^q + \alpha\Big\{\min_{i\leq k}(t_i)^q + \Big((k-2)\min_{i\leq k-1}(t_i)\Big)^q\Big\} \\
&\geq \Big(\sum_{i=1}^k t_i\Big)^q + \alpha\Big((k-1)\min_{i\leq k}(t_i)\Big)^q,
\end{aligned}$$

using the inequality $a^q + b^q \geq (a+b)^q$ at the last line. □

Proof of Proposition 6.1. The idea is to cut the two trajectories $(S_n)_{n\geq 0}$ and $(\widetilde{S}_n)_{n\geq 0}$ realizing the event $\mathscr{A}(k,L,\delta,t)$ into excursions in a natural way, and then realizing some surgery, to compare the probability of the event to the product of the probabilities of realizing a certain intersection inside k different cubes. Now let us proceed with the details. Fix $x_1,\dots,x_k\in\mathbb{Z}^d$, with $\|x_i-x_j\|\geq L^2t^{1/d}$, for all $i\neq j$. For $1\leq i\leq k$, set $Q_i:=Q(x_i,Lt^{1/d})$, and $\overline{Q}_i:=Q(x_i,L^2t^{1/d})$. We denote with ∂Q, the boundary of Q, that is the sites not in Q but at graph distance 1 from Q. Assume to simplify notation that all the x_i belong to $\lfloor L^2t^{1/d}\rfloor\mathbb{Z}^d$ (if not one can always replace them by the closest points on this lattice, and increase the side-length of the cubes Q_i, and reduce the one of the $\overline{Q}_i$, both by an innocuous factor 2). Finally to simplify also the discussion below, we further assume that the origin does not belong to any of the cubes $\overline{Q}_i$ (minor modifications of the argument would be required otherwise, which we safely leave to the reader). Then define two sequences of stopping times $(s_\ell)_{\ell\geq 0}$ and $(\tau_\ell)_{\ell\geq 0}$ as follows. First $s_0=\tau_0=0$, and for $\ell\geq 1$,

$$\tau_\ell:=\inf\{n\geq s_{\ell-1}:S_n\in\bigcup_{i=1}^k Q_i\},\quad\text{and}\quad s_\ell:=\inf\{n\geq\tau_\ell:S_n\notin\bigcup_{i=1}^k\overline{Q}_i\}.$$

Let $\mathscr{N}:=\sum_{\ell=1}^\infty\mathbf{1}\{\tau_\ell<\infty\}$, be the total number of excursions. Let $\tau(\Lambda):=\inf\{n:S_n\in\Lambda\}$, for the hitting time of a subset $\Lambda\subseteq\mathbb{Z}^d$. It follows from (2.1) and (2.2), that for any $\ell\geq 1$,

$$\begin{aligned}\mathbb{P}(\tau_{\ell+1}<\infty\mid\tau_\ell<\infty)&\leq\sup_{1\leq i\leq k}\sup_{y\in\partial\overline{Q}_i}\mathbb{P}_y(\tau(\cup_{i=1}^k Q_i)<\infty)\\&\leq\sup_{1\leq i\leq k}\sup_{y\in\partial\overline{Q}_i}\sum_{j=1}^k\mathbb{P}_y(\tau(Q_j)<\infty)\leq\frac{C\cdot k}{L^{d-2}},\end{aligned}$$

for some constant $C>0$. Recalling that $k\leq L$, we deduce that for some constant $C_0>0$, and all t large enough,

$$\mathbb{P}\left(\mathscr{N}\geq\frac{C_0t^{1-\frac{2}{d}}}{\log L}\right)\leq\exp(-2\mathscr{I}_\infty\cdot t^{1-\frac{2}{d}}).\tag{6.7}$$

Now, let $i(\ell)$ be the index of the cube to which $S(\tau_\ell)$ belongs, when τ_ℓ is finite: that is $S(\tau_\ell)\in Q_{i(\ell)}$. Define further $\ell_1,\dots,\ell_k$ inductively by $\ell_1=1$, and for $j\geq 1$,

$$\ell_{j+1}=\inf\left\{\ell>\ell_j:i(\ell)\notin\{i(\ell_1),\dots,i(\ell_j)\}\right\}.$$

This induces a permutation $\sigma\in\mathfrak{S}_k$, defined by $\sigma(j):=i(\ell_j)$, which represents the order of first visits of the cubes by the walk. Now, the harmonic probability measure μ_i of Q_i reads as follows.

$$\mu_i(z):=\frac{\mathbb{P}_z(\mathscr{R}[1,\infty)\cap Q_i=\varnothing)}{\mathrm{cap}(Q_i)}\quad\forall z\in Q_i.$$

The harmonic measure is related to the hitting distribution (see Proposition 6.5.4 in [12]).

(6.8)
$$\forall y \notin \overline{Q}_i, \quad \forall z \in Q_i, \quad \mathbb{P}_y(S_{\tau(Q_i)} = z \mid \tau(Q_i) < \infty) = \mu_i(z) \left[1 + \mathscr{O}\left(\frac{Lt^{1/d}}{\|y - x_i\|} \right) \right].$$

Combining it with (2.1) and (2.2), this yields for some constant $c_1 > 0$, for any $1 \le i, j \le k$, and any $z \in Q_j$,

$$\sup_{y \notin \overline{Q}_j} \mathbb{P}_y(\tau(Q_j) < \infty, S_{\tau(Q_j)} = z) \le \frac{c_1}{L^{d-2}} \mu_j(z), \tag{6.9}$$

and when $i \neq j$, we also get for $z \in Q_j$

$$\sup_{y \in \partial \overline{Q}_i} \mathbb{P}_y(\tau(Q_j) < \infty, S_{\tau(Q_j)} = z) \le \frac{c_1}{L^{d-2}} \mu_j(z) \cdot (L^2 t^{1/d})^{d-2} G(x_j - x_i). \tag{6.10}$$

Define analogously $\widetilde{\tau}_\ell$, $\widetilde{s}_\ell, \widetilde{i}(\cdot), \ldots$, for the walk $\widetilde{S}$. Then for $1 \le j \le k$, set

$$\mathscr{I}_j := \Big| \Big(\bigcup_{\ell : i(\ell) = j} \mathscr{R}[\tau_\ell, s_\ell] \Big) \cap \Big(\bigcup_{\ell : \widetilde{i}(\ell) = j} \widetilde{\mathscr{R}}[\widetilde{\tau}_\ell, \widetilde{s}_\ell] \Big) \Big|,$$

the number of intersections of the two walks inside the Q_j. Note that by construction,

$$\mathscr{I}_j = |\mathscr{R}_\infty \cap \widetilde{\mathscr{R}}_\infty \cap Q_j|, \quad \text{for all } 1 \le j \le k.$$

Let now $t_1, \ldots, t_k$, and n, m be some fixed positive integers. Then consider two fixed sequences of indices $(i_1, \ldots, i_n)$ and $(\widetilde{i}_1, \ldots, \widetilde{i}_m)$, taking values in $\{1, \ldots, k\}$, such that all $j \in \{1, \ldots, k\}$ appear at least once in the two sequences. This induces two permutations $\sigma, \widetilde{\sigma} \in \mathfrak{S}_k$, as defined above (one for each sequence). Then set

$$G_\sigma(x_1, \ldots, x_k) := (L^2 t^{1/d})^{k(d-2)} \cdot G(x_{\sigma(1)}) \prod_{j=1}^{k-1} G(x_{\sigma(j+1)} - x_{\sigma(j)}).$$

Let also for $1 \le j \le k$,

$$n_j := \sum_{\ell=1}^{n} \mathbf{1}\{i_\ell = j\}, \quad \text{and} \quad m_j := \sum_{\ell=1}^{m} \mathbf{1}\{\widetilde{i}_\ell = j\}.$$

Then applying (6.9), and (6.10) at indices $\{i_\ell, \widetilde{i}_{\widetilde{\ell}},\ \ell = 1, \ldots, n, \widetilde{\ell} = 1, \ldots, m\}$, shows that

$$\mathbb{P}\left(\begin{array}{c} \mathscr{N} = n, \quad \widetilde{\mathscr{N}} = m, \quad \mathscr{I}_j \ge t_j,\ \forall j = 1, \ldots, k \\ i(\ell) = i_\ell\ \forall \ell \le n, \quad \text{and} \quad \widetilde{i}(\widetilde{\ell}) = \widetilde{i}_{\widetilde{\ell}}\ \forall \widetilde{\ell} \le m \end{array} \right)$$

$$\le \Big(\frac{c_1}{L^{d-2}} \Big)^{n+m} \left(\prod_{j=1}^{k} \mathbb{P}_{\mu_j, n_j, m_j}(\mathscr{I}_j \ge t_j) \right) G_\sigma(x_1, \ldots, x_k) G_{\widetilde{\sigma}}(x_1, \ldots, x_k). \tag{6.11}$$

where for all $1 \le j \le k$, $\mathbb{P}_{\mu_j, n_j, m_j}$ denotes the law of the walk conditionally on $(S(\tau_\ell))_{\ell : i_\ell = j}$, and $(\widetilde{S}(\widetilde{\tau}_\ell))_{\ell : \widetilde{i}_\ell = j}$, being independent and identically distributed with

joint law μ_j, or equivalently the law of $n_j + m_j$ independent excursions starting from law μ_j.

Our next task is to bound the probabilities $\mathbb{P}_{\mu_j,n_j,m_j}(\mathscr{I}_j \geq t_j)$, using (1.2). Proposition 6.5.1 in [12] shows that for some constant $c > 0$, for any $1 \leq j \leq k$, and $y \notin \overline{Q}_j$,

$$\mathbb{P}_y(\tau(Q_j) < \infty) = c\frac{\mathrm{cap}(Q_j)}{\|y - x_j\|^{d-2}}\left[1 + \mathscr{O}\left(\frac{Lt^{1/d}}{\|y-x_j\|}\right)\right],$$

where $\mathrm{cap}(Q_j)$ denotes the capacity of the box Q_j, for which all we need to know is that it is of order $L^{d-2}t^{1-2/d}$. When combined with (6.8) this yields the existence of a constant $c_2 > 0$, such that for all $1 \leq j \leq k$, and all $z \in Q_j$,

$$\inf_{y\in\partial\overline{Q}_j} \mathbb{P}_y(\tau(Q_j) < \tau(Q(x_j, L^3t^{1/d})), S_{\tau(Q_j)} = z) \geq \frac{c_2}{L^{d-2}}\mu_j(z). \tag{6.12}$$

Now let $x \in \mathbb{Z}^d$, be such that the origin belongs to $\partial Q(x, L^2t^{1/d})$. The above inequality (6.12) shows that for any $1 \leq j \leq k$, and any integers n_j, m_j,

(6.13)

$$\mathbb{P}(|\mathscr{R}_{\tau(Q(x,L^3t^{1/d})} \cap \widetilde{\mathscr{R}}_{\widetilde{\tau}(Q(x,L^3t^{1/d})} \cap Q(x, Lt^{1/d})| \geq t_j) \geq \left(\frac{c_2}{L^{d-2}}\right)^{n_j+m_j} \mathbb{P}_{\mu_j,n_j,m_j}(\mathscr{I}_j \geq t_j).$$

On the other hand, Lemmas 2.1 and 2.2 show that for some constant $b > 0$,

$$\mathbb{P}(\tau(Q(x, L^3t^{1/d})) > bt) \leq \exp(-2\mathscr{I}_\infty t^{1-\frac{2}{d}}),$$

at least for t large enough. Thus if $t_j \geq \delta t$, we get with (1.2), that at least for t large enough, the left-hand side of (6.13) is bounded above by $2\mathbb{P}(|\mathscr{R}_{bt} \cap \widetilde{\mathscr{R}}_{bt}| \geq t_j)$. When combined with (6.11), this shows that for some constant $b > 0$, for all $t_j \geq \delta t$,

$$\begin{aligned}
&\mathbb{P}\left(\begin{array}{c} \mathscr{N} = n, \quad \widetilde{\mathscr{N}} = m, \quad \mathscr{I}_j \geq t_j,\ \forall j = 1,\dots,k \\ i(\ell) = i_\ell\ \forall \ell \leq n, \quad \text{and} \quad \widetilde{i}(\ell) = \widetilde{i}_\ell\ \forall \ell \leq m \end{array}\right) \\
&\leq 2^k\left(\frac{c_1}{c_2}\right)^{n+m}\left(\prod_{j=1}^k \mathbb{P}(|\mathscr{R}_{bt} \cap \widetilde{\mathscr{R}}_{bt}| \geq t_j)\right) G_\sigma(x_1,\dots,x_k) G_{\widetilde{\sigma}}(x_1,\dots,x_k) \qquad (6.14) \\
&\leq 2^k\left(\frac{c_1}{c_2}\right)^{n+m}\left(\prod_{j=1}^k \mathbb{P}(|\mathscr{R}_{b't_j} \cap \widetilde{\mathscr{R}}_{b't_j}| \geq t_j)\right) \max_{\sigma\in\mathfrak{S}_k} G_\sigma(x_1,\dots,x_k)^2,
\end{aligned}$$

with $b' = b/\delta$. The number of possible sequences $(i_\ell)_{\ell\leq n}$ and $(\widetilde{i}_\ell)_{\ell\leq m}$, are bounded respectively by k^n and k^m (where k is the number of boxes), so that as we sum (6.14) over them, we get

$$\begin{aligned}
&\mathbb{P}\left(\mathscr{N} = n, \widetilde{\mathscr{N}} = m, \mathscr{I}_j \geq t_j, \forall j = 1,\dots,k\right) \\
&\qquad \leq 2^k\left(\frac{kc_1}{c_2}\right)^{n+m}\left(\prod_{j=1}^k \mathbb{P}(|\mathscr{R}_{b't_j} \cap \widetilde{\mathscr{R}}_{b't_j}| \geq t_j)\right) \max_{\sigma\in\mathfrak{S}_k} G_\sigma(x_1,\dots,x_k)^2.
\end{aligned}$$

Summing then over all $n,m \le N_0 := \lfloor \frac{C_0 t^{1-\frac{2}{d}}}{\log L} \rfloor$, with C_0 as in (6.7), we get

$$\mathbb{P}\Big(\mathscr{N} \le N_0, \widetilde{\mathscr{N}} \le N_0, \mathscr{I}_j \ge t_j, \forall j = 1,\dots,k\Big)$$

$$\le 2^k N_0^2 (\frac{kc_1}{c_2})^{2N_0} \left(\prod_{j=1}^{k} \mathbb{P}(|\mathscr{R}_{b't_j} \cap \widetilde{\mathscr{R}}_{b't_j}| \ge t_j)\right) \max_{\sigma\in\mathfrak{S}_k} G_\sigma(x_1,\dots,x_k)^2. \tag{6.15}$$

Now letting $r := \lfloor L^2 t^{1/d} \rfloor$, we get using (2.1),

$$\sum_{x_1,\dots,x_k \in r\mathbb{Z}^d} \max_{\sigma\in\mathfrak{S}_k} G_\sigma(x_1,\dots,x_k)^2 \le \sum_{\sigma\in\mathfrak{S}_k} \sum_{x_1,\dots,x_k\in r\mathbb{Z}^d} G_\sigma(x_1,\dots,x_k)^2 \le C^k k!.$$

Thus summing over all $x_1,\dots,x_k \in r\mathbb{Z}^d$ in (6.15), and using (6.7), we get

$$\sum_{x_1,\dots,x_k\in r\mathbb{Z}^d} \mathbb{P}(\mathscr{I}_j \ge t_j, \forall j = 1,\dots,k)$$

$$\le (2C)^k (k!) N_0^2 (\frac{kc_1}{c_2})^{2N_0} \cdot \prod_{j=1}^{k} \mathbb{P}(|\mathscr{R}_{b't_j} \cap \widetilde{\mathscr{R}}_{b't_j}| \ge t_j) + \exp(-2\mathscr{I}_\infty t^{1-\frac{2}{d}}).$$

Finally, by using (1.2) and Lemma 6.3 (with $q = 1-\frac{2}{d}$), and then summing over all possible $t_1,\dots,t_k \ge \delta t$, satisfying $t_1+\cdots+t_k = t$, we conclude the proof of the proposition. □

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