

Bounded Harmonic Functions for the Heckman–Opdam Laplacian

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We describe the set of bounded harmonic functions for the Heckman–Opdam Laplacian when the multiplicity function is larger than $1/2$. We prove that this set is a vector space of dimension the cardinality of the Weyl group. We give some consequences in terms of the associated hypergeometric functions.

1 Introduction

In this paper, we will consider the operator $\mathcal{L}$ (called the Heckman–Opdam Laplacian) on $\mathbb{R}^n$ defined, for f a C^2 function, by

$$\begin{aligned} \mathcal{L} f(x) = & \Delta f(x) + \sum_{\alpha \in \mathcal{R}^+} k_{\alpha} \coth \frac{\langle \alpha, x \rangle}{2} \partial_{\alpha} f(x) \\ & - \sum_{\alpha \in \mathcal{R}^+} k_{\alpha} \frac{|\alpha|^2}{4 \sinh^2 \frac{\langle \alpha, x \rangle}{2}} \{f(x) - f(r_{\alpha} x)\}. \end{aligned} \quad (1)$$

Here, Δ is the usual Euclidean Laplacian, $\mathcal{R}$ is a root system, $\mathcal{R}^+$ is its positive part, the r_{α} 's are the orthogonal reflections associated with the roots, and k is a positive function invariant under the action of the r_{α} 's (see Section 2). We denote by W the Weyl group, i.e. the finite group generated by the r_{α} 's. We denote by L the restriction of $\mathcal{L}$ to the set of

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W -invariant functions. A simpler formula for L is given by

$$Lf(x) = \Delta f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \coth \frac{\langle \alpha, x \rangle}{2} \partial_\alpha f(x). \quad (2)$$

Our main results are the following two theorems.

Theorem 1.1. Assume that $k \geq 1/2$. Then the set of bounded W -invariant harmonic functions for the Heckman–Opdam Laplacian is exactly the set of constant functions. In other words, the Poisson boundary of L is trivial. $\square$

Theorem 1.2. Assume that $k \geq 1/2$. Then the set of bounded harmonic functions for the Heckman–Opdam Laplacian is a vector space of dimension $|W|$. In other words, the Poisson boundary of $\mathcal{L}$ is W . $\square$

In the next section, we will give a precise definition for the terminology “harmonic function.” We shall also discuss some consequences of our results in terms of the Heckman–Opdam hypergeometric functions, which are particular eigenfunctions of the operator $\mathcal{L}$.

The first result (Theorem 1.1) was already known for values of k , corresponding to the case of symmetric spaces of the noncompact type G/K (see [8]). The second result (Theorem 1.2) is new even for these particular values of k , but should also be compared to the situation on symmetric spaces. There, according to the fundamental work of Furstenberg [6] (see also [7]), the Poisson boundary of the Laplace–Beltrami operator is K/M . But as it has already been observed so far (see in particular the introduction of [10]), in the Heckman–Opdam (also called trigonometric Dunkl) theory, the group W often plays the same role as K or K/M . First geometrically, since there is a kind of Cartan decomposition: any $x \in \mathbb{R}^d$ can be uniquely decomposed as $w \cdot x^W$, with x^W the radial part of x (lying in the positive Weyl chamber) and $w \in W$. In representation theory also [10]: briefly, if $\mathcal{H}$ is the graded Hecke algebra generated by W and the Dunkl–Cherednik operators (see Section 2), then $(\mathcal{H}, W)$ shares some properties of the Gelfand pair (G, K) , just as in any irreducible finite-dimensional $\mathcal{H}$ -module the subspace of W -invariant vectors is at most one-dimensional. So in some sense Theorem 1.2 is another manifestation (say, at an analytical or probabilistic level) of the strong analogy between W and K .

We should add that the hypothesis $k > 0$ is probably sufficient to get the results of Theorems 1.1 and 1.2. Here, we restrict ourselves to the case $k \geq 1/2$, because, in this case, the stochastic process associated with L (or $\mathcal{L}$) a.s. never hits the walls (the hyperplanes orthogonal to the roots, which correspond to the singularities of L), and we need it to be sure that the coupling we use is well defined.

The paper is organized as follows. In Section 2, we recall all necessary definitions. In Section 3, we prove Theorem 1.1 by using the probabilistic technique of mirror coupling. In Section 4, we prove Theorem 1.2 by extending the coupling to the non-radial process. Our main tool for this is the skew-product representation from Chybiryakov [3], which we have to adapt to our setting.

2 Preliminaries

Let $\mathfrak{a}$ be a Euclidean vector space of dimension n , equipped with an inner product $\langle \cdot, \cdot \rangle$, and denote by $\mathfrak{h} := \mathfrak{a} + i\mathfrak{a}$ its complexification. We consider $\mathcal{R} \subset \mathfrak{a}$ an integral root system (see [1]). We choose a subset of positive roots $\mathcal{R}^+$. Let $\alpha^\vee = 2\alpha/|\alpha|^2$ be the coroot associated with a root α and let

$$r_\alpha(x) = x - \langle \alpha^\vee, x \rangle \alpha$$

be the corresponding orthogonal reflection. Remember that W denotes the Weyl group associated with $\mathcal{R}$, i.e. the group generated by the r_α 's. Let $k : \mathcal{R} \rightarrow [1/2, +\infty)$ be a multiplicity function, which by definition is W -invariant. We set

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} k_\alpha \alpha.$$

Let

$$\mathfrak{a}_+ = \{x \mid \forall \alpha \in \mathcal{R}^+, \langle \alpha, x \rangle > 0\},$$

be the positive Weyl chamber. Let also $\overline{\mathfrak{a}_+}$ be its closure, $\partial \mathfrak{a}_+$ be its boundary, and $\mathfrak{a}_{\text{reg}}$ be the subset of regular elements in $\mathfrak{a}$, i.e. those elements which belong to no hyperplane $\{\alpha = 0\}$. As recalled in the introduction, any $x \in \mathfrak{a}$ can be uniquely decomposed as $x = w x^W$, with $x^W \in \overline{\mathfrak{a}_+}$ and $w \in W$. We call x^W the radial part of x and w its angular part.

For $\xi \in \mathfrak{a}$, let T_ξ be the Dunkl–Cherednik operator [2]. It is defined, for $f \in C^1(\mathfrak{a})$, and $x \in \mathfrak{a}_{\text{reg}}$, by

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \frac{\langle \alpha, \xi \rangle}{1 - e^{-\langle \alpha, x \rangle}} \{f(x) - f(r_\alpha x)\} - \langle \rho, \xi \rangle f(x).$$

The Dunkl–Cherednik operators form a commutative family of differential-difference operators (see [2] or [10]). The Heckman–Opdam Laplacian $\mathcal{L}$ is also given by the formula

$$\mathcal{L} + |\rho|^2 = \sum_{i=1}^n T_{\xi_i}^2,$$

where $\{\xi_1, \dots, \xi_n\}$ is any orthonormal basis of $\mathfrak{a}$.

Let $\lambda \in \mathfrak{h}$. We denote by F_λ the unique (see [9], [10]) analytic W -invariant function on $\mathfrak{a}$, which satisfies the differential equations

$$p(T_\xi)F_\lambda = p(\lambda)F_\lambda \quad \text{for all } W\text{-invariant polynomials } p$$

and which is normalized by $F_\lambda(0) = 1$ (in particular, $\mathcal{L}F_\lambda = (\langle \lambda, \lambda \rangle - |\rho|^2)F_\lambda$). We denote by G_λ the unique analytic function on $\mathfrak{a}$, which satisfies the differential-difference equations (see [10])

$$T_\xi G_\lambda = \langle \lambda, \xi \rangle G_\lambda \quad \text{for all } \xi \in \mathfrak{a}, \quad (3)$$

and which is normalized by $G_\lambda(0) = 1$. These functions are related by the formula

$$F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx), \quad (4)$$

for all $x \in \mathfrak{a}$ and all $\lambda \in \mathfrak{h}$.

It was shown in [13] that $\frac{1}{2}\mathcal{L}$ and $\frac{1}{2}L$ are generators of Feller semigroups that we shall denote, respectively, by $(P_t, t \geq 0)$ and $(P_t^W, t \geq 0)$. We will use the following definition for harmonic functions.

Definition 2.1. A bounded or nonnegative function $h : \mathfrak{a} \rightarrow \mathbb{R}$ is called harmonic if it is measurable and satisfies $P_t h = h$ for all $t > 0$. $\square$

Remark 2.1. It is well known that if h is a C^2 function such that $\mathcal{L}h = 0$, then h is harmonic in the sense of Definition 2.1. Inversely, Corollary 2.1 shows, when $k \geq 1/2$, that any bounded harmonic function is regular; thus, it satisfies $\mathcal{L}h = 0$. On the other hand, it is a general fact (which applies for any $k > 0$) that bounded W -invariant harmonic functions are regular in $\mathfrak{a}_+$, but we will not use this fact here. $\square$

Observe that by definition F_ρ is a W -invariant harmonic function. Moreover, it is known (see [13], Remark 3.1) that it is bounded. So Theorem 1.1 shows that in fact F_ρ is constant equal to 1. Similarly, the functions $G_{w\rho}$'s, for $w \in W$, are harmonic and also bounded. This last property follows from Formula (4), since the $G_{w\rho}$'s are real positive (see [13] Lemma 3.1). In fact, one has the following corollary.

Corollary 2.1. If $k \geq 1/2$, then any bounded harmonic function is a linear combination of the $G_{w\rho}$'s, $w \in W$. $\square$

Proof. The only thing to prove is that the $G_{w\rho}$'s are linearly independent. This results from the fact that they are all eigenfunctions of the Dunkl–Cherednik operators but for

different eigenvalues. More precisely, assume that for some real numbers $(c_w)_{w \in W}$, we have

$$\sum_{w \in W} c_w G_{w\rho} = 0.$$

By applying the operators $p(T_\xi)$, with p polynomial, we get

$$\sum_{w \in W} c_w p(w\rho) G_{w\rho} = 0 \quad \text{for all } p.$$

From this, and the fact that $G_{w\rho}(0) = 1$ for all w , it is easily seen that we must have $c_w = 0$ for all w . ■

3 The W -invariant Case: Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. For this, we will use the stochastic process $(X_t^W, t \geq 0)$ associated with L , called radial HO-process, and the so-called mirror coupling technique.

First, it is known [12] that X^W is a strong solution of the stochastic differential equation (SDE) :

$$X_t^W = x + B_t + V_t^1,$$

where $(B_t, t \geq 0)$ is a Brownian motion on $\mathfrak{a}$ and

$$V_t^1 := \sum_{\alpha \in \mathcal{R}^+} k_\alpha \alpha \int_0^t \coth \langle \alpha, X_s^W \rangle ds.$$

Moreover, when $k \geq 1/2$, X^W a.s. takes values in $\mathfrak{a}_+$, or in other words, it never reaches $\partial \mathfrak{a}_+$ (see [12]). Now, if $x, y \in \mathfrak{a}_+$, we define the couple $((X_t^W, Y_t^W), t \geq 0)$ as follows. Set $T = \inf\{s \mid X_s^W = Y_s^W\}$. Then by definition X^W as above, and (X^W, Y^W) is the unique solution of the SDE:

$$(X_t^W, Y_t^W) = (x, y) + (B_t, B'_t) + (V_t^1, V_t^2), \quad \text{for } t < T, \tag{5}$$

where $dB'_t = r_t dB_t$, with r_t the orthogonal reflection with respect to the hyperplane orthogonal to the vector $Y_t^W - X_t^W$ (in particular, Levy criterion shows that B' is a Brownian motion), and

$$V_t^2 := \sum_{\alpha \in \mathcal{R}^+} k_\alpha \alpha \int_0^t \coth \langle \alpha, Y_s^W \rangle ds.$$

For $t \geq T$, we set $Y_t^W = X_t^W$. The existence of this coupling is guaranteed by the fact that equation (5) has locally regular coefficients. We also define Z^W by

$$Z_t^W := Y_t^W - X_t^W,$$

and set $z_t^W = |Z_t^W|$. It is known [12] that a.s. $X_t^W/t \rightarrow \rho$, and thus that $\langle \alpha, X_t^W \rangle \sim \langle \rho, \alpha \rangle t$, for all $\alpha \in \mathcal{R}^+$. From this, we see that a.s. $\sup_{t \geq 0} |V_t^2 - V_t^1| < +\infty$. Then Tanaka formula ([11], p. 222) shows that

$$z_t^W = \gamma_t + v_t, \quad \text{for } t < T,$$

with γ a one-dimensional Brownian motion and a.s. $\sup_{t \geq 0} |v_t| < +\infty$. In particular, T is a.s. finite.

The end of the proof is routine now. Assume that h is a bounded W -invariant harmonic function. Then it is well known, and not difficult to show, that $(h(X_t^W), t \geq 0)$ and $(h(Y_t^W), t \geq 0)$ are bounded martingales. Thus, they are a.s. converging toward some limiting (random) values, respectively l and l' . Since a.s. $X_t^W = Y_t^W$ for t large enough, we have a.s. $l = l'$. Then usual properties of bounded martingales show that

$$h(x) = \mathbb{E}[l] = \mathbb{E}[l'] = h(y).$$

Since this holds for any $x, y \in \mathfrak{a}_+$, this proves well that h is constant. $\square$

4 The Non- W -invariant Case: Proof of Theorem 1.2

In order to prove Theorem 1.2, the first idea is to extend the previous coupling to the full process $(X_t, t \geq 0)$ with semigroup $(P_t, t \geq 0)$. For this, our tool will be the skew-product representation founded by Chybiryakov [3] (see [7] and [4] for the one-dimensional case). Actually, Chybiryakov dealt with Dunkl processes, so we shall first mention the changes needed to adapt his proof to the present setting, and then explain how to combine this representation with the coupling from the previous section.

4.1 Skew-product representation and extension of the coupling

The skew-product representation gives a constructive way to define X starting from X^W , by adding successive jumps in the direction of the roots. Let us sketch the main steps of the construction (for more details see [3]). First, one fixes arbitrarily an order for the

positive roots: $\alpha_1, \dots, \alpha_{|\mathcal{R}^+|}$. Then for each $j \in [1, |\mathcal{R}^+|]$, set

$$\mathcal{L}^j f(x) := Lf(x) - \sum_{i \leq j} c_{\alpha_i}(x) \{f(x) - f(r_{\alpha_i} x)\},$$

where for any root α ,

$$c_\alpha(x) := k_\alpha \frac{|\alpha|^2}{4 \sinh^2 \frac{\langle \alpha, x \rangle}{2}}.$$

Decide also that $\mathcal{L}^0 = L$. Set

$$\tilde{\mathcal{L}}^j f(x) := c_{\alpha_j}^{-1}(x) \mathcal{L}^j f(x),$$

and

$$\mathcal{L}^{j,j+1} f(x) := c_{\alpha_{j+1}}^{-1}(x) \mathcal{L}^j f(x).$$

The goal is to define inductively a sequence of processes $(X^j(t), t \geq 0)$, $j = 0, \dots, |\mathcal{R}^+|$, associated with the operators $\mathcal{L}^j$'s. First, X^0 is just the radial HO-process considered in the previous section. Next, assume that $\mathcal{L}^j$ is the generator of a Markov process $(X^j(t), t \geq 0)$. Then set

$$A_t^j = \int_0^t c_{\alpha_{j+1}}(X_s^j) ds,$$

and

$$\tau_t^j = \inf \{s \geq 0 \mid A_s^j > t\}.$$

Using the martingale problem characterization, one can see that the radial part of X^j is a radial HO-process. Thus, for all $\alpha \in \mathcal{R}^+$, $|\langle \alpha, X_t^j \rangle| \geq ct$, for t large enough and $c > 0$ some constant. In particular, the increasing process A^j is bounded. Set $T^j = \lim_{t \rightarrow +\infty} A_t^j$. Then observe that $\tau_t^j = +\infty$, when $t \geq T^j$. This is essentially the only difference with the Dunkl case considered in [3] (where A^j was not bounded and τ_t^j finite for all t). But one can still see that if

$$X^{j,j+1}(t) := X^j(\tau_t^j) \quad t < T^j,$$

then $X^{j,j+1}$, killed at time T^j , is a solution of the martingale problem associated with $\mathcal{L}^{j,j+1}$ (see for instance [5] Exercise 15, p. 263 and Section 6, p. 306). The next step is to add jumps to $X^{j,j+1}$ in the direction of the root α_{j+1} . Namely, one define a new process $\tilde{X}^j$, also denoted by $X^{j,j+1} *_{\alpha_{j+1}} N$ in ([3] Section 2.5), which is the solution of the martingale problem associated with $\tilde{\mathcal{L}}^{j+1}$. Roughly, $\tilde{X}^j$ is constructed by gluing several paths, all with law $X^{j,j+1}$ or $r_{\alpha_{j+1}} X^{j,j+1}$, such that for any two consecutive paths the starting point

of the second is the image of the end point of the first path by the reflection $r_{\alpha_{j+1}}$. The lengths of the paths are determined by independent exponentially distributed random variables. Here, the only minor change is that $\tilde{X}^j$ explodes at some time, say $\tilde{T}^j$. A change of variables shows that

$$\lim_{t \rightarrow \tilde{T}^j} \int_0^t c_{\alpha_{j+1}}^{-1}(\tilde{X}^j(s)) ds = +\infty.$$

So for any $t \geq 0$, one can define $\tilde{A}^j(t)$ as the solution of the equation

$$t = \int_0^{\tilde{A}^j(t)} c_{\alpha_{j+1}}^{-1}(\tilde{X}^j(s)) ds.$$

Differentiating this equation one get

$$\frac{d}{dt} \tilde{A}^j(t) = c_{\alpha_{j+1}}(\tilde{X}^j(\tilde{A}^j(t))).$$

Then set $X^{j+1}(t) = \tilde{X}^j(\tilde{A}^j(t))$, for all $t \geq 0$. The preceding equation gives

$$\tilde{A}^j(t) = \int_0^t c_{\alpha_{j+1}}(X^{j+1}(s)) ds,$$

which in turn shows that X^{j+1} is the solution of the martingale problem associated with $\mathcal{L}^{j+1}$, as required.

The point now is to combine this construction of $X = X^{|\mathbb{R}^+|}$ with the coupling of the radial process from Section 3. We first take (X^0, Y^0) with law given by this coupling. Then we define the sequence $((X^j(t), Y^j(t)), t \geq 0)$, $j = 1, \dots, |\mathcal{R}^+|$, simply by following the previous construction for the two coordinates. Actually, this coupling is interesting only when $X = X^{|\mathcal{R}^+|}$ and $Y = Y^{|\mathcal{R}^+|}$ never jump, but this is precisely what we need. Indeed, in this case, we have $X_t = X^0(t)$ and $Y_t = Y^0(t)$, for all $t \geq 0$, so they coincide a.s. after some finite time.

4.2 End of the proof

For any $x \in \mathfrak{a}$, we denote by $\mathbb{P}_x$ the law of $(X_t, t \geq 0)$ starting from x . For $\epsilon \in (0, 1)$, set

$$A_\epsilon := \{z \in \mathfrak{a} \mid \mathbb{P}_z[X \text{ never jumps}] \geq 1 - \epsilon\}.$$

We know that the process $(X_t, t \geq 0)$ can jump, so a priori $A_\epsilon \subsetneq \mathfrak{a}$. But we also know [12] that a.s. X eventually stops to jump after some finite random time. This implies that

$$\lim_{t \rightarrow +\infty} \mathbb{P}_x[X \text{ never jumps after time } t] = 1, \tag{6}$$

for all $x \in \mathfrak{a}$. But by using the Markov property, we have for all $t > 0$,

$$\begin{aligned} \mathbb{P}_x[X \text{ never jumps after time } t] &= \mathbb{E}_x[\mathbb{P}_{X_t}[X \text{ never jumps}]] \\ &= \int_{\mathfrak{a}} \mathbb{P}_z[X \text{ never jumps}] d\mu_t^x(z), \end{aligned} \quad (7)$$

where μ_t^x is the law of X_t under $\mathbb{P}_x$. So (6) and (7) imply that for all $x \in \mathfrak{a}$, $\mu_t^x(A_\epsilon) \rightarrow 1$, when $t \rightarrow +\infty$. In particular, A_ϵ is nonempty. Moreover, by invariance of $\mathcal{L}$ under W , we know that for any $w \in W$, the law of $(wX_t, t \geq 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{wx}$. In particular, for any $w \in W$ and any $\epsilon \in (0, 1)$, we have $w(A_\epsilon \cap \mathfrak{a}_+) = A_\epsilon \cap w\mathfrak{a}_+$. Thus, all these subsets of A_ϵ are nonempty as well.

Now, let h be some harmonic function. Fix $w \in W$, and take $x, y \in A_\epsilon \cap w\mathfrak{a}_+$. Consider the coupling $((X_t, Y_t), t \geq 0)$ as defined above. Since $(h(X_t), t \geq 0)$ and $(h(Y_t), t \geq 0)$ are bounded martingales, they converge a.s. toward some limits, respectively l and l' . We already saw that X^W and Y^W a.s. coincide after some time. So if both processes X and Y never jump, they must also coincide after some time, and in this case we have $l = l'$. Since $x, y \in A_\epsilon$, this shows that

$$|h(x) - h(y)| = |\mathbb{E}[l] - \mathbb{E}[l']| \leq 2C\epsilon,$$

where $C = \sup h$. In particular, by completeness of $\mathbb{R}$, for any sequence $(x_\epsilon)_{\epsilon \in (0,1)}$, such that $x_\epsilon \in A_\epsilon \cap w\mathfrak{a}_+$ for all $\epsilon \in (0, 1)$, the limit of $h(x_\epsilon)$ when ϵ tends to 0 exists, and is independent of the chosen sequence. Call l_w this limit.

For all $t \geq 0$, we denote by w_t the angular part of X_t . Since X eventually stops to jump, $(w_t, t \geq 0)$ a.s. converges, i.e. becomes stationary. Then for any $w \in W$, define the function h_w on $\mathfrak{a}$ by

$$h_w(x) = \mathbb{P}_x\left[\lim_{t \rightarrow +\infty} w_t = w\right].$$

By standard properties of Markov processes, we know that these functions are measurable, and Markov property implies that they are harmonic. Moreover, the above convergence result for harmonic functions shows that these functions h_w , $w \in W$ are linearly independent. Then set

$$\tilde{h}(x) := \sum_{w \in W} l_w h_w(x),$$

for all $x \in \mathfrak{a}$. All that remains to do now is to prove that $\tilde{h} = h$. Indeed, if this was true, this would prove that the vector space of bounded harmonic function has dimension $|W|$

as required. By using the martingale property, we have for any $t > 0$

$$|h(x) - \tilde{h}(x)| = |\mathbb{E}_x[h(X_t) - \tilde{h}(X_t)]| \leq \int_a^\infty |h(z) - \tilde{h}(z)| d\mu_t^x(z). \quad (8)$$

We have seen that for all $\epsilon \in (0, 1)$,

$$\mu_t^x(A_\epsilon) \rightarrow 1, \quad (9)$$

when $t \rightarrow +\infty$. But it is not difficult to see (by using the definition of the l_w 's), that for any $\epsilon' > 0$, there exists $\epsilon > 0$ such that

$$|h(z) - \tilde{h}(z)| \leq \epsilon' \quad \forall z \in A_\epsilon.$$

Since this holds for any $\epsilon' > 0$, (8) and (9) show that $h = \tilde{h}$ as required. $\square$

Remark 4.1. We have seen in the previous proof that the family $(h_w)_{w \in W}$ is a basis of the space of bounded harmonic functions. Since the family $(G_{wp})_{w \in W}$ is another basis, it would be interesting to know the coefficients relating these two bases.

We notice also that the h_w 's are minimal. This means that if h is a nonnegative harmonic function such that $h \leq h_w$ for some w , then there exists a constant $c \geq 0$ such that $h = ch_w$.

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