# The Heckman-Opdam Markov processes

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**Abstract**

We introduce and study the natural counterpart of the Dunkl Markov processes in a negatively curved setting. We give a semimartingale decomposition of the radial part, and some properties of the jumps. We prove also a law of large numbers, a central limit theorem, and the convergence of the normalized process to the Dunkl process. Eventually we describe the asymptotic behavior of the infinite loop as was done by Anker, Bougerol and Jeulin in the symmetric spaces setting in [1].

**Key Words:** Markov processes, Jump processes, root systems, Dirichlet forms, Dunkl processes, limit theorems.

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# 1 Introduction

In the last few years, some processes living in cones have played an important role in probability. The most famous example is maybe the planar Brownian motion with reflections on the boundary of the cone (see e.g. [26], [25] where it is linked with the Schramm's Stochastic Loewner Evolution, or [13]). But here we consider other processes which propagate in very special cones related with Riemannian geometry (we will precise this in a moment).

The cones that we consider are associated to a root system. Roughly speaking a root system is a set of vectors, satisfying a few conditions, in a Euclidean space (a precise definition will be given in the sequel). The set of hyperplane orthogonal to the vectors of the root system delimit cones, which are called the Weyl chambers. Usually one of the chamber is chosen arbitrarily and called the positive Weyl chamber. One of the first example of process with value in a Weyl chamber is the intrinsic Brownian motion (or IBM) introduced by Biane in [3]. It may be simply defined as the Brownian motion killed on the boundary of the positive Weyl chamber, and conditioned to never reach it. In fact it was realized then, that this process appears in many branches of the probability theory. First in dimension 1, it is the Bessel-3 process. But in higher rank it appears also for instance in the theory of random matrices. A well known example is when the root system is of type $A$ (a precise definition will be given in the next section), in which case the intrinsic Brownian motion is the process of eigenvalues of Hermitian matrices with Gaussian entries (the GUE ensemble) and trace zero. But it appears also in a lot of other areas, like interacting particle systems, directed percolation, or random permutations (see e.g. [28] for a survey of these links). Let us also mention the recent work of Biane, Bougerol and O'Connell [4], where the IBM appears as a natural generalization of the Bessel-3 process in dimension $n$, in the sense that it may be obtained by a transform of the Brownian motion in $\mathbb{R}^n$, which coincides in dimension 1 with the Pitman transform $2S - B$.

Now another way, more geometric and more interesting for our purpose, to define the IBM, is to consider the radial part of the Brownian motion on a complex Riemannian flat symmetric space. Let us immediately say that the reader does not need to be familiar with these notions (however well explained in [20]) in order to understand the content of that paper. We will just mention that these spaces are tangent spaces of so called Riemmanian symmetric spaces of noncompact type (see also [20]), which are Riemannian manifolds with negative curvature. So the natural counterpart of the IBM in a negative curvature setting is the radial part of the Brownian motion on symmetric spaces (of noncompact type). Such Brownian motion was also already well studied, e.g. in [2] or in [5] [6] where interesting links with the IBM are exhibited (they will be detailed in what follows).

More recently, Rösler [31] and Rösler, Voit [32] have introduced a new type of processes related to Weyl chambers of root systems, the Dunkl processes. These processes are Markov processes as well as martingales, but they no longer have continuous paths. They may jump from a chamber to another. Nevertheless the projection on the positive Weyl chamber (a proper definition in the sequel) of these processes, which is called the radial part or radial process, has continuous paths. In fact the Dunkl processes are indexed by a parameter $k$, and when $k = 1$ then their radial part coincide with the IBM. Moreover in dimension 1 (see [16] for a particular study of this case) the radial Dunkl process of parameter $k$ is the Bessel process of dimension $2k + 1$. This explains why (radial) Dunkl processes are also called (in higher dimension) generalized Bessel processes. Let us notice however, that for a general value of $k$, there is no more (evident) link with Riemannian geometry. These processes were then studied more deeply by Gallardo and Yor [16] [17] [18], and by Chybiryakov [11] [12], who have obtained many interesting properties, such as the time inversion property, a Wiener chaos decomposition, and a skew product decomposition.

In this paper, we introduce and study the natural counterpart of the Dunkl processes in the negatively curved setting, which we call the Heckman-Opdam processes. These processes are also discontinuous, while their projections in the positive Weyl chamber have continuous paths and coincide with the radial part of the Brownian motion on symmetric spaces (of noncompact type) for particular choices of the parameter. We will show that many known results in probability theory (see [1] [2] [5]) in the symmetric spaces setup, can be generalized to these new processes. In particular the main result of that paper is the following. Consider the bridge of a radial (for simplify) Heckamn-Opdam process of length $T$ around 0 (it starts from 0 and is conditioned to come back at 0 at time $T$). Then when $T$ tends to infinity this process converges in law (in the path space) to some process that we identify (the $F_0$-process, which is a relativized process in the sense of Doob). Moreover after suitable normalization this process converges to the IBM. This was proved for the values of $k$ corresponding to the geometric setting (when radial Heckamn-Opdam process coincide with radial part of BM on symmetric spaces) in [6] in dimension 1, and then in all dimensions in [5]. Here we extend (Proposition 7.1 and Theorem 7.1) this result to all strictly positive values of $k$ (and we prove an analogue result in the non radial case). In particular it is interesting to notice that the limiting process does not depend on $k$. So the IBM is in some sense attracting, and distinguished among the other (radial) Dunkl processes. Let us also mention that an analogue result has been proved in the context of homogenous trees [6], and in their natural generalization in higher dimension, the affine buildings (in the case of type $A$), by the author [34].

We detail now the organization of the paper. In the next section we define all the preliminary notions which are required to introduce the Heckman-Opdam and Dunkl processes. These processes are defined in section 3, where we present also some elementary properties. In section 4 we show that the radial process is solution of a stochastic differential equation. Since the coefficients of the SDE have singularities on the boundary of the Weyl chambers, this can not be deduced from classical theorems, and is proved by using the theory of Dirichlet processes. This result allows to prove then a law of large numbers and a central limit theorem for the radial process. In the following section, we study the jumps of the process and show in particular that when the value of $k$ is large enough, the process does not jump anymore after a finite random time. In this case we prove also a law of large numbers and a central limit theorem for the whole process (with jumps). In section 6, we prove that after suitable normalization, the Heckman-Opdam process of parameter $k$ converges to a Dunkl process with parameter $k'$ (for simplify let say that $k'$ is "almost" equal to $k$). Eventually in the last section we prove the main theorem of the paper which was described above.

# 2 Preliminaries

Let $\mathfrak{a}$ be a Euclidean vector space of dimension $n$, equipped with an inner product $(\cdot,\cdot)$. Let $\mathfrak{h} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{a}$. For $\alpha \in \mathfrak{a}$ let $\alpha^\vee = \frac{2}{|\alpha|^2}\alpha$, and let

$$r_\alpha(x) = x - (\alpha^\vee, x)\alpha,$$

be the corresponding orthogonal reflection. In the sequel we will identify any vector with the linear form it represents. In particular if $\alpha, u \in \mathfrak{a}$, then both notations $(\alpha, u)$ and $\alpha(u)$ are equivalent. Let $\mathcal{R} \subset \mathfrak{a}$ be an *integral* (or crystallographic) *root system*, which by definition (cf [7]) satisfies the following hypothesis

1. $\mathcal{R}$ is finite, does not contain 0 and generates $\mathfrak{a}$.
2. $\forall \alpha \in \mathcal{R}$, $r_\alpha(\mathcal{R}) = \mathcal{R}$.
3. $\forall \alpha \in \mathcal{R}$, $\alpha^\vee(\mathcal{R}) \subset \mathbb{Z}$.

The hyperplane $\{(\alpha, x) = 0\}$ with $\alpha \in \mathcal{R}$ are called the *walls*. The root system is *reducible* if it is the disjoint union of two root systems $\mathcal{R}_1$ and $\mathcal{R}_2$, such that for all $\alpha, \beta \in \mathcal{R}_1 \times \mathcal{R}_2$, $(\alpha, \beta) = 0$. If it is not reducible, we say that it is *irreducible*. In fact if the root system is reducible, then for our purpose the study in $\mathcal{R}$ is equivalent to independent studies in $\mathcal{R}_1$ and $\mathcal{R}_2$. Thus with no loss of generality we will assume here that the root system is irreducible. A remarkable property of the root systems is that for any $\alpha \in \mathcal{R}$, there exists at most one (distinct of $\alpha$) positive multiple of $\alpha$ in $\mathcal{R}$, which can be either $\alpha/2$ or $2\alpha$. Now if for any $\alpha \in \mathcal{R}$, $2\alpha \notin \mathcal{R}$, we say that the root system is *reduced*.

A system of *positive roots* $\mathcal{R}^+$ is defined as follows: first choose arbitrarily a vector $u \in \mathfrak{a}$, such that $(\alpha, u) \neq 0$ for all $\alpha \in \mathcal{R}$. Then $\mathcal{R}^+$ is the subset of roots $\alpha \in \mathcal{R}$ such that $(\alpha, u) > 0$. Naturally, up to isometry, $\mathcal{R}^+$ does not depend on the choice of $u$. The *simple roots* $\{\alpha_1, \dots, \alpha_n\}$ are defined as a basis of $\mathcal{R}^+$, in the sense that they lie in $\mathcal{R}^+$ and for all $\alpha \in \mathcal{R}^+$, there exist integers $m_1, \dots, m_n$, such that $\alpha = \sum_{i=1}^n m_i \alpha_i$. The *Weyl group* associated to $\mathcal{R}$ is by definition the group generated by the $r_\alpha$'s, with $\alpha \in \mathcal{R}$. If $C$ is a subset of $\mathfrak{a}$, we call *symmetric* of $C$ any image of $C$ under the action of $W$.

The (irreducible) reduced integral root systems have been classified (up to isometry) into types. There exist 4 infinite families called $(A_n)_n$, $(B_n)_n$, $(C_n)_n$ and $(D_n)_n$, plus a finite number of other types called exceptional (see e.g. [7] for more details). Let us describe more precisely for instance the type $A_n$:

*Type $A_n$:* Let $(e_1, \dots, e_{n+1})$ be the canonical basis of $\mathbb{R}^{n+1}$. Let $\mathfrak{a}$ be the hyperplane $\{(x, e_1 + \cdots + e_{n+1}) = 0\}$. Then the set of vectors $e_i - e_j$, for $i \neq j$, is a root system of $\mathfrak{a}$ of type $A_n$. Moreover a choice of positive roots is the set of vectors $e_i - e_j$, with $i < j$.

**Remark 2.1** There exists also a notion of root systems (not integral) where the condition 3 in the definition is not imposed. For instance in dimension 2, there exists an infinite family of such root systems, which are the dihedral systems $I_n$. The system $I_n$ is the set of vectors $e^{i2k\pi/n}u$, for $1 \leq k \leq n$, where $u$ is any vector of the complex plane. For non integral root systems the general theory of Heckman-Opdam does not work anymore. We could however define radial HO-processes as solution of an SDE (see Proposition 4.1 below), but there existence would be assured only until the first hitting time of the walls, and above all we could not prove many interesting properties. Therefore in this work, we will restrict us to the case of integral root systems. Let us mention that in the Dunkl theory (cf section 3.2) this restriction is not necessary.

A multiplicity function $k : \mathcal{R} \to \mathbb{R}^+$ is by definition a $W$-invariant function on $\mathcal{R}$. We recall that by the property 2 of the definition of $\mathcal{R}$, we know that $W$ sends $\mathcal{R}$ into $\mathcal{R}$. Even if it will be of no use here, we just mention that there is at most 3 orbits in $\mathcal{R}$ under the action of $W$, thus $k$ takes at most 3 distinct values. Moreover if the root system is reduced there is at most 2 orbits, and $k$ takes at most 2 distinct values. We will assume in this paper that $k(\alpha)$ (also denoted by $k_\alpha$ in the sequel) is strictly positive for all $\alpha \in \mathcal{R}^+$. Let us remark that the condition $k_\alpha + k_{2\alpha} > 0$ for all $\alpha \in \mathcal{R}$, is not more general, since if $k$ is null on some orbit (and not on all), we can remove it, and do the study on the remaining root system.

Let

$$\mathfrak{a}_+ = \{x \in \mathfrak{a} \mid \forall \alpha \in \mathcal{R}^+,\ (\alpha, x) > 0\},$$

be the positive Weyl chamber. We denote by $\overline{\mathfrak{a}_+}$ its closure, and by $\partial \mathfrak{a}_+$ its boundary. Let also $\mathfrak{a}_{\text{reg}}$ be the subset of regular elements in $\mathfrak{a}$, i.e. those elements which belong to no hyperplane $\{x \in \mathfrak{a} \mid (\alpha, x) = 0\}$.
For $\xi \in \mathfrak{a}$, let $T_\xi$ be the Dunkl-Cherednik operator. It is defined, for $f \in C^1(\mathfrak{a})$ and $x \in \mathfrak{a}_{\text{reg}}$, by

$$T_\xi f(x) = \partial_\xi f(x) - (\rho, \xi) f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \frac{(\alpha, \xi)}{1 - e^{-(\alpha, x)}} \{f(x) - f(r_\alpha x)\},$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} k_\alpha \alpha.$$

The Dunkl-Cherednik operators form a commutative family of differential-difference operators (see [10]). The Laplacian $\mathcal{L}$ is defined by

$$\mathcal{L} = \sum_{i=1}^{n} T_{\xi_i}^2,$$

where $\{\xi_1, \dots, \xi_n\}$ is any orthonormal basis of $\mathfrak{a}$ ($\mathcal{L}$ is independent of the chosen basis). The Laplacian $\mathcal{L}$ is given explicitly by (see [33]) :

$$\begin{aligned} \mathcal{L}f(x) &= \Delta f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \coth \frac{(\alpha, x)}{2} \partial_\alpha f(x) + |\rho|^2 f(x) \qquad (1) \\ &+ \sum_{\alpha \in \mathcal{R}^+} k_\alpha \frac{|\alpha|^2}{4 \sinh^2 \frac{(\alpha,x)}{2}} \{f(r_\alpha x) - f(x)\}, \end{aligned}$$

for $f \in C^2(\mathfrak{a})$ and $x \in \mathfrak{a}_{\text{reg}}$, where $\Delta$ denotes the Euclidean Laplacian. Let $\mu$ be the measure on $\mathfrak{a}$ given by

$$d\mu(x) = \delta(x) dx,$$

where

$$\delta(x) = \prod_{\alpha \in \mathcal{R}^+} |\sinh \frac{(\alpha, x)}{2}|^{2k_\alpha}.$$

Let $\lambda \in \mathfrak{h}$. We denote by $G_\lambda$ the unique analytic function on $\mathfrak{a}$, which satisfies the differential and difference equations

$$T_\xi G_\lambda = (\lambda, \xi) G_\lambda, \quad \forall \xi \in \mathfrak{a}$$

and which is normalized by $G_\lambda(0) = 1$ (see [29]). Let $F_\lambda$ be the function defined for $x \in \mathfrak{a}$ by

$$F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx).$$

These functions were introduced in [19]. Let $C_0(\mathfrak{a})$ be the space of continuous functions on $\mathfrak{a}$ which vanish at infinity, and let $C_0^2(\mathfrak{a})$ be its subset of twice differentiable functions (with analogue definitions for $\overline{\mathfrak{a}_+}$ in place of $\mathfrak{a}$). We denote by $\mathcal{C}(\mathfrak{a})$ the Schwartz space on $\mathfrak{a}$ adapted to the measure $\mu$, i.e. the space of infinitely differentiable functions $f$ on $\mathfrak{a}$ such that for any polynomial $p$, and any $N \in \mathbb{N}$,

$$\sup_{x \in \mathfrak{a}} (1 + |x|)^N e^{(\rho, x^+)} |p(\frac{\partial}{\partial x}) f(x)| < +\infty,$$

where $x^+$ is the unique symmetric of $x$ which lies in $\overline{\mathfrak{a}_+}$. In particular $\mathcal{C}(\mathfrak{a})$ is a subspace of $L^2(\mu)$ and is dense in $C_0(\mathfrak{a})$. We denote by $\mathcal{C}(\mathfrak{a})^W$ the subspace of $W$-invariant functions, which we identify with their restriction to $\overline{\mathfrak{a}_+}$. We have seen in [33] that

$$\mathcal{D} := \frac{1}{2}(\mathcal{L} - |\rho|^2),$$

densely defined on $\mathcal{C}(\mathfrak{a})$, has a closure on $C_0(\mathfrak{a})$, which generates a Feller semigroup $(P_t, t \geq 0)$. We have also obtained the formula for $f \in C_0(\mathfrak{a})$ and $x \in \mathfrak{a}$:

$$P_t f(x) = \int_{\mathfrak{a}} p_t(x,y) f(y) d\mu(y),$$

where $p_t(\cdot,\cdot)$ is the heat kernel. It is defined for $x, y \in \mathfrak{a}$ and $t > 0$, by

$$p_t(x,y) = \int_{i\mathfrak{a}} e^{-\frac{t}{2}(|\lambda|^2+|\rho|^2)} G_\lambda(x) G_\lambda(-y) d\nu(\lambda),$$

where $\nu$ is the asymmetric Plancherel measure. It is defined on $i\mathfrak{a}$ by

$$d\nu(\lambda) = \text{const} \cdot \prod_{\alpha \in \mathcal{R}^+} \frac{\Gamma((\lambda, \alpha^\vee) + k_\alpha + \frac{1}{2}k_{\frac{\alpha}{2}})\Gamma(-(\lambda, \alpha^\vee) + k_\alpha + \frac{1}{2}k_{\frac{\alpha}{2}} + 1)}{\Gamma((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{\alpha}{2}})\Gamma(-(\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{\alpha}{2}} + 1)} d\lambda,$$

where $\Gamma$ is the Gamma function and by convention $k_{\frac{\alpha}{2}} = 0$, if $\frac{\alpha}{2} \notin \mathcal{R}$. We denote by $D$ the differential part of $\mathcal{D}$, which is equal for $f \in \mathcal{C}(\mathfrak{a})^W$ and $x \in \mathfrak{a}_{\text{reg}}$ to

$$Df(x) = \frac{1}{2}\Delta f(x) + (\nabla \log \delta^{\frac{1}{2}}, \nabla f)(x).$$

It has also a closure on $C_0(\overline{\mathfrak{a}_+})$, which generates a Feller semigroup $(P_t^W, t \geq 0)$. It is associated to a kernel $p_t^W$, which is defined for $x, y \in \overline{\mathfrak{a}_+}$ and $t > 0$, by

$$p_t^W(x,y) = \int_{i\mathfrak{a}} e^{-\frac{t}{2}(|\lambda|^2+|\rho|^2)} F_\lambda(x) F_\lambda(-y) d\nu'(\lambda),$$

where $\nu'$ is the symmetric Plancherel measure defined by

$$d\nu'(\lambda) = \text{const} \cdot \prod_{\alpha \in \mathcal{R}^+} \frac{\Gamma((\lambda, \alpha^\vee) + k_\alpha + \frac{1}{2}k_{\frac{\alpha}{2}})\Gamma(-(\lambda, \alpha^\vee) + k_\alpha + \frac{1}{2}k_{\frac{\alpha}{2}})}{\Gamma((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{\alpha}{2}})\Gamma(-(\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{\alpha}{2}})} d\lambda.$$

# 3 Definition and first properties

## 3.1 The Heckman-Opdam processes

The Heckman-Opdam process (also denoted by HO-process) is defined as the càdlàg Feller process $(X_t, t \geq 0)$ on $\mathfrak{a}$ with semigroup $(P_t, t \geq 0)$. Remember that it is characterized as the unique (in law) solution of the martingale problem associated to $(\mathcal{D}, \mathcal{C}(\mathfrak{a}))$ on $C_0(\mathfrak{a})$, see e.g. Theorem 4.1 and Corollary 4.3 in [14]. Observe that, by elementary calculation, the generator of the HO-process is also the closure of $\mathcal{D}$ on $C_c^\infty(\mathfrak{a})$, the space of infinitely differentiable functions with compact support on $\mathfrak{a}$. The multiplicity $k$ is called the parameter of the HO-process. Moreover a.s., for any $t \geq 0$, $X_t \in \mathfrak{a}$, i.e. the exploding time of $(X_t, t \geq 0)$ is almost surely infinite. This results for example from Proposition 2.4 in [14]. Similarly, we define the radial process (or radial part of the HO-process), as the Feller process on $\overline{\mathfrak{a}_+}$ with semigroup $(P_t^W, t \geq 0)$. It is also characterized as the unique solution of the martingale problem associated to $(D, \mathcal{C}(\mathfrak{a})^W)$ on $C_0(\overline{\mathfrak{a}_+})$. Consider now the process $(X_t^W, t \geq 0)$ on $\overline{\mathfrak{a}_+}$, defined as the projection on $\overline{\mathfrak{a}_+}$ under the Weyl group $W$ (for any $t$, $X_t^W$ is the unique symmetric of $X_t$ which lies in $\overline{\mathfrak{a}_+}$).

**Proposition 3.1** *The process* $(X_t^W, t \geq 0)$ *is the radial process.*

**Proof:** Remember that if $f \in \mathcal{C}(\mathfrak{a})^W$, then $\mathcal{D}f = Df \in \mathcal{C}(\mathfrak{a})^W$. Thus for $f \in \mathcal{C}(\mathfrak{a})^W$ and $t \geq 0$,

$$f(X_t^W) - f(X_0^W) - \int_0^t Df(X_s^W)ds = f(X_t) - f(X_0) - \int_0^t \mathcal{D}f(X_s)ds.$$

Therefore $(f(X_t^W) - f(X_0^W) - \int_0^t Df(X_s^W)ds, t \geq 0)$ is a local martingale, and we conclude by the uniqueness of the martingale problem associated to $D$. □

Let us recall eventually that when $2k$ equals the multiplicity associated with a Riemannian symmetric space of noncompact type $G/K$, then the radial HO-process coincide with the radial part of the Brownian motion on this symmetric space (see [19], or [33]). For instance if $G = SL_n$, then $k = \frac{1}{2}$ in the real case, $k = 1$ in the complex case, and $k = 2$ in the quaternionic case.

## 3.2 The Dunkl processes

We recall now the definition of the Dunkl process and of its radial part. Let $\mathcal{R}'$ be a reduced root system (i.e. such that $\forall \alpha \in \mathcal{R}'$, $2\alpha \notin \mathcal{R}'$), but non necessarily integral (i.e. we do not assume condition 3 in the definition), and let $k'$ be a multiplicity function on $\mathcal{R}'$. The Dunkl Laplacian $\mathcal{L}'$ is defined for $f \in C^2(\mathfrak{a})$, and $x \in \mathfrak{a}_{\text{reg}}$ by

$$\begin{aligned} \mathcal{L}'f(x) &= \frac{1}{2}\Delta f(x) + \sum_{\alpha \in \mathcal{R}'^+} k'_\alpha \frac{1}{(\alpha, x)} \partial_\alpha f(x) \\ &+ \sum_{\alpha \in \mathcal{R}'^+} k'_\alpha \frac{1}{(\alpha, x)^2} \{f(r_\alpha x) - f(x)\}. \end{aligned}$$

It was proved by Rösler in [31] that $\mathcal{L}'$ defined on $\mathcal{S}(\mathfrak{a})$, the classical Schwartz space on $\mathfrak{a}$, is a closable operator, which generates a Feller semigroup on $C_0(\mathfrak{a})$. Naturally the Dunkl process is the Feller process defined by this semigroup. Now our proof (in [33]) that the operator $D$ defined on $\mathcal{C}(\mathfrak{a})^W$ is closable, also holds in the Dunkl setting. Thus we may define the radial Dunkl process as the Feller process on $\overline{\mathfrak{a}_+}$ with generator the closure of $(L', \mathcal{S}(\mathfrak{a})^W)$, where $L'$ is the differential part of $\mathcal{L}'$, and $\mathcal{S}(\mathfrak{a})^W$ is the subspace of $\mathcal{S}(\mathfrak{a})$ of $W$-invariant function (identified with their restriction to $\overline{\mathfrak{a}_+}$). In this way we get a new characterization of the radial Dunkl process as the unique solution of the martingale problem associated to $(L, \mathcal{S}(\mathfrak{a})^W)$ on $C_0(\overline{\mathfrak{a}_+})$. Naturally the analogue of Proposition 3.1 holds as well in the Dunkl setting. Thus our definition of the radial

Dunkl process agrees with the usual one (for instance in [32]). Eventually the intrinsic Brownian motion is by definition the radial Dunkl process of parameter $k' = 1$.

# 4 The radial HO-process as a Dirichlet process

The goal of this section is to obtain an explicit semimartingale decomposition of the radial HO-process. In the case of root systems of type $A$, it was obtained by Cépa and Lépingle (see [8] and [9] Theorem 2.2). We present here another approach, which is based on the theory of Dirichlet processes. Our reference for this theory will be [15].
We consider $(D, \mathcal{C}(\mathfrak{a})^W)$ as a symmetric operator on $L^2(\overline{\mathfrak{a}_+}, \mu)$ (simply denoted by $L^2$ in the sequel). We have seen in [33] that this operator is closable. We denote by $(D, \mathcal{D}_2(D))$ its closure. Its associated semigroup is just the extension of $(P_t^W, t \geq 0)$ on $L^2$. It is defined for $f, g \in L^2$ and $t \geq 0$ by

$$P_t^W f(x) = \int_{\overline{\mathfrak{a}_+}} p_t^W(x,y) f(y) d\mu(y).$$

We denote by $\mathcal{E}$ the associated Dirichlet form, and by $\mathcal{F}$ its domain ($\mathcal{D}_2(D) \subset \mathcal{F}$). It is determined for $f, g \in \mathcal{D}_2(D)$ by

$$\mathcal{E}(f,g) := -\int_{\overline{\mathfrak{a}_+}} f(x) Dg(x) d\mu(x) = -\int_{\overline{\mathfrak{a}_+}} Df(x) g(x) d\mu(x).$$

The fact that $\mathcal{E}$ is a regular Dirichlet form with special standard core the algebra $\mathcal{C}(\mathfrak{a})^W$, results from the density of this space in $C_0(\overline{\mathfrak{a}_+})$ (see Lemma 5.1 in [33]) and Theorem 3.1.2 in [15]. We have seen in [33] that when $f \in L^1(\mathfrak{a}, \mu)$, then $Gf : x \mapsto \int_0^\infty P_t^W f(x) dt$ is a.e. finite. In the terminology of [15], this means that the Dirichlet form $\mathcal{E}$ (or the semigroup $P_t^W$) is transient. This implies in particular that the process tends to infinity when $t \to \infty$. In the sequel we will prove a law of large numbers which makes this fact precise. It implies also that we may consider $\mathcal{F}$, equipped with its inner product $\mathcal{E}$, as a Hilbert space (see [15] chapter 2). For $i = 1, \ldots, n$, we denote by $\varphi_i : x \mapsto x_i$ the coordinate functions on $\overline{\mathfrak{a}_+}$. For $A > 0$, let $\varphi_i^A \in C^\infty(\overline{\mathfrak{a}_+})$ be a function which coincides with $\varphi_i$ on $\{|x| \leq A\}$, and which is null on $\{|x| \geq A+1\}$.

**Lemma 4.1** *For all $A > 0$, $\varphi_i^A \in \mathcal{F}$, and for all $v \in \mathcal{C}(\mathfrak{a})^W$,*

$$\mathcal{E}(\varphi_i^A, v) = -\int_{\overline{\mathfrak{a}_+}} D\varphi_i^A v d\mu.$$

**Proof:** Let $(u_n)_n \in \mathcal{C}(\mathfrak{a})^W$ which converges uniformly to $\varphi_i^A$ as in Lemma 5.1 in [33]. In particular we may assume that $|u_n - \varphi_i^A|_\infty \leq \frac{1}{n}$ for all $n$. Moreover the sequence $(|\nabla u_n|_\infty)_n$ is bounded, and there exists a constant $C > 0$

(independent of $n$), such that $|\Delta u_n(x)| \leq C$ or $\leq Cn$ if respectively $d(x, \partial\mathfrak{a}_+) \geq \frac{1}{n}$ or $d(x, \partial\mathfrak{a}_+) \leq \frac{1}{n}$. It implies in particular that

$$\begin{aligned} |Du_n(x)| &\leq \text{const}\left(\frac{1}{d(x,\partial\mathfrak{a}_+)} + \sum_{\alpha\in\mathcal{R}^+}\coth\frac{(\alpha,x)}{2}\right) \\ &\leq \text{const}\cdot\frac{1}{d(x,\partial\mathfrak{a}_+)}, \end{aligned}$$

for all $n$ and all $x \in \mathfrak{a}_+$, such that $|x| \leq A+1$. We will deduce from these estimates that $(u_n)_n$ is an $\mathcal{E}$-Cauchy sequence, which will imply that $\varphi_i^A \in \mathcal{F}$. Let $n < m$ be two integers. Since $u_n(x) = 0$ when $|x| \geq A+1$, we have

$$\mathcal{E}(u_n - u_m, u_n - u_m) = -\int_{|x|\leq A+1}(u_n - u_m)D(u_n - u_m)d\mu.$$

Moreover since $k$ is strictly positive, the function $x \mapsto \frac{1}{d(x,\partial\mathfrak{a}_+)}$ is $\mu$-integrable. Then we get that,

$$\mathcal{E}(u_n - u_m, u_n - u_m) \leq \text{const}\cdot\frac{1}{n}.$$

Thus $(u_n)_n$ is an $\mathcal{E}$-Cauchy sequence, which converges to $\varphi_i^A$. The last statement of the lemma follows from the dominated convergence theorem. □

The lemma implies (in the terminology of [15]), that the functions $\varphi_i$ are in $\mathcal{F}_{b,\text{loc}}$. Thanks to Theorem 5.5.1 of [15], there exist martingale additive functionals locally of finite energy $M^i$, and additive functionals locally of zero energy $N^i$ such that, for every $i$,

$$\varphi_i(X^W) = M^i + N^i. \tag{2}$$

For $A > 0$, we denote by $\nu_i^A$ the measure defined by $d\nu_i^A = -D\varphi_i^A d\mu$. Observe that $1_{(|x|\leq A)}d\nu_i^A(x) = \sum_\alpha k_\alpha \coth(\frac{\alpha}{2}, x)\varphi_i(\alpha)1_{(|x|\leq A)}d\mu(x)$. We denote by $\nu_i$ the Radon measure defined by $d\nu_i(x) = \sum_\alpha k_\alpha \coth(\frac{\alpha}{2}, x)\varphi_i(\alpha)d\mu(x)$. Thanks to Theorem 5.5.4 of [15], and the preceding lemma, we see that $N^i$ is an additive functional of bounded variation, and that it is the unique continuous additive functional associated to the measure $\nu^i$ ($\nu^i$ is called the Revuz measure of $N^i$). Moreover from Theorem 5.1.3 (*iii*) in [15], we get for all $i$ and all $t \geq 0$,

$$N_t^i = \sum_{\alpha\in\mathcal{R}^+} k_\alpha\phi_i(\alpha)\int_0^t \coth\frac{(\alpha, X_s^W)}{2}ds.$$

In the same way, it is immediate from Theorem 5.5.2 and the identity 3.2.14 in [15], that the Revuz measure of $< M^i >$ is $\mu$ for each $i$. Therefore $M := \sum_i M^i e_i$ is necessarily a Brownian motion on $\mathfrak{a}$ (we have denoted by $e_i$ the $i^{\text{th}}$ vector of the canonical basis). Now, since coth is positive on $(0, +\infty)$, and $X^W$ does not explode, we see with (2) that necessarily, for all $t > 0$, $\sum_{i=1}^n N_t^i e_i \in \overline{\mathfrak{a}_+}$ (in particular it does not explode). Thus for any $\alpha$, $(\int_0^t \coth\frac{(\alpha,X_s^W)}{2}ds, t \geq 0)$ is in fact a positive additive functional and its expectation is therefore finite for each time $t \geq 0$ and for q.e. starting point $x$ (q.e. stands for quasi everywhere, as explained in [15]). But it results from [9] Theorem 2.2 that it is in fact true for all $x \in \overline{\mathfrak{a}_+}$. Indeed there it is proved that

$$\mathbb{E}_x[\int_0^t |\nabla \log \delta^{\frac{1}{2}}(X_s^W)|ds] < +\infty.$$

But $\nabla \log \delta^{\frac{1}{2}}$ is equal to $k_\alpha \alpha \coth \frac{\alpha}{2} + z$, where $z$ lies in the cone, let say $C^*$, generated by the convex hull of $\mathcal{R}^+$. Thus (since $-\alpha \notin C^*$) there exists a constant $c > 0$ such that,

$$c \coth \frac{(\alpha, x)}{2} \leq d(0, k_\alpha \alpha \coth \frac{(\alpha, x)}{2} + C^*) \leq |\nabla \log \delta^{\frac{1}{2}}(x)|,$$

for all $x \in \overline{\mathfrak{a}_+}$. Finally we have proved the following result

**Proposition 4.1** *The radial Heckman-Opdam process $(X_t^W, t \geq 0)$ starting at $x \in \overline{\mathfrak{a}_+}$, is a continuous semimartingale, and is the unique solution of the following SDE*

$$X_t^W = x + \beta_t + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \frac{\alpha}{2} \int_0^t \coth \frac{(\alpha, X_s^W)}{2} ds, \ t \geq 0, \tag{3}$$

*where $(\beta_t, t \geq 0)$ is a Brownian motion on $\mathfrak{a}$. Moreover for any $t \geq 0$, any $x \in \overline{\mathfrak{a}_+}$ and any $\alpha \in \mathcal{R}^+$,*

$$\mathbb{E}_x \left[ \int_0^t \coth \frac{(\alpha, X_s^W)}{2} ds \right] < +\infty.$$

The uniqueness in law of the SDE (3) is just a consequence of the uniqueness of solutions to the martingale problem associated to $(D, \mathcal{C}(\mathfrak{a})^W)$. In fact there is also strong uniqueness. This results simply from the fact that coth is decreasing. Indeed if $(X, B)$ and $(X', B)$ are two solutions of (3), then for all $t \geq 0$,

$$\frac{d}{dt}(|X_t - X_t'|^2) = 2(X_t - X_t', \nabla \log \delta^{\frac{1}{2}}(X_t) - \nabla \log \delta^{\frac{1}{2}}(X_t')) \leq 0,$$

which proves that $X$ and $X'$ are indistinguishable. With Theorem (1.7) p. 368 in [30], this implies also that each solution is strong.

**Remark 4.1**

1. The finiteness of the expectation in the proposition will be used in the next section for the study of the jumps.

2. We could ask whether the processes considered by Cépa and Lepingle in [9] coincide with ours. In fact they prove existence of a solution for the same SDE but with an additional local time term. The question is therefore

to know if this local time is equal to 0. Cépa and Lépingle have proved this for root systems of type $A$. But we can prove it now for the other root systems. Indeed by the Itô formula, their process is a solution of the martingale problem associated to $D$, since for any $W$-invariant function $f$, $(\nabla f(x), n) = 0$, for all $x \in \partial\mathfrak{a}_+$ and all normal vector $n$. Thus they coincide with the radial HO-process whose local time on $\partial\mathfrak{a}_+$ is 0.

3. In fact Proposition 4.1 is also valid (with the same proof) in the Dunkl setting (with the function $x \mapsto x^{-1}$ in place of the function coth), where it was proved in the same time, but with a completely different method, by Chybiryakov (see [12]).

A first consequence of Proposition 4.1 is an absolute continuity relation between the laws of the radial HO-process and the corresponding radial Dunkl process. More precisely, let $\mathbb{P}^W$ be the law of $(X_t^W, t \geq 0)$ with parameter $k$ on $C(\mathbb{R}^+, \overline{\mathfrak{a}_+})$. Let $\mathcal{R}' := \{\frac{\sqrt{2}\alpha}{|\alpha|} \mid \alpha \in \mathcal{R}\}$, and if $\beta = \frac{\sqrt{2}\alpha}{|\alpha|} \in \mathcal{R}'$, let $k'_\beta := k_\alpha + k_{2\alpha}$. Let $\mathbb{Q}^W$ be the law of the radial Dunkl process $(Z_t^W, t \geq 0)$ associated to the root system $\mathcal{R}'$ and with parameter $k'$. Let $(L_t, t \geq 0)$ be the process defined by

$$L_t := \int_0^t \nabla \log \frac{\delta^{\frac{1}{2}}}{\pi}(Z_s^W) d\beta_s,\ t \geq 0,$$

where $(\beta_t, t \geq 0)$ is a Brownian motion under $\mathbb{Q}^W$, and

$$\pi(x) = \prod_{\beta \in \mathcal{R}'} (\beta, x)^{k'_\beta}.$$

As the function $x \mapsto \frac{1}{x} - \coth(x)$ is bounded on $\mathbb{R}$ we get that, for all $t \geq 0$, $\mathbb{Q}^W[\exp(\frac{1}{2} < L >_t)] < \infty$ (where $\mathbb{Q}^W[\dots]$ stands for the expectation under the law $\mathbb{Q}^W$). Thus $M := \exp(L - \frac{1}{2} < L >)$, the stochastic exponential of $L$, is a $\mathbb{Q}^W$-martingale. Moreover as mentioned in Remark 4.1 we have also an explicit decomposition of the radial Dunkl process as "Brownian motion plus a term with bounded variation". Therefore, by using the Girsanov theorem [30], we get that for any $t \geq 0$, if $(\mathcal{F}_t, t \geq 0)$ is the canonical filtration on $C(\mathbb{R}^+, \overline{\mathfrak{a}_+})$, then

$$\mathbb{P}^W_{|\mathcal{F}_t} = M_t \cdot \mathbb{Q}^W_{|\mathcal{F}_t}. \tag{4}$$

As a consequence we obtain for instance that when $k_\alpha + k_{2\alpha} \geq 1/2$, then the HO-process starting at any $x \in \mathfrak{a}_+$ a.s. does not touch the walls (i.e. the subspaces of the type $\{\alpha = 0\}$). This follows from the similar result for the Dunkl processes proved in [11]. Now if it starts at some $x \in \partial\mathfrak{a}_+$ then a.s., by the Markov property, it will never touch the walls in strictly positive times (observe that $\mu(\partial\mathfrak{a}_+) = 0$, thus at any $t > 0$, a.s. $X_t \in \mathfrak{a}_+$).

We will now prove a law of large numbers and a central limit theorem for the radial Heckman-Opdam processes. These results are well known in the setting of symmetric spaces of noncompact type (see for instance Babillot [2]).

**Proposition 4.2** *The radial process satisfies the law of large numbers*

$$\lim_{t\to\infty}\frac{X_t^W}{t}\to\rho\ a.s.,$$

*and there is the convergence in* $C(\mathbb{R}^+,\overline{\mathfrak{a}_+})$

$$(\frac{X_{tT}^W-\rho tT}{\sqrt{T}},t\geq 0)\xrightarrow[T\to\infty]{}(\beta_t,t\geq 0).$$

**Proof:** The first step is to prove that $(\alpha,X_t^W)\to+\infty$, when $t\to+\infty$, for all $\alpha\in\mathcal{R}^+$. In fact by linearity it is enough to prove this only for the simple roots. We denote by $\{\alpha_1,\dots,\alpha_n\}$ the set of simple roots. From Proposition 4.1 we see that the radial process (starting at $x$) satisfies for any $t\geq 0$,

$$X_t^W=x+\beta_t+\rho t+\frac{1}{2}\sum_{\alpha\in\mathcal{R}^+}k_\alpha\alpha\int_0^t[\coth\frac{(\alpha,X_s^W)}{2}-1]ds. \tag{5}$$

Let $u\in\overline{\mathfrak{a}_+}$. From (5) we get that for all $t\geq 0$, $(u,X_t^W)-(u,x)-(u,\beta_t)\geq(u,\rho)t$, because $\coth(x)\geq 1$ for $x\geq 0$. Thus $(u,X_t^W)\to+\infty$, when $t\to+\infty$. In particular $(\rho,X_t^W)\to+\infty$. It implies that $\max_{i=1,\dots,n}(\alpha_i,X_t^W)\to+\infty$. For $t>0$, let $i_1,\dots,i_n$ be such that $(\alpha_{i_1},X_t^W)\geq\dots\geq(\alpha_{i_n},X_t^W)$ (we forget the dependance in $t$ in the notation). We prove now that $(\alpha_{i_2},X_t^W)\to+\infty$. Let $\epsilon>0$ and let $T_0$ be such that $\coth(\alpha_{i_1},X_t^W)-1\leq\epsilon$ and $|x+\beta_t|\leq\epsilon t$ for $t\geq T_0$. Let $\mathcal{R}_2$ be the root system generated by $\{\alpha_{i_2},\dots,\alpha_{i_n}\}$ and let $\rho_2=\sum_{\alpha\in\mathcal{R}_2^+}k_\alpha\alpha$. Observe in particular that if $\alpha\in\mathcal{R}_2^+$, then $(\alpha,\rho_2)\geq 0$, whereas if $\alpha\neq\alpha_{i_1}$ and $\alpha\notin\mathcal{R}_2^+$, then $\alpha-\alpha_{i_1}\in\mathcal{R}^+$ and thus $\coth\frac{(\alpha,X_t^W)}{2}\leq 1+\epsilon$. Now from (5) we get for $t\geq T_0$,

$$(\rho_2,X_t^W)\geq((\rho,\rho_2)-\epsilon)t+f(t),$$

where $f(t)=\sum_{\alpha\in\mathcal{R}^+}k_\alpha(\alpha,\rho_2)\int_0^t[\coth\frac{(\alpha,X_s^W)}{2}-1]ds$. Hence by our choice of $\rho_2$, we have for $t\geq T_0$, $f'(t)\geq -C\epsilon$ for some constant $C>0$. Then we get another constant $C'>0$ such that $f(t)\geq -C'-C\epsilon t$ for $t\geq T_0$. Thus we conclude that $(\rho_2,X_t^W)\to+\infty$ and $(\alpha_{i_2},X_t^W)\to+\infty$. In the same way we deduce that $(\alpha_i,X_t^W)\to+\infty$ for all $1\leq i\leq n$. Eventually we get immediately the law of large numbers from (5).
For the second claim of the proposition, we will show that a.s.

$$|\frac{1}{\sqrt{T}}(X_{tT}^W-x-\beta_{tT}-tT\rho)|\xrightarrow[T\to\infty]{}0,$$

uniformly in $t\in\mathbb{R}^+$. Let $\epsilon>0$. By the first claim, we know that there is some $N$ such that for every $s\geq N$, $|\coth\frac{(\alpha,X_s^W)}{2}-1|\leq e^{-cs}$ for some strictly positive constant $c$. Then we have

$$\begin{aligned}\frac{1}{\sqrt{T}}\int_0^{tT}[\coth\frac{(\alpha,X_s^W)}{2}-1]ds &= \frac{1}{\sqrt{T}}\int_0^{N\wedge tT}[\coth\frac{(\alpha,X_s^W)}{2}-1]ds\\ &+ \frac{1}{\sqrt{T}}1_{(tT\geq N)}\int_N^{tT}[\coth\frac{(\alpha,X_s^W)}{2}-1]ds.\end{aligned}$$

But the both integrals can be made smaller than $\epsilon$ by choosing $T$ sufficiently large. The second claim follows using the scaling property of the Brownian motion. $\square$

# 5 Jumps of the process

We will now study the behavior of the jumps of the Heckman-Opdam process. We use essentially the same tool as in [18] for the Dunkl processes, i.e. we use the predictable compensators of some discontinuous functionals. However in our setting we obtain a more precise result when $k_\alpha + k_{2\alpha} \geq \frac{1}{2}$ for all $\alpha$. In fact in this case, there is almost surely a finite random time, after which the process does not jump anymore. This allows to prove for such multiplicity $k$ a law of large numbers and a central limit theorem for the HO-process.

Let us first recall the definition of the Lévy kernel $N(x, dy)$ of a homogeneous Markov process with a transition semigroup $(P_t)_{t\geq 0}$ and generator $\mathcal{D}$ (see Meyer [27]). It is determined, for any $x \in \mathbb{R}^d$ by:

$$\mathcal{D}f(x) = \lim_{t\to 0} \frac{P_t f(x)}{t} = \int_{\mathfrak{a}} N(x, dy) f(y),$$

for $f$ a function in the domain of the infinitesimal generator which vanishes in a neighborhood of $x$. The following lemma describes the Lévy kernel of the HO-process. It is an immediate consequence of the explicit expression (1) of $\mathcal{L}$. First we introduce some notation. If $I$ is a subset of $\mathcal{R}^+$, we denote by

$$\mathfrak{a}^I = \{x \in \mathfrak{a} \mid \forall \alpha \in I,\ (\alpha, x) = 0\},$$

the face associated to $I$. We also denote by $\mathcal{R}_I$ the set of positive roots which are orthogonal to $\mathfrak{a}^I$.

**Lemma 5.1** *The Lévy kernel of the HO-process has the following form*

$$N(x, dy) = \begin{cases} \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{|\alpha|^2}{8} \frac{\epsilon_{r_\alpha x}(dy)}{\sinh^2 \frac{(\alpha,x)}{2}} & if \quad x \in \mathfrak{a}_{reg} \\ \sum_{\alpha\in\mathcal{R}^+\smallsetminus\mathcal{R}_I} k_\alpha \frac{|\alpha|^2}{8} \frac{\epsilon_{r_\alpha x}(dy)}{\sinh^2 \frac{(\alpha,x)}{2}} & if \quad x \in \mathfrak{a}^I, \end{cases}$$

*where $I$ is a subset of $\mathcal{R}^+$, and for $x \in \mathfrak{a}$, $\epsilon_x$ is the Dirac measure in $x$.*

**Remark 5.1** The lemma implies that when there is a jump at a random time $s$, i.e. $X_s \neq X_{s-}$, then almost surely there exists $\alpha \in \mathcal{R}^+$ such that $X_s = r_\alpha X_{s-}$ (see [18]). In this case we have

$$\Delta X_s := X_s - X_{s-} = -(\alpha^\vee, X_{s-})\alpha.$$

Using the finiteness of the expectation of the time integrals appearing in (3), we can show that the sum over any time interval of the amplitudes of the jumps is finite.

**Proposition 5.1** *Let $(X_t, t \geq 0)$ be a Heckman-Opdam process. For every $t > 0$,*

$$\mathbb{E}\left[\sum_{s \leq t} |\Delta X_s|\right] < +\infty.$$

**Proof:** From the above remark we get

$$\sum_{s \leq t} |\Delta X_s| = \sum_{\alpha \in \mathcal{R}^+} \sum_{s \leq t} f_\alpha(X_{s-}, X_s),$$

where

$$f_\alpha(x, y) = \frac{2}{|\alpha|} |(\alpha, x)| 1_{(y = r_\alpha x \neq x)}.$$

Now, the positive discontinuous functional $\sum_{s \leq t} f_\alpha(X_{s-}, X_s)$ is compensated by the process $\int_0^t ds \int_{\mathfrak{a}} N(X_{s-}, dy) f_\alpha(X_{s-}, y)$. As a consequence, the proposition will be proved if we know that the expectation of the compensator is finite at all time $t \geq 0$. Thus we have to show that for every $\alpha \in \mathcal{R}^+$,

$$\mathbb{E}\left[\int_0^t \left| \frac{(\alpha, X_s)}{\sinh^2 \frac{(\alpha, X_s)}{2}} \right| ds\right] < +\infty.$$

But for every $x > 0$, $\frac{x}{\sinh^2 x} \leq 2 \coth x$. Therefore the above condition follows from Proposition 4.1. □

For $\alpha \in \mathcal{R}^+$, we denote by $(M_t^\alpha, t \geq 0)$ the process defined for $t \geq 0$ by:

$$M_t^\alpha = \sum_{s \leq t} -(\alpha^\vee, X_{s-}) 1_{(r_\alpha X_{s-} = X_s)} + \frac{k_\alpha}{4} \int_0^t \frac{(\alpha, X_s)}{\sinh^2 \frac{(\alpha, X_s)}{2}} ds. \tag{6}$$

By the martingale characterization of $(X_t, t \geq 0)$, we know that $(f(X_t), t \geq 0)$ is a local semimartingale for all $f \in C_c^\infty(\mathfrak{a})$. Thus $(X_t, t \geq 0)$ itself is a local semimartingale. In the next proposition we give its explicit decomposition. It is the analogue of a result of Gallardo and Yor on the decomposition of the Dunkl processes. The proof is very similar (and uses Proposition 5.1), so we refer to [18] for more details.

**Proposition 5.2** *We have the following semimartingale decomposition:*

$$X_t = X_0 + \beta_t + \sum_{\alpha \in \mathcal{R}^+} M_t^\alpha \alpha + A_t,$$

*for $t \geq 0$, where*

$$A_t = \sum_{\alpha \in \mathcal{R}^+} \frac{k_\alpha}{2} \alpha \int_0^t \left[\coth \frac{(\alpha, X_s)}{2} - \frac{(\alpha, X_s)}{2 \sinh^2 \frac{(\alpha, X_s)}{2}}\right] ds,$$

*and the $M^\alpha$'s are purely discontinuous martingales given by (6) which satisfy*

$$[M^\alpha, M^\beta]_t = 0, \ \ \textit{if } \alpha \neq \beta,$$

*and*

$$< M^\alpha >_t = \frac{k_\alpha}{4|\alpha|^2} \int_0^t \frac{(\alpha, X_s)^2}{\sinh^2 \frac{(\alpha, X_s)}{2}} ds.$$

Another interesting property of the jumps is that, when $k_\alpha + k_{2\alpha} \geq 1/2$ for all $\alpha$, and when the starting point lies in $\mathfrak{a}_{\mathrm{reg}}$, in which case the HO-process does not touch the walls, the number of jumps $N_t$ up to a fixed time $t$ is a.s. finite. Indeed otherwise the paths of the trajectories would not be càdlàg. Therefore the sequence of stopping times $T_n = \inf\{t > 0, N_t \geq n\}$ converges a.s. to $+\infty$ when $n$ tends to infinity. Thus $(N_t, t \geq 0)$ is a locally integrable (because locally finite) increasing process. We will deduce from this observation and a general result of [21] a more precise result. For $t \geq 0$, we denote by $w_t$ the element of $W$ such that $X_t = w_t X_t^W$.

**Proposition 5.3** *Assume that $k_\alpha + k_{2\alpha} \geq 1/2$ for all $\alpha$.*

1. *If the starting point lies in $\mathfrak{a}_{reg}$, then a.s.*

$$\sup_{t \geq 0} N_t < +\infty.$$

2. *For any starting point in $\mathfrak{a}$, $w_t$ converges a.s. to $w_\infty \in W$ when $t \to \infty$. If the process starts from zero, then the law of $w_\infty$ is the uniform probability on $W$.*

3. *When $T \to \infty$, the sequences $(\frac{1}{T} X_{tT}, t \geq 0)$ and $(\frac{1}{\sqrt{T}}(X_{tT} - w_{tT}\rho tT), t \geq 0)$ converge in law in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ respectively to $(w_\infty \rho t, t \geq 0)$, and to a Brownian motion $(\beta_t, t \geq 0)$.*

**Proof:** Let us begin with the first claim. As in the preceding proposition we use the following result of Meyer about the Lévy kernel: the positive discontinuous functional $(N_t, t \geq 0)$ can be compensated by the predictable process $(\sum_{\alpha \in \mathcal{R}^+} k_\alpha \frac{|\alpha|^2}{8} \int_0^t \frac{1}{\sinh^2 \frac{(\alpha, X_s)}{2}} ds, t \geq 0)$. Now from the law of large numbers (Proposition 4.2) we deduce that this compensator converges a.s. to a finite value when $t \to \infty$. Thus the corollary (5.20) p. 168 of [21] gives the result. Then the second point is simply a consequence of the first point and of the Markov property. The assertion that the limit law is uniform when the process starts from zero results from the fact that for any $w \in W$, $\mathcal{D}$ remains unchanged if we replace $\mathcal{R}^+$ by $w\mathcal{R}^+$. The first convergence result of the last point is straightforward with the second point and Proposition 4.2. For the second convergence result, we can use Proposition 5.2. Indeed it says that for all $t > 0$ and $T > 0$,

$$\begin{aligned}\frac{X_{tT} - w_{tT}\rho tT}{\sqrt{T}} &= \frac{\beta_{tT}}{\sqrt{T}} + \frac{1}{\sqrt{T}}\sum_{s\leq tT}\Delta X_s \\ &+ \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{\alpha}{2\sqrt{T}}\int_0^{tT}\left[\coth\frac{(\alpha,X_s)}{2} - \epsilon_{tT}^\alpha\right] ds,\end{aligned}$$

where the $\epsilon_{tT}^\alpha \in \{\pm 1\}$ are defined by $w_{tT}\rho = \sum_{\alpha\in\mathcal{R}^+} k_\alpha \epsilon_{tT}^\alpha \alpha$, or equivalently by $\epsilon_{tT}^\alpha \alpha \in w_{tT}\mathcal{R}^+$. But by the second point of the proposition, we know that a.s. there exists a random time after which the process stays in the same chamber, which is $w_\infty \mathfrak{a}_+$. Moreover for all $s > 0$ and all $\alpha \in \mathcal{R}^+$, $\epsilon_s^\alpha(\alpha, X_s) \geq 0$. Thus by Proposition 4.2 $\coth\frac{(\alpha,X_s)}{2} - \epsilon_{tT}^\alpha$ tends to 0 exponentially fast when $s \to \infty$ (and $s \leq tT$). Then a.s. $\sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{\alpha}{2\sqrt{T}}\int_0^{tT}\left[\coth\frac{(\alpha,X_s)}{2} - \epsilon_{tT}^\alpha\right] ds$ tends to 0 when $T \to \infty$, uniformly in $t \in \mathbb{R}^+$. In the same way, by Proposition 5.1, a.s. for any $A > 0$, $\sum_{s\leq A}|\Delta X_s| < +\infty$. By the second point we know that a.s. after some time there is no more jumps, thus a.s. $\frac{1}{\sqrt{T}}\sum_{s\leq tT}|\Delta X_s|$ tends to 0 when $T \to \infty$, uniformly in $t \in \mathbb{R}^+$. This proves the desired result by the scaling property of the Brownian motion. □

# 6 Convergence to the Dunkl processes

In this section we will show that when it is well normalized, the HO-process of parameter $k > 0$ converges to a certain Dunkl process $(Z_t, t \geq 0)$. The proof uses a general criteria for a sequence of Feller processes with jumps, which can be found in [14] for instance. Roughly speaking it states that it suffices to prove the convergence of the generator of these processes on a core of the limit. Let us notice that the convergence of the normalized radial HO-process to the corresponding radial Dunkl process is more elementary. It could be proved essentially by using that the laws of the radial HO-process and the radial Dunkl process are absolutely continuous, and that the normalized Radon-Nikodym derivative tends to 1. Let us also observe that the convergence of the normalized radial process has a natural geometric interpretation in the setting of symmetric spaces of noncompact type. Indeed in this setting the radial Dunkl process is just the radial part of the Brownian motion on the tangent space (or the Cartan motion group, see the more precise description by De Jeu [24], and in [1] in the complex case). From the analytic point of view, it also illustrates the more conceptual principle, that the Dunkl (also called rational) theory is the limit of the Heckman-Opdam (or trigonometric) theory, when "the curvature goes to zero".

We denote by $(X_t^T, t \geq 0)$ the normalized HO-process, which is defined for $t \geq 0$ and $T > 0$ by:

$$X_t^T = \sqrt{T} X_{\frac{t}{T}}.$$

We recall that $\mathcal{R}' = \{\frac{\sqrt{2}\alpha}{|\alpha|},\ \alpha \in \mathcal{R}\}$, and that for $\beta = \frac{\sqrt{2}\alpha}{|\alpha|} \in \mathcal{R}'$, $k'_\beta = k_\alpha + k_{2\alpha}$.

**Theorem 6.1** *When $T \to \infty$, the normalized HO-process $(X_t^T, t \geq 0)$ with parameter $k$ starting at 0 converges in distribution in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ to the Dunkl process $(Z_t, t \geq 0)$ associated with $\mathcal{R}'$ and with parameter $k'$ starting at 0.*

**Proof:** First it is well known that the process $(X_t^T, t \geq 0)$ is also a Feller process with generator $\mathcal{L}^T$ defined for $f \in C^2(\mathfrak{a})$ and $x \in \mathfrak{a}_{\text{reg}}$, by $\mathcal{L}^T f(x) = \frac{1}{T}(\mathcal{L}g)(\frac{x}{\sqrt{T}})$, where $g(x) = f(\sqrt{T}x)$. Thus a core of $\mathcal{L}^T$ is, like for $\mathcal{L}$ and $\mathcal{L}'$, the space $C_c^\infty(\mathfrak{a})$. Moreover it is straightforward that for any $f \in C_c^\infty(\mathfrak{a})$, $\mathcal{L}^T f$ converges uniformly on $\mathfrak{a}$ to $\mathcal{L}'f$. Therefore we can apply Theorem 6.1 p. 28 and Theorem 2.5 p. 167 in [14], which give the desired result. □

# 7 The $F_0$-process and its asymptotic behavior

We introduce here and study a generalization of the radial part of the Infinite Brownian Loop (abbreviated as I.B.L.) introduced in [1]. Let $\tilde{F}_0(x,t) := F_0(x)e^{\frac{|\rho|^2}{2}t}$, for $(x,t) \in \mathfrak{a} \times [0, +\infty)$. Then $\tilde{F}_0$ is harmonic for the operator $\partial_t + \mathcal{D}$ which is the generator of $(X_t, t)_{t \geq 0}$. We define now the processes $(Y_t, t)_{t \geq 0}$ as the relativized $\tilde{F}_0$-processes in the sense of Doob of $(X_t, t)_{t \geq 0}$. By abuse of language we will call $(Y_t, t \geq 0)$ the $F_0$-process. We denote by $(Y_t^W, t \geq 0)$ its radial part, that we will call the radial $F_0$-process. For particular values of $k$ it coincides with the radial part (in the Lie group terminology) of the I.B.L. on a symmetric space.

The goal of this section is to generalize some results of [1] and [5], for any $k > 0$. Essentially we first prove the convergence of the HO-bridge of length $T$ around 0, i.e. the HO-process conditioned to be equal to 0 at time $T$, to the $F_0$-process starting at 0, when $T$ tends to infinity. Then we prove the convergence of the normalized $F_0$-process to a process whose radial part is the intrinsic Brownian motion, but which propagates in a random chamber (independently and uniformly chosen). We begin by the following lemma:

**Lemma 7.1** *Let $x, a \in \overline{\mathfrak{a}_+}$. When $T \to \infty$,*

$$\frac{p_{T-t}^W(x,a)}{p_T^W(a,a)} \to \frac{F_0(x)}{F_0(a)} e^{\frac{t}{2}|\rho|^2}.$$

**Proof:** We need the integral formula of the heat kernel:

$$p_t^W(x,y) = \int_{i\mathfrak{a}} e^{-\frac{t}{2}(|\lambda|^2 + |\rho|^2)} F_\lambda(x) F_{-\lambda}(y) d\nu'(\lambda), \ x, y \in \mathfrak{a}_+.$$

We make the change of variables $u := \lambda(T-t)$ for $p_{T-t}^W$, and $v := \lambda T$ for $p_T^W$. Then we let $T$ tend to $+\infty$ and the result follows. □

**Proposition 7.1** *Let $(X_t^{0,T}, t \geq 0)$ be the HO-bridge around 0 of length $T$. Then when $T \to +\infty$, it converges in distribution in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ to the $F_0$-process starting at 0.*

**Proof:** We know that for any $t \geq 0$, and any bounded $\mathcal{F}_t$-measurable function $F$,

$$\begin{aligned}\mathbb{E}\left[F(X_s^{0,T}, 0 \leq s \leq t)\right] &= \mathbb{E}\left[F(X_s, 0 \leq s \leq t)\frac{p_{T-t}(X_t, 0)}{p_T(0,0)}\right] \\ &= \mathbb{E}\left[F(X_s, 0 \leq s \leq t)\frac{p^W_{T-t}(X^W_t, 0)}{p^W_T(0,0)}\right]\end{aligned}$$

The second equality results from the fact that $p_t(x,0) = \frac{1}{|W|}p_t^W(x,0)$, for all $x \in \mathfrak{a}$ and all $t \geq 0$ (see [33]). Moreover since $F_\lambda$ is bounded (cf [29]) and the measure $\nu'$ is positive, we see from the integral formula of $p_t^W$, that there exists a constant $C$ such that, $p_t^W(x,y) \leq C p_t^W(0,0)$, for all $t > 0$ and all $x, y \in \overline{\mathfrak{a}_+}$. It follows that $\frac{p^W_{T-t}(x,0)}{p^W_T(0,0)}$ is a bounded function of $(x,T) \in \overline{\mathfrak{a}_+} \times [1,\infty)$. Then we get the result from the preceding lemma and the dominated convergence theorem. □

**Remark 7.1** In the radial setting the preceding result holds for any starting point. More precisely, the radial HO-bridge of length $T$ around $a \in \overline{\mathfrak{a}_+}$ is defined as the radial HO-process starting from $a$ and conditioned to be equal to $a$ at time $T$. Then this process converges in law, when $T$ tends to infinity, to the radial $F_0$-process starting from $a$. The proof is the same than in the preceding proposition (with $X_s$ and $p_t$ replaced respectively by $X_s^W$ and $p_t^W$).

The next result is an important technical lemma:

**Lemma 7.2** *There exist two Bessel processes $(R_t, t \geq 0)$ and $(R'_t, t \geq 0)$ (not necessarily with the same dimension), such that a.s. $|R_0| = |R'_0| = |Y_0|$ and for any $t \geq 0$,*

$$R_t^2 \leq |Y_t|^2 \leq R_t'^2.$$

**Proof:** First the $F_0$ process and its radial part have the same norm, hence it suffices to prove the result for $(Y_t^W, t \geq 0)$. Next we can follow exactly the same proof as in [1]. We recall it for the convenience of the reader. We know that $(Y_t^W, t \geq 0)$ is solution of the SDE

$$Y_t^W = Y_0^W + \beta_t + \int_0^t \nabla \log(\delta^{\frac{1}{2}} F_0)(Y_s^W) ds,$$

where $(\beta_t, t \geq 0)$ is a Brownian motion. By Itô formula we get

$$|Y_t^W|^2 = |Y_0^W|^2 + 2\int_0^t (Y_s^W, d\beta_s) + tn + 2\int_0^t E[\log(\delta^{\frac{1}{2}} F_0)](Y_s^W) ds,$$

where $E = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ is the Euler operator on $\mathfrak{a}$. And it was shown in [33] that $E[\log(\delta^{\frac{1}{2}} F_0)]$ is positive and bounded on $\mathfrak{a}$. Thus we can conclude by using comparison theorems. □

**Corollary 7.1** *Let $(Y_t, t \geq 0)$ be the $F_0$-process. Then almost surely,*

$$\lim_{t\to\infty} \frac{|Y_t|}{t} = 0.$$

*More precisely (law of the iterated logarithm), a.s.*

$$\limsup_{t\to\infty} \frac{|Y_t|}{\sqrt{2t\log\log t}} = 1.$$

**Proof:** This follows from the preceding lemma and known properties of the Bessel processes. □

In the complex case, i.e. when $2k$ is equal to the multiplicity function on some complex Riemannian symmetric space of noncompact type (or equivalently when $\mathcal{R}$ is reduced and $k = 1$), then it was proved in [1] that the radial $F_0$-process coincides with the intrinsic Brownian motion. It was also proved in [1] that in the real case, i.e. for other choices of $k$, a normalization of the radial $F_0$-process converges to the intrinsic Brownian motion. The next theorem gives a generalization of this result for any multiplicity $k > 0$ and for the (non radial) $F_0$-process also. We denote by $(Y_t^T, t \geq 0)$ the process defined for $t \geq 0$ and $T > 0$, by

$$Y_t^T := \frac{1}{\sqrt{T}} Y_{tT},$$

and we denote by $(Y_t^{W,T}, t \geq 0)$ its radial part. Let $(I_t, t \geq 0)$ be the intrinsic Brownian motion starting from 0. We denote by $(I_t^*, t \geq 0)$ the continuous process starting from 0, whose radial part is the intrinsic Brownian motion, but which propagates in a random chamber $w\mathfrak{a}_+$, where $w$ is chosen independently and with respect to the uniform probability on $W$. This is a typical example of a strong Markov process which is not Feller (it does not satisfy Blumenthal's zero-one law). We have

**Theorem 7.1** *Let $k > 0$. The normalized $F_0$-process $(Y_t^T, t \geq 0)$ starting at any $x \in \mathfrak{a}$ converges in distribution in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ to $(I_t^*, t \geq 0)$.*

**Proof:** The first step is to prove that $(Y_t^{W,T}, t \geq 0)$ converges in law in $C(\mathbb{R}^+, \overline{\mathfrak{a}_+})$ to $(I_t, t \geq 0)$. Thanks to Lemma 7.2 we can use the same proof as for Theorem 5.5 in [1]. The result about the convergence of the semigroup needed in the proof was established in [33]. Now let $\mathbb{P}^0$ be the law of the $F_0$-process, and let $\mathbb{P}$ be the one of the HO-process. By definition we have the absolute continuity relation

$$\mathbb{P}^0_{|\mathcal{F}_t} = \frac{F_0(Y_t)}{F_0(x)} e^{\frac{|\rho|^2}{2}t} \cdot \mathbb{P}_{|\mathcal{F}_t}.$$

Since $F_0$ is bounded (cf [29]) Proposition 5.1 implies that for any $t > 0$, $\mathbb{E}^0[\sum_{s\leq t} |\Delta Y_s|] < +\infty$. Thus by Girsanov theorem (see [30]) and Proposition

5.2 we get the semimartingale decomposition of the $F_0$-process:

$$\begin{aligned} Y_t &= x + \beta_t + \underbrace{\sum_{\alpha\in\mathcal{R}^+} M_t^\alpha \alpha}_{:=M_t} + \int_0^t \nabla \log \delta^{\frac{1}{2}} F_0(Y_s)ds \\ &- \sum_{\alpha\in\mathcal{R}^+} \frac{k_\alpha}{2}\alpha \int_0^t \frac{(\alpha, Y_s)}{2\sinh^2\frac{(\alpha,Y_s)}{2}}ds, \end{aligned}$$

where $(\beta_t, t \geq 0)$ is a $\mathbb{P}^0$-Brownian motion and the $M^\alpha$'s are defined by (6) (with $Y_s$ in place of $X_s$). We set $M_t^T := \frac{1}{\sqrt{T}} M_{tT}$. By Proposition 5.2 we know that

$$< M^T >_t := \sum_{i=1}^n < M_i^T >_t = \sum_{\alpha\in\mathcal{R}^+} \frac{k_\alpha}{4} \int_0^t \frac{(\alpha, \sqrt{T} Y_s^T)^2}{\sinh^2 \frac{\sqrt{T}}{2}(\alpha, Y_s^T)} ds,$$

where $M_i^T$ is the $i^{\text{th}}$ coordinate of $M^T$ in the canonical basis. Now for all $w \in W$ the preceding sum remains unchanged if $\mathcal{R}^+$ is replaced by $w\mathcal{R}^+$. Therefore we can replace $Y_s^T$ by $Y_s^{W,T}$ in the last equality. Moreover $\sinh x \geq x + \frac{x^3}{6}$ on $\mathbb{R}^+$. Hence

$$< M^T >_t \leq \sum_{\alpha\in\mathcal{R}^+} k_\alpha \int_0^t \frac{1}{1 + \frac{T}{12}(\alpha, Y_s^{W,T})^2} ds.$$

Thus by using the first step, we see that for any fixed $t > 0$, $\mathbb{E}[< M^T >_t] \to 0$ when $T \to +\infty$. It implies by Doob's $L^2$-inequality (see [30]), that $(M_t^T, t \geq 0)$ converges in law in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ to 0. Now the triangular inequality implies that for any $A > 0$, $\epsilon > 0$ and $\alpha > 0$,

$$\begin{aligned} \mathbb{P}\left[\sup_{|s-t|\leq\epsilon,\ s\leq t\leq A} |Y_s^T - Y_t^T| \geq \alpha\right] &\leq \mathbb{P}\left[\sup_{|s-t|\leq\epsilon,\ s\leq t\leq A} |Y_s^{W,T} - Y_t^{W,T}| \geq \frac{\alpha}{2N}\right] \\ &+ \mathbb{P}\left[\sum_{s\leq A} |\Delta Y_s^T| \geq \frac{\alpha}{2}\right], \end{aligned}$$

where $N$ is the number of Weyl chambers. Thus using that $\Delta Y^T = \Delta M^T$, tightness of $(Y_t^{W,T}, t \geq 0)$, and standard results (see Theorem 3.21 p. 314 and Proposition 3.26 p. 315 in [22] for instance), we see that the sequence $(Y_t^T, t \geq 0)$ is C-tight in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$, i.e. it is tight and any limit law of a subsequence is supported on $C(\mathbb{R}^+, \mathfrak{a})$. By the first step each limit has a radial part equal to the intrinsic Brownian motion. Since we know that the intrinsic Brownian motion does not touch the walls (in strictly positive time), each limit process is necessarily of the type $(wI_t, t \geq 0)$ where $w$ is some random variable on $W$. Thus in order to identify the limit, we need to prove that the law of $w$ has to be the uniform probability on $W$, and that $w$ is independent of the radial part. For the law of $w$ first, let us just observe that when the process starts from 0, the result is immediate since by $W$-invariance of $\mathcal{D}$ and $F_0$, the law of the

$F_0$-process is $W$-invariant, and thus the law of any limit also. However when $x \neq 0$ we can not argue like this, and we need to prove for instance that the law of $Y_1^T$ converges to the law of $I_1^*$, when $T \to \infty$. Let $f : \mathfrak{a} \to \mathbb{R}$ be continuous and bounded. We have

$$\mathbb{E}_{\frac{x}{\sqrt{T}}}\left[f(\frac{Y_T}{\sqrt{T}})\right] = \int_{\mathfrak{a}} p_T(\frac{x}{\sqrt{T}}, \sqrt{T}y)\frac{F_0(\sqrt{T}y)}{F_0(\frac{x}{\sqrt{T}})}e^{\frac{|\rho|^2}{2}T}f(y)d\mu(\sqrt{T}y).$$

Then it results from the asymptotic of $p_T(\frac{x}{\sqrt{T}}, \sqrt{T}y)$ and of $F_0(\sqrt{T}y)$ proved in [33], that

$$p_T(\frac{x}{\sqrt{T}}, \sqrt{T}y)\frac{F_0(\sqrt{T}y)}{F_0(\frac{x}{\sqrt{T}})}e^{\frac{|\rho|^2}{2}T}d\mu(\sqrt{T}y) \to \text{const} \cdot e^{-\frac{|y|^2}{2}} \prod_{\alpha \in \mathcal{R}^+}(\alpha, y)^2 dy,$$

which gives the density of the law of $I_1^*$ (see [1] for instance for the law of $I_1$). We conclude by Sheffé's lemma. Now the only missing part is the independence of $w$ and $(I_t, t \geq 0)$. Observe first that since any limit process is adapted, $w$ is $\mathcal{F}_{0^+}$-measurable. On the space $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$, we denote by $(\mathcal{F}_t^W)_{t\geq 0}$ the natural filtration of the radial process. We know that $(I_t, t \geq 0)$ is an $(\mathcal{F}_t^W)_{t\geq 0}$-Markov process. We will prove that it is also an $(\mathcal{F}_t)_{t\geq 0}$-Markov process. Indeed since it is a.s. continuous and equal to 0 at time 0, this will imply the independence with $w$. We know that $Y^{W,T}$ is an $(\mathcal{F}_t)_{t\geq 0}$-Markov process, since it is the projection of $Y^T$. We denote by $P^{W,T}$ the semigroup of $Y^{W,T}$, and by $Q^W$ the semigroup of $(I_t, t \geq 0)$. In particular we already know that for $t \geq 0$ and any continuous and bounded function $f$, $P_t^{W,T} f$ converges simply to $Q_t^W f$ when $T \to \infty$. For $s < t$, and $f$ and $g$ continuous and bounded functions, we have $\mathbb{E}[f(Y_t^{W,T})g(Y_s^T)] = \mathbb{E}[P_{t-s}^{W,T} f(Y_s^{W,T})g(Y_s^T)]$. For a suitable subsequence of $T$, the first term tends to $\mathbb{E}[f(I_t)g(\nu I_s)]$, and the second term tends to $\mathbb{E}[Q_{t-s}^W f(I_s)g(\nu I_s)]$, which implies the desired result. This finishes the proof of the theorem. □

We can now prove a generalization of a result of Bougerol and Jeulin [5]. Let $(R_t^{0,T}, 0 \leq t \leq 1)$ be the normalized HO-bridge of length $T$ around 0. It is defined for $t \geq 0$ by

$$R_t^{0,T} = \frac{1}{\sqrt{T}}X_t^{0,T}.$$

**Theorem 7.2** *Let $k > 0$. When $T \to \infty$, the process $(R_t^{0,T}, 0 \leq t \leq 1)$ converges in distribution in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ to the bridge $(I_t^{\{*,0,0,1\}}, 0 \leq t \leq 1)$ of length 1 associated to $(I_t^*, 0 \leq t \leq 1)$.*

**Proof:** Here again we can follow the same proof as in [5]. We just need to take care that the estimate of the heat kernel in Proposition 5.3 in [33], is uniform when $y$ lies in some compact subset of $\mathfrak{a}_+$. But this results directly from the proof of this proposition. □

**Remark 7.2** We can define similarly the normalized radial HO-bridge around any $a \in \overline{\mathfrak{a}_+}$. With the same proof, we can also prove that it converges to the bridge $(I_t^{\{0,0,1\}}, 0 \leq t \leq 1)$ of length 1 associated to the intrinsic Brownian motion starting from 0. Let us just notice that in dimension 1 this is the bridge of a Bessel-3, which is also the normalized Brownian excursion. Thus we do recover the result of [6].

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