Multivariable $(\varphi, \mathcal{O}_K^\times)$-modules and local-global compatibility

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Abstract

Let p be a prime number, K a finite unramified extension of $\mathbb{Q}_p$ and $\mathbb{F}$ a finite extension of $\mathbb{F}_p$. Using perfectoid spaces we associate to any finite-dimensional continuous representation $\overline{\rho}$ of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ an étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A^\otimes(\overline{\rho})$ over a completed localization A of $\mathbb{F}[\![\mathcal{O}_K]\!]$. We conjecture that one can also associate an étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A(\pi)$ to any smooth representation π of $\mathrm{GL}_2(K)$ occurring in some Hecke eigenspace of the mod p cohomology of a Shimura curve, and that moreover $D_A(\pi)$ is isomorphic (up to twist) to $D_A^\otimes(\overline{\rho})$, where $\overline{\rho}$ is the underlying 2-dimensional representation of $\mathrm{Gal}(\overline{K}/K)$. Using previous work of the same authors, we prove this conjecture when $\overline{\rho}$ is semi-simple and sufficiently generic.

Contents

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1 Introduction

Let p be a prime number. The main motivation of this work is the investigation of the (hoped for) mod p Langlands correspondence for $\mathrm{GL}_2(K)$, where K is a finite unramified extension of $\mathbb{Q}_p$. The case $K = \mathbb{Q}_p$ is now well known ([Bre03], [Col10a], [Eme]), whereas the case $K \neq \mathbb{Q}_p$ is still resisting after more than 10 years ([BP12]). An important aspect of the $\mathrm{GL}_2(\mathbb{Q}_p)$-case is the construction by Colmez in *loc. cit.* of an exact functor from the category of admissible finite length mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to the category of finite-dimensional continuous mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. The construction of this functor uses, as an intermediate step, Fontaine's category of (φ, Γ)-modules. In a previous article ([BHH$^+$21]), we constructed an exact functor $D_A^{\text{ét}}$ from a "good" subcategory of admissible mod p representations of $\mathrm{GL}_2(K)$ to a category of étale multivariable $(\varphi, \mathcal{O}_K^\times)$-modules. These multivariable $(\varphi, \mathcal{O}_K^\times)$-modules are A-modules with additional structures, where A is a ring obtained as a completed localization of the Iwasawa algebra of $\mathcal{O}_K$. In this work we propose a construction of a functor $D_A^{\otimes}$ from the category of continuous mod p representations of $\mathrm{Gal}(\overline{K}/K)$ to the category of étale multivariable $(\varphi, \mathcal{O}_K^\times)$-modules. This construction is based on the equivalence, also due to Fontaine ([Fon90]), between mod p representations of $\mathrm{Gal}(\overline{K}/K)$ and Lubin–Tate étale $(\varphi, \mathcal{O}_K^\times)$-modules. One of the main obstructions to pass from Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-modules to multivariable $(\varphi, \mathcal{O}_K^\times)$-modules over A lies in the comparison between the $\mathcal{O}_K^\times$-action on A and the $\mathcal{O}_K^\times$-action on (some tensor power of) the structural ring of the Lubin–Tate group. To solve this problem, we need to work at a perfectoid level and use the "Abel–Jacobi map" considered by Fargues in [Far20]. We then prove, under some conditions, that the two functors $D_A^{\text{ét}}$ and $D_A^{\otimes}$ satisfy a local-global compatibility property in the completed cohomology of a tower of Shimura curves.

We now describe in more detail the content of this article.

Let F be a totally real number field and let X_U be the smooth projective Shimura curve over F associated to a quaternion algebra D of center F (which splits at one infinite place) and to a compact open subgroup U of $(D \otimes_F \mathbb{A}_F^\infty)^\times$. For v a place of F above p which splits D and $\mathbb{F}$ a finite extension of $\mathbb{F}_p$ ("sufficiently large", as usual), consider the admissible smooth representation of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$

$$\pi \overset{\text{def}}{=} \varinjlim_{U_v} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\Big(\overline{r}, H^1_{\text{ét}}(X_{U^v U_v} \times_F \overline{F}, \mathbb{F})\Big), \tag{1}$$

where U^v is a fixed compact open subgroup of $(D \otimes_F \mathbb{A}_F^{\infty,v})^\times$, $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ is an absolutely irreducible continuous Galois representation such that $\pi \neq 0$, and where the inductive limit runs over compact open subgroups U_v of $(D \otimes_F F_v)^\times \cong \mathrm{GL}_2(F_v)$. In this introduction, we moreover assume for simplicity that v is the only p-adic place of F and that we are in a "multiplicity 1" situation, which then roughly means that U^v is "as big as possible" (in general, one needs to take into account the action of certain operators, which requires mild assumptions on F, D and $\overline{r}$, see (72)).

We know that the isomorphism class of π always determines the one of $\overline{r}_v \overset{\mathrm{def}}{=} \overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$, see [BD14], [Sch18]. We also expect that π is always of finite length, which is known in several cases, see [HW22], [BHH$^+$21]. However, the representation π is still not understood when $F_v \neq \mathbb{Q}_p$, in particular we have the key question:

Question 1.1. Assume $F_v \neq \mathbb{Q}_p$, does π only depend on $\overline{r}_v$?

Question 1.1, as routine as it may seem at first, has unfortunately proven to be surprisingly difficult, and there is not one single instance of a π as in (1) for which we know the answer. For instance the mod p étale cohomology of the Drinfeld tower in dimension 1, which provides a smooth representation of $\mathrm{GL}_2(F_v)$ only depending on $\overline{r}_v$, cannot give rise to representations like π as soon as $F_v \neq \mathbb{Q}_p$, see [CDN] (together with [Sch15], [Wu]). On the other hand, we know that, for F_v unramified and most $\overline{r}_v$, the diagram $(\pi^{I_1} \hookrightarrow \pi^{K_1})$ (where $K_1 \overset{\mathrm{def}}{=} 1 + pM_2(\mathcal{O}_{F_v}) \subseteq I_1 \overset{\mathrm{def}}{=}$ pro-p-Iwahori) only depends on $\overline{r}_v$, and this is a really non-trivial fact, see [DL21]. We do not answer Question 1.1 in this work, but we provide one further step towards the understanding of the representation π, and certainly Question 1.1 was a motivation. More precisely, we completely describe the multivariable étale $(\varphi, \mathcal{O}_{F_v}^\times)$-module $D_A(\pi)$ associated to π in [BHH$^+$21, §3] when F_v is unramified and $\overline{r}_v$ is semi-simple sufficiently generic, in particular we prove that it only depends on $\overline{r}_v$, and we provide a precise conjecture on what $D_A(\pi)$ should be for all $\overline{r}_v$ (and F_v unramified), crucially using perfectoid spaces. As an intermediate result, we construct a new fully faithful functor from continuous representations of $\mathrm{Gal}(\overline{F}_v/F_v)$ over $\mathbb{F}$ to a certain category of multivariable étale $(\varphi_q, \mathcal{O}_{F_v}^\times)$-modules.

Let us first recall the definition of these modules. Let K be a finite unramified extension of $\mathbb{Q}_p^\times$ of degree $f \geq 1$, then we can write the Iwasawa algebra $\mathbb{F}[\![\mathcal{O}_K]\!]$ as $\mathbb{F}[\![Y_\sigma,\ \sigma : \mathbb{F}_q \hookrightarrow \mathbb{F}]\!]$ for $Y_\sigma \overset{\mathrm{def}}{=} \sum_{\lambda \in \mathbb{F}_q^\times} \sigma(\lambda)^{-1}[\lambda] \in \mathbb{F}[\![\mathcal{O}_K]\!]$, where $q \overset{\mathrm{def}}{=} p^f$ and $[\lambda] \in \mathcal{O}_K$ is the multiplicative representative of λ (seen in $\mathbb{F}[\![\mathcal{O}_K]\!]$). We then define A to be the completion of $\mathbb{F}[\![\mathcal{O}_K]\!][1/Y_\sigma,\ \sigma : \mathbb{F}_q \hookrightarrow \mathbb{F}]$ for the $(Y_\sigma)_\sigma$-adic topology (in a suitable sense), see (16) for the precise definition. In fact A is isomorphic to the Tate algebra $\mathbb{F}(\!(Y_\sigma)\!)\langle (Y_{\sigma'}/Y_\sigma)^{\pm 1}, \sigma' \neq \sigma \rangle$ for any choice of σ, see Lemma 2.6.1. It is endowed with an $\mathbb{F}$-linear Frobenius φ coming from the multiplication by p on $\mathcal{O}_K$ and with a commuting continuous action of $\mathcal{O}_K^\times$ coming from its action on $\mathbb{F}[\![\mathcal{O}_K]\!]$ (by multiplication on $\mathcal{O}_K$). Then an étale $(\varphi, \mathcal{O}_K^\times)$-module over A is by definition a finite free A-module endowed with a semi-linear Frobenius φ whose image generates everything and a commuting continuous semi-linear action of $\mathcal{O}_K^\times$. Replacing φ on A by $\varphi_q \overset{\mathrm{def}}{=} \varphi^f$, we define in the same way étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A. When $f = 1$, the two definitions recover Fontaine's classical $(\varphi, \mathbb{Z}_p^\times)$-modules (or (φ, Γ)-modules) in characteristic p.

Now let π be an admissible smooth representation of $\mathrm{GL}_2(\mathcal{O}_K)$ over $\mathbb{F}$. We endow

$\pi^\vee \overset{\text{def}}{=} \mathrm{Hom}_{\mathbb{F}}(\pi,\mathbb{F})$ with the $\mathfrak{m}_{I_1}$-adic topology, where $\mathfrak{m}_{I_1}$ is the maximal ideal of the Iwasawa algebra $\mathbb{F}[\![I_1]\!]$. In particular we can see $\pi^\vee$ as an $\mathbb{F}[\![\mathcal{O}_K]\!]$-module via $\mathbb{F}[\![\mathcal{O}_K]\!] \cong \mathbb{F}[\![\left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right)]\!] \subseteq \mathbb{F}[\![I_1]\!]$. We define

$$D_A(\pi) \overset{\text{def}}{=} \Big(\mathbb{F}[\![\mathcal{O}_K]\!][1/Y_\sigma, \sigma : \mathbb{F}_q \hookrightarrow \mathbb{F}] \otimes_{\mathbb{F}[\![\mathcal{O}_K]\!]} \pi^\vee\Big)^\wedge,$$

where the completion is for the tensor product topology, see [BHH$^+$21, §3.1.1] or §3.1. Even though $D_A(\pi)$ is an A-module endowed with a semi-linear action of $\mathcal{O}_K^\times$ (coming from the action of $\left(\begin{smallmatrix} \mathcal{O}_K^\times & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on $\pi^\vee$), it is not clear if it has good properties in general (it might not have a Frobenius φ, it might not be of finite type, etc.). But we know that $D_A(\pi)$ is an étale $(\varphi, \mathcal{O}_K^\times)$-module of rank 2^f for some of the π in (1) when $K \overset{\text{def}}{=} F_v$ is unramified, see [BHH$^+$21, §1.3][1] together with Remark 2.6.2. In fact we conjecture in this paper that $D_A(\pi)$ is always an étale $(\varphi, \mathcal{O}_K^\times)$-module over A (hence equal to $D_A(\pi)^{\text{ét}}$) of rank 2^f for all representations π in (1) (when F_v is unramified).

In order to state our main result, we need some preliminaries. Fix $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{F})$ for $n \geq 1$ a continuous representation, then for any $\sigma : \mathbb{F}_q \hookrightarrow \mathbb{F}$ we can associate to $\overline{\rho}$ a Lubin–Tate $(\varphi_q, \mathcal{O}_K^\times)$-module. Recall that it is an n-dimensional $\mathbb{F}(\!(T_{K,\sigma})\!)$-vector space $D_{K,\sigma}(\overline{\rho})$ equipped with a semi-linear endomorphism φ_q whose image generates $D_{K,\sigma}(\overline{\rho})$ and a commuting continuous action of $\mathcal{O}_K^\times$. Here φ_q is $\mathbb{F}$-linear and satisfies $\varphi_q(T_{K,\sigma}) = T_{K,\sigma}^q$, and the action of $\mathcal{O}_K^\times$ on $\mathbb{F}(\!(T_{K,\sigma})\!)$ is given by the Lubin–Tate power series associated to the choice of logarithm $\sum_{n\geq 0} p^{-n} T_{K,\sigma}^{q^n}$ composed with $\sigma : \mathbb{F}_q \hookrightarrow \mathbb{F}$ on the coefficients. Recall we have $\mathbb{F}(\!(T_{K,\sigma})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma}^{q-1})\!)} D_{K,\sigma}(\overline{\rho})^{[\mathbb{F}_q^\times]} \xrightarrow{\sim} D_{K,\sigma}(\overline{\rho})$.

Assume now that $\overline{\rho}$ is absolutely semi-simple and define

$$D_{A,\sigma}(\overline{\rho}) \overset{\text{def}}{=} A \otimes_{\mathbb{F}(\!(T_{K,\sigma}^{q-1})\!)} D_{K,\sigma}(\overline{\rho})^{[\mathbb{F}_q^\times]}, \tag{2}$$

where the embedding $\mathbb{F}(\!(T_{K,\sigma}^{q-1})\!) \hookrightarrow A$ sends $T_{K,\sigma}^{q-1}$ to $\varphi(Y_\sigma)/Y_\sigma \in A$. We endow $D_{A,\sigma}(\overline{\rho})$ with $\varphi_q \overset{\text{def}}{=} \varphi^f \otimes \varphi_q$. The embedding $\mathbb{F}(\!(T_{K,\sigma}^{q-1})\!) \hookrightarrow A$ does not commute with $\mathcal{O}_K^\times$, but one easily checks that, for $\overline{\rho}$ absolutely semi-simple, there exists a unique (in a certain sense) continuous semi-linear action of $\mathcal{O}_K^\times$ on $D_{A,\sigma}(\overline{\rho})$ which commutes with φ_q and makes $D_{A,\sigma}(\overline{\rho})$ an étale $(\varphi_q, \mathcal{O}_K^\times)$-module over A of rank $\dim_{\mathbb{F}} \overline{\rho}$, see Lemma 2.2.1. Moreover there is a canonical isomorphism $\mathrm{id} \otimes \varphi : A \otimes_{\varphi, A} D_{A,\sigma\circ\varphi}(\overline{\rho}) \xrightarrow{\sim} D_{A,\sigma}(\overline{\rho})$ of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A, where $\sigma \circ \varphi \overset{\text{def}}{=} \sigma((-)^p)$. We then define:

$$D_A^{\otimes}(\overline{\rho}) \overset{\text{def}}{=} \bigotimes_{A, \sigma : \mathbb{F}_q \hookrightarrow \mathbb{F}} D_{A,\sigma}(\overline{\rho}) \tag{3}$$

endowed with the "diagonal" action of $\mathcal{O}_K^\times$. Using the isomorphism $\mathrm{id} \otimes \varphi$, we can define a canonical endomorphism $\varphi : D_A^{\otimes}(\overline{\rho}) \to D_A^{\otimes}(\overline{\rho})$ which cyclically permutes

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[1] Note that, with the notation of [BHH$^+$21, §3.1.2], $D_A(\pi)$ is equal to its étale quotient $D_A(\pi)^{\text{ét}}$ in our case, see [BHH$^+$21, Rem. 3.3.5.4(ii)].

the factors $D_{A,\sigma}(\overline{\rho})$, is semi-linear with respect to φ on A and is such that $\varphi^f = \varphi_q \otimes \cdots \otimes \varphi_q$. It is then clear that $D_A^{\otimes}(\overline{\rho})$ is an étale $(\varphi, \mathcal{O}_K^\times)$-module over A of rank $(\dim_{\mathbb{F}} \overline{\rho})^f$. The following theorem is our main result:

Theorem 1.2 (Corollary 3.1.4)**.** *Assume that $\overline{r}_v$ is absolutely semi-simple and sufficiently generic* (*see (75)*), *and assume standard technical assumptions on the global setting* (*see §3.1 for precise statements*). *Then there is an isomorphism of étale $(\varphi, \mathcal{O}_K^\times)$-modules $D_A(\pi) \cong D_A^{\otimes}(\overline{r}_v(1))$ over A, where $\overline{r}_v(1)$ is the usual Tate twist of $\overline{r}_v$.*

The proof of Theorem 1.2 is a long explicit computation of the dual étale $(\varphi, \mathcal{O}_K^\times)$-module $\mathrm{Hom}_A(D_A(\pi), A)$. Let us briefly indicate the various steps. We first describe $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$, which is not so hard, see Proposition 3.2.3. We then prove that there is a canonical injection

$$\mathrm{Hom}_A(D_A(\pi), A) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$$

induced by a non-zero continuous morphism $\mu : A \to \mathbb{F}$ uniquely determined (up to scalar in $\mathbb{F}^\times$) by the condition $\mu \circ \psi \in \mathbb{F}^\times \mu$, where $\psi : A \to A$ is a left inverse of φ, see Lemma 3.2.1, Proposition 3.3.1 and (87). To each Serre weight σ of $\overline{r}_v^\vee$ we then associate in (105) a certain projective system $x_\sigma = (x_{\sigma,k})_{k \geq 0}$, where $x_{\sigma,k} \in \pi[\mathfrak{m}_{I_1}^{kf+1}]$, and we prove via Proposition 3.2.3 that x_σ lies in $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$, see Lemma 3.4.7 and Proposition 3.5.1. Then the key calculation is to prove that x_σ actually also lies in the submodule $\mathrm{Hom}_A(D_A(\pi), A)$, and that the 2^f-tuple $(x_\sigma)_{\sigma \in W(\overline{r}_v^\vee)}$ even forms an A-basis of the free A-module $\mathrm{Hom}_A(D_A(\pi), A)$, see Theorem 3.7.1. For that we prove a crucial finiteness result (Proposition 3.6.1) using the technical – but important – computations in [BHH$^+$21, §3.2] that we need to strengthen, see §3.4. Once all this is done, it is easy to derive the explicit actions of $\mathcal{O}_K^\times$ and φ on $\mathrm{Hom}_A(D_A(\pi), A)$, see Proposition 3.8.3 and Lemma 3.10.1. We can then at last compare the two $(\varphi, \mathcal{O}_K^\times)$-modules $D_A(\pi)$ and $D_A^{\otimes}(\overline{r}_v(1))$ and prove that they are isomorphic, see Theorem 3.10.2. The same proof works verbatim for quaternion algebras D which are definite at all infinite places (and split at v) and the representations π of $\mathrm{GL}_2(K) = \mathrm{GL}_2(F_v)$ defined analogously to (1).

There is no doubt to us that there should exist a more conceptual proof of Theorem 1.2 which will hopefully avoid both the genericity assumptions on $\overline{r}_v$ and the technical computations. At present however, we do not know how to do this. But the first issue is to find a more conceptual definition of $D_{A,\sigma}(\overline{\rho})$ and of $D_A^{\otimes}(\overline{\rho})$. Indeed, it is likely that, when $\overline{\rho}$ is not semi-simple, the recipe (2) does not work in general because there might not always exist a continuous semi-linear action of $\mathcal{O}_K^\times$ on $A \otimes_{\mathbb{F}(\!(T_{K,\sigma}^{q-1})\!)} D_{K,\sigma}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ which commutes with $\varphi^f \otimes \varphi_q$ (or such an action might not be unique). Using perfectoid spaces we give below a functorial construction of an étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D_{A,\sigma}(\overline{\rho})$, and subsequently of an étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A^{\otimes}(\overline{\rho})$, which works for all $\overline{\rho}$.

The first step is to replace the ring A by its perfectoid version

$$A_\infty \overset{\text{def}}{=} \mathbb{F}(\!(Y_\sigma^{1/p^\infty})\!)\left\langle (Y_{\sigma'}/Y_\sigma)^{\pm 1/p^\infty}, \sigma' \neq \sigma \right\rangle \tag{4}$$

which is a perfectoid Tate algebra over the perfectoid field $\mathbb{F}(\!(Y_\sigma^{1/p^\infty})\!)$ (for any σ). Using the equivalence between finite étale A-algebras and finite étale A_∞-algebras together with the equivalence between locally constant sheaves of $\mathbb{F}_q$-vector spaces and Frobenius-equivariant vector bundles on a normal irreducible scheme over $\mathbb{F}_q$, it is not hard to check that the extension of scalars $(-) \mapsto (-) \otimes_A A_\infty$ induces an equivalence of categories between étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A and étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A_∞, and similarly with $(\varphi, \mathcal{O}_K^\times)$ instead of $(\varphi_q, \mathcal{O}_K^\times)$, see Corollary 2.6.6. Hence we may as well look for a definition of $D_{A_\infty,\sigma}(\overline{\rho})$ and $D_{A_\infty}^{\otimes}(\overline{\rho})$.

It is now convenient to fix an embedding $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ and set $\sigma_i \overset{\text{def}}{=} \sigma_0 \circ \varphi^i$ for $i \in \mathbb{Z}$. The second step is to consider the two perfectoid spaces

$$\begin{aligned} Z_{\mathrm{LT}} &\overset{\text{def}}{=} \underbrace{\operatorname{Spa}\left(\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!), \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}]\!]\right) \times_{\operatorname{Spa}(\mathbb{F})} \cdots \times_{\operatorname{Spa}(\mathbb{F})} \operatorname{Spa}\left(\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!), \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}]\!]\right)}_{f \text{ times}} \\ Z_{\mathcal{O}_K} &\overset{\text{def}}{=} \operatorname{Spa}\left(\mathbb{F}[\![Y_{\sigma_0}^{1/p^\infty}, \ldots, Y_{\sigma_{f-1}}^{1/p^\infty}]\!], \mathbb{F}[\![Y_{\sigma_0}^{1/p^\infty}, \ldots, Y_{\sigma_{f-1}}^{1/p^\infty}]\!]\right) \setminus V(Y_{\sigma_0}, \ldots, Y_{\sigma_{f-1}}), \end{aligned}$$

where Z_{LT} is endowed with an obvious action of $(K^\times)^f \rtimes \mathfrak{S}_f$ ($p \in K^\times$ acting via φ_q which is now bijective) and $Z_{\mathcal{O}_K}$ is endowed with an action of $K^\times$ (p acting via φ). It turns out that there is a morphism of perfectoid spaces (see the beginning of §2.4)

$$m : Z_{\mathrm{LT}} \longrightarrow Z_{\mathcal{O}_K}$$

such that $m \circ ((a_0, \ldots, a_{f-1}), w) = (\prod_i a_i) \circ m$ for $a_i \in K^\times$ and $w \in \mathfrak{S}_f$, a crucial fact that we learnt from [Far20]. Indeed, the sheaf on the perfectoid v-site over $\mathbb{F}$ represented by Z_{LT} sends a perfectoid $\mathbb{F}$-algebra R to a subset of $(B^+(R)^{\varphi_q = p})^f$ stable under multiplication, where $B^+(R)$ is the (relative version of the) ring defined in [FF18, §1.10] (a certain completion of $W(R^\circ)[1/p]$, where $R^\circ \subseteq R$ is the subring of power-bounded elements). Likewise, the sheaf represented by $Z_{\mathcal{O}_K}$ sends R to a subset of $B^+(R)^{\varphi_q = p^f}$ stable under multiplication, see §2.3. The map m then is induced by the product map $(B^+(R)^{\varphi_q = p})^f \to B^+(R)^{\varphi_q = p^f}$ in the ring $B^+(R)$, which satisfies the above relation with respect to the various group actions.

Note that $\operatorname{Spa}(A_\infty, A_\infty^\circ)$ is an affinoid open subspace of $Z_{\mathcal{O}_K}$ by (4). Let $\Delta \overset{\text{def}}{=} \{(a_0, \ldots, a_{f-1}) \in (K^\times)^f, \prod a_i = 1\}$ and $\Delta_1 \overset{\text{def}}{=} \Delta \cap (\mathcal{O}_K^\times)^f$. The third step is to prove that the morphism m induces a commutative diagram of perfectoid spaces over $\mathbb{F}$:

$$\begin{tikzcd} Z_{\mathrm{LT}} \arrow[d, "m"] & m^{-1}(\operatorname{Spa}(A_\infty, A_\infty^\circ)) \arrow[l, hook'] \arrow[d, "m"] & \cong \underline{(\Delta/\Delta_1) \rtimes \mathfrak{S}_f} \arrow[dl] & \times \operatorname{Spa}(A'_\infty, (A'_\infty)^\circ)\ , \\ Z_{\mathcal{O}_K} & \operatorname{Spa}(A_\infty, A_\infty^\circ) \arrow[l, hook'] & & \end{tikzcd}$$

where the middle vertical morphism is a pro-étale $\Delta \rtimes \mathfrak{S}_f$-torsor and where $\mathrm{Spa}(A'_\infty, (A'_\infty)^\circ)$ is an explicit affinoid open subspace of Z_{LT} preserved by the action of Δ_1 which is itself a pro-étale Δ_1-torsor over $\mathrm{Spa}(A_\infty, A_\infty^\circ)$, see Proposition 2.4.4, Corollary 2.4.5 and Lemma 2.4.7.

Now let $\overline{\rho}$ be any finite-dimensional continuous representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$, then $\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!)} D_{K,\sigma_0}(\overline{\rho})$ is the space of global sections of a $K^\times$-equivariant vector bundle $\mathcal{V}_{\overline{\rho}}$ on $\mathrm{Spa}(\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!), \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}]\!])$. For $i \in \{0, \dots, f-1\}$ we define $\mathcal{V}_{\overline{\rho}}^{(i)} \overset{\mathrm{def}}{=} \mathrm{pr}_i^* \mathcal{V}_{\overline{\rho}}$, where $\mathrm{pr}_i : Z_{\mathrm{LT}} \to \mathrm{Spa}(\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!), \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}]\!])$ is the i-th projection. Then $\mathcal{V}_{\overline{\rho}}^{(i)}$ is a $(K^\times)^f$-equivariant vector bundle on Z_{LT}, and thus $\mathcal{V}_{\overline{\rho}}^{(i)}|_{\mathrm{Spa}(A'_\infty, (A'_\infty)^\circ)}$ is a Δ_1-equivariant vector bundle on $\mathrm{Spa}(A'_\infty, (A'_\infty)^\circ)$. By the third step and using [SW20, Lemma 17.1.8], we deduce that $\Gamma(\mathrm{Spa}(A'_\infty, (A'_\infty)^\circ), \mathcal{V}_{\overline{\rho}}^{(i)})^{\Delta_1}$ is an étale $(\varphi_q, \mathcal{O}_K^\times)$-module over A_∞ of rank $\dim_{\mathbb{F}} \overline{\rho}$, see Theorem 2.5.1 and §2.6. Hence by the first step $\Gamma(\mathrm{Spa}(A'_\infty, (A'_\infty)^\circ), \mathcal{V}_{\overline{\rho}}^{(i)})^{\Delta_1}$ is the extension of scalars of a unique étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D_A^{(i)}(\overline{\rho})$ over A of rank $\dim_{\mathbb{F}} \overline{\rho}$.

The following theorem sums up the main properties of the functor $\overline{\rho} \mapsto D_A^{(i)}(\overline{\rho})$.

Theorem 1.3. *Let $i \in \{0, \dots, f-1\}$.*

(i) *There is a functorial A-linear isomorphism $\phi_i : A \otimes_{\varphi,A} D_A^{(i)}(\overline{\rho}) \xrightarrow{\sim} D_A^{(i+1)}(\overline{\rho})$ which commutes with $(\varphi_q, \mathcal{O}_K^\times)$ and is such that $\phi_{f-1} \circ \phi_{f-2} \circ \cdots \circ \phi_0 : A \otimes_{\varphi^f,A} D_A^{(0)}(\overline{\rho}) \xrightarrow{\sim} D_A^{(0)}(\overline{\rho})$ is $\mathrm{id} \otimes \varphi_q$, see Corollary 2.6.7.*

(ii) *The functor $\overline{\rho} \mapsto D_A^{(i)}(\overline{\rho})$ from finite-dimensional continuous representations of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ to étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A is fully faithful, see Corollary 2.8.3.*

(iii) *The surjection $A \twoheadrightarrow \mathbb{F}(\!(T)\!)$ induced by the trace $\mathbb{F}[\![\mathcal{O}_K]\!] \twoheadrightarrow \mathbb{F}[\![\mathbb{Z}_p]\!] \cong \mathbb{F}[\![T]\!]$ gives an isomorphism of $(\varphi_q, \mathbb{Z}_p^\times)$-modules*

$$\mathbb{F}(\!(T)\!) \otimes_A D_A^{(i)}(\overline{\rho}) \cong D_{\sigma_{f-i}}(\overline{\rho}),$$

where $D_{\sigma_{f-i}}(\overline{\rho})$ is the usual (cyclotomic) $(\varphi_q, \mathbb{Z}_p^\times)$-module over $\mathbb{F}(\!(T)\!)$ associated to $\overline{\rho}$ using σ_{f-i} to embed $\mathbb{F}_q$ into $\mathbb{F}$, see Proposition 2.8.1.

(iv) *If $\overline{\rho}$ is absolutely semi-simple then there is an isomorphism of $(\varphi_q, \mathcal{O}_K^\times)$-modules over A*

$$D_A^{(i)}(\overline{\rho}) \cong D_{A,\sigma_{f-i}}(\overline{\rho}),$$

where $D_{A,\sigma_{f-i}}(\overline{\rho})$ is as in (2), see Theorem 2.9.5.

Because of Theorem 1.3(iv) it is natural to rename $D_A^{(i)}(\overline{\rho})$ as $D_{A,\sigma_{f-i}}(\overline{\rho})$ for any $\overline{\rho}$. Using Theorem 1.3(i) we can then associate to any $\overline{\rho}$ an étale $(\varphi, \mathcal{O}_K^\times)$-module

$D_A^{\otimes}(\overline{\rho})$ over A of rank $(\dim_{\mathbb{F}} \overline{\rho})^f$ by exactly the same formula as in (3). Note that by Theorem 1.3(iii) $\mathbb{F}(\!(T)\!) \otimes_A D_A^{\otimes}(\overline{\rho})$ can be identified with the (φ, Γ)-module of the tensor induction from K to $\mathbb{Q}_p$ of $\overline{\rho}$.

We can now state our conjecture:

Conjecture 1.4 (Conjecture 3.1.2)**.** *For any π as in (1) (with $F_v = K$ unramified) there is an isomorphism of étale $(\varphi, \mathcal{O}_K^{\times})$-modules $D_A(\pi) \cong D_A^{\otimes}(\overline{r}_v(1))$ over A.*

By Theorem 1.3(iv) we see that Theorem 1.2 proves special cases of Conjecture 1.4 (but recall that our somewhat technical proof of Theorem 1.2 does not use perfectoids). Note that Conjecture 1.4 implies (the analogue of) [BHH$^+$21, Conjecture 1.2.5] for the representations π in (1). It is also reminiscent of the plectic structure of the local Galois action at p on the ℓ-adic cohomology ($\ell \neq p$) of certain Shimura varieties recently proven in [LH], where the above map m also plays a key role.

We finish this introduction by going back to Question 1.1 assuming Conjecture 1.4. The image of the natural map $\pi^{\vee} \to D_A(\pi) \cong D_A^{\otimes}(\overline{r}_v(1))$ is a compact $\mathbb{F}[\![\mathcal{O}_K]\!]$-submodule $D_A(\pi)^{\natural}$ which generates $D_A^{\otimes}(\overline{r}_v(1))$ over A and is preserved by $\mathcal{O}_K^{\times}$ and the operator ψ, with moreover $\psi : D_A(\pi)^{\natural} \twoheadrightarrow D_A(\pi)^{\natural}$ surjective. Assuming there is an admissible smooth representation of $\mathrm{GL}_2(K)$ naturally associated to $\overline{r}_v$, and that this representation is π (as is the case when $K = \mathbb{Q}_p$), one could hope to "guess" what $D_A(\pi)^{\natural}$ is inside $D_A^{\otimes}(\overline{r}_v(1))$, as the latter is pretty explicit, at least when $\overline{r}_v$ is semi-simple and sufficiently generic. However, even in the simplest case where K is quadratic (unramified) and $\overline{r}_v$ is the direct sum of two characters, where we know that π is semi-simple ([BHH$^+$21]), it seems impossible to find $D_A(\pi)^{\natural}$ "by hand" (there exists a natural explicit generating compact $\mathbb{F}[\![\mathcal{O}_K]\!]$-submodule in $D_A^{\otimes}(\overline{r}_v(1))$ which is preserved by $\mathcal{O}_K^{\times}$ and ψ with ψ surjective, but we can prove that it cannot be $D_A(\pi)^{\natural}$). Going back to perfectoids, one could hope to find instead a natural $\mathbb{F}[\![Y_{\sigma_0}^{1/p^{\infty}}, \dots, Y_{\sigma_{f-1}}^{1/p^{\infty}}]\!]$-submodule $D_{A_{\infty}}(\pi)^{\natural}$ inside $D_{A_{\infty}}^{\otimes}(\overline{r}_v(1)) = A_{\infty} \otimes_A D_A^{\otimes}(\overline{r}_v(1))$ and from there go to $D_A(\pi)^{\natural}$ in a similar way as what was done by Colmez when $K = \mathbb{Q}_p$ in [Col10b, §IV.2]. However, even though there is a natural candidate, namely the $\mathbb{F}[\![Y_{\sigma_0}^{1/p^{\infty}}, \dots, Y_{\sigma_{f-1}}^{1/p^{\infty}}]\!]$-submodule

$$\begin{aligned}&\Gamma\Big(Z_{\mathrm{LT}}, \mathcal{V}_{\overline{r}_v(1)}^{(0)} \otimes_{\mathcal{O}_{Z_{\mathrm{LT}}}} \cdots \otimes_{\mathcal{O}_{Z_{\mathrm{LT}}}} \mathcal{V}_{\overline{r}_v(1)}^{(f-1)}\Big)^{\Delta \rtimes \mathfrak{S}_f}\\ &\qquad \subseteq \Gamma\Big(m^{-1}(\mathrm{Spa}(A_{\infty}, A_{\infty}^{\circ})), \mathcal{V}_{\overline{r}_v(1)}^{(0)} \otimes_{\mathcal{O}_{Z_{\mathrm{LT}}}} \cdots \otimes_{\mathcal{O}_{Z_{\mathrm{LT}}}} \mathcal{V}_{\overline{r}_v(1)}^{(f-1)}\Big)^{\Delta \rtimes \mathfrak{S}_f} \cong D_{A_{\infty}}^{\otimes}(\overline{r}_v(1)),\end{aligned}$$

computations for $f = 2$ show no evidence for this submodule to be large enough (or even non-zero when $\overline{r}_v$ is irreducible).

We fix a few general notation (most of them have already been introduced above, but we recall them). We fix K a finite unramified extension of $\mathbb{Q}_p$ of residue field $\mathbb{F}_q = \mathbb{F}_{p^f}$, so $\mathcal{O}_K = W(\mathbb{F}_q)$ and $K = \mathcal{O}_K[1/p]$. We normalize the local reciprocity map so that it sends $p \in K^\times$ to (the image of) the geometric Frobenius $x \mapsto x^{-q}$. We fix an algebraic closure $\overline{K}$ of K with ring of integers $\mathcal{O}_{\overline{K}}$ and maximal ideal $\mathfrak{m}_{\overline{K}}$. We denote by $\mathbb{F}$ the coefficients, which is a finite extension of $\mathbb{F}_q$ that we always tacitly assume to be "large enough". We fix an embedding $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ (which is sometimes omitted from the notation when the context is clear) and we let $\sigma_i \overset{\text{def}}{=} \sigma_0 \circ \varphi^i$ for φ the Frobenius on $\mathbb{F}_q$ (i.e. $\varphi(x) = x^p$) and $i \in \mathbb{Z}$.

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2 Étale $(\varphi,\mathcal{O}_K^\times)$-modules and Galois representations

In this section we functorially associate to any finite-dimensional continuous representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ an étale $(\varphi_q,\mathcal{O}_K^\times)$-module $D_{A,\sigma}(\overline{\rho})$ of rank $\dim_{\mathbb{F}}\overline{\rho}$ over the ring A of [BHH+21, §3.1.1] (depending on an embedding $\sigma : \mathbb{F}_q \hookrightarrow \mathbb{F}$) and an étale $(\varphi,\mathcal{O}_K^\times)$-module $D_A^{\otimes}(\overline{\rho})$ of rank $(\dim_{\mathbb{F}}\overline{\rho})^f$ over A. We prove various properties of these modules and we make them explicit when $\overline{\rho}$ is absolutely semi-simple.

If X is an adic space over $\mathbb{F}$, we denote by h_X the functor $\mathrm{Hom}_{\mathrm{Spa}(\mathbb{F},\mathbb{F})}(-,X)$ from the category of adic spaces over $\mathbb{F}$ to the category of sets. If R is an adic Huber ring, i.e. a topological ring whose topology is I-adic for a finitely generated ideal I (see for instance [SW20, §2.2]), we use the shorthand $\mathrm{Spa}(R)$ for the adic spectrum $\mathrm{Spa}(R,R)$. We denote by $\mathrm{Perf}_{\mathbb{F}}$ the category of perfectoid spaces over $\mathbb{F}$. For background on adic spaces or perfectoid spaces we refer (mostly without comment) to [Hub96], [Sch12] or [SW20].

Let A be a (commutative) ring and let φ be a ring endomorphism of A. We define a *φ-module over A* as a finite free A-module D endowed with a φ-semi-linear map $\varphi : D \to D$. We say that a φ-module over A is *étale* if the A-linear map $\mathrm{id}_A \otimes \varphi : A \otimes_{\varphi,A} D \to D$ is an isomorphism. Assume moreover that A is a topological ring and that there exists a continuous action of an abelian topological group Γ on A via endomorphisms commuting to φ. We define a *(φ,Γ)-module over A* as a φ-module D over A endowed with a continuous semi-linear action of Γ such that, for $a \in A$, $v \in D$ and $\gamma \in \Gamma$:

$$\varphi(\gamma(v)) = \gamma(\varphi(v)).$$

Moreover we say that a (φ,Γ)-module is *étale* if its underlying φ-module over A is so.

2.1 Review of Lubin–Tate and classical (φ,Γ)-modules

We review Lubin–Tate and classical (φ,Γ)-modules associated to a finite-dimensional continuous representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$.

We denote by G_{LT} the unique (up to isomorphism) Lubin–Tate formal $\mathcal{O}_K$-module over $\mathcal{O}_K$. Let T_K be a formal variable of G_{LT}, we have power series:

$$a_{\mathrm{LT}}(T_K) \in aT_K + T_K^2\mathcal{O}_K[\![T_K]\!] \text{ for } a \in \mathcal{O}_K.$$

Note that the commutativity of the action of $a \in \mathcal{O}_K$ with $[\mathbb{F}_q]$ implies in fact

$$a_{\mathrm{LT}}(T_K) \in aT_K + T_K^q\mathcal{O}_K[\![T_K^{q-1}]\!] \tag{5}$$

and recall that $p_{\mathrm{LT}}(T_K) \in T_K^q + p\mathcal{O}_K[\![T_K]\!]$. We denote by K_∞ the abelian extension of K generated by $\{x \in \mathfrak{m}_{\overline{K}},\ p_{\mathrm{LT}}^n(x) = 0 \text{ for some } n \geq 1\}$ and recall that we have the commutative diagram:

$$\begin{tikzcd}
\mathrm{Gal}(\overline{K}/K) \arrow[r, two heads] & \mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}} \arrow[r, two heads] & \mathrm{Gal}(K_\infty/K) \arrow[r, two heads] & \mathrm{Gal}(K(\sqrt[p^\infty]{1})/K) \\
& K^\times \cong p^{\mathbb{Z}} \times \mathcal{O}_K^\times \arrow[u, hook] \arrow[r, two heads] & \mathcal{O}_K^\times \arrow[u, "\wr"] \arrow[r, two heads] & \mathbb{Z}_p^\times \arrow[u, "\wr"]
\end{tikzcd} \tag{6}$$

where the left vertical injection is the local reciprocity map, the bottom left horizontal surjection is the projection sending p to 1, and the bottom right horizontal surjection is the norm map.

We define a continuous $\mathbb{F}$-linear endomorphism φ of $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T_K)\!)$ (the Frobenius) and a continuous $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q$-linear action commuting to φ of $\mathcal{O}_K^\times$ on $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T_K)\!)$ satisfying the following conditions for $\lambda \in \mathbb{F}$, $c_n \in \mathbb{F}_q$ and $a \in \mathcal{O}_K^\times$:

$$\begin{cases} \varphi\Big((\lambda \otimes (\sum_{n>-\infty} c_n T_K^n))\Big) = \lambda \otimes (\sum_{n>-\infty} c_n^p T_K^{np}) \\ a\Big((\lambda \otimes (\sum_{n>-\infty} c_n T_K^n))\Big) = \lambda \otimes (\sum_{n>-\infty} c_n (a_{\mathrm{LT}}(T_K))^n) \end{cases} \tag{7}$$

where we still denote by $a_{\mathrm{LT}}(T_K) \in \mathbb{F}_q[\![T_K]\!]$ the reduction mod p of $a_{\mathrm{LT}}(T_K) \in \mathcal{O}_K[\![T_K]\!]$.

Identifying $\mathrm{Gal}(K_\infty/K)$ with $\mathcal{O}_K^\times$, it follows from Wintenberger's theory of the field of norms ([Win83]) and Fontaine's theory of (φ,Γ)-modules ([Fon90]) that there is a rank-preserving (covariant) equivalence of categories compatible with tensor products between the category of finite-dimensional continuous representations of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ and the category of étale $(\varphi,\mathcal{O}_K^\times)$-modules over $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T_K)\!)$. Note that the condition on the image of φ implies that the endomorphism φ of an étale $(\varphi,\mathcal{O}_K^\times)$-module over $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T_K)\!)$ is automatically injective.

The isomorphism

$$\begin{aligned} \mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T_K)\!) &\xrightarrow{\sim} \mathbb{F}(\!(T_{K,\sigma_0})\!) \times \mathbb{F}(\!(T_{K,\sigma_1})\!) \times \cdots \times \mathbb{F}(\!(T_{K,\sigma_{f-1}})\!) \\ \lambda \otimes (\textstyle\sum_{n>-\infty} c_n T_K^n) &\longmapsto \Big(\textstyle\sum_{n>-\infty} \lambda\sigma_0(c_n) T_{K,\sigma_0}^n, \ldots, \sum_{n>-\infty} \lambda\sigma_{f-1}(c_n) T_{K,\sigma_{f-1}}^n\Big) \end{aligned} \tag{8}$$

induces an analogous decomposition for any $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T_K)\!)$-module D_K:

$$D_K \xrightarrow{\sim} D_{K,\sigma_0} \times \cdots \times D_{K,\sigma_{f-1}}.$$

If D_K is an étale φ-module over $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T_K)\!)$, then φ induces a morphism (still denoted) $\varphi : D_{K,\sigma_i} \to D_{K,\sigma_{i-1}}$ such that $\varphi(\sum_{n>-\infty} c_n T_{K,\sigma_i}^n v) = \sum_{n>-\infty} c_n T_{K,\sigma_{i-1}}^{pn} \varphi(v)$ for $c_n \in \mathbb{F}$ and $v \in D_{K,\sigma_i}$. By a standard argument, the functor $D_K \longmapsto D_{K,\sigma_0}$ induces an equivalence of categories (compatible with tensor products) between the category of étale $(\varphi,\mathcal{O}_K^\times)$-modules over $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T_K)\!)$ and the category of $(\varphi_q,\mathcal{O}_K^\times)$-modules over $\mathbb{F}(\!(T_{K,\sigma_0})\!)$, where $\mathbb{F}(\!(T_{K,\sigma_0})\!)$ is endowed with an $\mathbb{F}$-linear endomorphism φ_q $(= \varphi^f)$

and a continuous $\mathbb{F}$-linear action commuting to φ_q of $\mathcal{O}_K^\times$ satisfying the conditions for $c_n \in \mathbb{F}$, and $a \in \mathcal{O}_K^\times$:

$$\begin{cases} \varphi_q\Big(\big(\sum_{n>-\infty} c_n T^n_{K,\sigma_0}\big)\Big) = \sum_{n>-\infty} c_n T^{nq}_{K,\sigma_0} \\ a\Big(\big(\sum_{n>-\infty} c_n T^n_{K,\sigma_0}\big)\Big) = \sum_{n>-\infty} c_n\big(a_{\mathrm{LT}}(T_{K,\sigma_0})\big)^n. \end{cases} \tag{9}$$

In (9) we have defined:

$$a_{\mathrm{LT}}(T_{K,\sigma_0}) \overset{\mathrm{def}}{=} \sigma_0(a_{\mathrm{LT}}(T_K)) \in \sigma_0(\overline{a})T_{K,\sigma_0} + T^q_{K,\sigma_0}\mathbb{F}[\![T^{q-1}_{K,\sigma_0}]\!],$$

where $\sigma_0(a_{\mathrm{LT}}(T_K))$ is the image of $a_{\mathrm{LT}}(T_K) \in \mathbb{F}_q[\![T_K]\!]$ via $\mathbb{F}_q[\![T_K]\!] \hookrightarrow \mathbb{F}[\![T_{K,\sigma_0}]\!]$, $\sum_{n>-\infty} c_n T_K^n \mapsto \sum_{n>-\infty} \sigma_0(c_n) T^n_{K,\sigma_0}$. If one chooses the embedding σ_i for some $i \in \{1, \dots, f-1\}$ instead of σ_0, one goes from D_{K,σ_0} to D_{K,σ_i} by the isomorphism

$$\mathrm{Id} \otimes \varphi^{f-i} : \mathbb{F}[\![T_{K,\sigma_i}]\!] \otimes_{\varphi^{f-i}, \mathbb{F}[\![T_{K,\sigma_0}]\!]} D_{K,\sigma_0} \xrightarrow{\sim} D_{K,\sigma_i}.$$

We can also work with the infinite Galois extension $K(\sqrt[p^\infty]{1})$ instead of K_∞ (see (6)). Let T be a coordinate of the formal group $\mathbb{G}_{\mathrm{m}}$. We endow the topological ring $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T)\!)$ with an $\mathbb{F}$-linear endomorphism φ and a continuous $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q$-linear action commuting to φ of $\mathbb{Z}_p^\times$ satisfying the following conditions for $\lambda \in \mathbb{F}$, $c_n \in \mathbb{F}_q$, and $a \in \mathbb{Z}_p^\times$:

$$\begin{cases} \varphi\Big(\big(\lambda \otimes (\sum_{n>-\infty} c_n T^n)\big)\Big) = \lambda \otimes (\sum_{n>-\infty} c_n^p T^{np}) \\ a\Big(\big(\lambda \otimes (\sum_{n>-\infty} c_n T^n)\big)\Big) = \lambda \otimes (\sum_{n>-\infty} c_n (a(T))^n). \end{cases} \tag{10}$$

We have as before a rank-preserving (covariant) equivalence of categories compatible with tensor products between the category of finite-dimensional continuous representations of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ and the category of étale $(\varphi, \mathbb{Z}_p^\times)$-modules over $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T)\!)$.

Here a standard choice is to take T such that $a(T) \in \overline{a}T + T^p\mathbb{F}_p[\![T^{p-1}]\!] \subseteq \overline{a}T + T^p\mathbb{F}_q[\![T^{p-1}]\!]$ is the reduction mod p of $(1+T)^a - 1 \in \mathbb{Z}_p[\![T]\!]$. Using a decomposition analogous to (8), we again have an equivalence (compatible with tensor products) between the category of étale $(\varphi, \mathbb{Z}_p^\times)$-modules over $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T)\!)$ and the category of étale $(\varphi_q, \mathbb{Z}_p^\times)$-modules over $\mathbb{F}(\!(T)\!)$, where $\mathbb{F}(\!(T)\!)$ is endowed with an $\mathbb{F}$-linear endomorphism φ_q $(= \varphi^f)$ and a continuous $\mathbb{F}$-linear action commuting to φ_q of $\mathbb{Z}_p^\times$ satisfying the conditions for $c_n \in \mathbb{F}$ and $a \in \mathbb{Z}_p^\times$:

$$\begin{cases} \varphi_q\Big(\big(\sum_{n>-\infty} c_n T^n\big)\Big) = \sum_{n>-\infty} c_n T^{nq} \\ a\Big(\big(\sum_{n>-\infty} c_n T^n\big)\Big) = \sum_{n>-\infty} c_n (a(T))^n. \end{cases} \tag{11}$$

If $\overline{\rho}$ is a finite-dimensional continuous representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$, we denote by:

$$\begin{cases} \begin{array}{ll} D_K(\overline{\rho}) & \text{its } (\varphi, \mathcal{O}_K^\times)\text{-module over } \mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T_K)\!) \\ D_{K,\sigma_0}(\overline{\rho}) & \text{the associated } (\varphi_q, \mathcal{O}_K^\times)\text{-module over } \mathbb{F}(\!(T_{K,\sigma_0})\!) \\ D(\overline{\rho}) & \text{its } (\varphi, \mathbb{Z}_p^\times)\text{-module over } \mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T)\!) \\ D_{\sigma_0}(\overline{\rho}) & \text{the associated } (\varphi_q, \mathbb{Z}_p^\times)\text{-module over } \mathbb{F}(\!(T)\!). \end{array} \end{cases}$$

We will mostly use $D_{K,\sigma_0}(\overline{\rho})$ and $D_{\sigma_0}(\overline{\rho})$ in the sequel.

We now relate $D_K(\overline{\rho})$ and $D(\overline{\rho})$, $D_{K,\sigma_0}(\overline{\rho})$ and $D_{\sigma_0}(\overline{\rho})$. In order to do so, we have to use the perfectoid versions of $\mathbb{F}_q(\!(T_K)\!)$, $\mathbb{F}_q(\!(T_{K,\sigma_0})\!)$, etc.

We let $\mathbb{F}_q[\![T_K^{p^{-\infty}}]\!]$ be the completion of the perfection $\cup_{n\geq 0}\mathbb{F}_q[\![T_K^{p^{-n}}]\!]$ of $\mathbb{F}_q[\![T_K]\!]$ with respect to the T_K-adic topology and $\mathbb{F}_q(\!(T_K^{p^{-\infty}})\!)$ the fraction field of $\mathbb{F}_q[\![T_K^{p^{-\infty}}]\!]$. Concretely:

$$\mathbb{F}_q[\![T_K^{p^{-\infty}}]\!] \cong \left\{\sum_{n\geq 0} c_n T_K^{\frac{d_n}{p^n}},\ c_n \in \mathbb{F}_q,\ d_n \in \mathbb{Z}_{\geq 0},\ \frac{d_n}{p^n} \to +\infty \text{ in } \mathbb{Q} \text{ when } n \to +\infty\right\}$$

and $\mathbb{F}_q(\!(T^{p^{-\infty}})\!) = \mathbb{F}_q[\![T_K^{p^{-\infty}}]\!][\frac{1}{T_K}]$. We define in a similar way $\mathbb{F}_q[\![T^{p^{-\infty}}]\!]$ and $\mathbb{F}_q(\!(T^{p^{-\infty}})\!)$. Let $\mathbb{C}_p = \mathcal{O}_{\mathbb{C}_p}[1/p]$, where $\mathcal{O}_{\mathbb{C}_p}$ is the p-adic completion of $\mathcal{O}_{\overline{K}}$ and $\mathbb{C}_p^\flat \overset{\text{def}}{=} \varprojlim_{x\mapsto x^p} \mathbb{C}_p$, then $\mathbb{C}_p^\flat$ is an algebraically closed field of characteristic p which is complete with respect to the valuation $v((x_m)_{m\geq 1}) \overset{\text{def}}{=} \mathrm{val}(x_1)$, where $x_m \in \mathbb{C}_p$, $x_m^p = x_{m-1}$ (for $m > 1$) and val is the usual p-adic valuation on $\mathbb{C}_p$ normalized by $\mathrm{val}(p) = 1$. Moreover its ring of integers $\{x \in \mathbb{C}_p^\flat,\ v(x) \geq 0\}$ is:

$$\mathcal{O}_{\mathbb{C}_p}^\flat \overset{\text{def}}{=} \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \xrightarrow{\sim} \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/(p) \cong \varprojlim_{x\mapsto x^p} \mathcal{O}_{\overline{K}}/(p) \cong \varprojlim_{x\mapsto x^q} \mathcal{O}_{\overline{K}}/(p)$$

(which Fontaine used to denote R) and $\mathbb{C}_p^\flat \cong \mathrm{Frac}(\mathcal{O}_{\mathbb{C}_p}^\flat)$. There is an action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, hence of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$, on $\mathbb{C}_p^\flat$ which preserves $\mathcal{O}_{\mathbb{C}_p}^\flat$. The following well-known theorem follows from the work of Wintenberger ([Win83]) and the Ax–Sen–Tate Theorem, see for instance [CE14, Cor. 3.4]:

Theorem 2.1.1. *We have isomorphisms of topological rings compatible with the action of $\mathcal{O}_K^\times$ (via (6)):*

$$\mathbb{F}_q[\![T_K^{p^{-\infty}}]\!] \cong {\mathcal{O}_{\mathbb{C}_p}^\flat}^{\mathrm{Gal}(\overline{K}/K_\infty)} \ \textit{and}\ \ \mathbb{F}_q(\!(T_K^{p^{-\infty}})\!) \cong {\mathbb{C}_p^\flat}^{\mathrm{Gal}(\overline{K}/K_\infty)}$$

and isomorphisms of topological rings compatible with the action of $\mathbb{Z}_p^\times$ (via (6)):

$$\mathbb{F}_q[\![T^{p^{-\infty}}]\!] \cong {\mathcal{O}_{\mathbb{C}_p}^\flat}^{\mathrm{Gal}(\overline{K}/K(\sqrt[p^\infty]{1}))} \ \textit{and}\ \ \mathbb{F}_q(\!(T^{p^{-\infty}})\!) \cong {\mathbb{C}_p^\flat}^{\mathrm{Gal}(\overline{K}/K(\sqrt[p^\infty]{1}))}.$$

In particular, $\mathbb{F}_q[\![T^{p^{-\infty}}]\!] \cong \mathbb{F}_q[\![T_K^{p^{-\infty}}]\!]^{\mathrm{Gal}(K_\infty/K(\sqrt[p^\infty]{1}))} \hookrightarrow \mathbb{F}_q[\![T_K^{p^{-\infty}}]\!]$ *and* $\mathbb{F}_q(\!(T^{p^{-\infty}})\!) \cong \mathbb{F}_q(\!(T_K^{p^{-\infty}})\!)^{\mathrm{Gal}(K_\infty/K(\sqrt[p^\infty]{1}))} \hookrightarrow \mathbb{F}_q(\!(T_K^{p^{-\infty}})\!)$.

By Theorem 2.1.1 we have in particular an embedding $\mathbb{F}_q[\![T]\!] \hookrightarrow \mathbb{F}_q[\![T_K^{p^{-\infty}}]\!]$.

Proposition 2.1.2. *Let* $\overline{\rho}$ *be a finite-dimensional continuous representation of* $\mathrm{Gal}(\overline{K}/K)$ *over* $\mathbb{F}$*. There is a canonical* $\mathbb{F}_q(\!(T_K^{p^{-\infty}})\!)$*-linear isomorphism which commutes with the actions of* $\mathcal{O}_K^\times$ *and* φ*:*

$$\mathbb{F}_q(\!(T_K^{p^{-\infty}})\!) \otimes_{\mathbb{F}_q(\!(T)\!)} D(\overline{\rho}) \xrightarrow{\sim} \mathbb{F}_q(\!(T_K^{p^{-\infty}})\!) \otimes_{\mathbb{F}_q(\!(T_K)\!)} D_K(\overline{\rho}) \tag{12}$$

where $\mathcal{O}_K^\times$*,* φ *act diagonally on each side,* $\mathcal{O}_K^\times$ *acting on* $D(\overline{\rho})$ *via the norm map* $\mathcal{O}_K^\times \twoheadrightarrow \mathbb{Z}_p^\times$*.*

Proof. Denote by $\mathbb{F}_q(\!(T_K)\!)^{\mathrm{sep}}$ (resp. $\mathbb{F}_q(\!(T)\!)^{\mathrm{sep}}$) the separable closure of $\mathbb{F}_q(\!(T_K)\!)$ (resp. $\mathbb{F}_q(\!(T)\!)$) in $\mathbb{C}_p^\flat$ via the canonical embeddings $\mathbb{F}_q(\!(T_K)\!) \subseteq \mathbb{F}_q(\!(T_K^{p^{-\infty}})\!) \hookrightarrow \mathbb{C}_p^\flat$ (resp. $\mathbb{F}_q(\!(T)\!) \subseteq \mathbb{F}_q(\!(T^{p^{-\infty}})\!) \hookrightarrow \mathbb{C}_p^\flat$) in Theorem 2.1.1, and recall that we have

$$\mathrm{Gal}(\mathbb{F}_q(\!(T_K)\!)^{\mathrm{sep}}/\mathbb{F}_q(\!(T_K)\!)) \cong \mathrm{Gal}(\overline{K}/K_\infty),\ \mathrm{Gal}(\mathbb{F}_q(\!(T)\!)^{\mathrm{sep}}/\mathbb{F}_q(\!(T)\!)) \cong \mathrm{Gal}(\overline{K}/K(\sqrt[p^\infty]{1}))$$

and

$$D_K(\overline{\rho}) = \Big(\mathbb{F}_q(\!(T_K)\!)^{\mathrm{sep}} \otimes_{\mathbb{F}_p} \overline{\rho}\Big)^{\mathrm{Gal}(\overline{K}/K_\infty)},\quad D(\overline{\rho}) = \Big(\mathbb{F}_q(\!(T)\!)^{\mathrm{sep}} \otimes_{\mathbb{F}_p} \overline{\rho}\Big)^{\mathrm{Gal}(\overline{K}/K(\sqrt[p^\infty]{1}))}.$$

Since we have for any integer $n \geq 1$:

$$H^1\Big(\mathrm{Gal}(\mathbb{F}_q(\!(T_K)\!)^{\mathrm{sep}}/\mathbb{F}_q(\!(T_K)\!)), \mathrm{GL}_n(\mathbb{F}_q(\!(T_K)\!)^{\mathrm{sep}})\Big) = 1$$

as follows by taking inductive limit from [Ser68, Prop. X.1.3], we have a canonical isomorphism

$$\mathbb{F}_q(\!(T_K)\!)^{\mathrm{sep}} \otimes_{\mathbb{F}_q(\!(T_K)\!)} D_K(\overline{\rho}) \xrightarrow{\sim} \mathbb{F}_q(\!(T_K)\!)^{\mathrm{sep}} \otimes_{\mathbb{F}_p} \overline{\rho} \tag{13}$$

that is compatible with the actions of φ and $\mathcal{O}_K^\times$, and likewise with $\mathbb{F}_q(\!(T)\!)^{\mathrm{sep}}$, $\mathbb{F}_q(\!(T)\!)$ and $D(\overline{\rho})$. Tensoring (13) by $\mathbb{C}_p^\flat$ over $\mathbb{F}_q(\!(T_K)\!)^{\mathrm{sep}}$, resp. its analogue over $\mathbb{F}_q(\!(T)\!)^{\mathrm{sep}}$, we obtain a canonical isomorphism

$$\mathbb{C}_p^\flat \otimes_{\mathbb{F}_q(\!(T_K)\!)} D_K(\overline{\rho}) \xrightarrow{\sim} \mathbb{C}_p^\flat \otimes_{\mathbb{F}_p} \overline{\rho} \xleftarrow{\sim} \mathbb{C}_p^\flat \otimes_{\mathbb{F}_q(\!(T)\!)} D(\overline{\rho})$$

compatible with the actions of φ and $\mathcal{O}_K^\times$. Taking invariants under $\mathrm{Gal}(\overline{K}/K_\infty)$, which acts trivially on $D_K(\overline{\rho})$, $D(\overline{\rho})$, and remembering ${\mathbb{C}_p^\flat}^{\mathrm{Gal}(\overline{K}/K_\infty)} = \mathbb{F}_q(\!(T_K^{p^{-\infty}})\!)$ from Theorem 2.1.1 we obtain the desired isomorphism (12). □

Let $\mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!)$ resp. $\mathbb{F}(\!(T^{p^{-\infty}})\!)$ be the completion of the perfection of $\mathbb{F}(\!(T_{K,\sigma_0})\!)$ resp. $\mathbb{F}(\!(T_{\sigma_0})\!)$. Applying $\mathbb{F} \otimes_{\sigma_0,\mathbb{F}_q} (-)$ to Theorem 2.1.1, we deduce embeddings $\mathbb{F}(\!(T)\!) \hookrightarrow \mathbb{F}(\!(T^{p^{-\infty}})\!) \hookrightarrow \mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!)$ and $\mathbb{F}[\![T]\!] \hookrightarrow \mathbb{F}[\![T^{p^{-\infty}}]\!] \hookrightarrow \mathbb{F}[\![T_{K,\sigma_0}^{p^{-\infty}}]\!]$.

Corollary 2.1.3. *There is a canonical* $\mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!)$*-linear isomorphism which commutes with the action of* $\mathcal{O}_K^\times$ *and* φ_q*:*

$$\mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T)\!)} D_{\sigma_0}(\overline{\rho}) \xrightarrow{\sim} \mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!)} D_{K,\sigma_0}(\overline{\rho})$$

where $\mathcal{O}_K^\times$*,* φ_q *act diagonally on each side,* $\mathcal{O}_K^\times$ *acting on* $D_{\sigma_0}(\overline{\rho})$ *via the norm map* $\mathcal{O}_K^\times \twoheadrightarrow \mathbb{Z}_p^\times$*.*

Proof. This follows from Proposition 2.1.2 and a discussion analogous to the one following the proof of *loc. cit.* (the details of which are left to the reader). ☐

Remark 2.1.4. Arguing as in the proofs of Theorem 2.6.4 and Corollary 2.6.6 below, the functor $D_{K,\sigma_0} \mapsto \mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!)} D_{K,\sigma_0}$ in fact still induces an equivalence of categories from the category of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over $\mathbb{F}(\!(T_{K,\sigma_0})\!)$ to the category of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over $\mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!)$. Likewise with the functor $D_{\sigma_0} \mapsto \mathbb{F}(\!(T^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T)\!)} D_{\sigma_0}$ and étale $(\varphi_q, \mathbb{Z}_p^\times)$-modules.

We finally recall a convenient explicit presentation of $D_{K,\sigma_0}(\overline{\rho})$ for $\overline{\rho}$ absolutely semi-simple (or equivalently for $\overline{\rho}$ absolutely irreducible).

For simplicity, we now choose the formal variable T_K such that $a_{\mathrm{LT}}(T_K) = aT_K$ when $a \in [\mathbb{F}_q]$ (so $a(T_{K,\sigma_0}) = \sigma_0(a)T_{K,\sigma_0}$ for $a \in [\mathbb{F}_q]$); for instance, this holds if T_K is such that the logarithm of the Lubin–Tate group G_{LT} ([Lan90, §6]) is the series $\sum_{n\geq 0} p^{-n}T_K^{q^n}$. Note that in that case $\mathbb{F}(\!(T_{K,\sigma_0})\!)^{[\mathbb{F}_q^\times]} = \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$. We recall the following straightforward lemma.

Lemma 2.1.5. *Let $\overline{\rho}$ be a finite-dimensional continuous representation of* $\mathrm{Gal}(\overline{K}/K)$ *over $\mathbb{F}$. Denote by $D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ the $\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$-vector subspace of $D_{K,\sigma_0}(\overline{\rho})$ fixed by $[\mathbb{F}_q^\times] \subseteq \mathcal{O}_K^\times$. Then $D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ is preserved by φ_q and the action of $\mathcal{O}_K^\times$, and we have an $\mathbb{F}(\!(T_{K,\sigma_0})\!)$-linear isomorphism compatible with φ_q and $\mathcal{O}_K^\times$:*

$$\mathbb{F}(\!(T_{K,\sigma_0})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]} \xrightarrow{\sim} D_{K,\sigma_0}(\overline{\rho})$$

where the actions of φ_q and $\mathcal{O}_K^\times$ on the left-hand side are the diagonal ones.

Proof. It is enough to prove that the morphism in the statement is an isomorphism, everything else being trivial. It is enough to prove

$$H^1\Big([\mathbb{F}_q^\times], \mathrm{GL}_n(\mathbb{F}(\!(T_{K,\sigma_0})\!))\Big) = 1$$

for any integer $n \geq 1$. But this is again the generalization of Hilbert 90 applied to the Galois extension $\mathbb{F}(\!(T_{K,\sigma_0})\!)/\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$ (which has Galois group $[\mathbb{F}_q^\times]$), see for instance [Ser68, Prop. X.1.3]. ☐

We now give explicitly $D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ for an absolutely irreducible $\overline{\rho}$.

For $\lambda \in \mathbb{F}^\times$ denote by $\mathrm{unr}(\lambda)$ the unramified character of $\mathrm{Gal}(\overline{K}/K)$ sending the Frobenius $x \mapsto x^q$ to λ^{-1}. For $f' \geq 1$ denote by $\omega_{f'} : I_K \to \mathbb{F}_{p^{f'}}^\times$ Serre's fundamental character of level f', where $I_K \subseteq \mathrm{Gal}(\overline{K}/K)$ is the inertia subgroup. We also denote by ω_f (instead of $\sigma_0 \circ \omega_f$) the composition

$$I_K \xrightarrow{\omega_f} \mathbb{F}_q^\times \overset{\sigma_0}{\hookrightarrow} \mathbb{F}^\times \tag{14}$$

and again ω_f its unique extension to $\mathrm{Gal}(\overline{K}/K)$ such that $\omega_f(p) = 1$ (via local class field theory). Recall that $\omega_f : \mathrm{Gal}(\overline{K}/K) \to \mathbb{F}^\times$ is the composition by $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ of the mod p Lubin–Tate character of $\mathrm{Gal}(\overline{K}/K)$. For $d \in \mathbb{Z}_{\geq 1}$, it goes back to Serre that any absolutely irreducible d-dimensional representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ is isomorphic to $(\mathrm{ind}\, \omega_{df}^h) \otimes \mathrm{unr}(\lambda)$ for some $\lambda \in \mathbb{F}^\times$ and some positive integer h which is not of the form $m\frac{q^d-1}{q^{d'}-1}$ for some $m \in \mathbb{Z}_{\geq 1}$ and some $d' \in \{1, \dots, d-1\}$, where $\mathrm{ind}\, \omega_{df}^h$ is the induction from $\mathrm{Gal}(\overline{K}/K_d)$ to $\mathrm{Gal}(\overline{K}/K)$ of the mod p Lubin–Tate character of $\mathrm{Gal}(\overline{K}/K_d)$ (seen with values in $\mathbb{F}$ via any embedding $\mathbb{F}_{q^d} \hookrightarrow \mathbb{F}$ lifting σ_0), where K_d is the unramified extension of K of degree d. Equivalently $\mathrm{ind}\, \omega_{df}^h$ is the unique representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ with determinant $\omega_f^h \cdot \mathrm{unr}(-1)^{d-1}$ such that $(\mathrm{ind}\, \omega_{df}^h)|_{I_K} \cong \omega_{df}^h \oplus \omega_{df}^{qh} \oplus \cdots \oplus \omega_{df}^{q^{d-1}h}$ (for any choice of embedding $\mathbb{F}_{q^d} \hookrightarrow \mathbb{F}$). Note that

$$(\mathrm{ind}\, \omega_{df}^h) \otimes \mathrm{unr}(\lambda) \cong (\mathrm{ind}\, \omega_{df}^{h'}) \otimes \mathrm{unr}(\lambda')$$

if and only if $h' \equiv q^i h \bmod q^d - 1$ for some $i \in \{0, \dots, d-1\}$ and $\lambda^d = \lambda'^d$.

For $a \in \mathcal{O}_K^\times$, we set:

$$f_a^{\mathrm{LT}} \overset{\mathrm{def}}{=} f_a^{\mathrm{LT}}(T_{K,\sigma_0}) \overset{\mathrm{def}}{=} \frac{\sigma_0(\overline{a}) T_{K,\sigma_0}}{a(T_{K,\sigma_0})} \in 1 + T_{K,\sigma_0}\mathbb{F}[\![T_{K,\sigma_0}]\!].$$

Note that $f_a^{\mathrm{LT}} = 1$ if $a \in [\mathbb{F}_q^\times]$ and that (5) implies

$$f_a^{\mathrm{LT}} \in 1 + T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!].$$

Lemma 2.1.6. *Let $\overline{\rho}$ be an absolutely irreducible continuous representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ and write $\overline{\rho} = (\mathrm{ind}\, \omega_{df}^h) \otimes \mathrm{unr}(\lambda)$ for some d, h, λ as above. Then $D_{K,\sigma_0}(\overline{\rho}) \cong \mathbb{F}(\!(T_{K,\sigma_0})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ (Lemma 2.1.5), where $D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ is explicitly described as follows:*

$$\begin{cases} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]} &= \bigoplus_{i=0}^{d-1} \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!) e_i \\ \varphi_q(e_i) &= e_{i+1}, \ i < d-1 \\ \varphi_q(e_{d-1}) &= \dfrac{\lambda^d}{T_{K,\sigma_0}^{h(q-1)}} e_0 \\ a(e_i) &= {f_a^{\mathrm{LT}}}^{\frac{hq^i(q-1)}{q^d-1}} e_i, \ a \in \mathcal{O}_K^\times. \end{cases} \tag{15}$$

Moreover a basis $(e_0, \dots, e_{d-1}) = (e_0, \varphi_q(e_0), \dots, \varphi_q^{d-1}(e_0))$ as in (15) is uniquely determined up to a scalar in $\mathbb{F}^\times$. Finally, if $h' = q^j h + m(q^d - 1)$ for some $j \in \{0, \dots, d-1\}$ and some $m \in \mathbb{Z}$, then the unique basis $(e'_i)_i = (e'_0, \varphi_q(e'_0), \dots, \varphi_q^{d-1}(e'_0))$ in (15) corresponding to h' is given by $e'_0 = \frac{1}{T_{K,\sigma_0}^{m(q-1)}} e_j$ (up to a scalar in $\mathbb{F}^\times$).

Proof. The first statement is [PS, Cor. 3.4]. We prove the uniqueness of the basis (e_i) in (15) (up to scalar). Let $(f_0, \dots, f_{d-1})$ be another basis of $D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ satisfying

(15), it is enough to prove $f_0 \in \mathbb{F}e_0$. Write $f_0 = \sum_{i=0}^{d-1} x_i e_i$ for some $x_i \in \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$. Since $\varphi_q^d(f_0) = \frac{\lambda^d}{T_{K,\sigma_0}^{h(q-1)}} f_0$ and $\varphi_q^d(e_i) = \frac{\lambda^d}{T_{K,\sigma_0}^{q^i h(q-1)}} e_i$, we deduce $\frac{1}{T_{K,\sigma_0}^{h(q-1)}} x_i = \frac{1}{T_{K,\sigma_0}^{q^i h(q-1)}} \varphi_q^d(x_i)$ for $i \in \{0, \dots, d-1\}$, i.e. $\varphi_q^d(x_i) = T_{K,\sigma_0}^{h(q-1)(q^i-1)} x_i$. This easily implies $x_i \in \mathbb{F}T_{K,\sigma_0}^{m_i}$, where $m_i \overset{\text{def}}{=} \frac{h(q-1)(q^i-1)}{q^d-1} \in \mathbb{Z}_{\geq 0}$. If $x_i \neq 0$, since $(q-1) | m_i$ in $\mathbb{Z}$, we obtain $h = \frac{m_i}{q-1} \frac{q^d-1}{q^i-1}$ for some $i \in \{1, \dots, d-1\}$ which contradicts the assumption on h. Hence $x_i = 0$ for all $i \neq 0$ and thus $f_0 \in \mathbb{F}e_0$. The last statement is an easy check that is left to the reader. □

Remark 2.1.7. One can prove that the action of $a \in \mathcal{O}_K^\times$ in (15) is the unique semi-linear action on $D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ which commutes with φ_q and is such that $a(e_i) \in e_i + T_{K,\sigma_0}^{q-1} \sum_{j=0}^{d-1} \mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!] e_j$ for all i. The argument is the same as in the proof of Lemma 2.2.1 below.

As a special case of Lemma 2.1.6 we have:

Lemma 2.1.8. *Let* $\chi : \mathrm{Gal}(\overline{K}/K) \to \mathbb{F}^\times$ *be a continuous character and write* $\chi = \omega_f^{h_\chi} \mathrm{unr}(\lambda_\chi)$ *for* $h_\chi \in \mathbb{Z}_{\geq 0}$ *and* $\lambda_\chi \in \mathbb{F}^\times$, *then* (*for* $a \in \mathcal{O}_K^\times$)*:*

$$\begin{cases} D_{K,\sigma_0}(\chi)^{[\mathbb{F}_q^\times]} & = & \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!) e_\chi \\ \varphi_q(e_\chi) & = & \dfrac{\lambda_\chi}{T_{K,\sigma_0}^{h_\chi(q-1)}} e_\chi \\ a(e_\chi) & = & (f_a^{\mathrm{LT}})^{h_\chi} e_\chi \end{cases}$$

If $\overline{\rho}$ is any finite-dimensional continuous representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$, write $D_{K,\sigma_0}(\overline{\rho})(\chi)$ for $D_{K,\sigma_0}(\overline{\rho}) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!)} D_{K,\sigma_0}(\chi)$ with tensor product structures. Then we have $D_{K,\sigma_0}(\overline{\rho} \otimes \chi) \cong D_{K,\sigma_0}(\overline{\rho})(\chi)$ as follows from the compatibility of $D_{K,\sigma_0}(-)$ with tensor products.

2.2 The $(\varphi_q, \mathcal{O}_K^\times)$-module over A of a semi-simple Galois representation

To an arbitrary semi-simple $\overline{\rho}$ we associate by an elementary recipe an étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D_{A,\sigma_0}(\overline{\rho})$ over A (depending on the fixed choice of the embedding σ_0).

Let $N_0 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ and $\mathfrak{m}_{N_0}$ the maximal ideal of $\mathbb{F}_q[\![N_0]\!]$. Recall that $\mathbb{F}_q[\![N_0]\!] = \mathbb{F}_q[\![Y_0, \dots, Y_{f-1}]\!]$ and $\mathfrak{m}_{N_0} = (Y_0, \dots, Y_{f-1})$, where

$$Y_i \overset{\text{def}}{=} \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{-p^i} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} \in \mathbb{F}_q[\![N_0]\!].$$

As in [BHH⁺21, §3.1.1] consider the multiplicative system

$$S \overset{\text{def}}{=} \{(Y_0 \cdots Y_{f-1})^k,\ k \geq 0\} \subseteq \mathbb{F}_q[\![N_0]\!]$$

and $A_q \overset{\text{def}}{=} \widehat{\mathbb{F}_q[\![N_0]\!]_S}$ the completion of the localization $\mathbb{F}_q[\![N_0]\!]_S$ with respect to the ascending filtration ($n \in \mathbb{Z}$):

$$F_n(\mathbb{F}_q[\![N_0]\!]_S) \overset{\text{def}}{=} \sum_{k\geq 0} \frac{1}{(Y_0 \cdots Y_{f-1})^k} \mathfrak{m}_{N_0}^{kf-n} = \bigcup_{k\geq 0} \frac{1}{(Y_0 \cdots Y_{f-1})^k} \mathfrak{m}_{N_0}^{kf-n} \tag{16}$$

where $\mathfrak{m}_{N_0}^m \overset{\text{def}}{=} \mathbb{F}_q[\![N_0]\!]$ if $m \leq 0$ (see [BHH⁺21, §3.1.1]). We denote by F_nA_q ($n \in \mathbb{Z}$) the induced ascending filtration on A_q and endow A_q with the associated topology ([LvO96, §I.3]). The ring A_q contains $\mathbb{F}_q[\![N_0]\!]$ and the $\mathbb{F}_q$-linear action of $\mathcal{O}_K^\times$ on $\mathbb{F}_q[\![N_0]\!]$ (induced by the multiplication on $\mathcal{O}_K \cong N_0$) canonically extends by continuity to A_q (but not to $\mathbb{F}_q[\![N_0]\!]_S$ as it does not preserve S). We will write this action of $\mathcal{O}_K^\times$ on $\mathbb{F}_q[\![N_0]\!]$ and A_q as $a(x)$ for $(a, x) \in \mathcal{O}_K^\times \times A_q$. In fact using $a - [\overline{a}] \in p\mathcal{O}_K$ one has for $a \in \mathcal{O}_K^\times$ and $i \in \mathbb{Z}$:

$$a(Y_i) \in \overline{a}^{p^i} Y_i + \mathfrak{m}_{N_0}^p \subseteq \mathfrak{m}_{N_0}$$

which implies

$$a(Y_i) \in \overline{a}^{p^i} Y_i \Big(1 + \frac{1}{Y_i}\mathfrak{m}_{N_0}^p\Big) \subseteq \overline{a}^{p^i} Y_i (1 + F_{1-p}A_q) \subseteq A_q^\times. \tag{17}$$

We define a Frobenius φ on $\mathbb{F}_q[\![N_0]\!]$ by the usual Frobenius $c \mapsto c^p$ on the coefficients $\mathbb{F}_q$ and the multiplication by p on $N_0 \cong \mathcal{O}_K$, i.e. by

$$\varphi\Big(\sum_{\underline{n}} c_{\underline{n}} Y_0^{n_0} \cdots Y_{f-1}^{n_{f-1}}\Big) \overset{\text{def}}{=} \sum_{\underline{n}} c_{\underline{n}}^p Y_0^{pn_0} \cdots Y_{f-1}^{pn_{f-1}},$$

where $\underline{n} \overset{\text{def}}{=} (n_0, \dots, n_{f-1}) \in \mathbb{Z}_{\geq 0}^f$ and $c_{\underline{n}} \in \mathbb{F}_q$. It canonically extends by continuity to A_q and obviously commutes with the action of $\mathcal{O}_K^\times$ on $\mathbb{F}_q[\![N_0]\!]$, hence on A_q.

Let A be the complete filtered ring in [BHH⁺21, §3.1.1]. Recall that A is defined similarly to A_q replacing $\mathbb{F}_q[\![N_0]\!]$ by $\mathbb{F}[\![N_0]\!]$ *except* that the Frobenius φ on $\mathbb{F}[\![N_0]\!]$ is now $\mathbb{F}$-linear. As in (8), we have an isomorphism $\mathbb{F} \otimes_{\mathbb{F}_p} A_q \xrightarrow{\sim} \underbrace{A \times A \times \cdots \times A}_{f \text{ times}}$ which sends $\lambda \otimes \sum_{\underline{n}} c_{\underline{n}} Y_0^{n_0} \cdots Y_{f-1}^{n_{f-1}} \in \mathbb{F} \otimes_{\mathbb{F}_p} A_q$ to:

$$\Big(\lambda \sum_{\underline{n}} \sigma_0(c_{\underline{n}}) Y_{\sigma_0}^{n_0} \cdots Y_{\sigma_{f-1}}^{n_{f-1}}, \lambda \sum_{\underline{n}} \sigma_1(c_{\underline{n}}) Y_{\sigma_1}^{n_0} \cdots Y_{\sigma_0}^{n_{f-1}}, \dots, \lambda \sum_{\underline{n}} \sigma_{f-1}(c_{\underline{n}}) Y_{\sigma_{f-1}}^{n_0} \cdots Y_{\sigma_{f-2}}^{n_{f-1}}\Big)$$

where we set for $\sigma : \mathbb{F}_q \hookrightarrow \mathbb{F}$:

$$Y_\sigma \overset{\text{def}}{=} \sum_{\lambda \in \mathbb{F}_q^\times} \sigma(\lambda)^{-1} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} \in \mathbb{F}[\![N_0]\!] \subseteq A. \tag{18}$$

It induces an analogous decomposition for any $\mathbb{F} \otimes_{\mathbb{F}_p} A_q$-module D_{A_q}:

$$D_{A_q} \xrightarrow{\sim} D_{A,\sigma_0} \times \cdots \times D_{A,\sigma_{f-1}}. \tag{19}$$

We extend $\mathbb{F}$-linearly the Frobenius φ and the action of $\mathcal{O}_K^\times$ from A_q to $\mathbb{F} \otimes_{\mathbb{F}_p} A_q$. Note that we have $\varphi(Y_{\sigma_i}) = Y_{\sigma_{i-1}}^p$ for $i \in \mathbb{Z}$ (see [BHH$^+$21, §3.1.1], where φ on A is denoted ϕ). We let $\varphi_q \overset{\text{def}}{=} \varphi^f$ on A. As in §2.1, the functor $D_{A_q} \mapsto D_{A,\sigma_0}$ induces an equivalence of categories between the category of étale $(\varphi, \mathcal{O}_K^\times)$-modules over $\mathbb{F} \otimes_{\mathbb{F}_p} A_q$ and the category of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A.

The embedding of $\mathbb{F}$-algebras

$$\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!) \hookrightarrow A, \qquad \sum_{n>-\infty} c_n T_{K,\sigma_0}^{n(q-1)} \longmapsto \sum_{n>-\infty} c_n \Big(\frac{\varphi(Y_{\sigma_0})}{Y_{\sigma_0}}\Big)^n \tag{20}$$

trivially commutes with φ_q and $[\mathbb{F}_q^\times]$ (the latter acting trivially on both sides). For $\overline{\rho}$ an absolutely semi-simple finite-dimensional continuous representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$, we define:

$$D_{A,\sigma_0}(\overline{\rho}) \overset{\text{def}}{=} A \otimes_{\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]} \tag{21}$$

where $D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ is as in Lemma 2.1.5. It follows from its definition that $D_{A,\sigma_0}(\overline{\rho})$ is an étale φ_q-module over A if it is endowed with the endomorphism $\varphi_q \overset{\text{def}}{=} \varphi_q \otimes \varphi_q$.

For $a \in \mathcal{O}_K^\times$, we set (see (17)):

$$f_{a,\sigma_0} \overset{\text{def}}{=} f_{a,\sigma_0}(Y_{\sigma_0}, \dots, Y_{\sigma_{f-1}}) \overset{\text{def}}{=} \frac{\sigma_0(\overline{a})Y_{\sigma_0}}{a(Y_{\sigma_0})} \in 1 + F_{1-p}A. \tag{22}$$

Lemma 2.2.1. *Let $\overline{\rho}$ be an absolutely irreducible continuous representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ and $(e_0, \dots, e_{d-1})$ a basis of $D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ as in Lemma 2.1.6. Then we have:*

$$\begin{cases} D_{A,\sigma_0}(\overline{\rho}) &= \bigoplus_{i=0}^{d-1} A(1 \otimes e_i) \\ \varphi_q(1 \otimes e_i) &= 1 \otimes e_{i+1}, \ i < d-1 \\ \varphi_q(1 \otimes e_{d-1}) &= \lambda^d \Big(\dfrac{Y_{\sigma_0}}{\varphi(Y_{\sigma_0})}\Big)^h (1 \otimes e_0). \end{cases}$$

Moreover there is a unique structure of $(\varphi_q, \mathcal{O}_K^\times)$-module over A on $D_{A,\sigma_0}(\overline{\rho})$ such that

$$a(1 \otimes e_i) \in 1 \otimes e_i + \sum_{j=0}^{d-1} (F_{1-p}A)(1 \otimes e_j) \text{ for all } i \text{ and } a \in \mathcal{O}_K^\times.$$

This action of $\mathcal{O}_K^\times$ is explicitly given by ($i \in \{0, \dots, d-1\}$, $a \in \mathcal{O}_K^\times$):

$$a(1 \otimes e_i) = \Big(\frac{f_{a,\sigma_0}}{\varphi(f_{a,\sigma_0})}\Big)^{\frac{hq^i}{1-q^d}} (1 \otimes e_i) \in (1 + F_{q^i(1-p)}A)(1 \otimes e_i) \tag{23}$$

and does not depend (up to isomorphism) on the choice of the basis $(e_i)_i$ of $D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$.

Proof. The first part of the statement follows from the definition of $D_{A,\sigma_0}(\overline{\rho})$ in (21). Fix $a \in \mathcal{O}_K^\times$ and write $a(1\otimes e_0) = \sum_{i=0}^{d-1} C_i(1\otimes e_i)$ for some $C_0 \in 1 + F_{1-p}A$ and $C_i \in F_{1-p}A$ if $i \neq 0$. Assume $C_i \neq 0$ for some $i \neq 0$ and let $m_i \geq p-1$ be the maximal integer such that $C_i \in F_{-m_i}A \setminus F_{-(m_i+1)}A$. Since $a(1\otimes e_0)$ and the $1\otimes e_j$ are fixed by $[\mathbb{F}_q^\times]$, the constants C_j are also fixed by $[\mathbb{F}_q^\times]$ in A for all j, and thus in particular by $[\mathbb{F}_p^\times]$, from which it is an exercise to deduce that we must have $(p-1)|m_i$. Since $\varphi_q^d(1\otimes e_j) = \lambda^d\Big(\frac{Y_{\sigma_0}}{\varphi(Y_{\sigma_0})}\Big)^{q^j h}(1\otimes e_j)$ for all j, the equality $a(\varphi_q^d(e_0)) = \varphi_q^d(a(e_0))$ yields for $j \in \{0,\dots,d-1\}$ (using $\sigma_0(\overline{a})^{q-1} = 1$):

$$C_j = \Big(\frac{f_{a,\sigma_0}}{\varphi(f_{a,\sigma_0})}\Big)^h \Big(\frac{Y_{\sigma_0}}{\varphi(Y_{\sigma_0})}\Big)^{(q^j-1)h} \varphi_q^d(C_j) \tag{24}$$

which implies in particular $-m_i = (q^i-1)h(p-1) - q^d m_i$, i.e. $(q^i-1)h(p-1) = (q^d-1)m_i$, i.e. $h = \frac{q^d-1}{q^i-1}\frac{m_i}{p-1}$, which contradicts the assumption on h since $\frac{m_i}{p-1} \in \mathbb{Z}$. Hence we must have $C_i = 0$ if $i \neq 0$. When $i = 0$, (24) is just $C_0 = \Big(\frac{f_{a,\sigma_0}}{\varphi(f_{a,\sigma_0})}\Big)^h \varphi_q^d(C_0)$, which has a unique solution in $1 + F_{1-p}A$ given by:

$$\begin{aligned} C_0 = \prod_{n=0}^{+\infty} \varphi_q^{nd}\Big(\Big(\frac{f_{a,\sigma_0}}{\varphi(f_{a,\sigma_0})}\Big)^h\Big) = \prod_{n=0}^{+\infty}\Big(\frac{f_{a,\sigma_0}}{\varphi(f_{a,\sigma_0})}\Big)^{q^{nd}h} &= \Big(\frac{f_{a,\sigma_0}}{\varphi(f_{a,\sigma_0})}\Big)^{h(1+q^d+q^{2d}+\cdots)} \\ &= \Big(\frac{f_{a,\sigma_0}}{\varphi(f_{a,\sigma_0})}\Big)^{\frac{h}{1-q^d}} \end{aligned}$$

where the second equality uses $x^{q^{nd}} = x$ if $x \in \mathbb{F}_q$. Then (23) immediately follows, from which the continuity of the action of $\mathcal{O}_K^\times$ is clear (as it is continuous on A). If one changes the basis $(e_i)_i$, or equivalently by (the last statement in) Lemma 2.1.6 changes the integer h, the last statement easily follows from the last statement of Lemma 2.1.6. □

Remark 2.2.2.

(i) The uniqueness of the action of $\mathcal{O}_K^\times$ in the proof of Lemma 2.2.1 works just assuming $a(1\otimes e_i) \in 1\otimes e_i + \sum_{j=0}^{d-1}(F_{-1}A)(1\otimes e_j)$ (and lands automatically in $1\otimes e_i + \sum_{j=0}^{d-1}(F_{1-p}A)(1\otimes e_j)$).

(ii) Definition (21) does not need the semi-simplicity of $\overline{\rho}$, but we will only use it in that case, see also Remark 2.9.6 below.

Let us finally make twists explicit. Let $\chi : \mathrm{Gal}(\overline{K}/K) \to \mathbb{F}^\times$ be a continuous character and write $\chi = \omega_f^{h_\chi}\mathrm{unr}(\lambda_\chi)$ for $h_\chi \in \mathbb{Z}_{\geq 0}$ and $\lambda_\chi \in \mathbb{F}^\times$, then (using Lemma

2.1.8) the étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D_{A,\sigma_0}(\chi)$ is explicitly given by ($a \in \mathcal{O}_K^\times$):

$$\left\{\begin{array}{rcl} D_{A,\sigma_0}(\chi) & = & A(1\otimes e_\chi) \\ \varphi_q(1\otimes e_\chi) & = & \lambda_\chi\Big(\dfrac{Y_{\sigma_0}}{\varphi(Y_{\sigma_0})}\Big)^{h_\chi}(1\otimes e_\chi) \\ a(1\otimes e_\chi) & = & \Big(\dfrac{f_{a,\sigma_0}}{\varphi(f_{a,\sigma_0})}\Big)^{\frac{h_\chi}{1-q}}(1\otimes e_\chi). \end{array}\right. \tag{25}$$

One has an action of $\mathcal{O}_K^\times$ on $D_{A,\sigma_0}(\overline{\rho}\otimes\chi) \overset{\text{def}}{=} D_{A,\sigma_0}(\overline{\rho})\otimes_A D_{A,\sigma_0}(\chi)$ by taking the tensor product action. We leave to the reader the exercise to check that, when $\overline{\rho}\otimes\chi \cong \overline{\rho}'\otimes\chi'$, then $D_{A,\sigma_0}(\overline{\rho}\otimes\chi) \cong D_{A,\sigma_0}(\overline{\rho}'\otimes\chi')$ as $(\varphi_q, \mathcal{O}_K^\times)$-modules over A.

2.3 A reminder on p-divisible groups and K-vector spaces

We review some results on constructions of Fargues and Fontaine ([FF18]) related to p-divisible groups (in a relative context, see for example [LB18, §5.1]) and we define the important perfectoid spaces Z_{LT} and $Z_{\mathcal{O}_K}$ over $\mathbb{F}$.

Let R be a perfectoid $\mathbb{F}$-algebra and ϖ a pseudo-uniformizer of R. As usual we denote by R° the subring of power-bounded elements in R and by $R^{\circ\circ} \subseteq R^\circ$ the subset of topologically nilpotent elements (i.e. those $a \in R$ such that a^n converges to 0 in R). We fix a power-multiplicative norm $|\cdot|$ on R defining the topology of R. Such a norm exists and can be explicitly given by

$$|a| = \inf\{2^{\frac{m}{n}}, (m,n) \in \mathbb{Z}\times\mathbb{Z}_{>0}, \varpi^m a^n \in R^\circ\} \in \mathbb{Q}_{\geq 0} \tag{26}$$

(so $|a| \leq 1 \Leftrightarrow a \in R^\circ$). We endow the Witt vectors $W(R^\circ)$ with the $(p,[\varpi])$-adic topology (where $[\cdot]$ is the multiplicative representative) and write $Y_{\mathrm{Spa}(R,R^\circ)} \overset{\text{def}}{=} \mathrm{Spa}(W(R^\circ)) \setminus V(p[\varpi])$. Let $\mathbf{B}^+(R) \overset{\text{def}}{=} \mathcal{O}^+_{Y_{\mathrm{Spa}(R,R^\circ)}}(Y_{\mathrm{Spa}(R,R^\circ)})$, the global sections of the sheaf $\mathcal{O}^+_{Y_{\mathrm{Spa}(R,R^\circ)}}$ on the adic space $Y_{\mathrm{Spa}(R,R^\circ)}$. This is a Fréchet K-algebra which can be more explicitly described as the completion of $W(R^\circ)[1/p]$ for the family of norms $|\cdot|_\rho$, $0<\rho<1$ defined by

$$\Big|\sum_{n>>-\infty}[x_n]p^n\Big|_\rho \overset{\text{def}}{=} \sup_{n\in\mathbb{Z}}(|x_n|\rho^n). \tag{27}$$

It is endowed with a continuous K-semi-linear endomorphism defined by

$$\varphi\left(\sum_n [x_n]p^n\right) \overset{\text{def}}{=} \sum_n [x_n^p]p^n$$

and we define $\varphi_q \overset{\text{def}}{=} \varphi^f$ which is K-linear.

When $1 \leq d \leq f$, it follows from [FS, Prop. II.2.5(iv)] that the functor $R \mapsto \mathbf{B}^+(R)^{\varphi_q = p^d}$ is "represented" by the topological ring $\mathbb{F}[\![x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]\!]$, the completion of the perfection of $\mathbb{F}[x_0, \dots, x_{d-1}]$ for the $(x_0, \dots, x_{d-1})$-topology (we put "" as $\mathbb{F}[\![x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]\!]$ is not a perfectoid ring). More precisely, if R is a perfectoid $\mathbb{F}$-algebra and $(r_0, \dots, r_{d-1}) \in (R^{\circ\circ})^d$, let

$$F(r_0, \dots, r_{d-1}) \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}} \sum_{i=0}^{d-1} [r_i^{p^{-i-nf}}] p^{i+nd} \in \mathbf{B}^+(R)^{\varphi_q = p^d} \tag{28}$$

then we have:

Lemma 2.3.1. *Let $1 \leq d \leq f$. For each perfectoid $\mathbb{F}$-algebra R, the following functorial map is a bijection:*

$$\begin{array}{rcl} \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}-\mathrm{alg}}(\mathbb{F}[\![x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]\!], R) \cong (R^{\circ\circ})^d & \longrightarrow & \mathbf{B}^+(R)^{\varphi_q = p^d} \\ (r_0, \dots, r_{d-1}) & \longmapsto & F(r_0, \dots, r_{d-1}). \end{array}$$

Proof. Mutatis mutandis, this is the proof of [FF18, Prop. 4.2.1]. □

Remark 2.3.2. If R is a Huber ring over $\mathbb{F}$ and $R^+ \subseteq R^\circ$ is an open and integrally closed subring ((R, R^+) is then called a *Huber pair*), we have $R^{\circ\circ} \subseteq R^+$ so that, by [Hub94, Prop. 2.1(i)]

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Spa}(\mathbb{F})}\Big(\mathrm{Spa}(R, R^+), \mathrm{Spa}(\mathbb{F}[\![x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]\!])\Big) \\ &\qquad\qquad \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}-\mathrm{alg}}(\mathbb{F}[\![x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]\!], R). \end{aligned}$$

Thus Lemma 2.3.1 and [SW20, Lemma 18.1.1] imply that the functor $(R, R^+) \mapsto \mathbf{B}^+(R)^{\varphi_q = p^d}$ can be extended to a sheaf on the site $\mathrm{Perf}_{\mathbb{F}}$ of perfectoid spaces over $\mathbb{F}$ endowed with either the pro-étale topology or the v-topology.

Remark 2.3.3. Let (R, R^+) be a perfectoid Huber pair over $\mathbb{F}$. If $x \in \mathrm{Spa}(R, R^+)$, then its residue field $k(x)$ is a perfectoid field containing $\mathbb{F}$ (see for example [Sch12, Cor. 6.7(ii)]). If $z \in \mathbf{B}^+(R)^{\varphi_q = p^d}$, we let z_x be its image in $\mathbf{B}^+(k(x))^{\varphi_q = p^d}$. Then the functorial bijection of Lemma 2.3.1 induces a functorial bijection:

$$\Big(\mathrm{Spa}(\mathbb{F}[\![x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]\!]) \backslash V(x_0, \dots, x_{d-1})\Big)(R, R^+) \cong \{z \in B^+(R)^{\varphi_q = p^p},\ z_x \neq 0\ \forall\, x\}.$$

The following remark will be used in §2.9.

Remark 2.3.4. There exists a norm $|\cdot|_1$ on $\mathbf{B}^+(R)$ which induces on $W(R^\circ)[1/p]$ the norm

$$\Bigg|\sum_{n >> -\infty} [x_n] p^n\Bigg|_1 = \sup\{|x_n|,\ n \in \mathbb{Z}\} \in [0, 1] \subseteq \mathbb{R}_{\geq 0}$$

and is such that $|x|_1 = \lim_{\substack{\rho<1 \\ \rho\to 1}} |x|_\rho$ (see [FF18, Prop. 1.10.5 & Prop. 1.6.16]). Equivalently, there exists a valuation $v_0 : \mathbf{B}^+(R) \to [0, +\infty]$ such that

$$\forall\ x = \sum_{n >> -\infty} [x_n]p^n \in W(R^\circ)[1/p], \quad v_0(x) = \inf\{v(x_n),\ n \in \mathbb{Z}\},$$

where v is the valuation $-\log|\cdot|$ on R. This description implies that if $(x_{-n})_{n\geq 0}$ is a sequence of elements of $R^{\circ\circ}$ such that $\sum_{n\leq 0}[x_n]p^n \in \mathbf{B}^+(R)$ and such that there exists $0 \leq c < 1$ with $|x_{-n}| \leq c$ for all $n \geq 0$, then

$$\left|\sum_{n\leq 0}[x_n]p^n\right|_1 \leq c.$$

Note that $|\cdot|_1 : \mathbf{B}^+(R) \to [0, 1]$ is not continuous since, for instance, $|p^n|_1 = 1$ for $n \in \mathbb{Z}$ although $p^n \to 0$ in $\mathbf{B}^+(R)$ when $n \to +\infty$ (in fact $|\cdot|_1$ induces the discrete topology on $K \subseteq \mathbf{B}^+(R)$).

Now we review the interpretation of $\mathbf{B}^+(-)^{\varphi_q = p^d}$ in terms of p-divisible groups in the two extreme cases $d = 1$ and $d = f$.

The case $d = 1$ Let G_{LT} be the Lubin–Tate formal group of §2.1. As at the end of *loc. cit.* we choose an isomorphism $G_{\mathrm{LT}} \cong \mathrm{Spf}(\mathcal{O}_K[\![T_K]\!])$ such that the logarithm map $\log_{G_{\mathrm{LT}}} : G_{\mathrm{LT}}^{\mathrm{rig}} \to \mathbb{G}_{a,K}^{\mathrm{rig}}$ (where $\mathbb{G}_{a,K}$ is the additive formal group over $\mathcal{O}_K$ and “rig” the rigid analytic generic fiber) is given by the series $\sum_{n\geq 0} p^{-n}T_K^{q^n}$. Let $\widetilde{G}_{\mathrm{LT}} \overset{\mathrm{def}}{=} \varprojlim_p (G_{\mathrm{LT}} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathbb{F}_q)) \cong \mathrm{Spf}(\mathbb{F}_q[\![T_K^{1/p^\infty}]\!])$ be the universal cover of $G_{\mathrm{LT}} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathbb{F}_q)$ (see for instance [SW13, §3.1]). The action of $\mathcal{O}_K$ on G_{LT} extends to an action of K on $\widetilde{G}_{\mathrm{LT}}$. Note that if R is a perfectoid $\mathbb{F}$-algebra, we have $\widetilde{G}_{\mathrm{LT}}(R) \xrightarrow{\sim} (G_{\mathrm{LT}} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathbb{F}_q))(R) \cong G_{\mathrm{LT}}(R)$ (see for example [SW13, Prop. 3.1.3(iii)]) so that $G_{\mathrm{LT}}(R)$ already has a structure of a K-vector space. By [FF18, Prop. 4.4.5] or [FS, Prop. II.2.2], for each perfectoid $\mathbb{F}$-algebra R, we have an isomorphism of K-vector spaces $\widetilde{G}_{\mathrm{LT}}(R^\circ) \xrightarrow{\sim} \mathbf{B}^+(R)^{\varphi_q = p}$ given by

$$r \in R^{\circ\circ} \cong \widetilde{G}_{\mathrm{LT}}(R^\circ) \longmapsto F(r) \overset{\mathrm{def}}{=} \sum_{n\in\mathbb{Z}} [r^{q^{-n}}]p^n \in \mathbf{B}^+(R)^{\varphi_q = p} \tag{29}$$

(this is the isomorphism of Lemma 2.3.1 when $d = 1$, where the variable x_0 in *loc. cit.* is denoted T_K). Note that on the left-hand side, the K-linear structure is given by (for $r \in R^{\circ\circ}$)

$$\begin{cases} \forall\ n \in \mathbb{Z}, & p^n(r) = r^{q^n}, \\ \forall\ a \in \mathcal{O}_K, & a(r) = a_{\mathrm{LT}}(r), \end{cases} \tag{30}$$

where we view the coefficients of the power series a_{LT} in $\mathbb{F}$ via $\mathcal{O}_K \twoheadrightarrow \mathbb{F}_q \overset{\sigma_0}{\hookrightarrow} \mathbb{F}$. We let

$$Z_{\mathrm{LT}} \overset{\mathrm{def}}{=} ((\widetilde{G}_{\mathrm{LT}} \times_{\mathrm{Spf}(\mathbb{F}_q)} \mathrm{Spf}(\mathbb{F}))^{\mathrm{ad}} \setminus \{0\})^f, \tag{31}$$

where $(\widetilde{G}_{\mathrm{LT}} \times_{\mathrm{Spf}(\mathbb{F}_q)} \mathrm{Spf}(\mathbb{F}))^{\mathrm{ad}}$ is the adic space associated to the formal scheme $\widetilde{G}_{\mathrm{LT}} \times_{\mathrm{Spf}(\mathbb{F}_q)} \mathrm{Spf}(\mathbb{F})$ and $\{0\}$ is the closed analytic subspace image of the 0-section, i.e. f-times the fiber product of $(\widetilde{G}_{\mathrm{LT}} \times_{\mathrm{Spf}(\mathbb{F}_q)} \mathrm{Spf}(\mathbb{F}))^{\mathrm{ad}} \setminus \{0\}$ over $\mathrm{Spa}(\mathbb{F})$ (still using σ_0). Using obvious notation, we have an isomorphism of adic spaces

$$Z_{\mathrm{LT}} \cong \mathrm{Spa}(\mathbb{F}[\![T_{K,0}^{1/p^\infty}, \dots, T_{K,f-1}^{1/p^\infty}]\!]) \setminus V(T_{K,0} \cdots T_{K,f-1}) \tag{32}$$

and there is an action of $(K^\times)^f$ on Z_{LT} given by

$$\forall\ \underline{a} = (a_0, \dots, a_{f-1}) \in (K^\times)^f, \quad \underline{a}(T_{K,i}) = a_{i,\mathrm{LT}}(T_{K,i}).$$

The case $d = f$ Let $\mathcal{G}_{f,f}$ be the p-divisible group over $\mathbb{F}_p$ defined in [FF18, §4.3.2] (with $\mathcal{O} = \mathbb{Z}_p$) as the kernel of $V^f - 1$ on the group scheme of Witt covectors CW (we use without comment the notation of *loc. cit.*, for instance V is the Verschiebung, F is the Frobenius, see [FF18, §1.10.2] for CW, etc.). The base change of $\mathcal{G}_{f,f}$ to $\mathbb{F}$ is endowed with an additional structure of functor in $\mathcal{O}_K$-modules. Namely if R is an $\mathbb{F}$-algebra, then $CW(R)$ is an $\mathcal{O}_K = W(\mathbb{F}_q)$-module via $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ and the action of $\mathcal{O}_K$ on $CW(R)$ commutes with V^f and F^f (but not with V and F).

As $\ker(V-1) \subseteq \ker(V^f - 1)$, there is a natural injection of p-divisible groups $\mathcal{G}_{1,1} \hookrightarrow \mathcal{G}_{f,f}$ which induces a morphism of p-divisible groups over $\mathbb{F}$ with $\mathcal{O}_K$-action

$$\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{G}_{1,1,\mathbb{F}} \longrightarrow \mathcal{G}_{f,f,\mathbb{F}}. \tag{33}$$

Lemma 2.3.5. *The map* (33) *is an isomorphism of p-divisible groups over $\mathbb{F}$ with $\mathcal{O}_K$-action.*

Proof. In this proof we will use (contravariant) Dieudonné Theory $\mathbb{D}(-)$ for p-divisible groups over $\mathbb{F}_p$. Recall that it yields free $\mathbb{Z}_p$-modules, and that when the p-divisible group is over $\mathbb{F}$ it yields free $W(\mathbb{F})$-modules. The map (33) corresponds to a non-zero map of Dieudonné modules which is both $\mathcal{O}_K$-linear and $W(\mathbb{F})$-linear:

$$\begin{aligned}\mathbb{D}(\mathcal{G}_{f,f,\mathbb{F}}) \longrightarrow \mathbb{D}(\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{G}_{1,1,\mathbb{F}}) &\cong \mathrm{Hom}_{\mathbb{Z}_p-\mathrm{mod}}(\mathcal{O}_K, \mathbb{D}(\mathcal{G}_{1,1,\mathbb{F}})) \\ &\cong \mathrm{Hom}_{\mathbb{Z}_p-\mathrm{mod}}(\mathcal{O}_K, W(\mathbb{F})) \otimes_{W(\mathbb{F})} \mathbb{D}(\mathcal{G}_{1,1,\mathbb{F}})\end{aligned} \tag{34}$$

where $\mathcal{O}_K$ acts on the right-hand side via its natural action on $\mathrm{Hom}_{\mathbb{Z}_p-\mathrm{mod}}(\mathcal{O}_K, W(\mathbb{F}))$. Note that $\mathbb{D}(\mathcal{G}_{f,f,\mathbb{F}}) = W(\mathbb{F}) \otimes_{\mathbb{Z}_p} \mathbb{D}(\mathcal{G}_{f,f})$, where the Dieudonné module $\mathbb{D}(\mathcal{G}_{f,f})$ has a $\mathbb{Z}_p$-basis $(e_0, e_1 \overset{\mathrm{def}}{=} V(e_0), \dots, e_{f-1} \overset{\mathrm{def}}{=} V^{f-1}(e_0))$ such that $F(e_i) = pe_{i-1}$ for all $0 \leq i \leq f-1$ (see [FF18, §4.3.2], we write $e_0 : \mathcal{G}_{f,f} \hookrightarrow CW$ for the canonical embedding e of *loc. cit.* and we use the convention that $i = i + f$). Moreover the action of $\mathcal{O}_K$ on $\mathcal{G}_{f,f,\mathbb{F}}$ induces an action of $\mathcal{O}_K$ on $\mathbb{D}(\mathcal{G}_{f,f,\mathbb{F}}) = W(\mathbb{F}) \otimes_{\mathbb{Z}_p} \mathbb{D}(\mathcal{G}_{f,f})$ such that $a(1 \otimes e_i) = \varphi^{-i}(a) \otimes e_i$ for $a \in \mathcal{O}_K$, where φ is the absolute Frobenius on $W(\mathbb{F})$ and $\mathcal{O}_K$ is seen in $W(\mathbb{F})$ via $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$. Using the $\mathcal{O}_K$- and $W(\mathbb{F})$-linearities, and the commutativity with F, one checks that there is an isomorphism $W(\mathbb{F}) \cong \mathbb{D}(\mathcal{G}_{1,1,\mathbb{F}})$ such that the map (34) is given by

$$\sum_{i=0}^{f-1} \lambda_i \otimes e_i \longmapsto \left(a \longmapsto \sum_{i=0}^{f-1} \lambda_i \varphi^{-i}(a)\right) \in \mathrm{Hom}_{\mathbb{Z}_p-\mathrm{mod}}(\mathcal{O}_K, W(\mathbb{F}))$$

(in particular, e_0 maps to the inclusion $\mathcal{O}_K \hookrightarrow W(\mathbb{F})$). To conclude the proof we need to show that the elements $a \mapsto \varphi^{-i}(a)$, $i \in \{0, \dots, f-1\}$, generate the $W(\mathbb{F})$-module $\mathrm{Hom}_{\mathbb{Z}_p-\mathrm{mod}}(\mathcal{O}_K, W(\mathbb{F}))$. This can be checked after reduction mod p and we have to prove that the elements $a \mapsto a^{p^i}$, $i \in \{0, \dots, f-1\}$, generate the $\mathbb{F}$-vector space $\mathrm{Hom}_{\mathbb{F}_p-\mathrm{vs}}(\mathbb{F}_q, \mathbb{F})$, which is a consequence of the linear independence of characters. $\square$

By [FF18, Prop 4.4.5] (replacing $\overline{\mathbb{F}}_p$ by $\mathbb{F}$, the field F by a perfectoid $\mathbb{F}$-algebra R and where the variable x_i of *loc. cit.* is reindexed x_{f-i} here for $i \in \{1, \dots, f-1\}$, x_0 being unchanged) there exists a coordinate z (resp. coordinates $x_0, \dots, x_{f-1}$) on the formal group $\mathcal{G}_{1,1,\mathbb{F}}$ (resp. $\mathcal{G}_{f,f,\mathbb{F}}$) such that the following map is an isomorphism of $\mathbb{Q}_p$-vector spaces (resp. K-vector spaces) for any perfectoid $\mathbb{F}$-algebra R:

$$\begin{array}{c}
\gamma_1 : \begin{array}{ccc} \mathcal{G}_{1,1,\mathbb{F}}(R) & \overset{\sim}{\to} & \mathbf{B}^+(R)^{\varphi=p} \\ z & \overset{\sim}{\to} & \sum_{n\in\mathbb{Z}}[z^{p^{-n}}]p^n \end{array} \\
\left(\text{resp. } \gamma_f : \begin{array}{ccc} \mathcal{G}_{f,f,\mathbb{F}}(R) & \overset{\sim}{\to} & \mathbf{B}^+(R)^{\varphi_q=p^f} \\ (x_0, \dots, x_{f-1}) & \overset{\sim}{\to} & \sum_{i=0}^{f-1}\sum_{n\in\mathbb{Z}}[x_i^{p^{-i-nf}}]p^{i+nf} \end{array}\right)
\end{array} \tag{35}$$

(we use $\widetilde{\mathcal{G}}_{1,1,\mathbb{F}}(R) \overset{\sim}{\to} \mathcal{G}_{1,1,\mathbb{F}}(R)$ by [SW13, Prop. 3.1.3(iii)] for the structure of $\mathbb{Q}_p$-vector space on $\mathcal{G}_{1,1,\mathbb{F}}(R)$, likewise with $\mathcal{G}_{f,f,\mathbb{F}}(R)$). Moreover these isomorphisms are given by the composition of the isomorphisms in the following diagram (we only give γ_f and refer to *loc. cit.* for the notation):

$$\begin{tikzcd}
 & \mathrm{Hom}_{W(\mathbb{F})[F]}(\mathbb{D}(\mathcal{G}_{f,f,\mathbb{F}}), BW(R)) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{Hom}_{W(\mathbb{F})[F]}(\mathbb{D}(\mathcal{G}_{f,f,\mathbb{F}}), \mathbf{B}^+(R)) \arrow[r, equal] & \mathbf{B}^+(R)^{\varphi_q=p^f} \\
\mathcal{G}_{f,f,\mathbb{F}}(R) \arrow[r, "\cong"] & \mathrm{Hom}_{W(\mathbb{F})[F]}(\mathbb{D}(\mathcal{G}_{f,f,\mathbb{F}}), CW(R)) & &
\end{tikzcd} \tag{36}$$

where the vertical isomorphism is a consequence of [FF18, Prop. 4.4.2] and the first top horizontal isomorphism a consequence of [FF18, Prop. 4.2.1]. We deduce from (36) the commutativity of the following diagram of $\mathbb{Q}_p$-vector spaces:

$$\begin{tikzcd}
\mathcal{G}_{1,1,\mathbb{F}}(R) \arrow[r, "\gamma_1"] \arrow[d] & \mathbf{B}^+(R)^{\varphi=p} \arrow[d] \\
\mathcal{G}_{f,f,\mathbb{F}}(R) \arrow[r, "\gamma_f"] & \mathbf{B}^+(R)^{\varphi_q=p^f}
\end{tikzcd}$$

and thus the commutativity of the following diagram of K-vector spaces:

$$\begin{tikzcd}
\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{G}_{1,1,\mathbb{F}}(R) \arrow[r, "\mathrm{Id}_{\mathcal{O}_K}\otimes\gamma_1"] \arrow[d] & \mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathbf{B}^+(R)^{\varphi=p} \arrow[d, "\cong"] \\
\mathcal{G}_{f,f,\mathbb{F}}(R) \arrow[r, "\gamma_f"] & \mathbf{B}^+(R)^{\varphi_q=p^f}.
\end{tikzcd} \tag{37}$$

Let $\widehat{\mathbb{G}}_{m,\mathbb{F}_p}$ be the multiplicative formal group over $\mathbb{F}_p$ and $\widehat{\mathbb{G}}_{m,\mathbb{F}}$ its base change over $\mathbb{F}$, we have $\mathcal{G}_{1,1} \cong \widehat{\mathbb{G}}_{m,\mathbb{F}_p}$ (see [FF18, Ex. 4.4.7]) and isomorphisms of p-divisible groups over $\mathbb{F}$ with $\mathcal{O}_K$-action

$$\mathrm{Hom}_{\mathbb{Z}_p-\mathrm{mod}}(\mathcal{O}_K, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}}_{m,\mathbb{F}} \cong \mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{G}_{1,1,\mathbb{F}} \xrightarrow{\sim} \mathcal{G}_{f,f,\mathbb{F}} \tag{38}$$

using the isomorphism of $\mathcal{O}_K$-modules

$$\mathcal{O}_K \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p-\mathrm{mod}}(\mathcal{O}_K, \mathbb{Z}_p), \ a \mapsto \mathrm{Tr}_{K/\mathbb{Q}_p}(a\cdot) \tag{39}$$

and Lemma 2.3.5. Here, the $\mathcal{O}_K$-action on the left-hand side of (38) is via the action of $\mathcal{O}_K$ on $\mathrm{Hom}_{\mathbb{Z}_p-\mathrm{mod}}(\mathcal{O}_K, \mathbb{Z}_p)$ given by $a(\lambda)=\lambda(a-)$ ($a\in\mathcal{O}_K$, $\lambda\in\mathrm{Hom}_{\mathbb{Z}_p-\mathrm{mod}}(\mathcal{O}_K, \mathbb{Z}_p)$). Using

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}-\mathrm{alg}}(\mathbb{F}[\![\mathcal{O}_K]\!], A) \cong \mathrm{Hom}_{\mathbb{Z}_p-\mathrm{mod}}(\mathcal{O}_K, A^{\circ\circ}) \cong \mathrm{Hom}_{\mathbb{Z}_p-\mathrm{mod}}(\mathcal{O}_K, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A^{\circ\circ}$$

for any complete topological $\mathbb{F}$-algebra A, we deduce from (38) an isomorphism of formal modules over $\mathbb{F}$ with $\mathcal{O}_K$-action

$$\mathcal{G}_{f,f,\mathbb{F}} \cong \mathrm{Spf}(\mathbb{F}[\![\mathcal{O}_K]\!]), \tag{40}$$

where $\mathcal{O}_K$ acts (continuously) on $\mathbb{F}[\![\mathcal{O}_K]\!]$ by multiplication on itself. It follows that $\widetilde{\mathcal{G}}_{f,f,\mathbb{F}} \overset{\mathrm{def}}{=} \varprojlim_p \mathcal{G}_{f,f,\mathbb{F}}$ is represented by the formal scheme $\mathrm{Spf}(\mathbb{F}[\![K]\!])$, where $\mathbb{F}[\![K]\!]$ is the $\mathfrak{m}_{\mathcal{O}_K}$-adic completion of $\mathbb{F}[K] \otimes_{\mathbb{F}[\mathcal{O}_K]} \mathbb{F}[\![\mathcal{O}_K]\!]$ ($\mathfrak{m}_{\mathcal{O}_K}$ being the maximal ideal of $\mathbb{F}[\![\mathcal{O}_K]\!]$). It also follows from the formula for γ_f in (35) that there exist elements $X_0, \dots, X_{f-1} \in \mathbb{F}[\![\mathcal{O}_K]\!]$ satisfying $\mathbb{F}[\![\mathcal{O}_K]\!] = \mathbb{F}[\![X_0, \dots, X_{f-1}]\!]$ such that we have isomorphisms $\mathcal{G}_{f,f,\mathbb{F}}(R) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}-\mathrm{alg}}(\mathbb{F}[\![\mathcal{O}_K]\!], R) \cong \mathbf{B}^+(R)^{\varphi_q=p^f}$ for any perfectoid $\mathbb{F}$-algebra R, where the second isomorphism is given by (where $X_i \mapsto x_i \in R^{\circ\circ}$)

$$(x_0, \dots, x_{f-1}) \in (R^{\circ\circ})^f \longmapsto \sum_{i=0}^{f-1} \sum_{n\in\mathbb{Z}} [x_i^{p^{-i-nf}}] p^{i+nf} = F(x_0, \dots, x_{f-1}) \in \mathbf{B}^+(R)^{\varphi_q=p^f}.$$

We then easily check that, in the coordinates X_i, the action of $K^\times$ on $\mathbb{F}[\![K]\!]$ has the following properties

$$\begin{cases} \forall\ 0 \le i \le f-1,\ \forall\ n \in \mathbb{Z}, & p^n(X_i) = X_{i-n}^{p^n} \\ \forall\ 0 \le i \le f-1,\ \forall\ a \in \mathbb{F}_q^\times, & [a](X_i) = \sigma_0(a)^{p^i} X_i \end{cases} \tag{41}$$

(with the usual convention that $X_{i+f} = X_i$). Finally, we let

$$Z_{\mathcal{O}_K} \overset{\text{def}}{=} \widetilde{\mathcal{G}}^{\mathrm{ad}}_{f,f,\mathbb{F}} \setminus \{0\}, \tag{42}$$

where $\widetilde{\mathcal{G}}^{\mathrm{ad}}_{f,f,\mathbb{F}}$ is the adic space over $\mathbb{F}$ associated to the formal scheme $\widetilde{\mathcal{G}}_{f,f,\mathbb{F}}$. We have an isomorphism

$$Z_{\mathcal{O}_K} \cong \mathrm{Spa}(\mathbb{F}[\![X_0^{1/p^\infty}, \dots, X_{f-1}^{1/p^\infty}]\!]) \setminus V(X_0, \dots, X_{f-1}). \tag{43}$$

Note that the adic spaces Z_{LT} and $Z_{\mathcal{O}_K}$ are both in $\mathrm{Perf}_{\mathbb{F}}$.

2.4 An analogue of the Abel–Jacobi map

We define and study certain open subspaces of the perfectoid spaces Z_{LT} and $Z_{\mathcal{O}_K}$ of §2.3, as well as a canonical map $m : Z_{\mathrm{LT}} \longrightarrow Z_{\mathcal{O}_K}$ preserving these subspaces.

For any perfectoid $\mathbb{F}$-algebra R, the product in the ring $\mathbf{B}^+(R)$ induces a functorial map:

$$m_R : \begin{array}{ccc} (\mathbf{B}^+(R)^{\varphi_q = p})^f & \longrightarrow & \mathbf{B}^+(R)^{\varphi_q = p^f} \\ (z_1, \dots, z_f) & \longmapsto & z_1 \cdots z_f. \end{array} \tag{44}$$

Using Remark 2.3.3, the fact that each $\mathbf{B}^+(k)$ is a domain for k a perfectoid field (see [FF18, Thm. 6.2.1 & Thm. 3.6.1]) and [SW20, Prop. 8.2.8(2)], the family of maps (m_R) induces a morphism of perfectoid spaces over $\mathbb{F}$

$$m : Z_{\mathrm{LT}} \longrightarrow Z_{\mathcal{O}_K}. \tag{45}$$

The map m_R being compatible with the actions of $(K^\times)^f$ (on the source) and $K^\times$ (on the target), we deduce that m is compatible with the actions of $(K^\times)^f$ and $K^\times$ on Z_{LT} and $Z_{\mathcal{O}_K}$, i.e. $m \circ (a_0, \dots, a_{f-1}) = (\prod_i a_i) \circ m$. For $0 \leq i \leq f-1$ let j_i be the morphism $K^\times \to (K^\times)^f$ sending a to the f-uple with 1 at all entries except at the i-th entry where it is a, then for all $a \in K^\times$ and $0 \leq i \leq f-1$, we have in particular

$$m \circ j_i(a) = a \circ m : Z_{\mathrm{LT}} \to Z_{\mathcal{O}_K}. \tag{46}$$

Remark 2.4.1. The map m can be seen as an analogue of the Abel–Jacobi map (cf. [Far20]). Namely the sheaf on the pro-étale site of $\mathrm{Perf}_{\mathbb{F}}$ associated to the quotient presheaf $(\mathbf{B}^+(-)^{\varphi_q = p} \setminus \{0\})/K^\times$ is isomorphic to the pro-étale sheaf $\mathrm{Div}^1_{\mathbb{F}}$ of degree 1 divisors on the relative Fargues–Fontaine curve over $\mathbb{F}$ and likewise $(\mathbf{B}^+(-)^{\varphi_q = p^f} \setminus \{0\})/K^\times$ is isomorphic to the pro-étale sheaf $\mathrm{Div}^f_{\mathbb{F}}$ of degree f divisors. The map m induces a morphism of pro-étale sheaves $(\mathrm{Div}^1_{\mathbb{F}})^f \to \mathrm{Div}^f_{\mathbb{F}}$ which is given by the sum of divisors, cf. [Far20, §2.4].

The group $\mathfrak{S}_f$ acts on the left on $(K^\times)^f$ by permutation of coordinates:

$$\forall\ \sigma \in \mathfrak{S}_f,\ \forall\ (a_i)_{0\leq i\leq f-1} \in (K^\times)^f, \quad \sigma(a_i) \overset{\text{def}}{=} (a_{\sigma^{-1}(i)}).$$

The group $\mathfrak{S}_f$ acts likewise on Z_{LT} by permuting the factors $(\widetilde{G}_{\mathrm{LT}} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathbb{F}) \setminus \{0\})^{\mathrm{ad}}$ so that the action of $(K^\times)^f$ on Z_{LT} extends to an action of the semi-direct product $(K^\times)^f \rtimes \mathfrak{S}_f$. Let Δ be the kernel of the multiplication $(K^\times)^f \to K^\times$ and $\Delta_1 \overset{\text{def}}{=} \Delta \cap (\mathcal{O}_K^\times)^f$. Then $\Delta \rtimes \mathfrak{S}_f$ is a subgroup of $(K^\times)^f \rtimes \mathfrak{S}_f$ and the map m is invariant under the action of $\Delta \rtimes \mathfrak{S}_f$. By [Far20, Lemme 7.6][2], the map m induces an isomorphism of pro-étale sheaves on $\mathrm{Perf}_{\mathbb{F}}$

$$\Delta \rtimes \mathfrak{S}_f \backslash Z_{\mathrm{LT}} \xrightarrow{\sim} Z_{\mathcal{O}_K}. \tag{47}$$

We let $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ be the open subspace of $\mathrm{Spa}(\mathbb{F}[\![K]\!])$ defined by the relations

$$|X_0| = \cdots = |X_{f-1}| \neq 0$$

(it is open as it is the intersection over $i \in \{0,\dots,f-1\}$ of the rational open subsets $U(\frac{X_0,\dots,X_{f-1}}{X_i})$ of $\mathrm{Spa}(\mathbb{F}[\![K]\!])$). Note that we have $Z_{\mathcal{O}_K}^{\mathrm{gen}} \subseteq Z_{\mathcal{O}_K}$. We also define $Z_{\mathrm{LT}}^{\mathrm{gen}} \overset{\text{def}}{=} m^{-1}(Z_{\mathcal{O}_K}^{\mathrm{gen}})$, an open subspace of Z_{LT}, so that $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ and $Z_{\mathrm{LT}}^{\mathrm{gen}}$ are both in $\mathrm{Perf}_{\mathbb{F}}$. We now give explicit descriptions of $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ and $Z_{\mathrm{LT}}^{\mathrm{gen}}$.

We start with $Z_{\mathcal{O}_K}^{\mathrm{gen}}$. We denote by $A_\infty \overset{\text{def}}{=} \mathcal{O}_{Z_{\mathcal{O}_K}}(Z_{\mathcal{O}_K}^{\mathrm{gen}})$ the ring of global sections on $Z_{\mathcal{O}_K}^{\mathrm{gen}}$.

Lemma 2.4.2. *The following statements hold.*

(i) *The ring A_∞ is the perfectoid $\mathbb{F}$-algebra*

$$\mathbb{F}(\!(X_0^{1/p^\infty})\!)\left\langle \left(\frac{X_i}{X_0}\right)^{\pm 1/p^\infty},\ 1 \leq i \leq f-1 \right\rangle.$$

(ii) *We have $Z_{\mathcal{O}_K}^{\mathrm{gen}} = \mathrm{Spa}(A_\infty, A_\infty^\circ)$, in particular $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ is affinoid perfectoid.*

(iii) *There exists a multiplicative norm $|\cdot|$ on A_∞ such that $|X_0| = p^{-1}$ inducing the topology of A_∞.*

(iv) *Any quasi-compact open subset of $Z_{\mathcal{O}_K}$ whose points of rank 1 are exactly the points of $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ of rank 1 is necessarily $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ itself.*

---

[2] Note that [Far20, Lemme 7.6] extends scalars to $\overline{\mathbb{F}}_q$, however the proof works the same without extending scalars as it is based on the proof of [Far20, Prop. 2.18] where one does not extend scalars.

Proof. Define the adic spaces

$$T^{\mathrm{gen}} \overset{\mathrm{def}}{=} \{|X_0| = \cdots = |X_{f-1}| \neq 0\} \subseteq T \overset{\mathrm{def}}{=} \mathrm{Spa}(\mathbb{F}[\![X_0, \ldots, X_{f-1}]\!]).$$

It is enough to prove (i), (ii) and (iii) replacing everywhere $Z^{\mathrm{gen}}_{\mathcal{O}_K} \subseteq \mathrm{Spa}(\mathbb{F}[\![K]\!])$ by $T^{\mathrm{gen}} \subseteq T$ (i.e. completed perfection will not change the arguments in the proof below). Moreover, as the map $T = \mathrm{Spa}(\mathbb{F}[\![X_0, \ldots, X_{f-1}]\!]) \to \mathrm{Spa}(\mathbb{F}[\![X_0^{1/p^\infty}, \ldots, X_{f-1}^{1/p^\infty}]\!])$ is a homeomorphism, it is also enough to prove (iv) with T^{gen} and $T \setminus V(X_0, \ldots, X_{f-1})$.

We first show the analogue (iii). Let $S \overset{\mathrm{def}}{=} \mathbb{F}(\!(X_0)\!)\Big\langle \Big(\frac{X_i}{X_0}\Big)^{\pm 1},\ 1 \leq i \leq f-1 \Big\rangle$ that we endow with the X_0-adic topology (it is a Tate algebra), then the norm in (iii) is the unique multiplicative extension to S of the Gauss norm on the restricted power series $\mathbb{F}(\!(X_0)\!)\Big\langle \Big(\frac{X_i}{X_0}\Big),\ 1 \leq i \leq f-1 \Big\rangle$ (which is well-known to be multiplicative). Note that $S^\circ = \mathbb{F}[\![X_0]\!]\Big\langle \Big(\frac{X_i}{X_0}\Big)^{\pm 1},\ 1 \leq i \leq f-1 \Big\rangle$ is the unit ball for this norm.

Let us prove (the analogues of) (i) and (ii). Looking at continuous valuations, it is clear that the morphism of adic spaces $\mathrm{Spa}(S, S^\circ) \to T$ factors as $\mathrm{Spa}(S, S^\circ) \to T^{\mathrm{gen}} \subseteq T$. In order to prove that the morphism of adic spaces $\mathrm{Spa}(S, S^\circ) \to T^{\mathrm{gen}}$ is an isomorphism, it is enough to prove that it induces an isomorphism $\mathrm{Spa}(S, S^\circ)(W) \xrightarrow{\sim} T^{\mathrm{gen}}(W)$ for any analytic adic space W over $\mathbb{F}$, and it is enough to take $W = \mathrm{Spa}(R, R^+)$ for an arbitrary complete analytic Huber pair (R, R^+) over $\mathbb{F}$ (the case R Tate would be enough). Then this easily follows from the definitions of T and S.

Let us finally prove (the analogue of) (iv). First note that $T \setminus V(X_0, \ldots, X_{f-1})$ is the analytic locus of the adic space T, the only non-analytic point of T being the unique (rank 0) valuation with kernel the maximal ideal of the local ring $\mathbb{F}[\![X_0, \ldots, X_{f-1}]\!]$. Let U be a quasi-compact open subset of $T \setminus V(X_0, \ldots, X_{f-1})$ whose points of rank 1 are the points of T^{gen} of rank 1. For $i \in \{0, \ldots, f-1\}$ consider the open subset U_i of T defined by $|X_j| \leq |X_i| \neq 0$ for all j, or equivalently (by the same argument as for the proof of (i))

$$U_i = \mathrm{Spa}\Big(\mathbb{F}(\!(X_i)\!)\left\langle \frac{X_j}{X_i}, j \neq i \right\rangle, \mathbb{F}[\![X_i]\!]\left\langle \frac{X_j}{X_i}, j \neq i \right\rangle\Big) \subseteq T \setminus V(X_0, \ldots, X_{f-1}).$$

Then $U \cap U_i$ and $T^{\mathrm{gen}} = \cap_j U_j$ are two open subsets of U_i with the same points of rank 1, and thus *a fortiori* with the same points with residue field being a finite extension of $\mathbb{F}(\!(X_i)\!)$. Let $U_i^{\mathrm{rig}} \subseteq U_i$ (resp. $(T^{\mathrm{gen}})^{\mathrm{rig}} \subseteq T^{\mathrm{gen}}$) be the subset of points of U_i (resp. T^{gen}) with residue field being a finite extension of $\mathbb{F}(\!(X_i)\!)$, then U_i^{rig} (resp. $(T^{\mathrm{gen}})^{\mathrm{rig}}$) can be identified with the affinoid rigid analytic space over $\mathbb{F}(\!(X_i)\!)$ corresponding to U_i (resp. T^{gen}) by [Hub96, (1.1.11)(a)], and we have $U \cap U_i^{\mathrm{rig}} = (T^{\mathrm{gen}})^{\mathrm{rig}}$. Note that U, U_i and T^{gen} are quasi-compact (U by assumption, U_i, T^{gen} as they are affinoid). As T is a quasi-separated adic space (being spectral as the adic space associated to a Huber pair, see for instance [Mor, Cor. III.2.4]), the open subset $U \cap U_i$ is still quasi-compact. As U_i^{rig} is quasi-separated, we deduce $U \cap U_i = T^{\mathrm{gen}}$ from $U \cap U_i^{\mathrm{rig}} =$

$(T^{\mathrm{gen}})^{\mathrm{rig}}$ by [Hub96, (1.1.11)] (see also [Sch12, Thm. 2.21]). Since $U = \cup_i(U \cap U_i)$ (as $U \subseteq T \setminus V(X_0, \dots, X_{f-1})$), we finally obtain $U = T^{\mathrm{gen}}$ in T. $\square$

Lemma 2.4.3. *The following statements hold.*

(i) *The open subset* $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ *of* $Z_{\mathcal{O}_K}$ *is stable under the action of* $K^\times$.

(ii) *The open subset* $Z_{\mathrm{LT}}^{\mathrm{gen}}$ *of* Z_{LT} *is stable under the action of* $(K^\times)^f \rtimes \mathfrak{S}_f$.

Proof. (ii) can be easily deduced from (i) and $Z_{\mathrm{LT}}^{\mathrm{gen}} \overset{\mathrm{def}}{=} m^{-1}(Z_{\mathcal{O}_K}^{\mathrm{gen}})$, so we only prove (i).

The fact that $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ is stable under the actions of p and p^{-1} on $Z_{\mathcal{O}_K}$ is a direct computation on $\mathbb{F}[\![X_0^{1/p^\infty}, \dots, X_{f-1}^{1/p^\infty}]\!] \hookrightarrow A_\infty^\circ$ using (41). Let us show that $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ is stable under the action of $\mathcal{O}_K^\times$. It follows from Lemma 2.4.2(iv) that it is sufficient to check that $Z_{\mathcal{O}_K}^{\mathrm{gen}}(C, \mathcal{O}_C)$ is stable under the action of $\mathcal{O}_K^\times$ on $\mathrm{Spa}(\mathbb{F}[\![K]\!])(C, \mathcal{O}_C) = \mathbf{B}^+(C)^{\varphi_q = p^f}$ for C a perfectoid field containing $\mathbb{F}$ (using $Z_{\mathcal{O}_K}(C, C^+) \xrightarrow{\sim} Z_{\mathcal{O}_K}(C, \mathcal{O}_C)$ for any open bounded valuation subring $C^+ \subseteq C$). We will use Newton polygons as in [FF18, §1.6.3]. Recall that $\mathbf{B}^+(C)^{\varphi_q = p^f}$ is the set of converging power series in $\mathbf{B}^+(C)$:

$$F(x_0, \dots, x_{f-1}) = \sum_{n \in \mathbb{Z}} \sum_{i=0}^{f-1} [x_i^{p^{-i-nf}}] p^{i+nf}$$

where $|x_i| < 1$ for all i with $|\cdot|$ a fixed power-multiplicative norm on C (e.g. as in (26)). A point $x \in \mathbf{B}^+(C)^{\varphi_q = p^f}$ is in $Z_{\mathcal{O}_K}^{\mathrm{gen}}(C, \mathcal{O}_C)$ if and only if $0 \neq |x_0| = \cdots = |x_{f-1}| < 1$, equivalently if and only if its Newton polygon (see [FF18, Déf. 1.5.2, Déf. 1.6.18, Déf. 1.6.21 & Ex. 1.6.22]) has slopes $\{cp^n,\ n \in \mathbb{Z}\}$, where $c \overset{\mathrm{def}}{=} (p-1)v(x_0) \in \mathbb{Q}_{>0}$ and $v : C \to \mathbb{Q}_{\geq 0}$ is defined as in [FF18, §1.1] by $|\cdot| = q^{-v(\cdot)}$. As the Newton polygon of x only depends on the norms $|x|_\rho$ for $0 < \rho < 1$ (see [FF18, Ex. 1.6.22] and (27) for $|\cdot|_\rho$), it is enough to show that $|x|_\rho$ does not change if we multiply x by an element of $\mathcal{O}_K^\times$. This follows from the multiplicativity of $|\cdot|_\rho$ (see [FF18, Prop. 1.4.9]) and the fact that $|\cdot|_\rho$ induces the p-adic norm on K. $\square$

From Lemma 2.4.3 we deduce a continuous action of $K^\times$ on the topological $\mathbb{F}$-algebra A_∞. We denote by φ the endomorphism of A_∞ induced by the action of $p \in K^\times$. It is $\mathbb{F}$-linear and satisfies (see (41))

$$\varphi(X_i) = X_{i-1}^p \text{ for } 0 \leq i \leq f-1 \tag{48}$$

(with $X_{-1} = X_{f-1}$ as usual). We also note $\varphi_q \overset{\mathrm{def}}{=} \varphi^f$ (which coincides with $x \mapsto x^q$ on A_∞ when $\mathbb{F}_q = \mathbb{F}$).

We now give an explicit description of $Z_{\mathrm{LT}}^{\mathrm{gen}}$.

Recall first that if a locally profinite group H acts continuously on a perfectoid space X' over $\mathbb{F}$, a morphism $X' \to X$ in $\mathrm{Perf}_{\mathbb{F}}$ (H acting trivially on X) is a pro-étale H-torsor if there exists a pro-étale cover $Y \to X$ in $\mathrm{Perf}_{\mathbb{F}}$ such that there is an isomorphism $X' \times_X Y \cong \underline{H} \times Y$ in $\mathrm{Perf}_{\mathbb{F}}$, where $\underline{H}$ is the sheaf on $\mathrm{Perf}_{\mathbb{F}}$ defined by $\underline{H}(T) \overset{\mathrm{def}}{=} \mathrm{Cont}(|T|, H)$, $|T|$ being the underlying topological space of the perfectoid space T (note that $\underline{H} \times Y$ is perfectoid by [Sch, Lemma 10.13]).

Let $\mathbb{Z}^f/\mathbb{Z}$ be the additive group quotient of $\mathbb{Z}^f$ by the diagonal embedding of $\mathbb{Z}$ into $\mathbb{Z}^f$. If $\underline{n} = (n_0, \dots, n_{f-1}) \in \mathbb{Z}^f/\mathbb{Z}$ we let $U_{\underline{n}}$ be the open affinoid perfectoid subspace of $Z_{\mathrm{LT}} \subseteq \mathrm{Spf}(\mathbb{F}[\![T_{K,0}, \dots, T_{K,f-1}]\!])$ defined by the relations

$$|T_{K,i}|^{p^{n_j}} = |T_{K,j}|^{p^{n_i}} \neq 0, \qquad \forall\ 0 \leq i, j \leq f-1$$

or equivalently $|T_{K,i}| = |T_{K,0}|^{p^{n_i - n_0}}$ for $0 \leq i \leq f-1$. Note that $U_{\underline{n}}$ is well-defined as it only depends on the class of $\underline{n}$ in $\mathbb{Z}^f/\mathbb{Z}$, and that $U_{\underline{n}}$ is disjoint from $U_{\underline{n}'}$ if $\underline{n} \neq \underline{n}'$ in $\mathbb{Z}^f/\mathbb{Z}$. The group $\mathfrak{S}_f$ acts on $\mathbb{Z}^f/\mathbb{Z}$ by permutation, for $\sigma \in \mathfrak{S}_f$ and $\underline{n} \in \mathbb{Z}^f/\mathbb{Z}$ we have

$$\sigma(\underline{n}) \overset{\mathrm{def}}{=} (n_{\sigma^{-1}(i)})_{0 \leq i \leq f-1}$$

and we check that $\sigma(U_{\underline{n}}) = U_{\sigma(\underline{n})}$. Moreover, if $\underline{a} = (a_0, \dots, a_{f-1}) \in (K^\times)^f$, we also easily check that (where v_p is the unique valuation on K with $v_p(p) = 1$):

$$\underline{a}(U_{\underline{n}}) = U_{\underline{n} + f v_p(\underline{a})}.$$

Proposition 2.4.4. *Let $\underline{n}_0 = (0, 1, \dots, f-1)$, we have in Z_{LT}*

$$Z_{\mathrm{LT}}^{\mathrm{gen}} = \coprod_{\sigma \in \mathfrak{S}_f} \coprod_{\underline{m} \in \mathbb{Z}^f/\mathbb{Z}} U_{\sigma(\underline{n}_0) + f\underline{m}}. \tag{49}$$

Moreover for each $U_{\underline{n}}$ in (49) the map $m : Z_{\mathrm{LT}}^{\mathrm{gen}} \to Z_{\mathcal{O}_K}^{\mathrm{gen}}$ restricts to a pro-étale Δ_1-torsor $m|_{U_{\underline{n}}} : U_{\underline{n}} \to Z_{\mathcal{O}_K}^{\mathrm{gen}}$.

Proof. We first check that the two sides of (49) have the same rank 1 points, i.e. the same $(C, \mathcal{O}_C)$-points, where C is a perfectoid field containing $\mathbb{F}$ (recall that $Z_{\mathrm{LT}}(C, C^+) \xrightarrow{\sim} Z_{\mathrm{LT}}(C, \mathcal{O}_C)$ for any open bounded valuation subring $C^+ \subseteq C$). We use Newton polygons and notation as in the proof of Lemma 2.4.3. If $(F(t_0), \dots, F(t_{f-1})) \in (\mathbf{B}^+(C)^{\varphi_q = p})^f$, the element $F(t_i)$ has slopes $\{(q-1)v(t_i)q^n,\ n \in \mathbb{Z}\}$ (see the references in *loc. cit.*) and recall that $(F(t_0), \dots, F(t_{f-1})) \in Z_{\mathrm{LT}}^{\mathrm{gen}}(C, \mathcal{O}_C)$ if and only if $F(t_0) \cdots F(t_{f-1}) \in \mathbf{B}^+(C)^{\varphi_q = p^f}$ lies in $Z_{\mathcal{O}_K}^{\mathrm{gen}}(C, \mathcal{O}_C)$. As the slopes of the Newton polygon of a product ab in $\mathbf{B}^+(C)$ is the union of the slopes of the Newton polygons of a and b (see [FF18, Prop. 1.6.20] for instance), we see that $F(t_0) \cdots F(t_{f-1}) \in Z_{\mathcal{O}_K}^{\mathrm{gen}}(C, \mathcal{O}_C)$ if and only if there exists $c \in \mathbb{Q}_{>0}$ such that $\bigcup_i \{(q-1)v(t_i)q^n,\ n \in \mathbb{Z}\} = \{cp^n,\ n \in \mathbb{Z}\}$ (see the proof of Lemma 2.4.3). Equivalently $F(t_0) \cdots F(t_{f-1}) \in Z_{\mathcal{O}_K}^{\mathrm{gen}}(C, \mathcal{O}_C)$ if and only if there exist $c \in \mathbb{Q}_{>0}$, $\sigma \in \mathfrak{S}_f$ and $m_0, \dots, m_{f-1} \in \mathbb{Z}$ such that $v(t_i) = cp^{\sigma^{-1}(i) + fm_i}$ for $0 \leq i \leq f-1$ if and only if there exist $\sigma \in \mathfrak{S}_f$ and $m_0, \dots, m_{f-1} \in \mathbb{Z}$ such that $v(t_i) = p^{(\sigma^{-1}(i)+fm_i)-(\sigma^{-1}(0)+fm_0)}v(t_0)$ for $0 \leq i \leq f-1$ if and only if $F(t_0) \cdots F(t_{f-1}) \in U_{\sigma(\underline{n}_0)+f\underline{m}}$.

For a point x of the analytic adic space Z_{LT} define $\tilde{x} \in Z_{\mathrm{LT}}$ as its maximal generization, then the corresponding valuation $|\cdot|_{\tilde{x}}$ is of rank 1, i.e. real valued (see for instance [Hub96, Lemma 1.1.10] or [Mor, Cor. II.2.4.8]). Thus one can define continuous maps as in [SW20, proof of Prop. 4.2.6]:

$$\kappa_{i,j} : Z_{\mathrm{LT}} \to]0, +\infty[, \ x \mapsto \kappa_{i,j}(x) \overset{\mathrm{def}}{=} \frac{\log(|T_{K,i}|_{\tilde{x}})}{\log(|T_{K,j}|_{\tilde{x}})}.$$

For $\underline{n} \in \mathbb{Z}^f/\mathbb{Z}$, define the closed subset of Z_{LT}

$$V_{\underline{n}} \overset{\mathrm{def}}{=} \kappa^{-1}(p^{n_0-n_1}, \dots, p^{n_0-n_{f-1}}),$$

where $\kappa = (\kappa_{0,1}, \dots, \kappa_{0,f-1})$. For $x \in U_{\underline{n}}$, we still have $\tilde{x} \in U_{\underline{n}}$ by [Hub96, Lemma 1.1.10(v)] applied to $X \overset{\mathrm{def}}{=} U_{\underline{n}} \hookrightarrow Y \overset{\mathrm{def}}{=} Z_{\mathrm{LT}}$, hence we have an inclusion of topological spaces $U_{\underline{n}} \subseteq V_{\underline{n}}$. Let us prove that the open subspace $Z_{\mathrm{LT}}^{\mathrm{gen}}$ of Z_{LT} is contained in $V \overset{\mathrm{def}}{=} \coprod_{\sigma \in \mathfrak{S}_f} \coprod_{\underline{m} \in \mathbb{Z}^f/\mathbb{Z}} V_{\sigma(\underline{n}_0)+f\underline{m}}$. Let $x \in Z_{\mathrm{LT}}^{\mathrm{gen}}$ of rank 1, then $x \in U_{\underline{n}} \subseteq V_{\underline{n}}$ for some $\underline{n}$ by the first paragraph. As $V_{\underline{n}}$ is closed, we have $\overline{\{x\}} \subseteq V_{\underline{n}}$. Now let $x \in Z_{\mathrm{LT}}^{\mathrm{gen}}$ be any point and $\tilde{x}$ its maximal generization (which is in $Z_{\mathrm{LT}}^{\mathrm{gen}}$ by [Hub96, Lemma 1.1.10(v)] applied to $Z_{\mathrm{LT}}^{\mathrm{gen}} \hookrightarrow Z_{\mathrm{LT}}$), then $\tilde{x}$ is of rank 1 and $x \in \overline{\{\tilde{x}\}}$, which implies $x \in V_{\underline{n}}$ for some $\underline{n}$, i.e. $Z_{\mathrm{LT}}^{\mathrm{gen}} \subseteq V$. As $Z_{\mathrm{LT}}^{\mathrm{gen}}$ is open in Z_{LT}, we have $Z_{\mathrm{LT}}^{\mathrm{gen}} \subseteq \mathring{V} \subseteq V$, where $\mathring{V}$ is the interior of the topological space V in Z_{LT} ($\mathring{V}$ is then open in the perfectoid space Z_{LT}, hence itself a perfectoid space). Let $x \in \mathring{V}$, then $x \in V_{\underline{n}}$ for some $\underline{n}$. But $V_{\underline{n}}$ is open in V as V is the inverse image by κ of a discrete set and $V_{\underline{n}}$ is the inverse image of a single, hence open, element in this discrete set. Hence there exists an open subset U of Z_{LT} such that $V_{\underline{n}} = U \cap V$. As $x \in U \cap \mathring{V}$ which is open in Z_{LT}, we deduce $x \in \mathring{V}_{\underline{n}}$ which proves that $\mathring{V} = \coprod_{\sigma \in \mathfrak{S}_f} \coprod_{\underline{m} \in \mathbb{Z}^f/\mathbb{Z}} \mathring{V}_{\sigma(\underline{n}_0)+f\underline{m}}$. Thus we finally have $Z_{\mathrm{LT}}^{\mathrm{gen}} \subseteq \coprod_{\sigma \in \mathfrak{S}_f} \coprod_{\underline{m} \in \mathbb{Z}^f/\mathbb{Z}} \mathring{V}_{\sigma(\underline{n}_0)+f\underline{m}}$ which implies

$$Z_{\mathrm{LT}}^{\mathrm{gen}} = \coprod_{\sigma \in \mathfrak{S}_f} \coprod_{\underline{m} \in \mathbb{Z}^f/\mathbb{Z}} (Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\sigma(\underline{n}_0)+f\underline{m}}) \tag{50}$$

as open (perfectoid) subspaces of Z_{LT}.

Now we go into group actions. Recall that the group $\Delta \rtimes \mathfrak{S}_f$ acts on Z_{LT}. It is not hard to check that $\Delta \rtimes \mathfrak{S}_f$ stabilizes V, more precisely $\sigma \in \mathfrak{S}_f$ sends $V_{\underline{n}}$ to $V_{\sigma(\underline{n})}$, $(p^{d_0}, \dots, p^{d_{f-1}}) \in \Delta \cap (p^{\mathbb{Z}})^f$ sends $V_{\underline{n}}$ to $V_{\underline{n}+f(d_0,\dots,d_{f-1})}$ and Δ_1 preserves each $V_{\underline{n}}$ (indeed, using that $f(\tilde{x}) = \widetilde{f(x)}$ for any $x \in Z_{\mathrm{LT}}$ and any endomorphism f of Z_{LT} by [Hub96, Lemma 1.1.10(iv)&(v)], it is enough to check this for rank 1 points, i.e. $(C, \mathcal{O}_C)$-points for perfectoid fields C containing $\mathbb{F}$, which is an easy exercise left to the reader). Then by continuity of the action of $\Delta \rtimes \mathfrak{S}_f$ the same holds for the interiors $\mathring{V}_{\underline{n}}$, and thus also for $Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}}$ by Lemma 2.4.3(ii). In particular, the group

$(\Delta \cap (p^{\mathbb{Z}})^f) \rtimes \mathfrak{S}_f$ permutes transitively the perfectoid spaces $Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}}$ for $\underline{n} \in \mathbb{Z}^f/\mathbb{Z}$ of the form $\sigma(\underline{n}_0) + f\underline{m}$ as in (50), and the group Δ_1 preserves each $Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}}$. Thus the associated sheaf $\underline{\Delta_1}$ acts on (the sheaf corresponding to) $Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}}$, and one easily checks that the group Δ_1 moreover acts freely on the $(C, \mathcal{O}_C)$-points of $Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}}$. By the proof of [Wei17, Prop. 4.3.2], $Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}}$ is a pro-étale Δ_1-torsor over $\Delta_1 \backslash (Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}})$, seen as a pro-étale sheaf on $\mathrm{Perf}_{\mathbb{F}}$. Since Δ_1 is a normal subgroup in $\Delta \rtimes \mathfrak{S}_f$, we deduce with (50) that $Z_{\mathrm{LT}}^{\mathrm{gen}}$ is a pro-étale $\Delta \rtimes \mathfrak{S}_f$-torsor over

$$\begin{aligned}
\Delta \rtimes \mathfrak{S}_f \backslash Z_{\mathrm{LT}}^{\mathrm{gen}} &\cong ((\Delta \cap (p^{\mathbb{Z}})^f) \rtimes \mathfrak{S}_f) \backslash (\Delta_1 \backslash Z_{\mathrm{LT}}^{\mathrm{gen}}) \\
&\cong ((\Delta \cap (p^{\mathbb{Z}})^f) \rtimes \mathfrak{S}_f) \backslash \Big(\coprod_{\sigma, \underline{m}} \Delta_1 \backslash (Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\sigma(\underline{n}_0) + f\underline{m}}) \Big) \\
&\cong \Delta_1 \backslash (Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}})
\end{aligned}$$

for each $\underline{n}$ of the form $\sigma(\underline{n}_0) + f\underline{m}$. Now, it follows from (47) (and Lemma 2.4.3) that we have an isomorphism $\Delta \rtimes \mathfrak{S}_f \backslash Z_{\mathrm{LT}}^{\mathrm{gen}} \xrightarrow{\sim} Z_{\mathcal{O}_K}^{\mathrm{gen}}$ of pro-étale sheaves, hence $\Delta_1 \backslash (Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}}) \cong Z_{\mathcal{O}_K}^{\mathrm{gen}}$ for each $\underline{n}$ of the form $\sigma(\underline{n}_0) + f\underline{m}$.

We now finish the proof. As $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ is affinoid perfectoid by Lemma 2.4.2(ii), each $Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}}$ is affinoid perfectoid by [SW20, Prop. 9.3.1], in particular is a quasi-compact open subset of Z_{LT}. The quasi-compact open subspaces $Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}}$ and $U_{\underline{n}}$ of $Z_{\mathrm{LT}} \subseteq \mathrm{Spa}(\mathbb{F}[\![K]\!]) \backslash V(T_{K,0})$ have the same points of rank 1 by the first paragraph of this proof, and we can then argue in a similar way as for the proof of Lemma 2.4.2(iv), applying the results in [Hub96, (1.1.11)] (or [Sch12, Thm. 2.21]) to the affinoid rigid analytic space over $\mathbb{F}(\!(T_{K,0})\!)$ associated to $\mathrm{Spa}(\mathbb{F}[\![T_{K,0}, \dots, T_{K,f-1}]\!]) \setminus V(T_{K,0})$ (recalling that $\mathrm{Spa}(\mathbb{F}[\![T_{K,0}, \dots, T_{K,f-1}]\!]) \to \mathrm{Spa}(\mathbb{F}[\![T_{K,0}^{1/p^\infty}, \dots, T_{K,f-1}^{1/p^\infty}]\!]) = Z_{\mathrm{LT}}$ is a homeomorphism). In particular, we obtain $U_{\underline{n}} = Z_{\mathrm{LT}}^{\mathrm{gen}} \cap \mathring{V}_{\underline{n}}$ for all $\underline{n} \in \mathbb{Z}^f/\mathbb{Z}$ of the form $\sigma(\underline{n}_0) + f\underline{m}$, which finishes the proof. $\square$

As a consequence of the above proof and of [Sch, Lemma 10.13], we also obtain:

Corollary 2.4.5. *The map* $m : Z_{\mathrm{LT}}^{\mathrm{gen}} \to Z_{\mathcal{O}_K}^{\mathrm{gen}}$ *is a pro-étale* $\Delta \rtimes \mathfrak{S}_f$*-torsor, in particular is a pro-étale cover.*

Remark 2.4.6. Note that $Z_{\mathrm{LT}}^{\mathrm{gen}}$ is not affinoid (contrary to $Z_{\mathcal{O}_K}^{\mathrm{gen}}$) as it is not quasi-compact.

Let us denote by $A'_\infty \overset{\mathrm{def}}{=} \mathcal{O}(U_{\underline{n}_0})$ the ring of global sections on $U_{\underline{n}_0}$. The following result on A'_∞ can be proved exactly as Lemma 2.4.2, and we leave the details to the reader.

Lemma 2.4.7. *The following statements hold.*

(i) *The ring* A'_∞ *is the perfectoid* $\mathbb{F}$*-algebra*

$$\mathbb{F}(\!(T_{K,0}^{1/p^\infty})\!)\Big\langle\left(\frac{T_{K,i}}{T_{K,0}^{p^i}}\right)^{\pm 1/p^\infty},\ 1\leq i\leq f-1\Big\rangle.$$

(ii) *We have* $U_{\underline{n}_0}=\mathrm{Spa}(A'_\infty,(A'_\infty)^\circ)$.

(iii) *There exists a multiplicative norm* $|\cdot|$ *on* A'_∞ *such that* $|T_{K,0}|=p^{-1}$ *inducing the topology of* A'_∞.

(iv) *Any quasi-compact open subset of* Z_{LT} *whose points of rank* 1 *are exactly the points of* $U_{\underline{n}_0}$ *of rank* 1 *is necessarily* $U_{\underline{n}_0}$ *itself.*

2.5 Equivariant vector bundles on $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ and $Z_{\mathrm{LT}}^{\mathrm{gen}}$

We show that continuous $(K^\times)^f\rtimes\mathfrak{S}_f$-equivariant vector bundles on $Z_{\mathrm{LT}}^{\mathrm{gen}}$ and étale $(\varphi,\mathcal{O}_K^\times)$-modules over A_∞ are the same thing.

Recall first that if X is an adic space with a left action of a group H, an H-equivariant vector bundle on X is a locally finite free $\mathcal{O}_X$-module $\mathcal{V}$ with a collection of $\mathcal{O}_X$-linear isomorphisms $(c_h:h^*\mathcal{V}\xrightarrow{\sim}\mathcal{V})_{h\in H}$ satisfying the relation $c_{h_2h_1}=c_{h_1}\circ h_1^*(c_{h_2})$ for all $h_1,h_2\in H$. This induces a *right* action of H on $\Gamma(X,\mathcal{V})$ given by

$$c_h^*:\Gamma(X,\mathcal{V})=\Gamma(X,h^*\mathcal{V})\xrightarrow{\sim}\Gamma(X,\mathcal{V}).$$

Now assume that X is perfectoid space (the only case we will use) and that H is a locally profinite topological group acting continuously on X. Let $\mathcal{V}$ be a vector bundle on X, for an open affinoid perfectoid subspace $U=\mathrm{Spa}(A,A^+)\subseteq X$, the finite projective A-module $\mathcal{V}(U)$ is endowed with the Banach topology given by the quotient topology of any surjection of A-modules $A^{\oplus d}\twoheadrightarrow\mathcal{V}(U)$. If $U\subseteq X$ is any open subspace, we endow $\mathcal{V}(U)\cong\varprojlim_{U'\subseteq U}\mathcal{V}(U')$ with the projective limit topology, where U' ranges over open affinoid subspaces of U, and we define $H_U\overset{\mathrm{def}}{=}\{h\in H,\ h(U)=U\}$, which is a closed subgroup of H by continuity of the action of H on X. We then define a *continuous* H-equivariant vector bundle on X as an H-equivariant vector bundle $\mathcal{V}$ on X such that for any open subspace $U\subseteq X$ the natural map $H_U\times\mathcal{V}(U)\to\mathcal{V}(U)$, $(h,s)\mapsto c_h^*(s)$ is continuous (for the product topology on the left).

By Lemma 2.4.2(i),(ii) and [KL15, Thm. 2.7.7], the functor of global sections induces an equivalence of categories from the category of vector bundles on $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ to the category of finite projective A_∞-modules. This equivalence is rank preserving and compatible with tensor products. As a finite projective A_∞-module is in fact always free (see [DH21, Thm. 2.19]) and as the action of $K^\times$ on $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ is continuous, we see that the functor of global sections induces an equivalence of categories which is rank preserving and compatible with tensor products from the category of continuous $K^\times$-equivariant vector bundles on $Z^{\mathrm{gen}}_{\mathcal{O}_K}$ to the category of étale $(\varphi, \mathcal{O}_K^\times)$-modules over A_∞, where φ on A_∞ is given by (48).

As $Z^{\mathrm{gen}}_{\mathrm{LT}}$ is perfectoid and as the fibered category of vector bundles on $\mathrm{Perf}_{\mathbb{F}}$ is a v-stack by [SW20, Lemma 17.1.8], we easily deduce from Corollary 2.4.5 an equivalence of categories between the category of (continuous) $\Delta \rtimes \mathfrak{S}_f$-equivariant vector bundles on $Z^{\mathrm{gen}}_{\mathrm{LT}}$ and the category of vector bundles on $Z^{\mathrm{gen}}_{\mathcal{O}_K}$ (the continuity condition is then automatic in that case, as $\Delta \rtimes \mathfrak{S}_f$ acts continuously on $Z^{\mathrm{gen}}_{\mathrm{LT}}$), hence also between the category of continuous $(K^\times)^f \rtimes \mathfrak{S}_f$-equivariant vector bundles on $Z^{\mathrm{gen}}_{\mathrm{LT}}$ and the category of continuous $K^\times$-equivariant vector bundles on $Z^{\mathrm{gen}}_{\mathcal{O}_K}$. In both cases this equivalence is given by the two functors $\mathcal{V} \mapsto (m_*\mathcal{V})^{\Delta \rtimes \mathfrak{S}_f}$ and $\mathcal{W} \mapsto m^*\mathcal{W}$, where $m : Z^{\mathrm{gen}}_{\mathrm{LT}} \to Z^{\mathrm{gen}}_{\mathcal{O}_K}$. If $\mathcal{V}$ is $(K^\times)^f \rtimes \mathfrak{S}_f$-equivariant, the $K^\times$-equivariant structure on $(m_*\mathcal{V})^{\Delta \rtimes \mathfrak{S}_f}$ can be made explicit as follows. For $a \in K^\times$ and any $i \in \{0, \ldots, f-1\}$ we have an isomorphism using the notation in (46)

$$a^* m_* \mathcal{V} \cong (a^{-1})_* m_* \mathcal{V} \cong (a^{-1}m)_* \mathcal{V} \cong (m j_i(a)^{-1})_* \mathcal{V} \cong m_*(j_i(a)^{-1})_* \mathcal{V} \cong m_* j_i(a)^* \mathcal{V}$$

(where the first isomorphism is $\mathrm{id} \in \mathrm{Hom}(m_*\mathcal{V}, m_*\mathcal{V}) = \mathrm{Hom}((a^{-1})^* a^* m_*\mathcal{V}, m_*\mathcal{V}) \cong \mathrm{Hom}(a^* m_*\mathcal{V}, (a^{-1})_* m_*\mathcal{V})$, the third comes from (46) and the last is analogous to the first). We then obtain an isomorphism of sheaves for $a \in K^\times$ and any $i \in \{0, \ldots, f-1\}$:

$$m_*(c_{j_i(a)}) : a^* m_* \mathcal{V} \cong m_* j_i(a)^* \mathcal{V} \xrightarrow{\sim} m_* \mathcal{V}$$

which preserves the subsheaf $(m_*\mathcal{V})^{\Delta \rtimes \mathfrak{S}_f}$ (as $\Delta \rtimes \mathfrak{S}_f$ is a normal subgroup of $(K^\times)^f \rtimes \mathfrak{S}_f$) and induces an isomorphism $m_*(c_{j_i(a)}) : a^*(m_*\mathcal{V})^{\Delta \rtimes \mathfrak{S}_f} \xrightarrow{\sim} (m_*\mathcal{V})^{\Delta \rtimes \mathfrak{S}_f}$ which does not depend on i.

We deduce from Proposition 2.4.4 that we have an isomorphism of perfectoid $\mathbb{F}$-algebras $A_\infty \xrightarrow{\sim} (A'_\infty)^{\Delta_1}$, and as above using [SW20, Lemma 17.1.8] that there is also an equivalence of categories between the category of Δ_1-equivariant vector bundles on $U_{\underline{n}_0}$ and the category of vector bundles on $Z^{\mathrm{gen}}_{\mathcal{O}_K}$. Using again [KL15, Thm. 2.7.7] and [DH21, Thm. 2.19], we deduce:

Theorem 2.5.1. *The functor $D_{A_\infty} \mapsto A'_\infty \otimes_{A_\infty} D_{A_\infty}$ induces an exact equivalence of categories which is rank preserving and compatible with tensor products from the category of finite free A_∞-modules to the category of finite free A'_∞-module with a semi-linear action of Δ_1. A quasi-inverse is given by $D_{A'_\infty} \mapsto D_{A'_\infty}^{\Delta_1}$.*

Let $\delta \in \mathfrak{S}_f$ be the cyclic permutation $i \mapsto i+1$ (with $f-1 \mapsto 0$). If $\sigma \in \mathfrak{S}_f$, let $p_\sigma \overset{\mathrm{def}}{=} (1, \ldots, p, \ldots, 1) \in (K^\times)^f$ with p at the $\sigma(0)$-th entry. From the discussion before Proposition 2.4.4 we get

$$(p_\sigma \circ \sigma)(U_{\underline{n}_0}) = p_\sigma(U_{\sigma(\underline{n}_0)}) = U_{\sigma\delta(\underline{n}_0)}.$$

In particular, $p_{\delta^{-1}} \circ \delta^{-1} : U_{\underline{n}_0} \xrightarrow{\sim} U_{\underline{n}_0}$ and we define an $\mathbb{F}$-linear continuous automorphism φ of $A'_\infty = \mathcal{O}(U_{\underline{n}_0})$ by

$$\varphi \overset{\text{def}}{=} (p_{\delta^{-1}} \circ \delta^{-1})^* = (\delta^{-1})^* \circ p_{\delta^{-1}}^*. \tag{51}$$

Using (30) and since $\delta^{-1}(0) = f-1$ this automorphism is easily checked to satisfy

$$\varphi(T_{K,i}) = T_{K,i+1} \text{ for } i \neq f-1 \text{ and } \varphi(T_{K,f-1}) = T_{K,0}^q. \tag{52}$$

In particular, φ^f on A'_∞ is $\mathbb{F}$-linear and such that $\varphi^f(T_{K,i}) = T_{K,i}^q$ for all i. Moreover if $\underline{a} \in (\mathcal{O}_K^\times)^f$, we have $\varphi \circ \underline{a} = \delta(\underline{a}) \circ \varphi$, where $\delta(\underline{a}) = (a_{i-1})_{0 \leq i \leq f-1}$ (with $a_{-1} = a_{f-1}$), in particular φ^f commutes with $(\mathcal{O}_K^\times)^f$. As $m : Z_{\mathrm{LT}}^{\mathrm{gen}} \to Z_{\mathcal{O}_K}^{\mathrm{gen}}$ is $K^\times$-equivariant and $\mathfrak{S}_f$-equivariant, the action of $K^\times$ on $Z_{\mathrm{LT}}^{\mathrm{gen}}$ being through j_i for any $0 \leq i \leq f-1$ and the action of $\mathfrak{S}_f$ on $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ being trivial, the isomorphism $A_\infty \xrightarrow{\sim} (A'_\infty)^{\Delta_1}$ commutes with the actions of φ and $\mathcal{O}_K^\times$ on both sides (see (48) for φ on A_∞).

The following result sums up the previous discussion and gives a more explicit way to compute the $(\varphi, \mathcal{O}_K^\times)$-module over A_∞ associated to a continuous $(K^\times)^f \rtimes \mathfrak{S}_f$-equivariant vector bundle on $Z_{\mathrm{LT}}^{\mathrm{gen}}$.

Corollary 2.5.2. *There is an equivalence of categories between the category of continuous $(K^\times)^f \rtimes \mathfrak{S}_f$-equivariant vector bundles on $Z_{\mathrm{LT}}^{\mathrm{gen}}$ and the category of étale $(\varphi, \mathcal{O}_K^\times)$-modules over A_∞. If $\mathcal{V}$ is a continuous $(K^\times)^f \rtimes \mathfrak{S}_f$-equivariant vector bundle on $Z_{\mathrm{LT}}^{\mathrm{gen}}$, its associated A_∞-module is $\Gamma(Z_{\mathcal{O}_K}^{\mathrm{gen}}, (m_*\mathcal{V})^{\Delta \rtimes \mathfrak{S}_f})$ which is isomorphic to $\Gamma(U_{\underline{n}_0}, \mathcal{V}|_{U_{\underline{n}_0}})^{\Delta_1}$. The action of $a \in \mathcal{O}_K^\times$ on $\Gamma(Z_{\mathcal{O}_K}^{\mathrm{gen}}, (m_*\mathcal{V})^{\Delta \rtimes \mathfrak{S}_f})$ is induced by the action of $(a, 1 \ldots, 1) = j_0(a)$ on $\Gamma(U_{\underline{n}_0}, \mathcal{V}|_{U_{\underline{n}_0}})$ and the action of φ on $\Gamma(Z_{\mathcal{O}_K}^{\mathrm{gen}}, (m_*\mathcal{V})^{\Delta \rtimes \mathfrak{S}_f})$ is induced by*

$$\begin{aligned}(\delta^{-1})^* \circ p_{\delta^{-1}}^* : \Gamma(U_{\underline{n}_0}, \mathcal{V}|_{U_{\underline{n}_0}}) &= \Gamma(U_{\delta^{-1}(\underline{n}_0)}, (p_{\delta^{-1}}^*\mathcal{V})|_{U_{\delta^{-1}(\underline{n}_0)}}) \\ &\cong \Gamma(U_{\underline{n}_0}, ((p_{\delta^{-1}} \circ \delta^{-1})^*\mathcal{V})|_{U_{\underline{n}_0}}) \xrightarrow{\sim} \Gamma(U_{\underline{n}_0}, \mathcal{V}|_{U_{\underline{n}_0}}).\end{aligned}$$

2.6 The $(\varphi_q, \mathcal{O}_K^\times)$-module over A of an arbitrary Galois representation

To an arbitrary $\overline{\rho}$ we functorially associate an étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D_A^{(i)}(\overline{\rho})$ over A for $i \in \{0, \ldots, f-1\}$.

Let $\overline{\rho}$ be a continuous representation of $\mathrm{Gal}(\overline{K}/K)$ on a finite-dimensional $\mathbb{F}$-vector space and $D_{K,\sigma_0}(\overline{\rho})$ its Lubin–Tate $(\varphi_q, \mathcal{O}_K^\times)$-module (see §2.1). The $(\varphi_q, \mathcal{O}_K^\times)$-module $\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!)} D_{K,\sigma_0}(\overline{\rho})$ is the space of global sections of a continuous $K^\times$-equivariant vector bundle $\mathcal{V}_{\overline{\rho}}$ on $\mathrm{Spa}(\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!), \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}]\!])$. For $i \in \{0, \ldots, f-1\}$ we define

$$\mathcal{V}_{\overline{\rho}}^{(i)} \overset{\text{def}}{=} \mathcal{O}_{Z_{\mathrm{LT}}} \otimes_{\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!), \iota_i} \mathcal{V}_{\overline{\rho}} \cong \mathcal{O}_{Z_{\mathrm{LT}}} \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!), \iota_i} D_{K,\sigma_0}(\overline{\rho}),$$

where ι_i denotes the $\mathbb{F}$-linear embedding $\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!) \hookrightarrow \mathcal{O}_{Z_{\rm LT}}$ corresponding to $T_{K,\sigma_0} \mapsto T_{K,i}$. Each $\mathcal{V}_{\overline{\rho}}^{(i)}$ is a Δ-equivariant vector bundle on $Z_{\rm LT}$ with $(a_0, \dots, a_{f-1}) \in \Delta \subseteq (K^\times)^f$ acting on $\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!)} D_{K,\sigma_0}(\overline{\rho})$ via a_i. In particular, $\mathcal{V}_{\overline{\rho}}^{(i)}|_{U_{\underline{n}_0}}$ is a Δ_1-equivariant vector bundle on $U_{\underline{n}_0}$ and $\Gamma(U_{\underline{n}_0}, \mathcal{V}_{\overline{\rho}}^{(i)}|_{U_{\underline{n}_0}}) = A'_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!),\iota_i} D_{K,\sigma_0}(\overline{\rho})$. We define for $i \in \{0, \dots, f-1\}$

$$D_{A_\infty}^{(i)}(\overline{\rho}) \overset{\rm def}{=} \Gamma(U_{\underline{n}_0}, \mathcal{V}_{\overline{\rho}}^{(i)}|_{U_{\underline{n}_0}})^{\Delta_1} = (A'_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!),\iota_i} D_{K,\sigma_0}(\overline{\rho}))^{\Delta_1} \tag{53}$$

which is a finite free A_∞-module of rank $\dim_{\mathbb{F}} \overline{\rho}$ by Theorem 2.5.1.

The endomorphism $x \otimes v \mapsto \varphi^f(x) \otimes \varphi_q(v)$ on $A'_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!),\iota_i} D_{K,\sigma_0}(\overline{\rho})$ (where $x \in A'_\infty$, $v \in D_{K,\sigma_0}(\overline{\rho})$ and see below (52) for φ^f on A'_∞) commutes with the action of Δ_1 and induces a φ_q-semi-linear automorphism of $D_{A_\infty}^{(i)}(\overline{\rho})$, which is thus naturally a φ_q-module (see below (48) for φ_q on A_∞). The action of $\mathcal{O}_K^\times$ on $A'_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!),\iota_i} D_{K,\sigma_0}(\overline{\rho})$ defined by $a(x \otimes v) \overset{\rm def}{=} j_i(a)(x) \otimes a(v)$ induces a continuous semi-linear action of $\mathcal{O}_K^\times$ on $D_{A_\infty}^{(i)}(\overline{\rho})$ (with respect to the action of $\mathcal{O}_K^\times$ on A_∞) which commutes with φ_q. In particular, $D_{A_\infty}^{(i)}(\overline{\rho})$ is naturally an étale $(\varphi_q, \mathcal{O}_K^\times)$-module over A_∞. Note that the functor $\overline{\rho} \mapsto D_{A_\infty}^{(i)}(\overline{\rho})$ from continuous representations of $\mathrm{Gal}(\overline{K}/K)$ on finite-dimensional $\mathbb{F}$-vector spaces to étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A_∞ is exact and $\mathbb{F}$-linear. We also have isomorphisms of functors for $0 \le i \le f-1$ (where $x \in A'_\infty$, $v \in D_{K,\sigma_0}(-)$):

$$\phi_i : \begin{array}{ccc} D_{A_\infty}^{(i)}(-) & \xrightarrow{\sim} & D_{A_\infty}^{(i+1)}(-) \\ x \otimes v & \longmapsto & \begin{cases} \varphi(x) \otimes v & \text{if } i < f-1 \\ \varphi(x) \otimes \varphi_q(v) & \text{if } i = f-1. \end{cases} \end{array} \tag{54}$$

We now show that étale φ_q-modules over A_∞, and hence étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A_∞, canonically descend to the ring A of §2.2. First we need an easy lemma.

Lemma 2.6.1. *The ring A of §2.2 can be identified with the ring of global sections of the structure sheaf $\mathcal{O}$ on the rational open subset of the adic space* $\mathrm{Spa}(\mathbb{F}[\![\mathcal{O}_K]\!])$ *defined by the relations*

$$|Y_{\sigma_0}| = \cdots = |Y_{\sigma_{f-1}}| \neq 0,$$

where the variables $Y_{\sigma_i} \in \mathbb{F}[\![\mathcal{O}_K]\!]$ are in (18).

Proof. Recall that A is by definition the completed localization $(\mathbb{F}[\![\mathcal{O}_K]\!]_{(Y_{\sigma_0} \cdots Y_{\sigma_{f-1}})})^\wedge = (\mathbb{F}[\![Y_{\sigma_0}, \dots, Y_{\sigma_{f-1}}]\!]_{(Y_{\sigma_0} \cdots Y_{\sigma_{f-1}})})^\wedge$, where the completion is for the $(Y_{\sigma_0}, \dots, Y_{\sigma_{f-1}})$-adic topology. Then using (for instance) [BHH+21, Rk. 3.1.1.3(iii)] one easily checks that

$$\begin{aligned} A &\cong \mathbb{F}[(Y_{\sigma_1}/Y_{\sigma_0})^{\pm 1}, \dots, (Y_{\sigma_{f-1}}/Y_{\sigma_0})^{\pm 1}][\![Y_{\sigma_0}]\!][1/Y_{\sigma_0}] \\ &\cong \mathbb{F}(\!(Y_{\sigma_0})\!)\langle (Y_{\sigma_1}/Y_{\sigma_0})^{\pm 1}, \dots, (Y_{\sigma_{f-1}}/Y_{\sigma_0})^{\pm 1} \rangle, \end{aligned}$$

where $\langle\ \rangle$ means, as usual, the corresponding Tate algebra with respect to the non-archimedean local field $\mathbb{F}(\!(Y_{\sigma_0})\!)$. This is exactly the Tate algebra of the statement. $\square$

Note that the open subset of Lemma 2.6.1 is stable under the endomorphisms deduced from the actions of p and $\mathcal{O}_K^\times$ on $\mathcal{O}_K$ by multiplication, in particular the $\mathbb{F}$-linear endomorphism φ on A sending Y_{σ_i} to $Y_{\sigma_{i-1}}$ (see §2.2) is the one deduced from the action of p.

Remark 2.6.2. It follows from Lemma 2.6.1 and [Lüt77, Satz 3, p. 131] (we thank Ofer Gabber for pointing out this reference to us) that any projective A-module of finite type is actually free.

Let $X_0, \dots, X_{f-1}$ be as at the end of §2.3, we have $\mathbb{F}[\![\mathcal{O}_K]\!] = \mathbb{F}[\![X_0, \dots, X_{f-1}]\!] = \mathbb{F}[\![Y_{\sigma_0}, \dots, Y_{\sigma_{f-1}}]\!]$, and from the second equality in (41) we deduce for $i \in \{0, \dots, f-1\}$

$$X_i = Y_{\sigma_i} + (\text{degree} \geq 2 \text{ in the variables } Y_{\sigma_j}). \tag{55}$$

This easily implies an isomorphism of completed localized rings

$$\Big(\mathbb{F}[\![X_0, \dots, X_{f-1}]\!]_{(X_0 \cdots X_{f-1})}\Big)^\wedge \cong \Big(\mathbb{F}[\![Y_{\sigma_0}, \dots, Y_{\sigma_{f-1}}]\!]_{(Y_{\sigma_0} \cdots Y_{\sigma_{f-1}})}\Big)^\wedge = A,$$

where the completion on the left-hand side is for the $(X_0 \cdots X_{f-1})$-adic topology. In other words we can use the variables X_i defined in §2.3 instead of the variables Y_{σ_i} to define the ring A. In particular, the perfectoid Tate algebra A_∞ in Lemma 2.4.2 is the completion of the perfection of A and the action of φ and $\mathcal{O}_K^\times$ on A_∞ are compatible with the corresponding actions on A.

We will use the following result:

Proposition 2.6.3. *Let X be a normal reduced and irreducible scheme over $\mathbb{F}_q$. There is an equivalence of categories between the category of locally constant étale sheaves of $\mathbb{F}_q$-vector spaces on X and the category of pairs $(\mathcal{V}, \phi)$, where $\mathcal{V}$ is a vector bundle on X and ϕ is an isomorphism $\varphi_q^*\mathcal{V} \xrightarrow{\sim} \mathcal{V}$ (where $\varphi_q(-) = (-)^q$). This equivalence is given by the two inverse functors $L \mapsto (\nu_*(L \otimes_{\mathbb{F}_q} \mathcal{O}_X), \mathrm{Id} \otimes \varphi)$ and $(\mathcal{V}, \phi) \mapsto (\nu^*\mathcal{V})^{\phi=1}$, where $\nu : X_{\text{ét}} \to X_{\text{zar}}$ is the restriction from the étale topos to the Zariski one. In particular, this equivalence is rank-preserving and compatible with tensor products.*

Proof. This is [Kat73, Prop. 4.1.1]. ☐

Let $A^{1/p^\infty} = \varinjlim_{x \mapsto x^p} A = \cup_{n \geq 0} \mathbb{F}[\![X_0^{1/p^n}, \dots, X_{f-1}^{1/p^n}]\!]$ be the perfection of the ring A. It follows from [AGV$^+$73, VIII Thm 1.1] that the étale topos of $\mathrm{Spec}(A)$ and the étale topos of $\mathrm{Spec}(A^{1/p^\infty})$ are equivalent. It follows from [SW20, Thm. 7.4.8] (more precisely the discussion following *loc. cit.*) that there is an equivalence of categories between the category of finite étale A^{1/p^∞}-algebras and the category of finite étale A_∞-algebras. Combining these two equivalences, it follows that the pullback functor induces an equivalence of categories from the category of locally constant étale sheaves of finite-dimensional $\mathbb{F}_q$-vector spaces on $\mathrm{Spec}(A)$ to the category of locally constant étale sheaves of finite-dimensional $\mathbb{F}_q$-vector spaces on $\mathrm{Spec}(A_\infty)$.

Theorem 2.6.4. *The functor $D_A \mapsto A_\infty \otimes_A D_A$ induces an equivalence of categories from the category of étale φ_q-modules over A to the category of étale φ_q-modules over A_∞.*

Proof. If $\mathbb{F} = \mathbb{F}_q$, this directly follows from Proposition 2.6.3, [KL15, Thm. 2.7.7] and the above discussion. In general, let A_q be the ring of §2.2, i.e. A_q is A but with $\mathbb{F}_q$ instead of $\mathbb{F}$, and $A_{q,\infty}$ its perfectoid version, then one can see an étale φ_q-module over A (resp. A_∞) as an étale φ_q-module over A_q (resp. $A_{q,\infty}$) together with the structure of an $\mathbb{F}$-vector space compatible with the action of $\mathbb{F}_q$ (seen in $\mathbb{F}$ via σ_0). We only prove essential surjectivity (full faithfulness being easy). Let D_{A_∞} be an étale φ_q-module over A_∞. By the equivalence of categories for $\mathbb{F} = \mathbb{F}_q$, there is an étale φ_q-module D_A over A_q, which is also an $\mathbb{F} \otimes_{\mathbb{F}_q} A_q = A$-module, such that

$$A_{q,\infty} \otimes_{A_q} D_A \cong (\mathbb{F} \otimes_{\mathbb{F}_q} A_{q,\infty}) \otimes_{\mathbb{F} \otimes_{\mathbb{F}_q} A_q} D_A = A_\infty \otimes_A D_A \xrightarrow{\sim} D_{A_\infty}.$$

As A_∞ is faithfully flat over A, we deduce that D_A is finite projective over A by faithfully flat descent of projectivity, hence is free by Remark 2.6.2, hence is an étale φ_q-module over A. □

Remark 2.6.5. We thank Laurent Berger for a discussion around Theorem 2.6.4, and Laurent Fargues for suggesting to use Proposition 2.6.3 in its proof (our first elementary proof was based on the operator ψ). Note that one can characterize the subspace $A^{1/p^\infty} \otimes_A D_A$ of an étale φ_q-module D_{A_∞} over A_∞ as the A-submodule of D_{A_∞} of elements $d \in D_{A_\infty}$ such that $\sum_{n \geq 0} A\varphi_q^n(d)$ is a finite type A-module.

Corollary 2.6.6. *The functor $D_A \mapsto A_\infty \otimes_A D_A$ induces an equivalence between the category of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules (resp. étale $(\varphi, \mathcal{O}_K^\times)$-modules) over A and the category of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules (resp. étale $(\varphi, \mathcal{O}_K^\times)$-modules) over A_∞. This equivalence is rank-preserving and compatible with tensor products.*

Proof. Let D_{A_∞} be an étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A_∞. Any $a \in \mathcal{O}_K^\times$ gives an isomorphism of étale φ_q-modules $\mathrm{id} \otimes a : a^* D_{A_\infty} \xrightarrow{\sim} D_{A_\infty}$ which canonically descends to an isomorphism of étale φ_q-modules $a^* D_A \xrightarrow{\sim} D_A$ by Theorem 2.6.4. Now let D_{A_∞} be an étale $(\varphi, \mathcal{O}_K^\times)$-modules over A_∞, then replacing φ by $\varphi_q \overset{\text{def}}{=} \varphi^f$, it is also an étale $(\varphi_q, \mathcal{O}_K^\times)$-module over A_∞. Let $\varphi^* D_{A_\infty} \overset{\text{def}}{=} A_\infty \otimes_{\varphi, A_\infty} D_{A_\infty}$, then $\mathrm{id} \otimes \varphi$ induces an isomorphism of étale φ_q-modules $\varphi^* D_{A_\infty} \xrightarrow{\sim} D_{A_\infty}$ which canonically descends to an isomorphism $\varphi^* D_A \xrightarrow{\sim} D_A$ by Theorem 2.6.4, giving the endomorphism φ on D_A. The action of $\mathcal{O}_K^\times$ canonically descends too by the first case of the proof and commutes with φ (using Theorem 2.6.4 again). The rest of the statement is easy and left to the reader. □

From (53), (54) and Corollary 2.6.6, we deduce:

Corollary 2.6.7. *For $i \in \{0, \dots, f-1\}$ there is a covariant exact $\mathbb{F}$-linear functor $\overline{\rho} \mapsto D_A^{(i)}(\overline{\rho})$ compatible with tensor products from continuous representations of* $\mathrm{Gal}(\overline{K}/K)$ *on finite-dimensional $\mathbb{F}$-vector spaces to étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A, these functors being related by functorial A-linear isomorphisms $\phi_i : A \otimes_{\varphi, A} D_A^{(i)}(\overline{\rho}) \xrightarrow{\sim} D_A^{(i+1)}(\overline{\rho})$ which commute with $(\varphi_q, \mathcal{O}_K^\times)$ and are such that $\phi_{f-1} \circ \phi_{f-2} \circ \cdots \circ \phi_0 : A \otimes_{\varphi^f, A} D_A^{(0)}(\overline{\rho}) \xrightarrow{\sim} D_A^{(0)}(\overline{\rho})$ is* $\mathrm{id} \otimes \varphi_q$.

Remark 2.6.8. One can check that $D_A^{(0)}(\overline{\rho}) \times D_A^{(f-1)}(\overline{\rho}) \times D_A^{(f-2)}(\overline{\rho}) \times \cdots \times D_A^{(1)}(\overline{\rho})$ can be given the structure of an étale $(\varphi, \mathcal{O}_K^\times)$-module over $\mathbb{F} \otimes_{\mathbb{F}_p} A_q$ in the sense of §2.2.

2.7 The $(\varphi, \mathcal{O}_K^\times)$-module over A associated to a Galois representation

To an arbitrary $\overline{\rho}$ we associate an étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A^{\otimes}(\overline{\rho})$ (which will be particularly important when $\dim_{\mathbb{F}} \overline{\rho} = 2$).

Keep the notation of §2.6 and let $\mathcal{V}_{\overline{\rho}}^{\boxtimes f} \overset{\mathrm{def}}{=} \bigotimes_{i=0}^{f-1} \mathrm{pr}_i^* \mathcal{V}_{\overline{\rho}}$ be the f-th "exterior tensor product" of $\mathcal{V}_{\overline{\rho}}$ on $Z_{\mathrm{LT}} = (\mathrm{Spa}(\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!), \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}]\!]))^f$, where

$$\mathrm{pr}_i : (\mathrm{Spa}(\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!), \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}]\!]))^f \twoheadrightarrow \mathrm{Spa}(\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!), \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}]\!])$$

is the i-th projection (so $\mathrm{pr}_i^* \mathcal{V}_{\overline{\rho}}$ is the sheaf $\mathcal{V}_{\overline{\rho}}^{(i)}$ of §2.6). As $\mathcal{V}_{\overline{\rho}}$ is a continuous $K^\times$-equivariant vector bundle, $\mathcal{V}_{\overline{\rho}}^{\boxtimes f}$ is naturally a continuous $(K^\times)^f$-equivariant vector bundle. We promote it to a (continuous) $(K^\times)^f \rtimes \mathfrak{S}_f$-equivariant vector bundle using the commutativity of the tensor product (where $\sigma \in \mathfrak{S}_f$):

$$c_\sigma : \sigma^* \mathcal{V}_{\overline{\rho}}^{\boxtimes f} = \sigma^* \Big(\bigotimes_{i=0}^{f-1} \mathrm{pr}_i^* \mathcal{V}_{\overline{\rho}} \Big) \cong \bigotimes_{i=0}^{f-1} \sigma^* \mathrm{pr}_i^* \mathcal{V}_{\overline{\rho}} \cong \bigotimes_{i=0}^{f-1} \mathrm{pr}_{\sigma^{-1}(i)}^* \mathcal{V}_{\overline{\rho}} \xrightarrow[\sim]{\sigma} \bigotimes_{i=0}^{f-1} \mathrm{pr}_i^* \mathcal{V}_{\overline{\rho}} = \mathcal{V}_{\overline{\rho}}^{\boxtimes f}.$$

We define $D_{A_\infty}^{\otimes}(\overline{\rho})$ as the A_∞-module with a continuous semi-linear action of $K^\times$ obtained as the global sections of the continuous $K^\times$-equivariant vector bundle on $Z_{\mathcal{O}_K}^{\mathrm{gen}}$ corresponding to $\mathcal{V}^{\boxtimes f}|_{Z_{\mathrm{LT}}^{\mathrm{gen}}}$, more concretely (see §2.5):

$$D_{A_\infty}^{\otimes}(\overline{\rho}) \overset{\mathrm{def}}{=} \Gamma\Big(Z_{\mathcal{O}_K}^{\mathrm{gen}}, (m_*(\mathcal{V}_{\overline{\rho}}^{\boxtimes f}|_{Z_{\mathrm{LT}}^{\mathrm{gen}}}))^{\Delta \rtimes \mathfrak{S}_f}\Big) = \Gamma(Z_{\mathrm{LT}}^{\mathrm{gen}}, \mathcal{V}_{\overline{\rho}}^{\boxtimes f})^{\Delta \rtimes \mathfrak{S}_f}.$$

This is an étale $(\varphi, \mathcal{O}_K^\times)$-module over A_∞ (recall φ is bijective).

Using Corollary 2.5.2 and §2.6, we can give a more explicit description of $D_{A_\infty}^{\otimes}(\overline{\rho})$. Note that we have:

$$D_{A_\infty}^{\otimes}(\overline{\rho}) = \Gamma(U_{\underline{n}_0}, \mathcal{V}_{\overline{\rho}}^{\boxtimes f}|_{U_{\underline{n}_0}})^{\Delta_1}$$

and that the vector bundle $\mathcal{V}_{\overline{\rho}}^{\boxtimes f}$ is isomorphic to the tensor product

$$\mathcal{V}_{\overline{\rho}}^{(0)} \otimes_{\mathcal{O}_{Z_{\mathrm{LT}}}} \cdots \otimes_{\mathcal{O}_{Z_{\mathrm{LT}}}} \mathcal{V}_{\overline{\rho}}^{(f-1)}.$$

As the equivalence with vector bundles on $Z_{\mathcal{O}_K}^{\mathrm{gen}}$, i.e. finite free A_∞-modules, is compatible with tensor products (see §2.5), we deduce an isomorphism of A_∞-modules

$$D_{A_\infty}^{\otimes}(\overline{\rho}) \cong (A'_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!),\iota_0} D_{K,\sigma_0}(\overline{\rho}))^{\Delta_1} \otimes_{A_\infty} \cdots \otimes_{A_\infty} (A'_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!),\iota_{f-1}} D_{K,\sigma_0}(\overline{\rho}))^{\Delta_1}.$$

Lemma 2.7.1. *There is a functorial isomorphism of étale* $(\varphi, \mathcal{O}_K^\times)$*-modules over* A_∞

$$D_{A_\infty}^{\otimes}(\overline{\rho}) \xrightarrow{\sim} \bigotimes_{i=0}^{f-1} D_{A_\infty}^{(i)}(\overline{\rho}),$$

where the automorphism φ *on the right-hand side is given by* (*see (54) for* ϕ_i)

$$\varphi(v_0 \otimes \cdots \otimes v_{f-1}) = \phi_{f-1}(v_{f-1}) \otimes \phi_0(v_0) \otimes \cdots \otimes \phi_{f-2}(v_{f-2})$$

(*and the action of* $\mathcal{O}_K^\times$ *is as defined in §2.6 on each factor* $D_{A_\infty}^{(i)}(\overline{\rho})$).

Proof. Recall that $\delta \in \mathfrak{S}_f$ sends i to $i+1$. Let $\alpha_i : (\delta^{-1})^*\mathcal{V}_{\overline{\rho}}^{(i-1)} \xrightarrow{\sim} \mathcal{V}_{\overline{\rho}}^{(i)}$ be the tautological isomorphism deduced from the identifications

$$(\delta^{-1})^*\mathcal{V}_{\overline{\rho}}^{(i-1)} = (\delta^{-1})^*\mathrm{pr}_{i-1}^*\mathcal{V}_{\overline{\rho}} \cong (\mathrm{pr}_{i-1} \circ \delta^{-1})^*\mathcal{V}_{\overline{\rho}} = \mathrm{pr}_i^*\mathcal{V}_{\overline{\rho}} = \mathcal{V}_{\overline{\rho}}^{(i)}.$$

Recall that $p_{\delta^{-1}} \in (K^\times)^f$ is defined in §2.5 and let $\beta_i : p_{\delta^{-1}}^*\mathcal{V}_{\overline{\rho}}^{(i)} \xrightarrow{\sim} \mathcal{V}_{\overline{\rho}}^{(i)}$ be the isomorphism of sheaves on Z_{LT} defined by ($f \in \mathcal{O}_{Z_{\mathrm{LT}}}$, $v \in \mathcal{V}_{\overline{\rho}}$ and compare with (52)):

$$f \otimes v \longmapsto \begin{cases} f(p_{\delta^{-1}}(-)) \otimes v & \text{if } i \neq f-1 \\ f(p_{\delta^{-1}}(-)) \otimes \varphi_q(v) & \text{if } i = f-1. \end{cases}$$

We obtain isomorphisms of sheaves on Z_{LT} for $i \in \{0, \ldots, f-1\}$

$$\alpha_i \circ (\delta^{-1})^*(\beta_{i-1}) : \varphi^*\mathcal{V}_{\overline{\rho}}^{(i-1)} \overset{(51)}{\cong} ((\delta^{-1})^* \circ p_{\delta^{-1}}^*)\mathcal{V}_{\overline{\rho}}^{(i-1)} \xrightarrow{\sim} (\delta^{-1})^*\mathcal{V}_{\overline{\rho}}^{(i-1)} \xrightarrow{\sim} \mathcal{V}_{\overline{\rho}}^{(i)}. \tag{56}$$

The isomorphism $c_{p_{\delta^{-1}} \circ \delta^{-1}} : \varphi^*\mathcal{V}_{\overline{\rho}}^{\boxtimes f} \xrightarrow{\sim} \mathcal{V}_{\overline{\rho}}^{\boxtimes f}$ (with the notation as at the beginning of §2.5) is easily checked to decompose as a tensor product

$$\bigotimes_{i=0}^{f-1} \Big(\alpha_i \circ (\delta^{-1})^*(\beta_i)\Big) : \varphi^*\mathcal{V}_{\overline{\rho}}^{\boxtimes f} \cong \bigotimes_{i=0}^{f-1} \varphi^*\mathcal{V}_{\overline{\rho}}^{(i-1)} \xrightarrow{\sim} \bigotimes_{i=0}^{f-1} \mathcal{V}_{\overline{\rho}}^{(i)}.$$

Taking global sections on $U_{\underline{n}_0}$ and Δ_1-invariants, we obtain the desired formula. □

From Lemma 2.7.1 and Corollary 2.6.6 we deduce $D^{\otimes}_{A_\infty}(\overline{\rho}) \cong A_\infty \otimes_A D^{\otimes}_A(\overline{\rho})$ for a unique étale $(\varphi, \mathcal{O}_K^\times)$-module $D^{\otimes}_A(\overline{\rho})$ over A such that

$$D^{\otimes}_A(\overline{\rho}) \cong \bigotimes_{i=0}^{f-1} D^{(i)}_A(\overline{\rho}) \tag{57}$$

with the same φ and action of $\mathcal{O}_K^\times$ on the right-hand side as in Lemma 2.7.1.

Remark 2.7.2. Note that, for $0 \leq i < f-1$, the isomorphism ϕ_i in (54) is induced by the natural A_∞-linear isomorphism $\varphi^* D^{(i)}_{A_\infty}(-) \cong D^{(i+1)}_{A_\infty}(-)$, whereas ϕ_{f-1} coincides with the A_∞-linear isomorphism

$$\varphi^* D^{(f-1)}_{A_\infty}(-) \cong \varphi^*((\varphi^{f-1})^* D^{(0)}_{A_\infty}(-)) = \varphi_q^* D^{(0)}_{A_\infty}(-) \longrightarrow D^{(0)}_{A_\infty}(-)$$

induced by the φ_q-semi-linear automorphism φ_q of $D^{(0)}_{A_\infty}(-)$. Therefore the isomorphism class of the $(\varphi, \mathcal{O}_K^\times)$-module $D^{\otimes}_{A_\infty}(\overline{\rho})$ (equivalently of $D^{\otimes}_A(\overline{\rho})$) is completely characterized by the isomorphism class of the $(\varphi_q, \mathcal{O}_K^\times)$-module $D^{(0)}_{A_\infty}(\overline{\rho})$ (equivalently of $D^{(0)}_A(\overline{\rho})$).

2.8 Relation to classical (φ, Γ)-modules

We show that the étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D^{(0)}_A(\overline{\rho})$ is related in a simple way to the (usual) étale $(\varphi_q, \mathbb{Z}_p^\times)$-module $D_{\sigma_0}(\overline{\rho})$ of §2.1 and derive some consequences.

As in [BHH⁺21, §3.1.3], let $\mathrm{tr} : A \twoheadrightarrow \mathbb{F}(\!(T)\!)$ be the ring surjection induced by the trace $\mathrm{tr} : \mathbb{F}[\![\mathcal{O}_K]\!] \to \mathbb{F}[\![\mathbb{Z}_p]\!] \cong \mathbb{F}[\![T]\!]$. Since the map tr commutes with φ (hence φ_q) and the action of $\mathbb{Z}_p^\times$, we deduce that $\mathbb{F}(\!(T)\!) \otimes_A D^{(0)}_A(\overline{\rho})$ is an étale $(\varphi_q, \mathbb{Z}_p^\times)$-module.

Proposition 2.8.1. *We have a functorial isomorphism of $(\varphi_q, \mathbb{Z}_p^\times)$-modules*

$$\mathbb{F}(\!(T)\!) \otimes_A D^{(0)}_A(\overline{\rho}) \cong D_{\sigma_0}(\overline{\rho}),$$

where $D_{\sigma_0}(\overline{\rho})$ is as in §2.1.

Proof. The trace $\mathrm{tr} : \mathbb{F}[\![K]\!] \to \mathbb{F}[\![\mathbb{Q}_p]\!] \cong \mathbb{F}[\![T^{p^{-\infty}}]\!]$ induces a ring surjection $\mathrm{tr} : A_\infty \twoheadrightarrow \mathbb{F}(\!(T^{p^{-\infty}})\!)$ commuting (in an obvious way) with $\mathrm{tr} : A \twoheadrightarrow \mathbb{F}(\!(T)\!)$. Using Corollary 2.6.6 it is enough to prove $\mathbb{F}(\!(T^{p^{-\infty}})\!) \otimes_{A_\infty} D^{(0)}_{A_\infty}(\overline{\rho}) \cong \mathbb{F}(\!(T^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T)\!)} D_{\sigma_0}(\overline{\rho})$.

For any perfectoid $\mathbb{F}$-algebra R we have a commutative diagram

$$\begin{tikzcd}
\mathbf{B}^+(R)^{\varphi_q = p} \arrow[r, hook] \arrow[d] & (\mathbf{B}^+(R)^{\varphi_q = p})^f \arrow[d, "m_R"] \\
\mathbf{B}^+(R)^{\varphi = p} \arrow[r, hook] & \mathbf{B}^+(R)^{\varphi_q = p^f}
\end{tikzcd} \tag{58}$$

where the top horizontal injection sends $x \in \mathbf{B}^+(R)^{\varphi_q=p}$ to $(x, \varphi(x), \dots, \varphi^{f-1}(x)) \in (\mathbf{B}^+(R)^{\varphi_q=p})^f$, the left vertical map sends $x \in \mathbf{B}^+(R)^{\varphi_q=p}$ to $x\varphi(x)\cdots\varphi^{f-1}(x) \in \mathbf{B}^+(R)^{\varphi=p}$ and where the bottom horizontal injection is the canonical injection. Note that the left vertical map commutes with the action of K, where K acts on $\mathbf{B}^+(R)^{\varphi=p}$ via $\mathrm{Norm}_{K/\mathbb{Q}_p} : K \to \mathbb{Q}_p$. As at the beginning of §2.4, using Remark 2.3.3 and [SW20, Prop. 8.2.8(2)], we deduce from (58) a corresponding commutative diagram of perfectoid spaces over $\mathbb{F}$:

$$\begin{tikzcd}
(\widetilde{G}_{\mathrm{LT}} \times_{\mathrm{Spf}(\mathbb{F}_q)} \mathrm{Spf}(\mathbb{F}) \setminus \{0\})^{\mathrm{ad}} \arrow[r, hook] \arrow[d] & Z_{\mathrm{LT}} \arrow[d] \\
Z_{\mathbb{Z}_p} \arrow[r] & Z_{\mathcal{O}_K}
\end{tikzcd} \tag{59}$$

where the top horizontal map is $r \mapsto (r, r^p, \dots, r^{p^{f-1}})$ on the coordinates and the right vertical map is the map m in (45). From the discussion above, the map $Z_{\mathbb{Z}_p} \to Z_{\mathcal{O}_K}$ commutes with the action of $K^\times$, where $K^\times$ acts on $Z_{\mathbb{Z}_p}$ via $\mathrm{Norm}_{K/\mathbb{Q}_p}$. Also, it follows from the end of §2.3 (see in particular (37), (38), (39) and (40)) that the bottom horizontal map is induced by the morphism $\mathbb{F}[\![K]\!] \to \mathbb{F}[\![\mathbb{Q}_p]\!]$ deduced from the trace $\mathrm{Tr}_{K/\mathbb{Q}_p} : K \to \mathbb{Q}_p$. Hence we deduce from (59) a commutative diagram of perfectoid rings over $\mathbb{F}$:

$$\begin{tikzcd}
A'_\infty \arrow[r, two heads] & \mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!) \\
A_\infty \arrow[u, hook] \arrow[r, two heads, "\mathrm{tr}"] & \mathbb{F}(\!(T^{p^{-\infty}})\!) \arrow[u, hook]
\end{tikzcd} \tag{60}$$

where the top horizontal surjection sends $T_{K,i}^{p^{-n}}$ to $T_{K,\sigma_0}^{p^{i-n}}$ for $i \in \{0, \dots, f-1\}$. Let us prove that the right vertical injection coincides with the one above Corollary 2.1.3. As it commutes with $\mathcal{O}_K^\times$ (acting on $\mathbb{F}(\!(T^{p^{-\infty}})\!)$ via the norm $\mathcal{O}_K^\times \twoheadrightarrow \mathbb{Z}_p^\times$) we deduce from Theorem 2.1.1 (and (6)) that it induces an injection of perfectoid fields $\mathbb{F}(\!(T^{p^{-\infty}})\!) \hookrightarrow \mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!)^{\mathrm{Gal}(K_\infty/K(\sqrt[p^\infty]{1}))} \cong \mathbb{F}(\!(T^{p^{-\infty}})\!)$. But since this injection commutes with the action of $\mathbb{Z}_p^\times$, one easily checks that it must be an isomorphism (any continuous $\mathbb{F}$-algebra homomorphism $\mathbb{F}(\!(\mathbb{Q}_p)\!) \to \mathbb{F}(\!(\mathbb{Q}_p)\!)$ commuting with the action of $\mathbb{Z}_p^\times$ sends $[1] \in \mathbb{F}(\!(\mathbb{Q}_p)\!)$ to $[\lambda] \in \mathbb{F}(\!(\mathbb{Q}_p)\!)$ for some $\lambda \in \mathbb{Q}_p^\times$).

Now let $\overline{\rho}$ be a continuous representation of $\mathrm{Gal}(\overline{K}/K)$ on a finite-dimensional $\mathbb{F}$-vector space, using the isomorphism $A'_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!),\iota_0} D_{K,\sigma_0}(\overline{\rho}) \cong A'_\infty \otimes_{A_\infty} D_{A_\infty}^{(0)}(\overline{\rho})$ from Theorem 2.5.1 we deduce from (60):

$$\begin{aligned}
\mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!)} D_{K,\sigma_0}(\overline{\rho}) &\cong \mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!) \otimes_{A'_\infty} \Big(A'_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!),\iota_0} D_{K,\sigma_0}(\overline{\rho})\Big) \\
&\cong \mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!) \otimes_{A'_\infty} \Big(A'_\infty \otimes_{A_\infty} D_{A_\infty}^{(0)}(\overline{\rho})\Big) \\
&\cong \mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T^{p^{-\infty}})\!)} \Big(\mathbb{F}(\!(T^{p^{-\infty}})\!) \otimes_{A_\infty} D_{A_\infty}^{(0)}(\overline{\rho})\Big).
\end{aligned}$$

By Corollary 2.1.3 we also have

$$\mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0})\!)} D_{K,\sigma_0}(\overline{\rho}) \cong \mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T^{p^{-\infty}})\!)} \Big(\mathbb{F}(\!(T^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T)\!)} D_{\sigma_0}(\overline{\rho})\Big).$$

Since the action of $\mathrm{Gal}(K_\infty/K(\sqrt[p^\infty]{1})) \cong \mathrm{Gal}(\mathbb{F}(\!(T_{K,\sigma_0}^{p^{-\infty}})\!)/\mathbb{F}(\!(T^{p^{-\infty}})\!))$ is trivial on both $\mathbb{F}(\!(T^{p^{-\infty}})\!)\otimes_{A_\infty} D_{A_\infty}^{(0)}(\overline{\rho})$ and $\mathbb{F}(\!(T^{p^{-\infty}})\!)\otimes_{\mathbb{F}(\!(T)\!)} D_{\sigma_0}(\overline{\rho})$, we deduce $\mathbb{F}(\!(T^{p^{-\infty}})\!)\otimes_{A_\infty} D_{A_\infty}^{(0)}(\overline{\rho}) \cong \mathbb{F}(\!(T^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T)\!)} D_{\sigma_0}(\overline{\rho})$ by Galois descent. All the above isomorphisms are functorial in $\overline{\rho}$. ☐

We can also consider the tensor product $\mathbb{F}(\!(T)\!) \otimes_A D_A^{\otimes}(\overline{\rho})$ for $\mathrm{tr} : A \twoheadrightarrow \mathbb{F}(\!(T)\!)$. It is obviously an étale $(\varphi, \mathbb{Z}_p^\times)$-module.

Corollary 2.8.2. *The* $(\varphi, \mathbb{Z}_p^\times)$*-module* $\mathbb{F}(\!(T)\!) \otimes_A D_A^{\otimes}(\overline{\rho})$ *is the* $(\varphi, \mathbb{Z}_p^\times)$*-module of the tensor induction* $\mathrm{ind}_K^{\otimes \mathbb{Q}_p}\overline{\rho}$.

Proof. This easily follows from (57), Proposition 2.8.1, Corollary 2.6.7 and the "tensor product version" of [Bre11, Lemma 3.6] (which we leave to the reader). ☐

Proposition 2.8.1 also enables to prove the following full faithfulness statement.

Corollary 2.8.3. *For* $i \in \{0, \dots, f-1\}$ *the functor* $\overline{\rho} \mapsto D_A^{(i)}(\overline{\rho})$ *from continuous representations of* $\mathrm{Gal}(\overline{K}/K)$ *on finite-dimensional* $\mathbb{F}$*-vector spaces to étale* $(\varphi_q, \mathcal{O}_K^\times)$*-modules over* A *is fully faithful.*

Proof. By Corollary 2.6.7 it is enough to prove the statement for $i = 0$. We have morphisms:

$$\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)}(\overline{\rho}, \overline{\rho}') \longrightarrow \mathrm{Hom}_{(\varphi_q, \mathcal{O}_K^\times)}(D_A^{(0)}(\overline{\rho}), D_A^{(0)}(\overline{\rho}')) \longrightarrow \mathrm{Hom}_{(\varphi_q, \mathbb{Z}_p^\times)}(D_{\sigma_0}(\overline{\rho}), D_{\sigma_0}(\overline{\rho}')) \tag{61}$$

where we use Proposition 2.8.1 for the second. By the theory of $(\varphi_q, \mathbb{Z}_p^\times)$-modules (see e.g. §2.1), we know that the composition of the two morphisms is bijective. Hence the first morphism is injective. Let us prove that the second morphism is also injective. Let $f : D_A^{(0)}(\overline{\rho}) \to D_A^{(0)}(\overline{\rho}')$ mapping to 0 i.e. $f(D_A^{(0)}(\overline{\rho})) \subseteq \mathfrak{p} D_A^{(0)}(\overline{\rho}')$, where $\mathfrak{p} \overset{\mathrm{def}}{=} \mathrm{Ker}(\mathrm{tr} : A \twoheadrightarrow \mathbb{F}(\!(T)\!))$ (a maximal ideal of the noetherian domain A). Using the fact that $D_A^{(0)}(\overline{\rho})$ is étale and that f commutes with φ_q, we derive $f(D_A^{(0)}(\overline{\rho})) \subseteq \varphi_q^n(\mathfrak{p}) D_A^{(0)}(\overline{\rho}')$ for any $n \geq 0$. For those n such that $x \mapsto x^{q^n}$ is $\mathbb{F}$-linear, the map φ_q^n on A is just $x \mapsto x^{q^n}$, hence $\varphi_q^n(\mathfrak{p}) \subseteq \mathfrak{p}^{q^n}$ for those n, and thus $f(D_A^{(0)}(\overline{\rho})) \subseteq (\cap_{m \geq 0} \mathfrak{p}^m) D_A^{(0)}(\overline{\rho}') = 0$. As the first and last $\mathbb{F}$-vector spaces in (61) have the same dimension, we finally have

$$\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)}(\overline{\rho}, \overline{\rho}') \xrightarrow{\sim} \mathrm{Hom}_{(\varphi_q, \mathcal{O}_K^\times)}(D_A^{(0)}(\overline{\rho}), D_A^{(0)}(\overline{\rho}')) \xrightarrow{\sim} \mathrm{Hom}_{(\varphi_q, \mathbb{Z}_p^\times)}(D_{\sigma_0}(\overline{\rho}), D_{\sigma_0}(\overline{\rho}'))$$

whence the result. ☐

Remark 2.8.4.

(i) We do not expect the functor $\overline{\rho} \mapsto D_A^{(i)}(\overline{\rho})$ to be essentially surjective (for any i). It is probably an interesting question to characterize its essential image.

(ii) It is *not true* that the functor $\overline{\rho} \mapsto D_A^{\otimes}(\overline{\rho})$ is fully faithful, as in general the isomorphism class of the $(\varphi, \mathcal{O}_K^\times)$-module $D_A^{\otimes}(\overline{\rho})$ does not determine the one of the Galois representation $\overline{\rho}$. For instance, one can check by an explicit computation using Theorem 2.9.5 below that, if $f = 2$ and $\overline{\rho} \cong (\mathrm{ind}\, \omega_4^h) \otimes \mathrm{unr}(\lambda)$ is irreducible, $D_A^{\otimes}(\overline{\rho})$ only sees λ^4, i.e. does not distinguish $\overline{\rho}$ and $(\mathrm{ind}\, \omega_4^h) \otimes \mathrm{unr}(\lambda')$, where $\lambda'^2 = -\lambda^2$. However, one can also check (again using Theorem 2.9.5) that, at least when $\overline{\rho}$ is 2-dimensional and semi-simple, $D_A^{\otimes}(\overline{\rho})$ determines $\overline{\rho}$ if $\overline{\rho}$ is split or if $\det(\overline{\rho})(p) = 1$.

2.9 An explicit computation in the semi-simple case

When $\overline{\rho}$ is semi-simple we show that the explicit étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D_{A,\sigma_0}(\overline{\rho})$ defined in §2.2 is isomorphic to the $(\varphi_q, \mathcal{O}_K^\times)$-module $D_A^{(0)}(\overline{\rho})$ defined in §2.6.

It follows from (55) that for all $a \in \mathcal{O}_K^\times$ and $0 \leq i \leq f-1$, we have (using as usual $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$) $a(X_i) = \overline{a}^{p^i} X_i$ modulo terms of degree ≥ 2. Therefore we have the following result:

Lemma 2.9.1. *For $0 \leq i \leq f-1$, we have $a(X_i) \in \overline{a}^{p^i} X_i (1 + A^{\circ\circ})$.*

We define

$$f_{a,0}^X \overset{\mathrm{def}}{=} \frac{\overline{a} X_0}{a(X_0)} \in 1 + F_{-1}A = 1 + A^{\circ\circ} \subseteq 1 + A_\infty^{\circ\circ}$$

(note that by (55) $f_{a,0}^X$ in fact coincides with f_{a,σ_0} in (22) up to a factor in $1 + F_{-2}A$).

Lemma 2.9.2. *There exists $u \in \mathcal{O}(U_{\underline{n}_0})^{(1+p\mathcal{O}_K)^f}$ such that*

$$u^{q-1} = \frac{X_{f-1}^p}{X_0} \overset{(48)}{=} \frac{\varphi(X_0)}{X_0} \in A \subseteq A_\infty = \mathcal{O}(U_{\underline{n}_0})^{\Delta_1} \subseteq \mathcal{O}(U_{\underline{n}_0})^{(1+p\mathcal{O}_K)^f}.$$

Moreover we have

$$\begin{cases} \forall\ \underline{a} = (a_0, \ldots, a_{f-1}) \in \Delta_1 & \underline{a}(u) = \overline{a_0} u \\ \forall\ a \in \mathcal{O}_K^\times & (a, 1, \ldots, 1)(u) = \overline{a} \left(\frac{f_{a,0}^X}{\varphi(f_{a,0}^X)} \right)^{\frac{1}{q-1}} u \end{cases}$$

noting that $\left(\frac{f_{a,0}^X}{\varphi(f_{a,0}^X)} \right)^{\frac{1}{q-1}}$ is well-defined in $1 + F_{-1}A \subseteq 1 + A_\infty^{\circ\circ}$ since $\frac{f_{a,0}^X}{\varphi(f_{a,0}^X)} \in 1 + F_{-1}A$.

Proof. Let $|\cdot|$ be a multiplicative norm on $A'_\infty = \mathcal{O}(U_{\underline{n}_0})$ such that $|T_{K,i}| = |T_{K,0}^{p^i}| = p^{-p^i}$ for $0 \le i \le f-1$ whose existence comes from Lemma 2.4.7(iii)&(i). Let $|\cdot|_1$ be the associated norm on $\mathbf{B}^+(A'_\infty)$ defined in Remark 2.3.4. As $|\cdot|$ is multiplicative, the same proof as in [FF18, Prop. 1.4.9] shows that $|\cdot|_1$ is multiplicative.

By definition of the map $m_{A'_\infty}$ in (44), we have the relation in $\mathbf{B}^+(A'_\infty)$

$$\prod_{i=0}^{f-1}\left(\sum_{n\in\mathbb{Z}}[T_{K,i}^{q^{-n}}]p^n\right) = \sum_{n\in\mathbb{Z}}\sum_{i=0}^{f-1}[X_i^{p^{-nf-i}}]p^{nf+i} = F(X_0,\dots,X_{f-1}). \tag{62}$$

For $c \in \mathbb{R}_{>0}$ let $\mathfrak{p}_c$ be the ideal of $\mathbf{B}^+(A'_\infty)$

$$\mathfrak{p}_c \overset{\text{def}}{=} \{x \in \mathbf{B}^+(A'_\infty),\ |x|_1 < p^{-c}\} \subseteq \mathbf{B}^+(A'_\infty)$$

(note that it is an ideal as $|\cdot|_1$ is multiplicative and with values in $[0,1] \subseteq \mathbb{R}_{\ge 0}$). Let $c = 1 + p + \cdots + \cdots + p^{f-1}$. As $|T_{K,i}^{q^n}| = p^{-p^iq^n} \le p^{-q^n} < p^{-c}$ for $n \ge 1$, we have $\left|\sum_{n\le -1}[T_{K,i}^{q^{-n}}]p^n\right|_1 \le p^{-q^n} < p^{-c}$, see Remark 2.3.4, hence we obtain from (62)

$$\prod_{i=0}^{f-1}\left(\sum_{n\ge 0}[T_{K,i}^{q^{-n}}]p^n\right) - F(X_0,\dots,X_{f-1}) \in \mathfrak{p}_c$$

and we deduce from Lemma 2.9.3 below applied to the element

$$x \overset{\text{def}}{=} \prod_{i=0}^{f-1}\Big(\sum_{n\ge 0}[T_{K,i}^{q^{-n}}]p^n\Big) - F(X_0,\dots,X_{f-1}) = \sum_{n\in\mathbb{Z}}[x_n]p^n \in \mathbf{B}^+(A'_\infty)$$

that we have

$$\sum_{n\ge 0}[x_n]p^n = \prod_{i=0}^{f-1}\left(\sum_{n\ge 0}[T_{K,i}^{q^{-n}}]p^n\right) - \sum_{n\ge 0}\sum_{i=0}^{f-1}[X_i^{p^{-nf-i}}]p^{nf+i} \in \mathfrak{p}_c.$$

Note that the left-hand side is now in $W((A'_\infty)^\circ)$. As a consequence, we have

$$|x_0| = |T_{K,0}\cdots T_{K,f-1} - X_0| < p^{-c}$$

so that we can write in $(A'_\infty)^\circ$

$$X_0 = T_{K,0}\cdots T_{K,f-1}(1+w_0) \tag{63}$$

with $|w_0| < p^{-c+(1+p+\cdots+p^{f-1})} = 1$, i.e. $w_0 \in (A'_\infty)^{\circ\circ}$. Applying the automorphism φ^{-1} of A'_∞ to (63) and since φ^{-1} respects $(A'_\infty)^\circ$ and $(A'_\infty)^{\circ\circ}$ (as it is continuous) we obtain in $(A'_\infty)^\circ$

$$X_1^{p^{-1}} = T_{K,0}T_{K,1}\cdots T_{K,f-1}^{q^{-1}}(1+w_1)$$

with $w_1 \overset{\text{def}}{=} \varphi^{-1}(w_0) \in (A'_\infty)^{\circ\circ}$. We deduce the equality $X_0 X_1^{-p^{-1}} \in T_{K,f-1}^{1-q^{-1}}(1+(A'_\infty)^{\circ\circ})$ which gives (raising everything to the q)

$$X_0^{p^f} X_1^{-p^{f-1}} \in T_{K,f-1}^{q-1}(1+(A'_\infty)^{\circ\circ}). \tag{64}$$

If we apply $\varphi^{-(f-1)}$ to (64) we obtain

$$X_{f-1}^p X_0^{-1} \in T_{K,0}^{q-1}(1+(A'_\infty)^{\circ\circ}).$$

Using that $x \mapsto x^{q-1}$ is bijective on $1+(A'_\infty)^{\circ\circ}$, we see that there exists a unique $u \in T_{K,0}(1+(A'_\infty)^{\circ\circ})$ such that $u^{q-1} = X_{f-1}^p X_0^{-1}$.

As Δ_1 acts trivially on A_∞, we have $\underline{a}(u)^{q-1} = u^{q-1}$ for all $\underline{a} \in \Delta_1$. Therefore there exists a character χ of Δ_1 with values in $\mathbb{F}_q^\times \overset{\sigma_0}{\hookrightarrow} \mathbb{F}$ such that

$$\forall\ \underline{a} \in \Delta_1, \quad \underline{a}(u) = \chi(\underline{a})u.$$

Writing $u = T_{K,0}(1+w)$ with $w \in (A'_\infty)^{\circ\circ}$, this gives $\overline{a_0} T_{K,0} f_a^{\mathrm{LT}}(T_{K,0})^{-1}(1+a(w)) = \chi(\underline{a})u$, where $f_a^{\mathrm{LT}}(T_{K,0}) = \overline{a} T_{K,0}(a_{\mathrm{LT}}(T_{K,0}))^{-1} \in 1+(A'_\infty)^{\circ\circ}$. As $u \in T_{K,0}(1+(A'_\infty)^{\circ\circ})$ this implies

$$\chi(\underline{a})\overline{a_0}^{-1} \in (1+(A'_\infty)^{\circ\circ}) \cap \mathbb{F}_q^\times = \{1\}$$

which proves $\chi(\underline{a}) = \overline{a_0}$.

For the last relation, we have

$$\begin{aligned}((a,1,\dots,1)(u))^{q-1} = (a,1,\dots,1)(u^{q-1}) = a(X_{f-1}^p X_0^{-1}) = \frac{a(\varphi(X_0))}{a(X_0)} = \frac{\varphi(a(X_0))}{a(X_0)}\\ = \frac{f_{a,0}^X}{\varphi(f_{a,0}^X)}\frac{\varphi(X_0)}{X_0} = \frac{f_{a,0}^X}{\varphi(f_{a,0}^X)}u^{q-1} = \left(\left(\frac{f_{a,0}^X}{\varphi(f_{a,0}^X)}\right)^{\frac{1}{q-1}} u\right)^{q-1}\end{aligned}$$

so that as above there is a character $\chi : \mathcal{O}_K^\times \to \mathbb{F}_q^\times \subseteq \mathbb{F}^\times$ such that $(a,1,\dots,1)(u) = \chi(a)\left(\frac{f_{a,0}^X}{\varphi(f_{a,0}^X)}\right)^{\frac{1}{q-1}} u$. But $u \in T_{K,0}(1+(A'_\infty)^{\circ\circ})$ implies $(a,1\dots,1)(u) \in \overline{a}T_{K,0}(1+(A'_\infty)^{\circ\circ})$ so that $\chi(a) = \overline{a}$ since $\left(\frac{f_{a,0}^X}{\varphi(f_{a,0}^X)}\right)^{\frac{1}{q-1}} \in 1+A_\infty^{\circ\circ}$. This finishes the proof. $\square$

Lemma 2.9.3. *Let R be a perfectoid $\mathbb{F}$-algebra and let $(x_n)_{\in\mathbb{Z}}$ a family of elements of $R^{\circ\circ}$ such that the series $\sum_{n\in\mathbb{Z}}[x_n]p^n$ converges to an element x in $\mathbf{B}^+(R)$. Assume that $|x|_1 < c$ for some $c \in [0,1[$. Then we have*

$$\left|\sum_{n<0}[x_n]p^n\right|_1 < c.$$

Proof. Recall that $|x|_1 = \lim\limits_{\substack{\rho<1 \\ \rho\to 1}} |x|_\rho$ (see the reference in Remark 2.3.4). Therefore we can find $0 < \rho < 1$ such that $|x|_\rho < c$. This implies $\sup_{n\in\mathbb{Z}}\{|x_n|\rho^n\} < c$ and thus for $n \le -1$, $|x_n|\rho^n < c$ which implies $|x_n| < c\rho < c$. The claim then follows from Remark 2.3.4 applied to $c\rho$. $\square$

Remark 2.9.4. An examination of the proof of Lemma 2.9.2 shows that u actually lies in $\mathbb{F}_q(\!(T_{K,0}^{1/p^\infty})\!)\langle\big(\frac{T_{K,i}}{T_{K,0}^{p^i}}\big)^{\pm 1/p^\infty}\rangle \subseteq A'_\infty = \mathcal{O}(U_{\underline{n}_0})$ (using $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ as usual). This implies in particular $\varphi^f(u) = u^q$ in A'_∞.

Let $v \overset{\text{def}}{=} uT_{K,0}^{-1}$. We have $v \in 1+\mathcal{O}(U_{\underline{n}_0})^{\circ\circ}$, so that, for each $r \in \mathbb{Z}_{(p)}$ ($= \mathbb{Z}$ localized at the prime ideal (p)), the element

$$v^r \overset{\text{def}}{=} \sum_{n\ge 0} \binom{r}{n}(v-1)^n \in 1 + \mathcal{O}(U_{\underline{n}_0})^{\circ\circ}$$

exists. By Lemma 2.9.2, we have

$$\forall\ \underline{a} \in \Delta_1\ \forall\ r \in \mathbb{Z}_{(p)}, \quad \underline{a}(v^r) = f_a^{\mathrm{LT}}(T_{K,0})^r v^r. \tag{65}$$

Now, let $\overline{\rho}$ be an absolutely irreducible continuous representation of $\mathrm{Gal}(\overline{K}/K)$ on a finite-dimensional $\mathbb{F}$-vector space and choose a basis $(e_0, \dots, e_{d-1})$ of the $\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$-module $D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ as in (15). We consider the associated étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D_{A,\sigma_0}(\overline{\rho}) = A\otimes_{\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ defined in Lemma 2.2.1, where A has the structure of $\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$-algebra given by (20).

Theorem 2.9.5. *Assume that $\overline{\rho}$ is absolutely irreducible. The étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D_A^{(0)}(\overline{\rho})$ in Corollary 2.6.7 is isomorphic to the étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D_{A,\sigma_0}(\overline{\rho})$ in Lemma 2.2.1.*

Proof. By Corollary 2.6.6, it is enough to prove the same statement after extending everywhere scalars from A to A_∞. Recall we have $D_{K,\sigma_0}(\overline{\rho}) = \mathbb{F}(\!(T_{K,\sigma_0})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ with basis $(1\otimes e_i)_{0\le i\le d-1}$ as in Lemma 2.1.6 and let $u \in T_{K,0}(1+(A'_\infty)^{\circ\circ})$ as in Lemma 2.9.2. Then using (53), (65) and the action of $\mathcal{O}_K^\times$ in (15) we obtain

$$A_\infty \otimes_A D_A^{(0)}(\overline{\rho}) = (A'_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!),\iota_0} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]})^{\Delta_1} = \bigoplus_{i=0}^{d-1} A_\infty(uT_{K,0}^{-1})^{-\frac{hq^i(q-1)}{q^d-1}}(1\otimes e_i).$$

Moreover it follows again from the last equality in Lemma 2.9.2 that we have in

$A'_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!),\iota_0} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ for $a \in \mathcal{O}_K^\times$

$$\begin{aligned} a((uT_{K,0}^{-1})^{-\frac{hq^i(q-1)}{q^d-1}}(1\otimes e_i)) &= \left(\left(\frac{f_{a,0}^X}{\varphi(f_{a,0}^X)}\right)^{\frac{1}{q-1}} \frac{\overline{a}T_{K,0}}{a(T_{K,0})}uT_{K,0}^{-1}\right)^{-\frac{hq^i(q-1)}{q^d-1}} f_a^{\mathrm{LT}}(T_{K,0})^{\frac{hq^i(q-1)}{q^d-1}}(1\otimes e_i) \\ &= \left(\frac{\varphi(f_{a,0}^X)}{f_{a,0}^X}\right)^{\frac{hq^i}{q^d-1}} (uT_{K,0}^{-1})^{-\frac{hq^i(q-1)}{q^d-1}}(1 \otimes e_i). \end{aligned}$$

We define an A_∞-linear isomorphism $A_\infty \otimes_A D_{A,\sigma_0}(\overline{\rho}) = A_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]} \xrightarrow{\sim} A_\infty \otimes_A D_A^{(0)}(\overline{\rho})$ by $1\otimes e_i \mapsto (uT_{K,0}^{-1})^{-\frac{hq^i(q-1)}{q^d-1}} \otimes e_i$ for $i \in \{0,\dots,d-1\}$. This isomorphism commutes with the actions of $\mathcal{O}_K^\times$ on both sides by the above computation (together with Lemma 2.2.1). It also commutes with φ_q, namely we have in $A'_\infty \otimes_{\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!),\iota_0} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ (using Remark 2.9.4):

$$\varphi_q\Big((uT_{K,0}^{-1})^{-\frac{hq^i(q-1)}{q^d-1}} \otimes e_i\Big) = (uT_{K,0}^{-1})^{-\frac{hq^{i+1}(q-1)}{q^d-1}} \otimes e_{i+1} \quad \text{for} \quad i < d-1$$

and (using the formula for u^{q-1} in Lemma 2.9.2)

$$\begin{aligned} \varphi_q\Big((uT_{K,0}^{-1})^{-\frac{hq^{d-1}(q-1)}{q^d-1}} \otimes e_{d-1}\Big) &= (uT_{K,0}^{-1})^{-h(q-1)}(uT_{K,0}^{-1})^{-\frac{h(q-1)}{q^d-1}} \otimes \lambda^d T_K^{-h(q-1)}e_0 \\ &= u^{-h(q-1)}\lambda^d\Big((uT_{K,0}^{-1})^{-\frac{h(q-1)}{q^d-1}} \otimes e_0\Big) \\ &= \lambda^d \left(\frac{\varphi(X_0)}{X_0}\right)^{-h} \Big((uT_{K,0}^{-1})^{-\frac{h(q-1)}{q^d-1}} \otimes e_0\Big). \quad \square \end{aligned}$$

Remark 2.9.6. Theorem 2.9.5 shows that, when $\overline{\rho}$ is absolutely semi-simple, one can obtain the étale φ_q-module $D_A^{(0)}(\overline{\rho})$ from the Lubin–Tate $(\varphi_q, \mathcal{O}_K^\times)$-module $D_{K,\sigma_0}(\overline{\rho}) = \mathbb{F}(\!(T_{K,\sigma_0})\!) \otimes_{\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ by the simple recipe (20). However, we do not expect this recipe to work in general when $\overline{\rho}$ is not semi-simple.

Define $D_{A,\sigma}(\overline{\rho})$ as $D_{A,\sigma_0}(\overline{\rho})$ (see §2.2) but using the embedding σ instead of σ_0. From §2.2 one easily checks that there are canonical A-linear isomorphisms for $i \in \mathbb{Z}$

$$\mathrm{Id} \otimes \varphi : A \otimes_{\varphi,A} D_{A,\sigma_i}(\overline{\rho}) \xrightarrow{\sim} D_{A,\sigma_{i-1}}(\overline{\rho}) \tag{66}$$

which commute with $\mathcal{O}_K^\times$ and φ_q on both sides. Comparing the isomorphism ϕ_i in Corollary 2.6.7 with the isomorphism (66) we see that we have for $i \in \{0,\dots,f-1\}$

$$D_A^{(i)}(\overline{\rho}) \cong D_{A,\sigma_{f-i}}(\overline{\rho}). \tag{67}$$

Using (67) and (57) we have therefore

$$D_A^{\otimes}(\overline{\rho}) \cong D_{A,\sigma_0}(\overline{\rho}) \otimes_A D_{A,\sigma_1}(\overline{\rho}) \otimes_A \cdots \otimes_A D_{A,\sigma_{f-1}}(\overline{\rho}). \tag{68}$$

When $\dim_{\mathbb{F}} \overline{\rho} = 1$, i.e. for $\chi : \mathrm{Gal}(\overline{K}/K) \to \mathbb{F}^\times$ a continuous character, we will need in §3 the (very simple) description of $D_A^{\otimes}(\chi)$.

Lemma 2.9.7. *Viewing χ as a character of $K^\times$ via the local reciprocity map, we have (for $a \in \mathcal{O}_K^\times$):*

$$\begin{cases} D_A^{\otimes}(\chi) & = & AF_\chi \\ \varphi(F_\chi) & = & \chi(p)F_\chi \\ a(F_\chi) & = & \chi(a)F_\chi. \end{cases}$$

In particular, $D_A^{\otimes}(\overline{\rho} \otimes \chi)$ equals $D_A^{\otimes}(\overline{\rho})$, but with the action of φ multiplied by $\chi(p)$ and the action of $a \in \mathcal{O}_K^\times$ multiplied by $\chi(a)$.

Proof. By (66) and (68) replacing $\overline{\rho}$ by χ we can describe $D_A^{\otimes}(\chi)$ as AE_χ, where $E_\chi \overset{\text{def}}{=} e_\chi \otimes \varphi(e_\chi) \otimes \cdots \otimes \varphi^{f-1}(e_\chi)$ with $\varphi^j(e_\chi) \in D_{A,\sigma_{f-j}}(\chi)$ (noting e_χ instead of $1 \otimes e_\chi$). Write $\chi = \omega_f^{h_\chi}\mathrm{unr}(\lambda_\chi)$ for $h_\chi \in \mathbb{Z}_{\geq 0}$ and $\lambda_\chi \in \mathbb{F}^\times$. Set $F_\chi \overset{\text{def}}{=} Y_{\sigma_0}^{h_\chi} E_\chi$, then one computes:

$$\begin{aligned} \varphi(F_\chi) = \varphi(Y_{\sigma_0})^{h_\chi}\varphi(E_\chi) &= \varphi(Y_{\sigma_0})^{h_\chi}\varphi^f(e_\chi) \otimes \varphi(e_\chi) \otimes \cdots \otimes \varphi^{f-1}(e_\chi) \\ &= \lambda_\chi \varphi(Y_{\sigma_0})^{h_\chi}\Big(\frac{Y_{\sigma_0}}{\varphi(Y_{\sigma_0})}\Big)^{h_\chi} E_\chi = \lambda_\chi F_\chi = \chi(p)F_\chi \end{aligned}$$

where the third equality follows from (25). An analogous computation using $a(Y_{\sigma_0}^{h_\chi}) = \sigma(\overline{a})^{h_\chi} Y_{\sigma_0}^{h_\chi} f_{a,\sigma_0}^{-h_\chi}$ and $a(\varphi^j(e_\chi)) = \big(\frac{\varphi^j(f_{a,\sigma_0})}{\varphi^{j+1}(f_{a,\sigma_0})}\big)^{\frac{h_\chi}{1-q}} \varphi^j(e_\chi)$ (see again (25)) gives $a(F_\chi) = \sigma_0(\overline{a})^{h_\chi} F_\chi$. But $\sigma_0(\overline{a})^{h_\chi} = \chi(a)$ (see (14)). The rest of the statement follows from the discussion after (25). $\square$

Appendix

We prove a lemma on sheaves on $\mathrm{Perf}_{\mathbb{F}}$ represented by (non-perfectoid) Huber rings $\mathbb{F}[\![x_1^{1/p^\infty}, \ldots, x_m^{1/p^\infty}]\!]$. The result is not needed in the text, but can be used to show that some maps in §2 between perfectoids come from maps between such Huber rings.

Lemma 2.9.8. *For $m \geq 1$, let $S_m \overset{\text{def}}{=} \mathbb{F}[\![x_1^{1/p^\infty}, \ldots, x_m^{1/p^\infty}]\!]$. For all integers $m \geq 1$ and $n \geq 1$, the map*

$$\mathrm{Hom}_{\mathrm{Spa}(\mathbb{F})}(\mathrm{Spa}(S_n), \mathrm{Spa}(S_m)) \longrightarrow \mathrm{Hom}(h_{\mathrm{Spa}(S_n)}|_{\mathrm{Perf}_{\mathbb{F}}}, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_{\mathbb{F}}})$$

is an isomorphism.

Remark 2.9.9. If X is an adic space over $\mathbb{F}$, note that $h_X|_{\mathrm{Perf}_{\mathbb{F}}}$ is the diamond over $\mathbb{F}$ associated to X by [SW20, Def. 10.1.1].

Proof. Recall first that for any integer $d \geq 1$ the adic space $\mathrm{Spa}(\mathbb{F}[\![t_1^{1/p^\infty}, \dots, t_d^{1/p^\infty}]\!]) \setminus V(t_1, \dots, t_d)$ is a perfectoid space, as it can be covered by the perfectoid affinoid open subspaces for $i \in \{1, \dots, d\}$

$$\mathrm{Spa}\Big(\mathbb{F}(\!(t_i^{p^{-\infty}})\!)\Big\langle\Big(\frac{t_j}{t_i}\Big)^{p^{-\infty}}, j \neq i\Big\rangle, \mathbb{F}[\![t_i^{p^{-\infty}}]\!]\Big\langle\Big(\frac{t_j}{t_i}\Big)^{p^{-\infty}}, j \neq i\Big\rangle\Big).$$

By [SW20, Lemma 18.1.1] the presheaf $h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_\mathbb{F}}$ is a sheaf for the v-topology on $\mathrm{Perf}_\mathbb{F}$. Let U be the open subset of $\mathrm{Spa}(S_n[\![t^{1/p^\infty}]\!])$ defined by $|x_i| \leq |t| \neq 0$, $i \in \{1, \dots, n\}$. Then U is affinoid perfectoid as a rational open subset of the perfectoid space $\mathrm{Spa}(S_n[\![t^{1/p^\infty}]\!]) \setminus V(x_1, \dots, x_n, t)$. Let us prove that the map $U \to h_{\mathrm{Spa}(S_n)}|_{\mathrm{Perf}_\mathbb{F}}$ is an epimorphism of v-sheaves on $\mathrm{Perf}_\mathbb{F}$ (here and in what follows we write U for $h_U|_{\mathrm{Perf}_\mathbb{F}}$). Let $\mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space and $|\cdot|$ a power-multiplicative norm on R, then $x \in h_{\mathrm{Spa}(S_n)}(\mathrm{Spa}(R, R^+))$ corresponds to some $y = (y_1, \dots, y_n) \in (R^{\circ\circ})^n$. If ϖ is a pseudo-uniformizer of R, there exists $r \in \mathbb{Q}_{>0}$ such that $|y_i| < |\varpi^r|$, $i \in \{1, \dots, n\}$. Then the unique $\tilde{x} \in \mathrm{Spa}(S_n[\![t^{1/p^\infty}]\!])(\mathrm{Spa}(R, R^+))$ above x obtained by sending t to ϖ^r lies in $U(\mathrm{Spa}(R, R^+))$. Therefore $U \to h_{\mathrm{Spa}(S_n)}|_{\mathrm{Perf}_\mathbb{F}}$ is a universal effective epimorphism in the category of v-sheaves on $\mathrm{Perf}_\mathbb{F}$ by [DGA+11, IV. Prop. 4.4.3] (together with [DGA+11, IV. Déf. 1.3]). This implies that $h_{\mathrm{Spa}(S_n)}|_{\mathrm{Perf}_\mathbb{F}}$ is the coequalizer in the category of v-sheaves on $\mathrm{Perf}_\mathbb{F}$ of the diagram

$$U \times_{h_{\mathrm{Spa}(S_n)}} U \rightrightarrows U.$$

Thus $\mathrm{Hom}(h_{\mathrm{Spa}(S_n)}|_{\mathrm{Perf}_\mathbb{F}}, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_\mathbb{F}})$ is the equalizer of

$$\mathrm{Hom}(U, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_\mathbb{F}}) \rightrightarrows \mathrm{Hom}(U \times_{h_{\mathrm{Spa}(S_n)}} U, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_\mathbb{F}}).$$

Using $\mathrm{Hom}(U, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_\mathbb{F}}) = h_{\mathrm{Spa}(S_m)}(U) = (\mathcal{O}(U)^{\circ\circ})^m$, we need to prove that the equalizer of this double map is exactly the subset $(S_n^{\circ\circ})^m \subseteq (\mathcal{O}(U)^{\circ\circ})^m$. One easily checks that it contains $(S_n^{\circ\circ})^m$, hence it is enough to prove that it is contained in $(S_n^{\circ\circ})^m$. Let $U_n \overset{\mathrm{def}}{=} \mathrm{Spa}(S_n) \setminus V(x_1, \dots, x_n)$ which is a perfectoid open subset of $\mathrm{Spa}(S_n)$. Then $U \times_{h_{\mathrm{Spa}(S_n)}} U_n = U \setminus V(x_1, \dots, x_n)$ is perfectoid as an open subset of the perfectoid space U. By base change ([Sch12, Prop. 6.18]) $U \times_{h_{\mathrm{Spa}(S_n)}} U \times_{h_{\mathrm{Spa}(S_n)}} U_n = (U \times_{h_{\mathrm{Spa}(S_n)}} U_n) \times_{U_n} (U \times_{h_{\mathrm{Spa}(S_n)}} U_n)$ is also in $\mathrm{Perf}_\mathbb{F}$. Moreover the map $U \times_{h_{\mathrm{Spa}(S_n)}} U_n \to U_n$ is an effective epimorphism in the category of v-sheaves on $\mathrm{Perf}_\mathbb{F}$ as the base change of a universal effective epimorphism. It is then not hard to check that it is a v-cover in the category $\mathrm{Perf}_\mathbb{F}$, i.e. any quasi-compact open subspace $V_n \subseteq U_n$ is the image of a quasi-compact open subspace of $U \times_{h_{\mathrm{Spa}(S_n)}} U_n$. As U_n is perfectoid and $X \mapsto \mathcal{O}(X)$ is a sheaf of rings on the v-site $\mathrm{Perf}_\mathbb{F}$ ([Sch, Thm. 8.7]), we have $\mathcal{O}(U_n) = \ker(\mathcal{O}(U \times_{\mathrm{Spa}(S_n)} U_n) \rightrightarrows \mathcal{O}(U \times_{h_{\mathrm{Spa}(S_n)}} U \times_{h_{\mathrm{Spa}(S_n)}} U_n))$. Denote by $(U_{n,i})_{i \in \{1, \dots, n\}}$ the affinoid open cover of U_n at the beginning of this proof. For each $i \in \{1, \dots, n\}$ there exists a finite union of affinoid open subsets $(V_{i,j})_j$ of $U \times_{\mathrm{Spa}(S_n)} U_n$ whose images cover $U_{n,i}$ in U_n, so that (by the open mapping theorem for Banach spaces) $\mathcal{O}(U_{n,i}) \subseteq \oplus_j \mathcal{O}(V_{i,j})$ is an embedding of Banach spaces. In particular, any element of $\mathcal{O}(U_{n,i})$ which is topologically nilpotent in $\oplus_j \mathcal{O}(V_{i,j})$ is topologically nilpotent in $\mathcal{O}(U_{n,i})$. Since the topology on $\mathcal{O}(U_n)$ is the one induced from the embedding $\mathcal{O}(U_n) \hookrightarrow \oplus_i \mathcal{O}(U_{n,i})$, we see that any element of $\mathcal{O}(U_n)$ which is topologically nilpotent in $\oplus_{i,j}\mathcal{O}(V_{i,j})$ is topologically nilpotent in $\oplus_i \mathcal{O}(U_{n,i})$ hence also in $\mathcal{O}(U_n)$, which implies

$$\mathcal{O}(U_n)^{\circ\circ} = \ker\Big(\mathcal{O}(U \times_{\mathrm{Spa}(S_n)} U_n)^{\circ\circ} \rightrightarrows \mathcal{O}(U \times_{h_{\mathrm{Spa}(S_n)}} U \times_{h_{\mathrm{Spa}(S_n)}} U_n)^{\circ\circ}\Big).$$

From the commutative diagram induced by the open embedding $U \times_{h_{\mathrm{Spa}(S_n)}} U_n \hookrightarrow U$:

$$\begin{tikzcd}
\mathrm{Hom}(U, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_{\mathbb{F}}}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathrm{Hom}(U \times_{h_{\mathrm{Spa}(S_n)}} U, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_{\mathbb{F}}})) \arrow[d] \\
\mathrm{Hom}(U \times_{h_{\mathrm{Spa}(S_n)}} U_n, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_{\mathbb{F}}}) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}(U \times_{h_{\mathrm{Spa}(S_n)}} U \times_{h_{\mathrm{Spa}(S_n)}} U_n, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_{\mathbb{F}}}))
\end{tikzcd}$$

we deduce that the equalizer of

$$\mathrm{Hom}(U, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_{\mathbb{F}}}) \rightrightarrows \mathrm{Hom}(U \times_{h_{\mathrm{Spa}(S_n)}} U, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_{\mathbb{F}}}))$$

lies in the preimage of $(\mathcal{O}(U_n)^{\circ\circ})^m \subseteq \mathrm{Hom}(U \times_{h_{\mathrm{Spa}(S_n)}} U_n, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_{\mathbb{F}}}) = (\mathcal{O}(U \times_{\mathrm{Spa}(S_n)} U_n)^{\circ\circ})^m$ in $(\mathcal{O}(U)^{\circ\circ})^m = \mathrm{Hom}(U, h_{\mathrm{Spa}(S_m)}|_{\mathrm{Perf}_{\mathbb{F}}})$. Using that the restriction map $(\mathcal{O}(U)^{\circ\circ})^m \to (\mathcal{O}(U \times_{\mathrm{Spa}(S_n)} U_n)^{\circ\circ})^m$ is injective by [SW20, Prop. 5.3.4], the result follows easily from the topological isomorphism $S_n \cong \mathcal{O}(U_n)^+$ (and thus $S_n^{\circ\circ} = \mathcal{O}(U_n)^{\circ\circ}$) which can be checked directly using the covering at the beginning of this proof. □

We can use Lemma 2.9.8 as follows. The functorial bijection of Lemma 2.3.1 induces a map of commutative rings

$$\begin{aligned}\mathrm{End}^{\mathrm{cont}}_{\mathbb{F}-\mathrm{alg}}(\mathbb{F}\llbracket x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}\rrbracket) \cong \mathrm{End}_{\mathrm{Spa}(\mathbb{F})}&(\mathrm{Spa}(\mathbb{F}\llbracket x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}\rrbracket)) \\ &\longrightarrow \mathrm{End}(\mathbf{B}^+(-)^{\varphi_q = p^d}),\end{aligned}$$

where the right-hand side means the endomorphisms as sheaves of sets on $\mathrm{Perf}_{\mathbb{F}}$. It follows from Lemma 2.9.8 that the last map is a bijection. In particular, using Lemma 2.3.1 and the above discussion, we see that the map m in (45) extends to a morphism between adic spaces over $\mathbb{F}$

$$m : (\widetilde{G}_{\mathrm{LT}} \times_{\mathrm{Spf}(\mathbb{F}_q)} \mathrm{Spf}(\mathbb{F}))^{\mathrm{ad},f} \longrightarrow \widetilde{\mathcal{G}}^{\mathrm{ad}}_{f,f,\mathbb{F}} \cong \mathrm{Spa}(\mathbb{F}\llbracket K \rrbracket).$$

Likewise the diagram (60) comes from a commutative diagram of Huber rings

$$\begin{tikzcd}
\mathbb{F}\llbracket T_{K,0}^{1/p^\infty}, \dots, T_{K,f-1}^{1/p^\infty}\rrbracket \arrow[r, two heads] & \mathbb{F}\llbracket T_{K,\sigma_0}^{p^{-\infty}}\rrbracket \\
\mathbb{F}\llbracket K \rrbracket \arrow[r] \arrow[u] & \mathbb{F}\llbracket \mathbb{Q}_p \rrbracket . \arrow[u]
\end{tikzcd}$$

3 Étale $(\varphi, \mathcal{O}_K^\times)$-modules and modular representations of GL_2

In this section we prove that the étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A(\pi)$ over A associated in [BHH$^+$21, §3] to certain automorphic admissible smooth representations π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ is isomorphic to (a certain twist of) the étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A^{\otimes}(\overline{\rho})$ of §2, where $\overline{\rho}$ is the underlying 2-dimensional representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$, which is assumed semi-simple and sufficiently generic. We conjecture that an analogous statement holds without these assumptions and for any automorphic admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$.

We let $I \overset{\mathrm{def}}{=} \left(\begin{smallmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{smallmatrix}\right)$ be the Iwahori subgroup of $\mathrm{GL}_2(\mathcal{O}_K)$, $K_1 \overset{\mathrm{def}}{=} \left(\begin{smallmatrix} 1+p\mathcal{O}_K & p\mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{smallmatrix}\right)$ the first congruence subgroup, $I_1 \overset{\mathrm{def}}{=} \left(\begin{smallmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{smallmatrix}\right)$ the pro-p radical of I and Z_1 the center of I_1. We recall from §2.2 that $N_0 = \left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right) \subseteq I_1$. If C is a pro-p group we denote by $\mathbb{F}[\![C]\!]$ its Iwasawa algebra over $\mathbb{F}$ (a local ring), and $\mathfrak{m}_C$ the maximal ideal of $\mathbb{F}[\![C]\!]$. If M is a filtered module in the sense of [LvO96, §I.2] with $(F_nM)_{n\in\mathbb{Z}}$ its ascending filtration, we define $\mathrm{gr}(M) \overset{\mathrm{def}}{=} \oplus_{n\in\mathbb{Z}} F_nM/F_{n-1}M$. When $R = \mathbb{F}[\![C]\!]$ and M is an R-module, the filtration $F_nM = \mathfrak{m}_R^{-n}M$ if $n \le 0$ and $F_nM = M$ if $n \ge 0$ is called the $\mathfrak{m}_R$-adic filtration on M.

3.1 A local-global compatibility conjecture for $(\varphi, \mathcal{O}_K^\times)$-modules over A

We conjecture that any automorphic smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ gives rise to an étale $(\varphi, \mathcal{O}_K^\times)$-module over A which is (up to twist) a direct sum of copies of the module $D_A^{\otimes}$ in §2.7 of the corresponding local Galois representation at p. We state our main results.

First, we quickly review the construction of the A-module $D_A(\pi)$ associated to certain smooth representations π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ in [BHH$^+$21, §3.1].

Let π be an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with a central character and endow the $\mathbb{F}$-linear dual $\pi^\vee$ with the $\mathfrak{m}_{I_1}$-adic filtration, or equivalently the $\mathfrak{m}_{I_1/Z_1}$-adic filtration (which, in general, *strictly* contains the $\mathfrak{m}_{N_0}$-adic filtration). We endow

$$(\pi^\vee)_{(Y_{\sigma_0}\cdots Y_{\sigma_{f-1}})} \overset{\mathrm{def}}{=} \mathbb{F}[\![N_0]\!]_{(Y_{\sigma_0}\cdots Y_{\sigma_{f-1}})} \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee$$

with the tensor product filtration (where the localization $\mathbb{F}[\![N_0]\!]_{(Y_{\sigma_0}\cdots Y_{\sigma_{f-1}})}$ is endowed with the filtration described by (16), replacing $\mathbb{F}_q$ by $\mathbb{F}$) and define $D_A(\pi)$ as the completion of $(\pi^\vee)_{(Y_{\sigma_0}\cdots Y_{\sigma_{f-1}})}$ for this filtration ([LvO96, §I.3.4]). Then $D_A(\pi)$ is a complete filtered A-module and the action of $\mathcal{O}_K^\times$ on $\pi^\vee$ extends by continuity to $D_A(\pi)$. Moreover the action $f \mapsto f \circ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on $\pi^\vee$ gives rise to a continuous A-linear morphism (see [BHH$^+$21, §3.1.2])

$$\beta : D_A(\pi) \longrightarrow A \otimes_{\varphi,A} D_A(\pi), \tag{69}$$

where φ on A is as in §2.2. We let $\mathcal{C}$ be the abelian category of those π such that $\mathrm{gr}(D_A(\pi))$ is a finitely generated $\mathrm{gr}(A)$-module. Then for any $\pi \in \mathcal{C}$, the A-module $D_A(\pi)$ is finite free (see [BHH$^+$21, Cor. 3.1.2.9] and Remark 2.6.2).

The following straightforward lemma will be used. For $\chi : K^\times \to \mathbb{F}^\times$ a smooth character, denote by $D_A(\chi)$ the rank 1 étale $(\varphi, \mathcal{O}_K^\times)$-module over A defined by Ae_χ with $\varphi(e_\chi) \overset{\text{def}}{=} \chi(p)e_\chi$ and $a(e_\chi) \overset{\text{def}}{=} \chi(a)e_\chi$ for $a \in \mathcal{O}_K^\times$.

Lemma 3.1.1. *Let $\chi : K^\times \to \mathbb{F}^\times$ be a smooth character and π in the category $\mathcal{C}$, then $D_A(\pi \otimes \chi) \cong D_A(\pi) \otimes_A D_A(\chi^{-1})$ with diagonal φ and action of $\mathcal{O}_K^\times$.*

Proof. This directly follows from the definitions of $D_A(\pi)$, of φ and of the action of $\mathcal{O}_K^\times$ on $D_A(\pi)$. ☐

For π in $\mathcal{C}$, when β is moreover an isomorphism, its inverse $\beta^{-1} = \mathrm{Id} \otimes \varphi$ makes $D_A(\pi)$ an étale $(\varphi, \mathcal{O}_K^\times)$-module.

We now go to the global setting.

We fix a totally real number field F that is unramified at p. We fix a quaternion algebra D of center F which is split at all places above p and at not more than one infinite place. When D is split at one infinite place we say that we are in the *indefinite case*, and in the *definite case* otherwise. For a compact open subgroup $U = \prod U_w \subseteq (D \otimes_F \mathbb{A}_F^\infty)^\times$ we let X_U be the associated smooth projective algebraic Shimura curve over F (see [BHH$^+$21, §8.1] and the references therein for more details).

Fix an absolutely irreducible continuous representation $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ and for a finite place w of F we write $\overline{r}_w \overset{\text{def}}{=} \overline{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$. We let S_D be the set of finite places where D ramifies, $S_{\overline{r}}$ the set of (finite) places where $\overline{r}$ is ramified and S_p the set of (finite) places above p. Finally, we fix a place $v \in S_p$. Let $\omega = \omega_1$ denote the mod p cyclotomic character.

For any compact open subgroup $U^v = \prod_{w \neq v} U_w \subseteq (D \otimes_F \mathbb{A}_F^{\infty,v})^\times$ we consider the following admissible smooth representation π of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$ with central character $(\omega\det(\overline{r}_v))^{-1}$:

$$\pi \overset{\text{def}}{=} \varinjlim_{U_v} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\left(\overline{r}, H^1_{\text{ét}}(X_{U^vU_v} \times_F \overline{F}, \mathbb{F})\right) \tag{70}$$

where the inductive limit runs over the compact open subgroups U_v of $(D \otimes_F F_v)^\times \cong \mathrm{GL}_2(F_v)$. In the definite case, we replace $\mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}(\overline{r}, H^1_{\text{ét}}(X_U \times_F \overline{F}, \mathbb{F}))$ by the

Hecke eigenspace $S(U,\mathbb{F})[\mathfrak{m}] \subseteq S(U,\mathbb{F}) \overset{\mathrm{def}}{=} \{f : D^\times\backslash(D\otimes_F \mathbb{A}_F^\infty)^\times/U \to \mathbb{F}\}$ associated to $\overline{r}$ (see [BHH$^+$21, §8.1]) and define analogously

$$\pi \overset{\mathrm{def}}{=} \varinjlim_{U_v} S(U^vU_v,\mathbb{F})[\mathfrak{m}]. \tag{71}$$

We also need the "multiplicity 1" variants of the representations π. For that, we need to assume that $p \geq 5$, that $\overline{r}|_{G_{F(\sqrt[5]{1})}}$ is absolutely irreducible, that the image of $\overline{r}(G_{F(\sqrt[5]{1})})$ in $\mathrm{PGL}_2(\mathbb{F})$ is not isomorphic to A_5, that $\overline{r}_w$ for $w \in S_p$ is generic in the sense of [BP12, Def. 11.7] (which implies $S_p \subseteq S_{\overline{r}}$) and that $\overline{r}_w$ is non-scalar if $w \in S_D$. Under these assumptions, a so-called "local factor" is defined in [BD14, §3.3] (in the indefinite case and when $\overline{r}_w$ is reducible for all $w \in S_p$) and in [EGS15, §6.5] (without these two conditions):

$$\pi \overset{\mathrm{def}}{=} \mathrm{Hom}_{U^v}\Big(\overline{M}^v, \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\big(\overline{r}, \varinjlim_{V} H^1_{\text{ét}}(X_V \times_F \overline{F},\mathbb{F})\big)\Big)[\mathfrak{m}'] \text{ (indefinite case)} \tag{72}$$

$$\pi \overset{\mathrm{def}}{=} \mathrm{Hom}_{U^v}\Big(\overline{M}^v, \varinjlim_{V} S(V,\mathbb{F})[\mathfrak{m}]\Big)[\mathfrak{m}'] \text{ (definite case)} \tag{73}$$

where the inductive limits run over the compact open subgroups V of $(D\otimes_F \mathbb{A}_F^\infty)^\times$, and where we refer to *loc. cit.* for the definitions of the compact open subgroup $U^v \subseteq (D\otimes_F \mathbb{A}_F^{\infty,v})^\times$, of the (finite-dimensional) irreducible smooth representation $\overline{M}^v$ of U^v over $\mathbb{F}$ and of the maximal ideal $\mathfrak{m}'$ in a certain Hecke algebra.

Conjecture 3.1.2. *Let π be as in (70), (71), (72) or (73) and assume $\pi \neq 0$. Then π is in the category $\mathcal{C}$, β in (69) is a bijection and we have an isomorphism of étale $(\varphi, \mathcal{O}_K^\times)$-modules $D_A(\pi) \cong D_A^{\otimes}(\overline{r}_v(1))^{\oplus r}$ for some integer $r \geq 1$ which is equal to 1 when π is as in (72) or (73).*

In the sequel, we prove Conjecture 3.1.2 for π as in (72) or (73) when $\overline{r}_v$ is semisimple and satisfies a strong genericity hypothesis (as defined below). We actually prove a purely local result for certain smooth representations π, that will ultimately include the representations in (72) and (73).

Let first $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation satisfying the genericity assumption of [BP12, Def. 11.7]. Let π be a smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ satisfying the following two conditions:

(i) there is an isomorphism of diagrams $(\pi^{I_1} \hookrightarrow \pi^{K_1}) \cong D(\overline{\rho})^{\oplus r}$ for some $r \in \mathbb{Z}_{\geq 1}$, where $D(\overline{\rho})$ is a diagram associated to $\overline{\rho}$ as in [BP12] or [BHH$^+$21, §3.2.1] with the constants ν_σ for $\sigma \in W(\overline{\rho})$ as in Remark 3.4.6 below;

(ii) for any character $\chi : I \to \mathbb{F}^\times$ appearing in $\pi[\mathfrak{m}_{I_1}]$ there is an equality of multiplicities $[\pi[\mathfrak{m}_{I_1}^3] : \chi] = [\pi[\mathfrak{m}_{I_1}] : \chi]$.

We moreover assume that $\overline{\rho}$ is of the following form *up to twist*:

$$\overline{\rho}|_{I_K} \cong \begin{cases} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} \oplus 1 & \text{if } \overline{\rho} \text{ is reducible} \\ \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j} \oplus \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^{j+f}} & \text{if } \overline{\rho} \text{ is irreducible} \end{cases} \tag{74}$$

where the integers r_i satisfy the following (strong) genericity condition:

$$\begin{aligned} \max\{12, 2f-1\} \leq\ & r_j \leq\ p - \max\{15, 2f+2\} \text{ if } j > 0 \text{ or } \overline{\rho} \text{ is reducible} \\ \max\{13, 2f\} \leq\ & r_0 \leq\ p - \max\{14, 2f+1\} \text{ if } \overline{\rho} \text{ is irreducible.} \end{aligned} \tag{75}$$

The following is the main result of §3.

Theorem 3.1.3 (See §3.10)**.** *Assume that $\overline{\rho}$ and π are as above with moreover $(\pi^{I_1} \hookrightarrow \pi^{K_1}) \cong D(\overline{\rho})$, i.e. $r = 1$. Then π is in the category $\mathcal{C}$, β in (69) is a bijection and we have an isomorphism of étale $(\varphi, \mathcal{O}_K^\times)$-modules $D_A(\pi) \cong D_A^\otimes(\overline{\rho}^\vee(1))$, where $\overline{\rho}^\vee(1)$ is the Cartier dual of $\overline{\rho}$.*

It implies the following special cases of Conjecture 3.1.2.

Corollary 3.1.4. *Let π be as in (72) or (73) and assume moreover that $\overline{r}_v$ satisfies (75), (74), and that the framed deformation ring $R_{\overline{r}_w}$ of $\overline{r}_w$ over $W(\mathbb{F})$ is formally smooth if $w \in (S_D \cup S_{\overline{r}}) \setminus S_p$. Then Conjecture 3.1.2 is true for π.*

Proof. By [DL21, Thm. 5.36] (and the references therein) π satisfies condition (i) above with $\overline{\rho} = \overline{r}_v^\vee$ and $r = 1$. By [BHH$^+$20, Thm. 8.3.14], [BHH$^+$20, Thm. 1.5] and [BHH$^+$20, Rem. 8.4.5] π satisfies condition (ii). Hence we can apply Theorem 3.1.3. ☐

Remark 3.1.5. Under the assumptions of Theorem 3.1.3 (but without assuming necessarily $r = 1$), we already knew the étale $(\varphi, \mathbb{Z}_p^\times)$-module $\mathbb{F}(\!(T)\!) \otimes_A D_A(\pi)$. Indeed, it follows from [BHH$^+$21, Cor. 3.3.2.4], [BHH$^+$21, Thm. 3.1.3.7], Remark 2.6.2 - and some unravelling of the definition of the functor V_{GL_2} of [BHH$^+$21, §2.1.1] using Lemma 3.1.1 and Lemma 2.9.7 - that $D_A(\pi)$ is free of rank $r2^f$ and $\mathbb{F}(\!(T)\!) \otimes_A D_A(\pi)$ is isomorphic to r copies of the $(\varphi, \mathbb{Z}_p^\times)$-module of the tensor induction $\mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\overline{\rho}^\vee(1))$ (compare with Corollary 2.8.2).

The rest of this paper is devoted to the proof of Theorem 3.1.3 and to the necessary material that needs to be introduced for that.

We fix $\overline{\rho}$ and π as in Theorem 3.1.3. Twisting both $\overline{\rho}$ and π using Lemma 2.9.7 and Lemma 3.1.1, we can and do assume *from now on* $\overline{\rho} \cong (\mathrm{ind}\, \omega_{2f}^h) \otimes \mathrm{unr}(\lambda)$ or $\overline{\rho} \cong \begin{pmatrix} \omega_f^h \mathrm{unr}(\lambda_0) & 0 \\ 0 & \mathrm{unr}(\lambda_1) \end{pmatrix}$ with $h = \sum_{j=0}^{f-1} p^j(r_j + 1)$.

3.2 Duality for étale $(\varphi, \mathcal{O}_K^\times)$-modules over A

If D is an étale $(\varphi, \mathcal{O}_K^\times)$-module over A we equip $\mathrm{Hom}_A(D, A)$ with the structure of an étale $(\varphi, \mathcal{O}_K^\times)$-module over A.

Fix D an étale $(\varphi, \mathcal{O}_K^\times)$-module over A and recall that D is finite free by Remark 2.6.2.

We first equip D with a left inverse $\psi : D \to D$ of φ, as follows. Fix a set of representatives $\{n\}$ of N_0/N_0^p including 1. Note that as D is étale, every element x of D can be uniquely written as $x = \sum_{N_0/N_0^p} \delta_n \varphi(x_n)$, where δ_n denotes the image of the element $[n] \in \mathbb{F}[\![N_0]\!]$ in A. Let $\psi : D \to D$ be defined by $\psi(x) \overset{\text{def}}{=} x_1$. The following easy lemma is left to the reader.

Lemma 3.2.1. *The map $\psi : D \to D$ is a left inverse of φ that is independent of any choices. We have $x = \sum_{N_0/N_0^p} \delta_n \varphi(\psi(\delta_n^{-1} x))$ for any $x \in D$. Moreover, the actions of ψ and $\mathcal{O}_K^\times$ commute.*

To define φ on $\mathrm{Hom}_A(D, A)$ recall that we have

$$\begin{aligned} \beta : D &\xrightarrow{\sim} A \otimes_\varphi D \\ x &\mapsto \sum_n \delta_n \otimes_\varphi \psi(\delta_n^{-1} x), \end{aligned} \tag{76}$$

where the sum runs over representatives $\{n\}$ of N_0/N_0^p. Now if M, N are A-modules with M finitely presented, we have for any A-algebra B a canonical isomorphism $B \otimes_A \mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_B(B \otimes_A M, B \otimes_A N)$, hence the A-linear dual of (76) gives rise to

$$A \otimes_\varphi \mathrm{Hom}_A(D, A) \xrightarrow{\sim} \mathrm{Hom}_A(D, A),$$

in other words we get a φ-linear endomorphism of $\mathrm{Hom}_A(D, A)$ that we also call φ (an étale Frobenius). Explicitly, this endomorphism is given by the formula

$$\begin{aligned} \mathrm{Hom}_A(D, A) &\to \mathrm{Hom}_A(D, A) \\ h &\mapsto \varphi(h) = (x \mapsto \sum_{N_0/N_0^p} \delta_n \varphi(h(\psi(\delta_n^{-1} x)))). \end{aligned} \tag{77}$$

By construction, it is independent of the choice of representatives.

Using Lemma 3.2.1 we can rewrite formula (77) as follows:

$$\varphi(h) : \sum_n \delta_n \varphi(x_n) \mapsto \sum_n \delta_n \varphi(h(x_n)). \tag{78}$$

We also define the action of $a \in \mathcal{O}_K^\times$ by the formula $a(h) \overset{\text{def}}{=} a \circ h \circ a^{-1}$.

Lemma 3.2.2. *With the definitions above,* $\mathrm{Hom}_A(D, A)$ *is an étale* $(\varphi, \mathcal{O}_K^\times)$*-module. Moreover, the natural pairing* $D \times \mathrm{Hom}_A(D, A) \to A$ *is equivariant for the actions of* φ *and* $\mathcal{O}_K^\times$*.*

Fix now a smooth representation π of $\mathrm{GL}_2(K)$ in the category $\mathcal{C}$ and endow the finite free A-module $D_A(\pi)$ with its filtration coming from the $\mathfrak{m}_{I_1}$-adic filtration on $\pi^\vee$, cf. §3.1. If D is an étale $(\varphi, \mathcal{O}_K^\times)$-module (endowed with its natural topology of finite free A-module), recall that $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D, \mathbb{F})$ means the continuous $\mathbb{F}$-linear morphisms $D \to \mathbb{F}$, or equivalently ($\mathbb{F}$ being endowed with the discrete topology) the $\mathbb{F}$-linear locally constant morphisms $D \to \mathbb{F}$. We give $\mathbb{F}$ the filtration such that $F_d\mathbb{F} = 0$ if and only if $d < 0$.

We write now Y_i for Y_{σ_i} (as in [BHH$^+$21, §3.1.1], note that there will be no confusion with the variables $Y_i \in A_q$ in §2.2 which are not used here) and $\underline{Y}^{(i_0,\dots,i_{f-1})}$ for $Y_0^{i_0} \cdots Y_{f-1}^{i_{f-1}} \in A$ (as in [BHH$^+$21, §3.2.2]). We also sometimes use the shorthand $\underline{Y}$ for $\underline{Y}^{\underline{1}} = \prod_{j=0}^{f-1} Y_j$.

Proposition 3.2.3. *There is an isomorphism of* $\mathbb{F}[\![N_0]\!]$*-modules between* $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ *and the set of sequences* $(x_k)_{k\geq 0}$ *such that* $x_k \in \pi$ *and*

- (i) $\underline{Y}x_k = x_{k-1}$ *for all* $k \geq 1$*;*
- (ii) *there exists* $d \in \mathbb{Z}$ *such that* $x_k \in \pi[\mathfrak{m}_{I_1}^{fk+d+1}]$ *for all* $k \geq 0$ *(where* $\pi[\mathfrak{m}_{I_1}^j] \overset{\mathrm{def}}{=} 0$ *for* $j \leq 0$*).*

A continuous $\mathbb{F}$*-linear map* $h : D_A(\pi) \to \mathbb{F}$ *corresponds to a sequence* $(x_k)_{k\geq 0}$ *as above if and only if*

$$h \circ \underline{Y}^{-\underline{k}} = \langle x_k, - \rangle \quad \text{on } \pi^\vee$$

for all $k \geq 0$*. Moreover,* h *is filtered of degree* d *if and only if* $x_k \in \pi[\mathfrak{m}_{I_1}^{fk+d+1}]$ *for all* $k \geq 0$*.*

Proof. Note that by definition we have

$$(\pi^\vee)_S \cong \varinjlim_{\substack{k\geq 0 \\ Y_0\cdots Y_{f-1}}} \pi^\vee \quad \text{and} \quad F_{-d-1}(\pi^\vee)_S \cong \varinjlim_{\substack{k\geq 0 \\ Y_0\cdots Y_{f-1}}} \mathfrak{m}_{I_1}^{fk+d+1}\pi^\vee,$$

so

$$(\pi^\vee)_S / F_{-d-1}(\pi^\vee)_S \cong \varinjlim_{\substack{k\geq 0 \\ Y_0\cdots Y_{f-1}}} \pi^\vee / \mathfrak{m}_{I_1}^{fk+d+1}\pi^\vee.$$

(Explicitly, the k-th map $\pi^\vee \to (\pi^\vee)_S$ is given by $(Y_0 \cdots Y_{f-1})^{-k}$.) Therefore, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F}) &= \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}((\pi^\vee)_S, \mathbb{F}) = \bigcup_{d \ge 0} \mathrm{Hom}_{\mathbb{F}}((\pi^\vee)_S / F_{-d-1}(\pi^\vee)_S, \mathbb{F}) \\ &= \bigcup_{d \ge 0} \mathrm{Hom}_{\mathbb{F}}((\pi^\vee)_S / F_{-d-1}(\pi^\vee)_S, \mathbb{F}) = \bigcup_{d \ge 0} \varprojlim_{\substack{k \ge 0 \\ Y_0 \cdots Y_{f-1}}} \mathrm{Hom}_{\mathbb{F}}(\pi^\vee / \mathfrak{m}_{I_1}^{fk+d+1} \pi^\vee, \mathbb{F}) \\ &= \bigcup_{d \ge 0} \varprojlim_{\substack{k \ge 0 \\ Y_0 \cdots Y_{f-1}}} \pi[\mathfrak{m}_{I_1}^{fk+d+1}].\end{aligned}$$

The final claims follow by unravelling these identifications. □

The A-action on sequences $(x_k)_{k \ge 0}$ by transport of structure will be made explicit in Lemma 3.8.1.

3.3 The continuous morphism $\mu : A \to \mathbb{F}$

For D an étale $(\varphi, \mathcal{O}_K^\times)$-module over A we relate $\mathrm{Hom}_A(D, A)$ to $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D, \mathbb{F})$ using a certain continuous morphism $\mu : A \to \mathbb{F}$.

Let us write $\mathbb{F}[\![N_0]\!] = \mathbb{F}[\![T_0, \ldots, T_{f-1}]\!]$ with $T_j \overset{\mathrm{def}}{=} [\alpha_j] - 1$, where $(\alpha_j)_{j \in \{0, \ldots, f-1\}}$ is a fixed $\mathbb{Z}_p$-basis of $\mathcal{O}_K$. Recall that A is endowed with a map $\psi : A \to A$ defined in §3.2, and which is a left inverse of $\varphi : A \to A$.

Proposition 3.3.1. *Up to scalar in $\mathbb{F}^\times$ there exists a unique $\mu \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(A, \mathbb{F})$ such that $\mu \circ \psi \in \mathbb{F}^\times \mu$. For such a μ we have $\mu \circ \psi = (-1)^{f-1} \mu$.*

It will be convenient for the proof to avoid using the variables Y_j. To obtain A from $\mathbb{F}[\![N_0]\!]$ it suffices to invert elements Z_j $(0 \le j \le f-1)$ such that $\mathrm{gr}(Z_j) = \mathrm{gr}(Y_j)$ in the graded ring and then complete. We will let Z_j be the unique linear combination of the $T_{j'}$ such that $\mathrm{gr}(Z_j) = \mathrm{gr}(Y_j)$. (Note that the Z_j are not canonical but depend on the choice of T_0, …, T_{f-1}.) There exists an element of $\mathrm{GL}_f(\mathbb{F})$ that relates the Z_j and the T_j. Hence we get the same description of A as in [BHH$^+$21, Rk. 3.1.1.3(iii)] with Z_j instead of Y_j, and also the valuation of an element of A is still given by the minimal total degree as a series in $\underline{Z}$. We note that $\varphi(T_j) = T_j^p$ and $\varphi(Z_j) = Z_{j-1}^p$ (because $\varphi(Z_j)$ is a homogeneous polynomial of degree p in the $T_{j'}$ and hence in the $Z_{j'}$, and since $\varphi(Y_j) = Y_{j-1}^p$).

Before starting the proof of Proposition 3.3.1 we note that $\mu \circ \psi = c\mu$ (with $c \in \mathbb{F}^\times$) is equivalent to the two conditions

$$\mu = c(\mu \circ \varphi), \tag{79}$$

$$\mu(\delta_n \varphi(x)) = 0 \qquad \forall\, n \in N_0 \setminus N_0^p, \forall\, x \in A. \tag{80}$$

This follows immediately from the definition of $\psi : D \to D$ in §3.2.

Proof (*Uniqueness*). Suppose that $\mu \circ \psi = c\mu$ for some $c \in \mathbb{F}^\times$. For the representatives $\{n\}$ of N_0/N_0^p we take $n = \begin{pmatrix} 1 & \sum_j i_j\alpha_j \\ & 1 \end{pmatrix}$ $(0 \le i_j \le p-1)$, so $\delta_n = \prod_j (1+T_j)^{i_j}$. By induction and (79)–(80) we have for any $\underline{0} \le \underline{i} \le \underline{p-1}$ that

$$\begin{aligned} \mu(\underline{T}^{\underline{i}}\varphi(x)) &= (-1)^{\|\underline{i}\|}\mu(\varphi(x)) \\ &= (-1)^{\|\underline{i}\|}c^{-1}\mu(x). \end{aligned} \tag{81}$$

Take now $x \in F_{f-1}A$. Then by iterating (81) we have

$$\mu(x) = c\mu(\underline{T}^{\underline{p-1}}\varphi(x)) = \cdots = c^n\mu(\underline{T}^{\underline{p^n-1}}\varphi^n(x)) = 0$$

for $n \gg 0$, since $\underline{T}^{\underline{p^n-1}}\varphi^n(x) \to 0$ in A as $n \to \infty$ if $x \in F_{f-1}A$ and μ is continuous. Hence

$$\mu(F_{f-1}A) = 0. \tag{82}$$

We claim that $\mu(\underline{Z}^{\underline{i}})$ for $\underline{i} \in \mathbb{Z}^f$ is an explicit multiple of $\mu(\underline{Z}^{-\underline{1}})$, only depending on c. To prove the claim, we may suppose that $\|\underline{i}\| \le -f$ by (82) and we will argue by descending induction on $\|\underline{i}\|$. Write $\underline{i} = \underline{r} + p\underline{s}$ with $\underline{0} \le \underline{r} \le \underline{p-1}$ and $\underline{s} \in \mathbb{Z}^f$. Hence $\mu(\underline{Z}^{\underline{i}}) = \mu(\underline{Z}^{\underline{r}}\underline{Z}^{p\underline{s}})$ and that can be expressed in terms of various $\mu(\underline{T}^{\underline{r}'}\underline{Z}^{p\underline{s}})$ with $\underline{r}' \ge \underline{0}$ and $\|\underline{r}'\| = \|\underline{r}\|$. Fix now one such term and write $\underline{r}' = \underline{r}'' + p\underline{r}'''$ with $\underline{0} \le \underline{r}'' \le \underline{p-1}$ and $\underline{0} \le \underline{r}'''$. Then we can express $\mu(\underline{T}^{\underline{r}'}\underline{Z}^{p\underline{s}}) = \mu(\underline{T}^{\underline{r}''}\underline{T}^{p\underline{r}'''}\underline{Z}^{p\underline{s}})$ in terms of various $\mu(\underline{T}^{\underline{r}''}\underline{Z}^{p\underline{t}})$ with $\|\underline{t}\| = \|\underline{s}\| + \|\underline{r}'''\|$. By (81) we are reduced to $\pm\mu(\underline{Z}^{p\underline{t}}) = \pm c^{-1}\mu(\underline{Z}^{\underline{t}'})$, where $\underline{t}'$ is a cyclic permutation of $\underline{t}$ and hence $\|\underline{t}'\| = \|\underline{t}\| = \|\underline{s}\| + \|\underline{r}'''\| = (\|\underline{i}\| - \|\underline{r}''\|)/p$.

From $\|\underline{r}''\| \le (p-1)f$ and $\|\underline{i}\| \le -f$ it follows that $\|\underline{i}\| \le \|\underline{t}'\|$ and moreover that equality can only hold if $\underline{r}'' = \underline{p-1}$ and $\|\underline{i}\| = -f$, in which case $\underline{r} = \underline{r}' = \underline{p-1}$ and $\underline{r}''' = \underline{0}$ (as $\|\underline{r}\| = \|\underline{r}''\| + p\|\underline{r}'''\| \le (p-1)f$). Thus $\|\underline{i}\| < \|\underline{t}'\|$ and we are done by induction, except possibly when $\|\underline{i}\| = -f$ and $\underline{i} \equiv -\underline{1} \pmod p$. Applying the same argument to $\mu(\underline{Z}^{\underline{t}'})$, we are done in the exceptional case except if $\underline{t}' \equiv -\underline{1} \pmod p$, which implies $\underline{s} = \underline{t} \equiv -\underline{1} \pmod p$ and hence $\underline{i} = \underline{p-1} + p\underline{s} \equiv -\underline{1} \pmod{p^2}$. By iterating we are left with the case $\underline{i} = -\underline{1}$, which completes the proof of the claim.

Finally we show that c is uniquely determined (assuming $\mu \ne 0$). Consider $\underline{i} = -\underline{1}$ above. Then

$$\mu(\underline{Z}^{-\underline{1}}) = \mu(\underline{Z}^{\underline{p-1}}\underline{Z}^{-\underline{p}}) = c'\mu(\underline{T}^{\underline{p-1}}\underline{Z}^{-\underline{p}}) = c'c^{-1}\mu(\underline{Z}^{-\underline{1}}),$$

where c' is the coefficient of $\underline{T}^{\underline{p-1}}$ in $\underline{Z}^{\underline{p-1}}$. Here, the second equality follows from the analysis in the preceding paragraph (the case $\|\underline{i}\| = -f$) that all other intervening terms $\underline{T}^{\underline{r}'}\underline{Z}^{-\underline{p}}$ with $\underline{r}' \ge \underline{0}$ and $\|\underline{r}'\| = \|\underline{p-1}\|$ lie in the kernel of μ (by (82)). The third equality follows from (81) with $\underline{i} = \underline{p-1}$. Hence $c = c'$ is uniquely determined. □

Proof (*Existence*). We define

$$\mu(x) \overset{\text{def}}{=} \varepsilon_{-\underline{1}}\Big(x\prod_j (1+T_j)^{-1}\Big) \tag{83}$$

for $x \in A$, where $\varepsilon_{-\underline{1}}(y)$ is the coefficient of $\underline{Z}^{-\underline{1}}$ in y for $y \in A$ (expanded in terms of the $\underline{Z}^{\underline{i}}$ as in [BHH+21, Rk. 3.1.1.3(iii)]. Then $\mu \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(A, \mathbb{F})$, as $\mu(F_0 A) = \{0\}$.

By (79)–(80) it suffices to show that for $\underline{1} \le \underline{i} \le \underline{p}$ we have

$$\varepsilon_{-\underline{1}}(\prod_j (1+T_j)^{i_j - 1} \varphi(x)) = 0 \ \text{ if } \underline{i} \neq \underline{p} \tag{84}$$

and

$$\varepsilon_{-\underline{1}}(\prod_j (1+T_j)^{p-1} \varphi(x)) = (-1)^{f-1} \varepsilon_{-\underline{1}}(x). \tag{85}$$

(This time we take representatives $n = \left(\begin{smallmatrix} 1 & \sum_j i_j \alpha_j \\ & 1 \end{smallmatrix}\right)$ with $1 \le i_j \le p$.)

Recalling that we can write

$$Z_j = \sum_i a_{ij} T_i \ \text{ for some } (a_{ij}) \in \mathrm{GL}_f(\mathbb{F}), \tag{86}$$

we deduce (84) and reduce (85) to showing that the coefficient of $\underline{Z}^{\underline{p-1}}$ in $\underline{T}^{\underline{p-1}}$ equals $(-1)^{f-1}$. From (86), by considering the action of φ and letting $a_i \overset{\text{def}}{=} a_{i0}$, we obtain that

$$Z_j = \sum_i a_i^{p^j} T_i \ \text{ with } (a_i^{p^j}) \in \mathrm{GL}_f(\mathbb{F}).$$

As $a_i^{p^f} = a_i$, the a_i are in the image of k in $\mathbb{F}$ and in fact they form an $\mathbb{F}_p$-basis of k. (If not, then $\sum_i \lambda_i a_i = 0$ for some $\lambda_i \in \mathbb{F}_p$ that are not all zero. This implies that $\sum_i \lambda_i a_i^{p^j} = 0$ for all $0 \le j \le f-1$, contradicting that $(a_i^{p^j}) \in \mathrm{GL}_f(\mathbb{F})$.)

Let us now work with formal variables $\underline{x} \overset{\text{def}}{=} (x_i)_{0 \le i \le f-1}$ and b_i $(0 \le i \le f-1)$.

Lemma 3.3.2. *The coefficient of* $\underline{x}^{\underline{p-1}} (= \prod_j x_j^{p-1})$ *in* $\prod_j (\sum_i b_i^{p^j} x_i)^{p-1}$ *equals*

$$\prod_{\underline{c} \in (\mathbb{F}_p^f - \{\underline{0}\})/\mathbb{F}_p^\times} (\sum_i c_i b_i)^{p-1} = (-1)^{(p^f-1)/(p-1)} \prod_{\underline{c} \in \mathbb{F}_p^f - \{\underline{0}\}} (\sum_i c_i b_i).$$

(Note that the first product does not depend on the choice of representatives, and for the equality note that $\prod_{x \in \mathbb{F}_p^\times} x = -1$.)

This lemma implies what we want: as the a_i form an $\mathbb{F}_p$-basis of k, the lemma (applied with $x_i = T_i$, $b_i = a_i$) shows that the coefficient of $\underline{T}^{\underline{p-1}}$ in $\underline{Z}^{\underline{p-1}}$ equals $-(-1)^{(p^f-1)/(p-1)} = (-1)^{f-1}$, as $\prod_{x \in k^\times} x = -1$.

To prove Lemma 3.3.2, we use the following.

Sublemma 3.3.3. *Suppose* $h \in \mathbb{F}[x_0, \dots, x_{f-1}]$. *Then the coefficient of* $\underline{x}^{\underline{p-1}}$ *in* h *is invariant under any linear change of variables over* $\mathbb{F}_p$, *i.e. is equal to the coefficient of* $\underline{y}^{\underline{p-1}}$ *in* h *if* $\underline{x}$ *and* $\underline{y}$ *are related by an element* γ *of* $\mathrm{GL}_f(\mathbb{F}_p)$.

(This is presumably well known. For the proof we may assume that h is a monomial and that γ is an elementary transformation, in which case it follows from the facts that $\mathbb{F}_p^\times$ is of order $p-1$ and that $\binom{r}{p-1} = 0$ for $p \le r \le 2p-2$.)

Let C denote the coefficient of $\underline{x}^{\underline{p-1}} (= \prod_j x_j^{p-1})$ in $\prod_j (\sum_i b_i^{p^j} x_i)^{p-1}$. Then $C \in \mathbb{F}[b_0, \dots, b_{f-1}]$ is a homogeneous polynomial of degree $p^f - 1$, which is clearly divisible by $\underline{b}^{\underline{p-1}}$. For any linear change of variables $x_i = \sum_i \lambda_{ij} y_j$ with $\lambda_{ij} \in \mathbb{F}_p$, Sublemma 3.3.3 then implies that $\prod_j (\sum_i b_i \lambda_{ij})^{p-1}$ divides C. In particular, $(\sum_i c_i b_i)^{p-1}$ divides C for each $c \in (\mathbb{F}_p^f - \{\underline{0}\})/\mathbb{F}_p^\times$. But the product of such polynomials is already of degree $p^f - 1$ and they are pairwise relatively prime, hence we are done by remarking that the coefficient of $\prod_i b_i^{p^i(p-1)}$ is the same on both sides. $\square$

Remark 3.3.4. Fix $\mu \neq 0$ as in Proposition 3.3.1. By uniqueness we must have $\mu \circ a^{-1} \in \mathbb{F}\mu$ for any $a \in \mathcal{O}_K^\times$. But it is easy to compute the scalar: by applying the explicit formula (83) to the element $\prod_j (1 + T_j)^{p-1} \underline{Z}^{\underline{-1}}$ we obtain

$$\mu \circ a^{-1} = N_{k/\mathbb{F}_p}(\overline{a})\mu \quad \forall\, a \in \mathcal{O}_K^\times.$$

Remark 3.3.5. Even though Z_j depends on the choice of T_j (i.e. our choice of basis of $\mathcal{O}_K$), if we write $\prod_j (1 + T_j) = h(\underline{Z})$ for some $h \in \mathbb{F}[\![Z_0, \dots, Z_{f-1}]\!]$, then h is independent of any choices. This is easy to see from our explicit formula for μ in Proposition 3.3.1.

Suppose $\mu \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(A, \mathbb{F})$ is non-zero such that $\mu \circ \psi = (-1)^{f-1}\mu$ and D is an étale $(\varphi, \mathcal{O}_K^\times)$-module over A. Then composition with μ induces an A-linear map

$$\mu_* : \mathrm{Hom}_A(D, A) \to \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D, \mathbb{F}). \tag{87}$$

Recall from Lemma 3.2.2 that $\mathrm{Hom}_A(D, A)$ is naturally an étale $(\varphi, \mathcal{O}_K^\times)$-module. The following lemma will allow us to calculate this structure on the level of $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D, \mathbb{F})$.

Lemma 3.3.6.

(i) *The map μ_* in* (87) *is injective.*

(ii) *We have $\mu_*(\varphi(h)) = (-1)^{f-1}\mu_*(h) \circ \psi$.*

(iii) *We have $\mu_*(a(h)) = N_{k/\mathbb{F}_p}(\overline{a})^{-1}\mu_*(h) \circ a^{-1}$ for $a \in \mathcal{O}_K^\times$.*

Proof. Part (iii) follows immediately from Remark 3.3.4.

For (i) we can reduce to $D = A$ by using that D is finite projective. Observe then that the kernel of μ_* is an $\mathcal{O}_K^\times$-stable ideal of A by (iii); but by [BHH$^+$21, Cor. 3.1.1.7] it is zero, as it cannot be all of A. (Alternatively part (i) also follows from the explicit formula for μ above.)

Part (ii) follows from the explicit formula (78) for φ on $\mathrm{Hom}_A(D, A)$ as well as the two conditions at the beginning of the proof of Proposition 3.3.1. $\square$

We make part (ii) more explicit. Suppose that $h \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ corresponds to a sequence $(x_k)_{k \geq 0}$ as in Proposition 3.2.3. Then $(-1)^{f-1} h \circ \psi$ corresponds to a sequence $(x'_k)_{k \geq 0}$ determined by the relation

$$x'_{pk} = (-1)^{f-1} \begin{pmatrix} p & \\ & 1 \end{pmatrix} x_k, \tag{88}$$

since $\psi \circ \underline{Y}^{-p\underline{k}} = \underline{Y}^{-\underline{k}} \circ \psi$ on $D_A(\pi)$.

Lemma 3.3.7. *Suppose that D is a finite projective A-module. Then the image of $\mu_* : \mathrm{Hom}_A(D, A) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D, \mathbb{F})$ consists precisely of all continuous $\mathbb{F}$-linear maps $h : D \to \mathbb{F}$ such that for all $M \in \mathbb{Z}$ and all $x \in D$ the set $X'_M \overset{\text{def}}{=} \{\underline{i} \in \mathbb{Z}^f : h(\underline{Z}^{\underline{i}} x) \neq 0, \|\underline{i}\| = M\}$ is finite.*

Equivalently, the image of $\mu_* : \mathrm{Hom}_A(D, A) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D, \mathbb{F})$ consists precisely of all continuous $\mathbb{F}$-linear maps $h : D \to \mathbb{F}$ such that for all $M \in \mathbb{Z}$ and all $x \in D$ the set $X_M \overset{\text{def}}{=} \{\underline{i} \in \mathbb{Z}^f : h(\underline{Y}^{\underline{i}} x) \neq 0, \|\underline{i}\| = M\}$ is finite.

Proof. For the first part it is easy to reduce to the case where $D = A$, using the compatibility of μ_* with direct sums $D = D_1 \oplus D_2$. If $h = \mu_*(a)$ for some $a \in A$ and $x \in A$, then we write $ax \prod_j (1 + T_j)^{-1} = \sum_{\underline{i}} \lambda_{\underline{i}} \underline{Z}^{\underline{i}}$ for $\lambda_{\underline{i}} \in \mathbb{F}$. Then $h(\underline{Z}^{\underline{i}} x) = \lambda_{-\underline{i}-\underline{1}}$ (by the explicit formula for μ_* in §3.3), so $h(\underline{Z}^{\underline{i}} x) \neq 0$ can only happen for finitely many $\underline{i}$ of any fixed degree $\|\underline{i}\| = M$. Conversely, if $h : A \to \mathbb{F}$ is continuous such that for all $M \in \mathbb{Z}$ the set $\{\underline{i} : h(\underline{Z}^{\underline{i}}) \neq 0, \|\underline{i}\| = M\}$ is finite, then by continuity of h and the finiteness assumption it follows that $a \overset{\text{def}}{=} (\prod_j (1 + T_j)) \sum_{\underline{i}} h(\underline{Z}^{\underline{i}}) \underline{Z}^{-\underline{i}-\underline{1}} \in A$, and by the explicit formula for μ_* we have $\mu_*(a) = h$.

To justify the second part, recall that Y_j, $Z_j \in \mathbb{F}[\![N_0]\!]$ with $\mathrm{gr}(Y_j) = \mathrm{gr}(Z_j)$, so $Z_j = Y_j \sum_{d=0}^{\infty} F_{d,j}$, where $Y_j F_{d,j}$ is a homogeneous polynomial in $Y_0, \dots, Y_{f-1}$ of degree $d + 1$ and $F_{0,j} = 1$. Define the subring

$$A_0 \overset{\text{def}}{=} \left\{ \sum_{d=0}^{\infty} \frac{F_d}{\underline{Y}^{\underline{d}}} : F_d \text{ a homog. poly. in } Y_0, \dots, Y_{f-1} \text{ of degree } d(f+1) \right\}$$

of A with maximal ideal $\mathfrak{m}_0$ defined by the condition $F_0 = 0$. The above observation then implies that $Z_j \in Y_j(1 + \mathfrak{m}_0)$ for any j, hence

$$\underline{Z}^{\underline{i}} \in \underline{Y}^{\underline{i}}(1 + \mathfrak{m}_0) \quad \forall\ \underline{i} \in \mathbb{Z}^f. \tag{89}$$

Also note that

$$A_0 = \left\{ \sum_{\underline{k} \in \mathbb{Z}^f;\ k_j \geq -\|\underline{k}\|\ \forall j} \lambda_{\underline{k}} \underline{Y}^{\underline{k}} : \lambda_{\underline{k}} \in \mathbb{F} \right\} \tag{90}$$

and that the condition $k_j \geq -\|\underline{k}\|$ for all j implies $k_j \leq f\|\underline{k}\|$ for all j (and $\|\underline{k}\| \geq 0$), so that there are only finitely many terms of any fixed degree.

Fix now $x \in D$ and suppose that the set $X_N = \{\underline{i} \in \mathbb{Z}^f : h(\underline{Y}^{\underline{i}}x) \neq 0, \|\underline{i}\| = N\}$ is finite for any $N \in \mathbb{Z}$. By continuity of h we know that $h(\underline{Y}^{\underline{i}}x) = 0$ for all $\|\underline{i}\| \geq e$ (some $e \in \mathbb{Z}$). Fix any $M \in \mathbb{Z}$ and suppose that $h(\underline{Z}^{\underline{i}}x) \neq 0$ and $\|\underline{i}\| = M$. By equations (89)–(90) we get that $h(\underline{Y}^{\underline{i}+\underline{k}}x) \neq 0$ for some $\underline{k} \in \mathbb{Z}^f$ such that $k_j \geq -\|\underline{k}\|$ for all j. In particular, $\|\underline{i}\| \leq \|\underline{i}\| + \|\underline{k}\| < e$, so

$$X'_M \subseteq \bigcup_{\substack{k_j \geq -\|\underline{k}\| \ \forall j \\ 0 \leq \|\underline{k}\| < e-M}} \Big(X_{M+\|\underline{k}\|} - \underline{k} \Big),$$

a finite union of finite sets. The converse direction follows by reversing the roles of Y_j and Z_j. $\square$

3.4 Some combinatorial lemmas and computations

We give several technical but important lemmas (some generalizing results in [BHH+21, §3.2]) involving the combined action of $\underline{Y}^{\underline{k}}$ (for some $\underline{k} \in \mathbb{Z}^f_{\geq 0}$) and $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in a representation π as at the end of §3.1.

We recall some notation and results from [BHH+21]. We fix $\overline{\rho}$ as at the end of §3.1. We identify $W(\overline{\rho})$ with the subsets of $\{0, \dots, f-1\}$ as in [Bre11, §2] and let J_σ be the subset associated to σ.

Let $\sigma \in W(\overline{\rho})$. Denote $\delta(\sigma) \overset{\text{def}}{=} \delta_{\text{red}}(\sigma)$ if $\overline{\rho}$ is reducible and $\delta(\sigma) \overset{\text{def}}{=} \delta_{\text{irr}}(\sigma)$ if $\overline{\rho}$ is irreducible the Serre weights defined in [Bre11, §5]. We fix a non-zero vector $v_\sigma \in \sigma^{N_0}$, and let $\chi_\sigma : H \to \mathbb{F}^\times$ be the H-eigencharacter of v_σ. As in [BP12, §2] we identify the irreducible constituents of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s)$ with the subsets of $\{0, \dots, f-1\}$ (for example $\emptyset$ corresponds to the socle σ of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s)$). We know that $\delta(\sigma)$ occurs in $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s)$ and we denote by $J^{\max}(\sigma) \subseteq \{0, \dots, f-1\}$ the associated subset. Precisely, using [BP12, Lemma 2.7] one checks that

$$J^{\max}(\sigma) = (J_\sigma \cup J_{\delta(\sigma)}) \setminus (J_\sigma \cap J_{\delta(\sigma)}).$$

By [BHH+21, Lemma 3.2.3.2], we have $|J^{\max}(\sigma)| = |J^{\max}(\delta(\sigma))|$. As a consequence, the quantity

$$m \overset{\text{def}}{=} |J^{\max}(\sigma)| \in \{0, \dots, f-1\}$$

depends only on the orbit of σ.

Write

$$\sigma = (s_0, \dots, s_{f-1}) \otimes \eta, \quad \delta(\sigma) = (s'_0, \dots, s'_{f-1}) \otimes \eta'.$$

Lemma 3.4.1. *The vector* $\prod_{j \in J^{\max}(\sigma)} Y_j^{s'_j} \prod_{j \notin J^{\max}(\sigma)} Y_j^{p-1} \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)(v_\sigma)$ *spans* $\delta(\sigma)^{N_0}$ *as* $\mathbb{F}$*-vector space. Hence there is a unique scalar* $\mu_\sigma \in \mathbb{F}^\times$ *such that*

$$v_{\delta(\sigma)} = \mu_\sigma \cdot \prod_{j \in J^{\max}(\sigma)} Y_j^{s'_j} \prod_{j \notin J^{\max}(\sigma)} Y_j^{p-1} \begin{pmatrix} p & \\ & 1 \end{pmatrix}(v_\sigma) \tag{91}$$

Proof. This is [BHH$^+$21, Prop. 3.2.3.1(i)]. $\square$

By [BHH$^+$21, Lemma 3.2.2.6(ii)], if $\underline{0} \leq \underline{i} \leq \underline{s}$, there is a unique H-eigenvector $\underline{Y}^{-\underline{i}}v_\sigma \in \sigma$ that is sent by $\underline{Y}^{\underline{i}}$ to v_σ. The following result is a generalization of [BHH$^+$21, Lemma 3.2.3.5].

Lemma 3.4.2. *Assume* $m > 0$. *Let* $\underline{k}, \underline{i} \in \mathbb{Z}_{\geq 0}^f$ *such that* $\|\underline{i}\| \leq f-1$ *and* $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)(\underline{Y}^{-\underline{i}}v_\sigma) \neq 0$.

(i) *We have*

$$\|\underline{k}\| \leq p\|\underline{i}\| + \sum_{j \in J^{\max}(\sigma)} s'_j + \sum_{j \notin J^{\max}(\sigma)} (p-1).$$

(ii) *If* $\|\underline{k}\| \geq p\|\underline{i}\| - (f-1) + \sum_{j \in J^{\max}(\sigma)} s'_j + \sum_{j \notin J^{\max}(\sigma)}(p-1)$, *then*

$$\mu_\sigma \cdot \underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)(\underline{Y}^{-\underline{i}}v_\sigma) = \underline{Y}^{-\underline{\ell}}v_{\delta(\sigma)} \in \delta(\sigma)$$

for some $\underline{\ell} \geq \underline{0}$ *with* $\|\underline{\ell}\| \leq f-1$. *More precisely,*

$$\begin{aligned} \ell_j &= i_{j+1}p + s'_j - k_j && \text{if } j \in J^{\max}(\sigma), \\ \ell_j &= i_{j+1}p + (p-1) - k_j && \text{if } j \notin J^{\max}(\sigma). \end{aligned}$$

Proof. Before starting the proof, we first remark that Lemma 3.2.3.3 and Lemma 3.2.3.4 of [BHH$^+$21] remain true if we replace the assumption $\|\underline{i}\| \leq m-1$ by $\|\underline{i}\| \leq f-1$ in the statements. Indeed, for Lemma 3.2.3.3, this new assumption $\|\underline{i}\| \leq f-1$ implies $i_j \leq f-1$, and so

$$2i_j + 1 \leq 2f - 1 \leq s_j$$

for all j (s_j is denoted t_j in *loc. cit.*) by the genericity assumption. Hence, [BHH$^+$20, Prop. 6.2.2] still applies and the rest of the proof of Lemma 3.2.3.3 works without change. The proof of Lemma 3.2.3.4 of [BHH$^+$21] also works through, because one checks that besides the citation to Lemma 3.2.3.3 the condition $\|\underline{i}\| \leq m-1$ is only used to deduce $\|\underline{i}\| \leq f-1$.

Now we prove the lemma, following the proof of [BHH$^+$21, Lemma 3.2.3.5]. We first prove by induction on $\|\underline{i}\| \leq f-1$ the following fact: if

$$\|\underline{k}\| \geq p\|\underline{i}\| - (f-1) + \sum_{j \in J^{\max}(\sigma)} s'_j + \sum_{j \notin J^{\max}(\sigma)} (p-1) \overset{\text{def}}{=} B$$

and $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)(\underline{Y}^{-\underline{i}}v_\sigma) \neq 0$, then $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)(\underline{Y}^{-\underline{i}}v_\sigma) = \underline{Y}^{\underline{k}'}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)(v_\sigma)$ for some $\underline{k}' \in \mathbb{Z}_{\geq 0}^f$ such that $k'_j = k_j - i_{j+1}p$ for all j. This is trivial if $\underline{i} = \underline{0}$, so we can assume $\underline{i} \neq \underline{0}$. Moreover, as in *loc. cit.*, by induction we are reduced to the case $k_j < p$ for all j. We make this assumption and derive below a contradiction (so this case cannot happen).

Define a set J as in *loc. cit.*, i.e.

$$J \overset{\mathrm{def}}{=} \{j \in J^{\max}(\sigma), i_{j+1} = 0\}. \tag{92}$$

As in *loc. cit.* we have

$$\|\underline{k}\| \leq (p-1)(f-|J|) + \sum_{j \in J}(s'_j + 2i_j) + |J \setminus (J^{\max}(\sigma)+1)| \overset{\mathrm{def}}{=} A$$

and to get a contradiction it is enough to show $A < B$, which is equivalent to

$$mp + |J \setminus (J^{\max}(\sigma)+1)| < (p-2)\|\underline{i}\| + (p-1)|J| + C + D, \tag{93}$$

where

$$C \overset{\mathrm{def}}{=} m - (f-1), \quad D \overset{\mathrm{def}}{=} 2\sum_{j \notin J} i_j + \sum_{j \in J^{\max}(\sigma) \setminus J} s'_j.$$

We have the following two cases.

- If $|J^{\max}(\sigma) \setminus J| > 0$, then as in *loc. cit.* $m \leq \|\underline{i}\| + |J|$, hence (93) is implied by

  $$mp + |J \setminus (J^{\max}(\sigma)+1)| < (p-2)\|\underline{i}\| + (p-2)(m - \|\underline{i}\|) + |J| + C + D,$$

  or equivalently

  $$m + (f-1) + |J \setminus (J^{\max}(\sigma)+1)| < |J| + D.$$

  This is slightly stronger than (140) of [BHH$^+$21], but one checks that the argument in *loc. cit.* still allows to conclude.

- If $J^{\max}(\sigma) = J$, then as in *loc. cit.* we have $|J \setminus (J^{\max}(\sigma)+1)| \leq f - m$ and $|J| = m$, and (93) is implied by

  $$mp + (f-m) < (p-2)\|\underline{i}\| + (p-1)m + C + D$$

  or equivalently

  $$2f - 1 < (p-2)\|\underline{i}\| + m + D.$$

  As $\|\underline{i}\| > 0$ and $D \geq 0$, the last inequality holds by our genericity condition (i.e. $p > 4f$).

This proves the desired fact. The rest of the proof is the same as the proof of [BHH$^+$21, Lemma 3.2.3.5] and we omit the details. (Several times $f - m = (f-1) - (m-1)$ has to be added or subtracted from expressions in the last three paragraphs of the proof in *loc. cit.* to account for the weaker lower bound in Lemma 3.4.2(ii).) □

Remark 3.4.3. Taking $\underline{i} = \underline{0}$ in Lemma 3.4.2, we get the following. If $\underline{Y}^{\underline{k}}\begin{pmatrix} p & \\ & 1 \end{pmatrix}(v_\sigma) \neq 0$ for some $\underline{k} \in \mathbb{Z}_{\geq 0}^f$ and if

$$\|\underline{k}\| \geq \sum_{j \in J^{\max}(\sigma)} s'_j + \sum_{j \notin J^{\max}(\sigma)} (p-1) - (f-1),$$

then $\mu_\sigma \cdot \underline{Y}^{\underline{k}}\begin{pmatrix} p & \\ & 1 \end{pmatrix}(v_\sigma) = \underline{Y}^{-\underline{\ell}} v_{\delta(\sigma)} \in \delta(\sigma)$ for some $\|\underline{\ell}\| \leq f-1$. More precisely,

$$\begin{aligned} \ell_j &= s'_j - k_j && \text{if } j \in J^{\max}(\sigma), \\ \ell_j &= (p-1) - k_j && \text{if } j \notin J^{\max}(\sigma). \end{aligned}$$

We will need the following analogue of Lemma 3.4.2. Define $\underline{c} \in \mathbb{Z}^f$ by $c_j \overset{\text{def}}{=} s'_j$ if $j \in J^{\max}(\sigma)$, and $c_j \overset{\text{def}}{=} p-1$ otherwise.

Lemma 3.4.4. *Assume $m > 0$. Let $\underline{i} \in \mathbb{Z}_{\geq 0}^f$ such that $\|\underline{i}\| \leq f-1$. Let $\underline{k} \in \mathbb{Z}_{\geq 0}^f$ and assume that there exists $0 \leq j_0 \leq f-1$ such that*

- (a) $k_{j_0} \leq p(i_{j_0+1} - 1)$ *(hence $i_{j_0+1} \geq 1$);*
- (b) $\|\underline{k}\| > p\|\underline{i}\| + \|\underline{c}\| - c_{j_0}$.

Then $\underline{Y}^{\underline{k}}\begin{pmatrix} p & \\ & 1 \end{pmatrix}(\underline{Y}^{-\underline{i}} v_\sigma) = 0$.

Proof. Assume for a contradiction that $\underline{Y}^{\underline{k}}\begin{pmatrix} p & \\ & 1 \end{pmatrix}(\underline{Y}^{-\underline{i}} v_\sigma) \neq 0$. As in the proof of [BHH+21, Lemma 3.2.3.5], by induction we are reduced to the case $k_j < p$ for all j; we make this assumption from now on. Note $\|\underline{c}\| = \sum_{j \in J^{\max}(\sigma)} s'_j + \sum_{j \notin J^{\max}(\sigma)} (p-1)$.

Let J be the set defined by (92). Then by (a) we have $j_0 \notin J$. As explained in the proof of Lemma 3.4.2, [BHH+21, Lemma 3.2.3.4] still applies, and we get (see the fourth paragraph of the proof of Lemma 3.2.3.5 of *loc. cit.*)

$$\sum_{j \neq j_0} k_j \leq (f-1-|J|)(p-1) + \sum_{j \in J} (s'_j + 2i_j) + |J \setminus (J^{\max}(\sigma)+1)| \overset{\text{def}}{=} A.$$

On the other hand, letting $\gamma \overset{\text{def}}{=} 1$ if $i_{j_0+1} > 1$ and $\gamma \overset{\text{def}}{=} 0$ if $i_{j_0+1} = 1$ we see that $k_{j_0} \leq (p-1)\gamma$ (using (a) when $\gamma = 0$), which together with condition (b) implies

$$\sum_{j \neq j_0} k_j > p\|\underline{i}\| - (p-1)\gamma + \sum_{j \in J^{\max}(\sigma)} s'_j + \sum_{j \notin J^{\max}(\sigma)} (p-1) - c_{j_0} \overset{\text{def}}{=} B.$$

To get a contradiction it is enough to show $A \leq B$.

A computation shows that $A \leq B$ is equivalent to

$$\begin{aligned} mp + |J \setminus (J^{\max}(\sigma)+1)| &\leq (p-2)\|\underline{i}\| + (p-1)|J| + (p-1)(1-\gamma) + C + D \\ &= (p-2)(\|\underline{i}\| + |J| + 1 - \gamma) + |J| + 1 - \gamma + C + D, \end{aligned} \tag{94}$$

where

$$C \overset{\text{def}}{=} m - c_{j_0}, \quad D \overset{\text{def}}{=} 2\sum_{j \notin J} i_j + \sum_{j \in J^{\max}(\sigma) \setminus J} s'_j.$$

If $j \in J^{\max}(\sigma) \setminus J$, then $i_{j+1} > 0$, so we obtain

$$\begin{aligned} |J^{\max}(\sigma) \setminus J| &\leq \sum_{J^{\max}(\sigma) \setminus (J \cup \{j_0\})} i_{j+1} + 1 = \left(\sum_{J^{\max}(\sigma) \setminus (J \cup \{j_0\})} i_{j+1} \right) + i_{j_0+1} + (1 - i_{j_0+1}) \\ &\leq \|\underline{i}\| + (1 - i_{j_0+1}). \end{aligned}$$

As $|J^{\max}(\sigma) \setminus J| = m - |J|$ and $i_{j_0+1} \geq \gamma + 1$, this means

$$m \leq \|\underline{i}\| + |J| + (1 - i_{j_0+1}) \leq \|\underline{i}\| + |J| - \gamma.$$

Thus, to show (94) it is enough to show

$$mp + |J \setminus (J^{\max}(\sigma)+1)| \leq (p-2)(m+1) + |J| + 1 - \gamma + C + D$$

or equivalently

$$2m + |J \setminus (J^{\max}(\sigma)+1)| \leq |J| + (p - 1 - \gamma + C) + D.$$

If $|J^{\max}(\sigma) \setminus J| > 0$, then it is true by [BHH+21, Eq. (140)] (and using $p-2+m-c_{j_0} \geq 0$ as $m \geq 1$). If $J^{\max}(\sigma) = J$, then again as in *loc. cit.*, we have $|J \setminus (J^{\max}(\sigma)+1)| \leq f - m$ and $|J| = m$, and (94) is implied by

$$mp + f - m \leq (p-2)\|\underline{i}\| + (p-1)(m+1-\gamma) + (m - c_{j_0}) + D,$$

equivalently,

$$\begin{aligned} f &\leq (p-2)\|\underline{i}\| + (p-1)(1-\gamma) + (m - c_{j_0}) + D \\ &= (p-2)(\|i\| - \gamma) + (p - 1 - c_{j_0}) + (m - \gamma) + D. \end{aligned}$$

This is true by our genericity condition: indeed, as $\|\underline{i}\| \geq \gamma + 1$, $m \geq 1$, $c_{j_0} \leq p - 1$, and $D \geq 0$, the above inequality is implied by $f \leq p - 2 \leq p - 1 - \gamma$. □

Now, fix $\sigma \in W(\overline{\rho})$ and define $\sigma_i \in W(\overline{\rho})$ inductively by $\sigma_0 \overset{\text{def}}{=} \sigma$ and $\sigma_i \overset{\text{def}}{=} \delta(\sigma_{i-1})$ for $i > 1$. Let $d \geq 1$ be the smallest integer such that $\sigma_d \cong \sigma_0$. For convenience, if $i \geq 0$ we set $\sigma_i \overset{\text{def}}{=} \sigma_{i'}$, where $i' \in \{0, \ldots, d-1\}$ is the unique integer such that $i \equiv i' \pmod{d}$. Write

$$\sigma_i = (s_0^{(i)}, \ldots, s_{f-1}^{(i)}) \otimes \eta_i.$$

For convenience, we introduce the following notation. For $i \geq 1$, define $\underline{c_i^\sigma} \in \mathbb{Z}_{\geq 0}^f$ by

$$c_{i,j}^\sigma \overset{\text{def}}{=} \begin{cases} s_j^{(i)} & \text{if } j \in J^{\max}(\sigma_{i-1}) \\ p-1 & \text{otherwise} \end{cases} \tag{95}$$

(in particular $\underline{0} \leq \underline{c_i^\sigma} \leq \underline{p-1}$). Define a shift function $\delta : \mathbb{Z}^f \to \mathbb{Z}^f$ by setting

$$\delta(\underline{i})_j \overset{\text{def}}{=} i_{j+1}, \quad \underline{i} = (i_j) \in \mathbb{Z}^f.$$

Note that δ does not change $\|\cdot\|$ and that $\underline{Y}^{p\delta(\underline{i})}\begin{pmatrix} p & \\ & 1 \end{pmatrix} = \begin{pmatrix} p & \\ & 1 \end{pmatrix}\underline{Y}^{\underline{i}}$. We inductively define $\underline{a_n^\sigma} \in \mathbb{Z}_{\geq 0}^f$ for $n \geq 0$ as follows: $\underline{a_0^\sigma} \overset{\text{def}}{=} \underline{0}$ and for $n \geq 1$,

$$\underline{a_n^\sigma} \overset{\text{def}}{=} p\delta(\underline{a_{n-1}^\sigma}) + \underline{c_n^\sigma}. \tag{96}$$

For example, $a_{1,j}^\sigma = c_{1,j}^\sigma = s_j^{(1)}$ if $j \in J^{\max}(\sigma)$ and $a_{1,j}^\sigma = c_{1,j}^\sigma = p-1$ if $j \notin J^{\max}(\sigma)$.

For $i \geq 0$ let

$$v_i \overset{\text{def}}{=} v_{\delta^i(\sigma)} \in \delta^i(\sigma)^{N_0} \setminus \{0\}$$

and $\mu_i \overset{\text{def}}{=} \mu_{\delta^i(\sigma)} \in \mathbb{F}^\times$, as defined in Lemma 3.4.1. Then by (91) we have

$$v_i = \mu_{i-1} \cdot \underline{Y}^{\underline{c_i^\sigma}}\begin{pmatrix} p & \\ & 1 \end{pmatrix}(v_{i-1}) \quad \forall\, i \geq 1. \tag{97}$$

Let

$$\lambda_\sigma \overset{\text{def}}{=} (-1)^{d(f-1)}\Big(\prod_{0 \leq i' \leq d-1} \prod_{j \in J^{\max}(\sigma_{i'})} (p-1-s_j^{(i'+1)})!\Big)^{-1} \nu_\sigma, \tag{98}$$

where $\nu_\sigma \in \mathbb{F}^\times$ is defined as before [BHH$^+$21, Prop. 3.2.4.2], i.e. the eigenvalue of the operator S^d defined in [Bre11, §4] acting on σ^{I_1}. Note that ν_σ depends only on the orbit of σ, and hence the same is true for λ_σ.

Lemma 3.4.5. *We have*

$$\prod_{i=0}^{d-1} \mu_i = \lambda_\sigma^{-1}.$$

Proof. This follows from [BHH$^+$21, Lemma 3.2.2.5] and the definition of ν_σ. □

Remark 3.4.6. When π moreover comes from cohomology, i.e. is as in (70) or (71), it is conjectured in [Bre11, §6] and proved in [DL21, Thm. 5.36] that

- if $\overline{\rho}$ is irreducible, then $\nu_\sigma = (-1)^{\frac{dh}{2f}(1+\sum_{j=0}^{f-1} r_j)}(-\det(\overline{\rho})(p))^{\frac{d}{2}}$;
- if $\overline{\rho}$ is reducible, then $\nu_\sigma = (-1)^{\frac{dh}{f}\sum_{j=0}^{f-1} r_j}\lambda_0^{|\overline{J_\sigma}|\frac{d}{f}}\lambda_1^{|J_\sigma|\frac{d}{f}}$, where $J_\sigma \subseteq \{0, 1, \ldots, f-1\}$ is the set corresponding to σ and $\overline{J_\sigma}$ denotes its complement.

Here, h is the number attached to σ in [Bre11, Lemma 6.2] (it is not the integer h of §3.1). By the proof of [Bre11, Lemma 6.2], we deduce

$$\lambda_\sigma = \begin{cases} (-1)^{d(f-1)}(-\det(\overline{\rho})(p))^{\frac{d}{2}} & \text{if } \overline{\rho} \text{ irreducible,} \\ (-1)^{d(f-1)}\lambda_0^{|\overline{J_\sigma}|\frac{d}{f}}\lambda_1^{|J_\sigma|\frac{d}{f}} & \text{if } \overline{\rho} \text{ reducible.} \end{cases} \tag{99}$$

The following result follows by induction from (97), as well as Lemma 3.4.5.

Lemma 3.4.7. *For all $n \geq 0$, we have $\Big(\prod\limits_{i=0}^{n-1}\mu_i\Big)\cdot \underline{Y}^{\underline{a_n^\sigma}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^n(v_0) = v_n$. In particular, for all $n \geq 0$, $\underline{Y}^{\underline{a_{nd}^\sigma}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd}(v_\sigma) = \lambda_\sigma^n v_\sigma$.*

Proposition 3.4.8. *Let $\underline{k} \in \mathbb{Z}_{\geq 0}^f$ and $n \geq 0$. If $\|\underline{k}\| \geq \|\underline{a_n^\sigma}\| - (f-1)$ and $\underline{Y}^{\underline{k}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^n(v_0) \neq 0$, then $\underline{k} = \underline{a_n^\sigma} - \underline{\ell}$ for some $\underline{\ell} \geq \underline{0}$ satisfying $\|\underline{\ell}\| \leq f-1$ and*

$$\Big(\prod_{i=0}^{n-1}\mu_i\Big)\cdot \underline{Y}^{\underline{k}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^n(v_0) = \underline{Y}^{-\underline{\ell}}v_n \in \sigma_n.$$

Proof. If $n = 0$, we necessarily have $\underline{k} = \underline{a_0^\sigma} = \underline{\ell} = \underline{0}$ and there is nothing to prove. Assume $n \geq 1$ and that the statement holds for $n-1$.

Let $\underline{k} \in \mathbb{Z}_{\geq 0}^f$ with $\|\underline{k}\| \geq \|\underline{a_n^\sigma}\| - (f-1)$. Write $\underline{k} = p\delta(\underline{k}') + \underline{k}''$, with $\underline{k}' \geq \underline{0}$ and $\underline{0} \leq \underline{k}'' \leq \underline{p-1}$. Recalling that $\|\delta(\cdot)\| = \|\cdot\|$, the assumption implies the following inequalities

$$p\|\underline{k}'\| + (p-1)f \geq \|\underline{k}\| > \|\underline{a_n^\sigma}\| - f \geq p\|\underline{a_{n-1}^\sigma}\| - f,$$

from which we deduce $\|\underline{k}'\| > \|\underline{a_{n-1}^\sigma}\| - f$, equivalently

$$\|\underline{k}'\| \geq \|\underline{a_{n-1}^\sigma}\| - (f-1).$$

We clearly have

$$\underline{Y}^{\underline{k}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^n(v_0) = \underline{Y}^{\underline{k}''}\begin{pmatrix} p & \\ & 1\end{pmatrix}\Big(\underline{Y}^{\underline{k}'}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{n-1}(v_0)\Big), \tag{100}$$

so in particular $\underline{Y}^{\underline{k}'}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{n-1}(v_0) \neq 0$. As $\|\underline{k}'\| \geq \|\underline{a_{n-1}^\sigma}\| - (f-1)$, by the inductive hypothesis there exists $\underline{\ell}' \geq \underline{0}$ with $\|\underline{\ell}'\| \leq f-1$ such that

$$\underline{k}' = \underline{a_{n-1}^\sigma} - \underline{\ell}' \quad\text{and}\quad \Big(\prod_{i=0}^{n-2}\mu_i\Big)\cdot \underline{Y}^{\underline{k}'}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{n-1}(v_0) = \underline{Y}^{-\underline{\ell}'}v_{n-1} \in \sigma_{n-1}. \tag{101}$$

We first assume $m > 0$ and claim that $\underline{\ell}' = \underline{0}$. Indeed, the relation $\|\underline{k}\| \geq \|\underline{a_n^\sigma}\| - (f-1)$ together with (101) gives

$$\|\underline{k}''\| \geq p\|\underline{\ell}'\| - (f-1) + \|\underline{c_n^\sigma}\|.$$

Lemma 3.4.2(ii) applied with $\sigma = \sigma_{n-1}$ (and genericity) shows that $k''_j \geq \ell'_{j+1}p$ for all j. However, by definition $0 \leq k''_j \leq p-1$, so we must have $\ell'_{j+1} = 0$ for all j. This proves the claim.

By the claim and by equations (100)–(101) we have $\underline{k}' = \underline{a^{\sigma}_{n-1}}$ and

$$\Big(\prod_{i=0}^{n-2} \mu_i\Big) \cdot \underline{Y}^{\underline{k}'} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{n-1} (v_0) = v_{n-1}, \quad \text{so} \quad \underline{Y}^{\underline{k}''} \begin{pmatrix} p & \\ & 1 \end{pmatrix} (v_{n-1}) \neq 0.$$

By the previous paragraph we have moreover that $\|\underline{k}''\| \geq \|\underline{c^{\sigma}_{n}}\| - (f-1)$. Remark 3.4.3 applied with $\sigma = \sigma_{n-1}$ gives $\mu_{n-1} \cdot \underline{Y}^{\underline{k}''} \begin{pmatrix} p & \\ & 1 \end{pmatrix} (v_{n-1}) = \underline{Y}^{-\underline{\ell}} v_n \in \sigma_n$ for some $\underline{\ell} \geq \underline{0}$ satisfying $\|\underline{\ell}\| \leq f-1$ and $\underline{\ell} = \underline{c^{\sigma}_{n}} - \underline{k}''$. As $\underline{k}' = \underline{a^{\sigma}_{n-1}}$ we deduce $\underline{\ell} = \underline{a^{\sigma}_{n}} - \underline{k}$ and the result follows.

Now we assume $m = 0$, equivalently $\sigma \cong \delta(\sigma)$. It is easy to see that this case only happens when $\overline{\rho}$ is reducible (split) and either $J_{\sigma} = \emptyset$ or $J_{\sigma} = \{0, \dots, f-1\}$. In this case we have $\underline{a^{\sigma}_{n}} = \underline{p^n - 1}$ for any $n \geq 0$, and Lemma 3.4.1 implies that $\underline{Y}^{\underline{a^{\sigma}_{n}}} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^n (v_0) \neq 0$. Using (100) and the fact $Y_j v_0 = 0$ for all j, an induction shows that if $k_j \geq p^n$ for some $0 \leq j \leq f-1$, then

$$\underline{Y}^{\underline{k}} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^n (v_0) = \underline{Y}^{\underline{k} - p^n \underline{k}'} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^n \Big(\underline{Y}^{\delta^{-n}(\underline{k}')} v_0\Big) = 0,$$

where $k' \in \mathbb{Z}^f_{\geq 0}$ is defined as: $k'_j = 1$ and $k'_{j'} = 0$ for $j' \neq j$. We deduce that $\underline{Y}^{\underline{k}} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^n (v_0) \neq 0$ if and only if $\underline{k} \leq \underline{a^{\sigma}_{n}}$, which implies the first assertion. The second assertion can be proved as above, noting that Remark 3.4.3 remains true when $m = 0$. ☐

Corollary 3.4.9. *Let $\underline{k} \in \mathbb{Z}^f_{\geq 0}$ and $n \geq 0$.*

(i) *If $\|\underline{k}\| > \|\underline{a^{\sigma}_{n}}\|$, then $\underline{Y}^{\underline{k}} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^n (v_0) = 0$.*

(ii) *If $\|\underline{k}\| = \|\underline{a^{\sigma}_{n}}\|$ and if $\underline{Y}^{\underline{k}} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^n (v_0) \neq 0$, then $\underline{k} = \underline{a^{\sigma}_{n}}$.*

Proof. It is a direct consequence of Proposition 3.4.8. ☐

3.5 The degree function on an admissible smooth representation of $\mathrm{GL}_2(K)$

We define and study a "degree function" on representations π as at the end of §3.1.

Let $\overline{\rho}$ and π be as in *loc. cit.* For $v \in \pi$, we define

$$\deg(v) \overset{\text{def}}{=} \min\{n \geq -1 : \ v \in \pi[\mathfrak{m}_{I_1}^{n+1}]\} \in \mathbb{Z}_{\geq -1}.$$

Fix $\sigma \in W(\overline{\rho})$ and let $v_{\sigma} \in \sigma^{N_0} \setminus \{0\}$. Define $\underline{a^{\sigma}_{n}} \in \mathbb{Z}^f_{\geq 0}$ as in §3.4.

Proposition 3.5.1. *For all $n \geq 0$ we have*

$$\deg\left(\begin{pmatrix} p & \\ & 1 \end{pmatrix}^{n}(v_\sigma)\right) = \|\underline{a_n^\sigma}\|.$$

Proof. Put $u_n \overset{\text{def}}{=} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{n}(v_\sigma)$ for simplicity. First, by the proof of [BHH$^+$20, Cor. 5.3.5], we know that as a $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$-module $\mathrm{gr}(\pi)$ is annihilated by the ideal J defined by $J \overset{\text{def}}{=} (y_jz_j, z_jy_j; 0 \leq j \leq f-1)$, so that $\mathrm{gr}(\pi)$ becomes a graded module over $R \overset{\text{def}}{=} \mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J$ which is commutative, isomorphic to $\mathbb{F}[y_j, z_j]/(y_jz_j; 0 \leq j \leq f-1)$, with y_j, z_j of degree -1. On the other hand, it is easy to check that u_n is annihilated by $\sum_{\lambda\in\mathbb{F}_q} \lambda^{-p^j}\begin{pmatrix} 1 & 0 \\ p[\lambda] & 1 \end{pmatrix} \in \mathbb{F}[\![I_1/Z_1]\!]$ (a lifting of z_j), hence $z_j\,\mathrm{gr}(u_n) = 0$ and consequently we observe that any element in $\langle R\cdot \mathrm{gr}(u_n)\rangle$ is annihilated by z_j.

Next we note the following fact: if $v \in \pi$ with $\deg(v) > 0$ and if $\mathrm{gr}(v)$ is annihilated by all z_j, then there exists some $i \in \{0, \dots, f-1\}$ such that $y_i\,\mathrm{gr}(v) \neq 0$. (If not, suppose $v \in \pi[\mathfrak{m}_{I_1}^{n+1}] \setminus \pi[\mathfrak{m}_{I_1}^{n}]$ for some $n \geq 1$, so R_1, the degree -1 part of R, annihilates v. But $R_1 = \mathfrak{m}_{I_1}/\mathfrak{m}_{I_1}^2$, so $\mathfrak{m}_{I_1}v \subseteq \pi[\mathfrak{m}_{I_1}^{n-1}]$, i.e. $v \in \pi[\mathfrak{m}_{I_1}^{n}]$, contradiction.) As a consequence, $Y_iv \neq 0$ and

$$\deg(Y_iv) = \deg(v) - 1;$$

moreover we have $\mathrm{gr}(Y_iv) = y_i\,\mathrm{gr}(v) \in \langle R\cdot \mathrm{gr}(v)\rangle$. Applying this fact to u_n (and to Y_iu_n, etc.) and using the observation of the last paragraph, we find that there exists $\underline{a_n'^\sigma} \in \mathbb{Z}_{\geq 0}^{f}$ such that $\underline{Y}^{\underline{a_n'^\sigma}}u_n$ is of degree 0, i.e. $\underline{Y}^{\underline{a_n'^\sigma}}u_n \in \pi^{I_1} \setminus \{0\}$ and

$$\deg(u_n) = \|\underline{a_n'^\sigma}\|.$$

On the one hand, we have $\|\underline{a_n'^\sigma}\| \leq \|\underline{a_n^\sigma}\|$ by Corollary 3.4.9(i) (as $\underline{Y}^{\underline{a_n'^\sigma}}u_n \neq 0$). On the other hand, we have $\deg(\begin{pmatrix} p & \\ & 1 \end{pmatrix}^{n}(v_\sigma)) \geq \|\underline{a_n^\sigma}\|$ by Lemma 3.4.7, so the result follows. ☐

If V is any admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$, we define $\deg(v)$ for $v \in V$ as above. On the other hand, by restricting V to N_0, we can also define

$$\deg'(v) \overset{\text{def}}{=} \min\{n \geq -1 : v \in V[\mathfrak{m}_{N_0}^{n+1}]\}.$$

This is well-defined as V is smooth. It is clear that $\deg(v) \geq \deg'(v)$.

We note the following consequence of the proof of Proposition 3.5.1 (it will not be used in this paper).

Corollary 3.5.2. *Let V be in the category $\mathcal{C}$ of §3.1 and assume that $\mathrm{gr}(V)$ is annihilated by the ideal J defined in the proof of Proposition 3.5.1. If $v \in V$ is an element fixed by $\begin{pmatrix} 1 & 0 \\ p\mathcal{O}_K & 1 \end{pmatrix}$, then there exists $\underline{k} \in \mathbb{Z}_{\geq 0}^{f}$ with $\|\underline{k}\| = \deg(v)$ such that $0 \neq \underline{Y}^{\underline{k}}v \in V^{I_1}$. Moreover, we have $\deg(v) = \deg'(v)$.*

3.6 A crucial finiteness result

We prove an important finiteness result (Proposition 3.6.1) which will be crucially used in §3.7 to construct elements of $\mathrm{Hom}_A(D_A(\pi), A)$.

Fix $\sigma \in W(\overline{\rho})$ and define $\sigma_i \in W(\overline{\rho})$, $v_i \in \sigma_i$ and $d \in \mathbb{Z}_{\geq 1}$ as in §3.4 (before Lemma 3.4.5). We have elements $\underline{c^\sigma_n}, \underline{a^\sigma_n} \in \mathbb{Z}^f_{\geq 0}$ defined for $n \geq 1$ (resp. $n \geq 0$) in (95) (resp. (96)). By induction we have

$$\underline{a^\sigma_n} = \sum_{i=0}^{n-1} p^i \delta^i(\underline{c^\sigma_{n-i}})$$

and as $\underline{c^\sigma_n}$ is periodic with period d, we deduce

$$\underline{a^\sigma_{nd'}} = p^{d'} \underline{a^\sigma_{(n-1)d'}} + \underline{a^\sigma_{d'}}, \tag{102}$$

where $d' \overset{\mathrm{def}}{=} df$ (so $\delta^{d'}$ is the identity).

We consider the following elements for $\underline{i} \in \mathbb{Z}^f$:

$$x_{\sigma,\underline{i}} \overset{\mathrm{def}}{=} \lambda_\sigma^n \underline{Y}^{\underline{a^\sigma_{nd}}-\underline{i}} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{nd} (v_\sigma), \tag{103}$$

where λ_σ is defined in (98) and $n \geq 0$ is chosen large enough so that $\underline{a^\sigma_{nd}} - \underline{i} \geq \underline{0}$. By Lemma 3.4.7 the definition is independent of the choice of n.

The following finiteness result is the main result of this section.

Proposition 3.6.1. *For any $M \in \mathbb{Z}$ the set $\{\underline{i} \in \mathbb{Z}^f : x_{\sigma,\underline{i}} \neq 0, \|\underline{i}\| = M\}$ is finite.*

For Lemmas 3.6.2 and 3.6.3 below, we assume $m = |J^{\max}(\sigma)| > 0$.

Lemma 3.6.2. *Let $\underline{k} \in \mathbb{Z}^f_{\geq 0}$ and $n \geq 1$. Assume that for some $0 \leq j_0 \leq f-1$,*

(a) $k_{j_0} \leq a^\sigma_{n,j_0} - p - c^\sigma_{n,j_0}$,

(b) $\|\underline{k}\| > \|\underline{a^\sigma_n}\| - c^\sigma_{n,j_0}$.

Then $\underline{Y}^{\underline{k}} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^n (v_0) = 0$.

Proof. Write $\underline{k} = p\delta(\underline{k}') + \underline{k}''$ with $\underline{k}', \underline{k}'' \geq \underline{0}$ and $\underline{k}'' \leq \underline{p-1}$. Condition (b) implies

$$p\|\underline{k}'\| + (p-1)f \geq \|\underline{k}\| > p\|\underline{a^\sigma_{n-1}}\| + \sum_{j \neq j_0} c^\sigma_{n,j}$$

and consequently

$$p\|\underline{k}'\| + pf > p\|\underline{a^\sigma_{n-1}}\|.$$

Thus, we have $\|\underline{k}'\| > \|\underline{a^\sigma_{n-1}}\| - f$.

Assume $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)^n(v_0) \neq 0$ for a contradiction. Then by the proof of Proposition 3.4.8 we also have $\underline{Y}^{\underline{k}'}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)^{n-1}(v_0) \neq 0$. Moreover, by Proposition 3.4.8, there exists $\underline{i} \geq \underline{0}$ with $\|\underline{i}\| \leq f-1$ such that $\underline{k}' = \underline{a^\sigma_{n-1}} - \underline{i}$ and

$$\Big(\prod_{i=0}^{n-2} \mu_i\Big) \cdot \underline{Y}^{\underline{k}'}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)^{n-1}(v_0) = \underline{Y}^{-\underline{i}} v_{n-1} \in \sigma_{n-1}.$$

Thus, condition (a) translates to

$$p(a^\sigma_{n-1,j_0+1} - i_{j_0+1}) + k''_{j_0} \leq a^\sigma_{n,j_0} - p - c^\sigma_{n,j_0}$$

from which we deduce $k''_{j_0} \leq p(i_{j_0+1} - 1)$ using (96), and we get a contradiction by Lemma 3.4.4 applied to $\underline{k}''$. Indeed, $\underline{Y}^{\underline{k}''}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)(\underline{Y}^{-\underline{i}} v_{n-1}) \neq 0$ and the equality $\underline{k}' = \underline{a^\sigma_{n-1}} - \underline{i}$ together with condition (b) imply

$$\|\underline{k}''\| > p\|\underline{i}\| + \|\underline{c^\sigma_n}\| - c^\sigma_{n,j_0}$$

which verifies the corresponding condition (b) of Lemma 3.4.4 (with $\sigma = \sigma_{n-1}$). $\square$

Recall that $d' = df$, that $\delta^{d'}$ is the identity, and that $\underline{c^\sigma_n}$ is periodic with period d.

Lemma 3.6.3. *Let* $\underline{k} \in \mathbb{Z}^f_{\geq 0}$ *and* $n' > n \geq 0$. *Assume that for some* $0 \leq j_0 \leq f-1$,

(a) $k_{j_0} \leq a^\sigma_{n'd',j_0} - a^\sigma_{nd',j_0} - p^{nd'}(p + c^\sigma_{d,j_0})$ *and*

(b) $\|\underline{k}\| > \|\underline{a^\sigma_{n'd'}}\| - \|\underline{a^\sigma_{nd'}}\| - p^{nd'} c^\sigma_{d,j_0} + f(p^{nd'} - 1)$.

Then $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)^{n'd'}(v_0) = 0$.

Proof. Applying Lemma 3.6.2 with $n \overset{\text{def}}{=} (n'-n)d'$, we see that if $\underline{k}' \in \mathbb{Z}^f_{\geq 0}$ such that $k'_{j_0} \leq a^\sigma_{(n'-n)d',j_0} - p - c^\sigma_{d,j_0}$ (recall that c^σ_{d,j_0} is periodic) and if $\|\underline{k}'\| > \|\underline{a^\sigma_{(n'-n)d'}}\| - c^\sigma_{d,j_0}$, then $\underline{Y}^{\underline{k}'}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)^{(n'-n)d'}(v_0) = 0$.

Write $\underline{k} = p^{nd'}\underline{k}' + \underline{k}''$ with $\underline{k}', \underline{k}'' \geq \underline{0}$ and $\underline{k}'' \leq \underline{p^{nd'} - 1}$. Note that $\underline{a^\sigma_{n'd'}} - \underline{a^\sigma_{nd'}} = p^{nd'}\underline{a^\sigma_{(n'-n)d'}}$ by (102). Firstly, by condition (a) we have

$$p^{nd'} k'_{j_0} \leq k_{j_0} \leq p^{nd'} a^\sigma_{(n'-n)d',j_0} - p^{nd'}(p + c^\sigma_{d,j_0})$$

and so $k'_{j_0} \leq a^{\sigma}_{(n'-n)d',j_0} - p - c^{\sigma}_{d,j_0}$. Secondly, as $f(p^{nd'}-1) - \|\underline{k}''\| \geq 0$, condition (b) implies that

$$p^{nd'}\|\underline{k}'\| > p^{nd'}\|\underline{a^{\sigma}_{(n'-n)d'}}\| - p^{nd'}c^{\sigma}_{d,j_0}$$

so that

$$\|\underline{k}'\| > \|\underline{a^{\sigma}_{(n'-n)d'}}\| - c^{\sigma}_{d,j_0}.$$

We then conclude that $\underline{Y}^{\underline{k}'}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{(n'-n)d'}(v_0) = 0$ as explained above, hence

$$\underline{Y}^{\underline{k}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{n'd'}(v_0) = \underline{Y}^{\underline{k}''}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd'}\underline{Y}^{\underline{k}'}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{(n'-n)d'}(v_0) = 0. \qquad \square$$

Proof of Proposition 3.6.1. If $m = 0$, then the end of the proof of Proposition 3.4.8 implies $x_{\sigma,\underline{i}} = 0$ if $i_j < 0$ for some $0 \leq j \leq f-1$, from which the result easily follows.

Assume $m > 0$ from now on, so that Lemma 3.6.3 applies. Fix any $M \in \mathbb{Z}$. We will show that the set $\{\underline{i} \in \mathbb{Z}^f : x_{\sigma,\underline{i}} \neq 0, \|\underline{i}\| = M\}$ is finite. Choose n large enough such that for all $0 \leq j \leq f-1$:

$$\|\underline{a^{\sigma}_{nd'}}\| + p^{nd'}c^{\sigma}_{d,j} - f(p^{nd'}-1) > M; \tag{104}$$

this is always possible because the left-hand side tends to infinity when $n \to \infty$ (recall that $c^{\sigma}_{d,j} > f$, by genericity).

Now pick any $\underline{i} \in \mathbb{Z}^f$ such that $\|\underline{i}\| = M$. Choose $n' > n$ large enough such that $\underline{a^{\sigma}_{n'd'}} \geq \underline{i}$, hence $x_{\sigma,\underline{i}} \in \mathbb{F}^{\times}\underline{Y}^{\underline{a^{\sigma}_{n'd'}}-\underline{i}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{n'd'}(v_0)$. By (104) and as $\|\underline{i}\| = M$, we get for all $0 \leq j \leq f-1$:

$$\|\underline{a^{\sigma}_{n'd'}}\| - \|\underline{i}\| > \|\underline{a^{\sigma}_{n'd'}}\| - (\|\underline{a^{\sigma}_{nd'}}\| + p^{nd'}c^{\sigma}_{d,j} - f(p^{nd'}-1)).$$

There are two cases:

- If $i_{j_0} \geq a^{\sigma}_{nd',j_0} + p^{nd'}(p + c^{\sigma}_{d,j_0})$ for some j_0, then $x_{\sigma,\underline{i}} = 0$ by Lemma 3.6.3 (applied to $\underline{k} \overset{\text{def}}{=} \underline{a^{\sigma}_{n'd'}} - \underline{i}$).
- Otherwise, we must have $i_j < a^{\sigma}_{nd',j} + p^{nd'}(p + c^{\sigma}_{d,j})$ for all j, and such a set (together with the restriction $\|\underline{i}\| = M$) is automatically finite. Note that the quantities $a^{\sigma}_{nd',j} + p^{nd'}(p + c^{\sigma}_{d,j})$ depend only on our fixed M, as n does. $\square$

3.7 An explicit basis of $\mathrm{Hom}_A(D_A(\pi), A)$

We exhibit an A-basis of $\mathrm{Hom}_A(D_A(\pi), A)$ and explicitly describe its image in the vector space $\mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}}(D_A(\pi), \mathbb{F})$ via the embedding (87).

Recall π and $\overline{\rho}$ are as in Theorem 3.1.3 with $\overline{\rho}$ as at the end of §3.1, in particular π^{I_1} is multiplicity-free for the action of I. For any $\sigma \in W(\overline{\rho})$ and our fixed choice of $v_\sigma \in \sigma^{N_0} \setminus \{0\}$ we define:

$$x_{\sigma,k} \overset{\mathrm{def}}{=} \lambda_\sigma^n \underline{Y}^{\underline{a}^\sigma_{nd} - \underline{k}} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{nd} v_\sigma \tag{105}$$

for $k \geq 0$ and any $n \gg_k 0$. This is well-defined by Lemma 3.4.7.

Recall from Proposition 3.5.1 that

$$\begin{pmatrix} p & \\ & 1 \end{pmatrix}^{nd} v_\sigma \in \pi[\mathfrak{m}_{I_1}^{\|\underline{a}^\sigma_{nd}\|+1}], \quad \text{so } x_{\sigma,k} \in \pi[\mathfrak{m}_{I_1}^{kf+1}],$$

hence by Proposition 3.2.3 the sequence $(x_{\sigma,k})_{k\geq 0}$ defines an element x_σ of $\mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}}(D_A(\pi), \mathbb{F})$ of degree 0.

Theorem 3.7.1. *The elements* $\{x_\sigma : \sigma \in W(\overline{\rho})\}$ *are contained in the image of the injection*

$$\mu_* : \mathrm{Hom}_A(D_A(\pi), A) \hookrightarrow \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}}(D_A(\pi), \mathbb{F})$$

and form an A-basis of $\mathrm{Hom}_A(D_A(\pi), A)$.

We first need a lemma. Note that π^{I_1} is multiplicity-free for the action of I, so there exist unique I-eigenvectors $v_\sigma^* \in (\pi^{I_1})^\vee = \mathrm{gr}_0(\pi^\vee)$ such that $\langle v_\sigma, v_{\sigma'}^* \rangle = \delta_{\sigma,\sigma'}$ (for σ, $\sigma' \in W(\overline{\rho})$). We already know that $D_A(\pi)$ is free by Remark 2.6.2. The following result only applies to our current π but is more precise.

Lemma 3.7.2. *Suppose that π is as above. Then* $\mathrm{gr}(D_A(\pi))$ *is a free* $\mathrm{gr}(A)$*-module with basis* $(v_\sigma^*)_{\sigma\in W(\overline{\rho})}$. *In particular, $D_A(\pi)$ is a filtered free A-module of rank 2^f.*

Proof. Recall from [BHH+21, §3.1] that $\mathrm{gr}(D_A(\pi))$ is obtained from $\mathrm{gr}(\pi^\vee)$ by localizing at $\prod_j y_j$. By localizing the surjection in [BHH+21, Thm. 3.3.2.1] at $\prod_j y_j$ and using [BHH+21, Lemma 3.3.1.3(i)] we obtain a surjection $\oplus_{\sigma\in W(\overline{\rho})} \mathrm{gr}(A) \twoheadrightarrow \mathrm{gr}(D_A(\pi))$ of $\mathrm{gr}(A)$-modules, sending the standard basis element indexed by σ on the left to v_σ^*. But $\mathrm{rk}_{\mathrm{gr}(A)}(\mathrm{gr}(D_A(\pi))) = \mathrm{rk}_A(D_A(\pi)) = 2^f$ by [BHH+21, Lemma 3.1.4.1] and [BHH+21, Cor. 3.3.2.4], hence the surjection $\oplus_{\sigma\in W(\overline{\rho})} \mathrm{gr}(A) \twoheadrightarrow \mathrm{gr}(D_A(\pi))$ is an isomorphism. By [LvO96, Thm. I.4.2.4(5)] we can lift it to an isomorphism $\oplus_{\sigma\in W(\overline{\rho})} A \xrightarrow{\sim} D_A(\pi)$ of filtered A-modules. □

Proof of Theorem 3.7.1. Fix any $\sigma \in W(\overline{\rho})$ and consider the continuous $\mathbb{F}$-linear map $h_\sigma (\overset{\mathrm{def}}{=} x_\sigma) : D_A(\pi) \to \mathbb{F}$ of degree 0 corresponding to the sequence $(x_{\sigma,k})_{k\geq 0}$. We endow $D_A(\pi)$ with its natural good filtration (coming from the $\mathfrak{m}_{I_1}$-adic filtration on $\pi^\vee$, cf. [BHH+21, §3.1.2]). To descend h_σ to $\mathrm{Hom}_A(D_A(\pi), A)$ we now check the second criterion in Lemma 3.3.7. Thus fix any $x \in D_A(\pi)$ and $M \in \mathbb{Z}$. By continuity there exists $e \in \mathbb{Z}$ such that $h_\sigma(F_e D_A(\pi)) = 0$. As $(\pi^\vee)_S$ is dense in $D_A(\pi)$ we can find

$\underline{\ell} \in \mathbb{Z}^f$ and $x^* \in \pi^\vee$ such that $x - \underline{Y}^{\underline{\ell}} x^* \in F_{e+\|\underline{i}\|} D_A(\pi)$. Then $h_\sigma((x - \underline{Y}^{\underline{\ell}} x^*)\underline{Y}^{\underline{i}}) = 0$ for all $\underline{i} \in \mathbb{Z}^f$ such that $\|\underline{i}\| = M$, so we may assume that $x = \underline{Y}^{\underline{\ell}} x^* \in (\pi^\vee)_S$.

As in §3.6 we define $x_{\sigma,\underline{i}} \overset{\text{def}}{=} \lambda_\sigma^n \underline{Y}^{\underline{a}^\sigma_{nd} - \underline{i}} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{nd} (v_\sigma)$ for $\underline{i} \in \mathbb{Z}^f$, where $n \gg_{\underline{i}} 0$. (In particular, $x_{\sigma,(k,\dots,k)} = x_{\sigma,k}$ for $k \geq 0$ and $\underline{Y}^{\underline{j}} x_{\sigma,\underline{i}} = x_{\sigma,\underline{i}-\underline{j}}$ for any $\underline{j} \geq \underline{0}$.) Explicitly,

$$h_\sigma \circ \underline{Y}^{-\underline{k}} = \langle x_{\sigma,k}, - \rangle \quad \text{on } \pi^\vee$$

for all $k \geq 0$, from which it follows from the properties of $(x_{\sigma,\underline{i}})_{\underline{i}}$ that

$$h_\sigma \circ \underline{Y}^{-\underline{i}} = \langle x_{\sigma,\underline{i}}, - \rangle \quad \text{on } \pi^\vee \tag{106}$$

for all $\underline{i} \in \mathbb{Z}^f$. This implies that

$$h_\sigma(\underline{Y}^{\underline{i}} x) = h_\sigma(\underline{Y}^{\underline{i}+\underline{\ell}} x^*) = x^*(x_{\sigma,-(\underline{i}+\underline{\ell})})$$

which can be non-zero for only finitely many $\underline{i}$ by Proposition 3.6.1. Thus h_σ indeed descends to an element H_σ of $\mathrm{Hom}_A(D_A(\pi), A)$.

For the final claim, first note that

$$\mathrm{gr}\Big(\mathrm{Hom}_A(D_A(\pi), A)\Big) \cong \mathrm{Hom}_{\mathrm{gr}(A)}\Big(\mathrm{gr}(D_A(\pi)), \mathrm{gr}(A)\Big)$$

by [LvO96, Lemma I.6.9] and Lemma 3.7.2. By [LvO96, Cor. I.4.2.5(2)] it then suffices to show that the $\mathrm{gr}(H_\sigma)$ ($\sigma \in W(\overline{\rho})$) form a basis of $\mathrm{Hom}_{\mathrm{gr}(A)}(\mathrm{gr}(D_A(\pi)), \mathrm{gr}(A))$. By Lemma 3.7.2, the $\mathrm{gr}(A)$-module $\mathrm{gr}(D_A(\pi))$ has basis v_σ^* ($\sigma \in W(\overline{\rho})$), so it will be enough to establish $\langle \mathrm{gr}(H_\sigma), v_{\sigma'}^* \rangle = \delta_{\sigma,\sigma'} \underline{y}^{-\underline{1}}$ for all σ, $\sigma' \in W(\overline{\rho})$.

By the explicit formula from the proof of Lemma 3.3.7 we know that

$$H_\sigma(x) = (\prod_j (1 + T_j)) \sum_{\underline{i}} h_\sigma(\underline{Z}^{\underline{i}} x) \underline{Z}^{-\underline{i}-\underline{1}} \quad \forall\, x \in D_A(\pi).$$

Consider the equality $\mu \circ H_\sigma = h_\sigma$. Note that H_σ is a filtered map of degree f, since h_σ is of degree 0, $\underline{Z}^{\underline{i}} \in F_{-\|\underline{i}\|} A$, and $\prod_j (1 + T_j) \in F_0 A$. Similarly, μ is a filtered map of degree $-f$. Therefore

$$\mathrm{gr}(\mu) \circ \mathrm{gr}(H_\sigma) = \mathrm{gr}(h_\sigma). \tag{107}$$

Recall that $\mathrm{gr}(A) = \mathbb{F}[y_0^{\pm 1}, \dots, y_{f-1}^{\pm 1}]$. Let $\overline{\varepsilon}_{\underline{i}} : \mathrm{gr}(A) \to \mathbb{F}$ be the map sending $\sum_{\underline{j} \in \mathbb{Z}^f} \lambda_{\underline{j}} \underline{y}^{\underline{j}}$ to $\lambda_{\underline{i}}$; it is $\mathbb{F}$-linear and of degree $\|\underline{i}\|$. By definition, $\mathrm{gr}(\mu) : \mathrm{gr}_f A \to \mathbb{F}$ sends $\mathrm{gr}(\prod_j (1 + T_j) \sum_{\|\underline{i}\| \geq -f} \lambda_{\underline{i}} \underline{Z}^{\underline{i}})$ to $\lambda_{-\underline{1}}$. As $\mathrm{gr}(Y_j) = \mathrm{gr}(Z_j)$, it follows that

$$\mathrm{gr}(\mu) = \overline{\varepsilon}_{-\underline{1}}. \tag{108}$$

On the other hand, relation (106) implies that

$$\mathrm{gr}(h_\sigma) \circ \underline{y}^{-\underline{i}} = \langle \mathrm{gr}(x_{\sigma,\underline{i}}), - \rangle \quad \text{on } \mathrm{gr}(\pi^\vee) \tag{109}$$

for all $\underline{i} \in \mathbb{Z}^f$. (They are graded maps of degree $\|\underline{i}\|$; we filter π as in §3.5.)

Using equations (107)–(109) we compute that

$$\begin{aligned}\overline{\varepsilon}_{\underline{i}-\underline{1}} \circ \mathrm{gr}(H_\sigma) &= \overline{\varepsilon}_{-\underline{1}} \circ \underline{y}^{-\underline{i}} \circ \mathrm{gr}(H_\sigma) = \mathrm{gr}(\mu) \circ \mathrm{gr}(H_\sigma) \circ \underline{y}^{-\underline{i}} \\ &= \mathrm{gr}(h_\sigma) \circ \underline{y}^{-\underline{i}} = \langle \mathrm{gr}(x_{\sigma,\underline{i}}), - \rangle \quad \text{on } \mathrm{gr}(\pi^\vee).\end{aligned}$$

As $\overline{\varepsilon}_{\underline{i}-\underline{1}} \circ \mathrm{gr}(H_\sigma)$ is a map of degree $\|\underline{i}\|$, if $(\overline{\varepsilon}_{\underline{i}-\underline{1}} \circ \mathrm{gr}(H_\sigma))(v_{\sigma'}^*) \neq 0$, then $\|\underline{i}\| = 0$. By the definition of $x_{\sigma,\underline{i}}$ and by Corollary 3.4.9 we know that $x_{\sigma,\underline{i}} = 0$ if $\|\underline{i}\| = 0$ and $\underline{i} \neq \underline{0}$. Therefore,

$$\mathrm{gr}(H_\sigma) = \langle \mathrm{gr}(x_{\sigma,\underline{0}}), - \rangle \underline{y}^{-\underline{1}} = \langle \mathrm{gr}(v_\sigma), - \rangle \underline{y}^{-\underline{1}} \quad \text{on } \mathrm{gr}(\pi^\vee),$$

as desired. □

3.8 The $\mathcal{O}_K^\times$-action on $\mathrm{Hom}_A(D_A(\pi), A)$

We make explicit the $\mathcal{O}_K^\times$-action on the elements x_σ of $\mathrm{Hom}_A(D_A(\pi), A)$ defined in §3.7.

We first consider more generally the actions of A and $\mathcal{O}_K^\times$ on $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ defined so that the map μ_* in (87) becomes A and $\mathcal{O}_K^\times$-linear (cf. Lemma 3.3.6).

Lemma 3.8.1. *Suppose that $h : D_A(\pi) \to \mathbb{F}$ is continuous of degree d, i.e. sending $F_{-d-1}D_A(\pi)$ to 0. Let h correspond to the sequence $(x_k)_{k \geq 0}$ as in Proposition 3.2.3, so $\underline{Y} x_{k+1} = x_k$ and $x_k \in \pi[\mathfrak{m}_{I_1}^{kf+d+1}]$.*

(i) *If $a \in A$, then $ah \overset{\mathrm{def}}{=} h \circ a$ corresponds to the sequence $(y_k)_{k \geq 0}$, where*

$$y_k = \underline{Y}^{\underline{\ell}-\underline{k}} a x_\ell \tag{110}$$

for $\ell \gg_k 0$.

(ii) *If $a \in \mathcal{O}_K^\times$, then $a(h) \overset{\mathrm{def}}{=} N_{k/\mathbb{F}_p}(\overline{a})^{-1}(h \circ \mathrm{diag}(a^{-1}, 1))$ corresponds to the sequence $(z_k)_{k \geq 0}$, where*

$$z_k = N_{k/\mathbb{F}_p}(\overline{a})^{-1} \begin{pmatrix} a & \\ & 1 \end{pmatrix} \frac{\underline{Y}^{\underline{\ell}}}{a^{-1}(\underline{Y}^{\underline{k}})} x_\ell = N_{k/\mathbb{F}_p}(\overline{a})^{-1} \frac{a(\underline{Y}^{\underline{\ell}})}{\underline{Y}^{\underline{k}}} \begin{pmatrix} a & \\ & 1 \end{pmatrix} x_\ell \tag{111}$$

for $\ell \gg_k 0$.

Remark 3.8.2. To explain the notation in equations (110), (111) we note that for $x \in \pi[\mathfrak{m}_{I_1}^e]$ $(e \geq 0)$ we can extend the action of $\mathbb{F}[\![N_0]\!]$ on x to an action of the ring $\mathbb{F}[\![N_0]\!] + F_{-e}A$ such that $F_{-e}A$ kills x (because $F_{-e}\mathbb{F}[\![N_0]\!] = \mathbb{F}[\![N_0]\!] \cap F_{-e}A$ kills x, by assumption). For (110) we note that $\underline{Y}^{-\underline{k}} a \in A = \mathbb{F}[\![N_0]\!]_S + F_{-d-1}A$, so $\underline{Y}^{\underline{\ell}-\underline{k}} a \in \mathbb{F}[\![N_0]\!] + F_{-\ell f-d-1}A$ for $\ell \gg_k 0$ and $x_\ell \in \pi[\mathfrak{m}_{I_1}^{\ell f+d+1}]$. Similarly for (111) we note that $\frac{a(\underline{Y}^{\underline{\ell}})}{\underline{Y}^{\underline{k}}} \in \mathbb{F}[\![N_0]\!] + F_{-\ell f-d-1}A$ for $\ell \gg_k 0$ (and $\begin{pmatrix} a & \\ & 1 \end{pmatrix}$ normalizes I_1).

Proof. For (i) we first note that $h(F_{-d-1}A \cdot \pi^\vee) \subseteq h(F_{-d-1}D_A(\pi)) = 0$, so $h \circ a'|_{\pi^\vee}$ only depends on a' modulo $F_{-d-1}A$. Writing $\underline{Y}^{-\underline{k}}a \in \underline{Y}^{-\underline{\ell}}b + F_{-d-1}A$ as above with $b \in \mathbb{F}[\![N_0]\!]$ and $\ell \gg_k 0$, we compute for $k \geq 0$,

$$h \circ a \circ \underline{Y}^{-\underline{k}}|_{\pi^\vee} = h \circ \underline{Y}^{-\underline{\ell}} \circ b|_{\pi^\vee} = \langle x_\ell, b(-)\rangle = \langle bx_\ell, -\rangle = \langle \underline{Y}^{\underline{\ell}-\underline{k}}ax_\ell, -\rangle \tag{112}$$

as functions $\pi^\vee \to \mathbb{F}$, as desired (keeping in mind Remark 3.8.2).

For (ii), first note that $a(h) \circ \underline{Y}^{-\underline{k}} = N_{k/\mathbb{F}_p}(\overline{a})^{-1}\Big(h \circ a^{-1}(\underline{Y}^{-\underline{k}}) \circ \operatorname{diag}(a^{-1}, 1)\Big)$. By (112) (applied with $k = 0$), $h \circ a^{-1}(\underline{Y}^{-\underline{k}})|_{\pi^\vee} = \langle \underline{Y}^{\underline{\ell}} \cdot a^{-1}(\underline{Y}^{-\underline{k}})x_\ell, -\rangle$ for $\ell \gg_k 0$ and the result follows. ☐

We now determine the $\mathcal{O}_K^\times$-action on the elements $x_\sigma \in \operatorname{Hom}_A(D_A(\pi), A)$ ($\sigma \in W(\overline{\rho})$), as defined in §3.7. By Lemma 3.3.6(iii) we can compute this action on the image of x_σ in $\operatorname{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ (i.e. before descending).

For $a \in \mathcal{O}_K^\times$ and $0 \leq i \leq f-1$ we put

$$f_{a,i} \overset{\mathrm{def}}{=} \frac{\overline{a}^{p^i}Y_i}{a(Y_i)} \in 1 + F_{1-p}A,$$

where we follow the convention in §3.2 of just writing an index i instead of an index σ_i (in particular $f_{a,0} = f_{a,\sigma_0}$ in (22)). Note that $\varphi(f_{a,i}) = f_{a,i-1}^p$. We also let $\chi_\sigma : \mathbb{F}_q^\times \to \mathbb{F}^\times$ denote the eigencharacter of $\operatorname{diag}(-,1)$ on σ^{I_1}.

Proposition 3.8.3. *For $a \in \mathcal{O}_K^\times$ we have*

$$a(x_\sigma) = N_{k/\mathbb{F}_p}(\overline{a})^{-1}\chi_\sigma(\overline{a})\left(\prod_{i=0}^{f-1} f_{a,i}^{-a^\sigma_{d',i}/(1-p^{d'})}\right)x_\sigma$$

in $\operatorname{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$*, where* $d' \overset{\mathrm{def}}{=} df$*.*

Proof. First note that we may apply any element of $\mathbb{F}[\![N_0]\!] + F_{-kf-1}A$ to (105) (with $F_{-kf-1}A$ killing both sides) by applying our convention in Remark 3.8.2 to both $x_{\sigma,k} \in \pi[\mathfrak{m}_{I_1}^{kf+1}]$ and $\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd'} v_\sigma \in \pi[\mathfrak{m}_{I_1}^{\|a^\sigma_{\underline{nd'}}\|+1}]$.

Let us now consider $N_{k/\mathbb{F}_p}(\overline{a})\chi_\sigma(\overline{a})^{-1}\left(\prod_{i=0}^{f-1} f_{a,i}^{a^\sigma_{d',i}/(1-p^{d'})}\right)a(x_\sigma)$. Combining both parts of Lemma 3.8.1 and the previous paragraph, we obtain that its k-th component is given by the following formulas (where $\ell \gg_k 0$ and $n \gg_\ell 0$):

$$\begin{aligned}
&\chi_\sigma(\overline{a})^{-1}\left(\prod_{i=0}^{f-1} f_{a,i}^{a^\sigma_{d',i}/(1-p^{d'})}\right)\frac{a(\underline{Y}^{\underline{\ell}})}{\underline{Y}^{\underline{k}}}\begin{pmatrix} a & \\ & 1\end{pmatrix}x_{\sigma,\ell}\\
&= \chi_\sigma(\overline{a})^{-1}\left(\prod_{i=0}^{f-1} f_{a,i}^{a^\sigma_{d',i}/(1-p^{d'})}\right)\frac{a(\underline{Y}^{\underline{\ell}})}{\underline{Y}^{\underline{k}}}\begin{pmatrix} a & \\ & 1\end{pmatrix}\lambda_\sigma^n \underline{Y}^{\underline{a}^\sigma_{\underline{nd'}}-\underline{\ell}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd'}v_\sigma\\
&= \chi_\sigma(\overline{a})^{-1}\left(\prod_{i=0}^{f-1} f_{a,i}^{a^\sigma_{d',i}/(1-p^{d'})}\right)\lambda_\sigma^n\frac{a(\underline{Y}^{\underline{a}^\sigma_{\underline{nd'}}})}{\underline{Y}^{\underline{k}}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd'}\begin{pmatrix} a & \\ & 1\end{pmatrix}v_\sigma\\
&= \left(\prod_{i=0}^{f-1} f_{a,i}^{a^\sigma_{d',i}/(1-p^{d'})}\right)\lambda_\sigma^n\frac{a(\underline{Y}^{\underline{a}^\sigma_{\underline{nd'}}})}{\underline{Y}^{\underline{k}}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd'}v_\sigma.
\end{aligned}$$

Recalling that $a(Y_i) = \overline{a}^{p^i}Y_i f_{a,i}^{-1}$ and $\underline{a}^\sigma_{\underline{nd'}} = \underline{a}^\sigma_{\underline{d'}}\frac{p^{nd'}-1}{p^{d'}-1}$ the formula simplifies to

$$\begin{aligned}
&= \left(\prod_{i=0}^{f-1} f_{a,i}^{a^\sigma_{d',i}/(1-p^{d'})}\right)\lambda_\sigma^n\left(\prod_{i=0}^{f-1}\overline{a}^{p^i a^\sigma_{nd',i}}\right)\underline{Y}^{\underline{a}^\sigma_{\underline{nd'}}-\underline{k}}\left(\prod_{i=0}^{f-1} f_{a,i}^{-a^\sigma_{nd',i}}\right)\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd'}v_\sigma\\
&= \left(\prod_{i=0}^{f-1} f_{a,i}^{p^{nd'}a^\sigma_{d',i}/(1-p^{d'})}\right)\left(\prod_{i=0}^{f-1}\overline{a}^{p^i a^\sigma_{nd',i}}\right)\lambda_\sigma^n\underline{Y}^{\underline{a}^\sigma_{\underline{nd'}}-\underline{k}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd'}v_\sigma.
\end{aligned}$$

Now $\prod_{i=0}^{f-1} f_{a,i}^{p^{nd'}a^\sigma_{d',i}/(1-p^{d'})}$ only matters modulo $F_{-kf-1}A$. But as $f_{a,i} \in 1 + F_{-(p-1)}A$ we have $f_{a,i}^{p^{nd'}a^\sigma_{d',i}/(1-p^{d'})} \in 1 + F_{-p^{nd'}(p-1)}A$, so for n sufficiently large we can omit this factor. In summary, the k-th component of $N_{k/\mathbb{F}_p}(\overline{a})\chi_\sigma(\overline{a})^{-1}\left(\prod_{i=0}^{f-1} f_{a,i}^{a^\sigma_{d',i}/(1-p^{d'})}\right)a(x_\sigma)$ is given by $\left(\prod_{i=0}^{f-1}\overline{a}^{p^i a^\sigma_{nd',i}}\right)\lambda_\sigma^n\underline{Y}^{\underline{a}^\sigma_{\underline{nd'}}-\underline{k}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd'}v_\sigma = \left(\prod_{i=0}^{f-1}\overline{a}^{p^i a^\sigma_{nd',i}}\right)x_{\sigma,k}$.

Finally notice that $\sum p^i a^\sigma_{nd',i} = (1 + p^{d'} + \cdots + p^{(n-1)d'})\sum p^i a^\sigma_{d',i} \equiv n\sum p^i a^\sigma_{d',i} \pmod{q-1}$, as $f \mid d'$. Since n (sufficiently large) was arbitrary above, we deduce that $\sum p^i a^\sigma_{d',i} \equiv 0 \pmod{q-1}$, and the result follows. $\square$

3.9 Combinatorics of modular Serre weights

We collect explicit formulas on Serre weights in $W(\overline{\rho})$ which will be used in §3.10.

We assume $f \geq 2$. We remark that if $f = 1$ and $\overline{\rho}$ is irreducible, some formulas need to be modified, e.g. Lemma 3.9.1(i). But the main result (Theorem 3.1.3) is known in this case, so it is harmless to exclude it.

Recall from §3.4 that we identify $W(\overline{\rho})$ with the set of subsets of $\{0, 1, \ldots, f-1\}$ as in [Bre11, §2] and let J_σ be the subset associated to $\sigma \in W(\overline{\rho})$. Precisely, if $\sigma = (s_0, \ldots, s_{f-1}) \otimes \eta$, we have $s_j \in \{p-2-r_j, p-3-r_j\}$ for $j \in J_\sigma$ if $j > 0$ or $\overline{\rho}$ is reducible, $s_0 \in \{p-2-r_0, p-1-r_0\}$ if $0 \in J_\sigma$ and $\overline{\rho}$ is irreducible.

Let $J_{\delta(\sigma)}$ be the subset of $\{0, 1, \ldots, f-1\}$ corresponding to $\delta(\sigma)$. We have the following explicit description (see [Bre11, §5]):

$$\begin{aligned} j \in J_{\delta(\sigma)}, j < f-1 \text{ (resp. } f-1 \in J_{\delta(\sigma)}) &\iff j+1 \in J_\sigma \text{ (resp. } 0 \notin J_\sigma) && \text{if } \delta = \delta_{\mathrm{irr}} \\ j \in J_{\delta(\sigma)} &\iff j+1 \in J_\sigma && \text{if } \delta = \delta_{\mathrm{red}}. \end{aligned}$$

Write $\delta(\sigma) = (s'_0, \ldots, s'_{f-1}) \otimes \eta'$. Recall that $\underline{c_1^\sigma} \in \mathbb{Z}^f_{\geq 0}$ is defined by: $c^\sigma_{1,j} = s'_j$ if $j \in J^{\max}(\sigma)$ and $c^\sigma_{1,j} = p-1$ if $j \notin J^{\max}(\sigma)$. The following lemma explicitly determines s'_{i-1} and $c^\sigma_{1,i-1}$ in terms of s_i.

Lemma 3.9.1.

(i) *Assume* $\overline{\rho}$ *is irreducible. If* $i = 0$, *then*

s_0	s'_{f-1}		$c^\sigma_{1,f-1}$
r_0	$p-2-r_{f-1}$	$f-1 \in J^{\max}(\sigma)$	$p-2-r_{f-1}$
r_0-1	$p-3-r_{f-1}$	$f-1 \notin J^{\max}(\sigma)$	$p-1$
$p-2-r_0$	r_{f-1}	$f-1 \notin J^{\max}(\sigma)$	$p-1$
$p-1-r_0$	$r_{f-1}+1$	$f-1 \in J^{\max}(\sigma)$	$r_{f-1}+1$

while if $1 \leq i \leq f-1$, *then*

s_i	$s'_{i-1},\ i=1$	$s'_{i-1},\ i>1$		$c^\sigma_{1,i-1}$
r_i	r_0-1	r_{i-1}	$i-1 \notin J^{\max}(\sigma)$	$p-1$
r_i+1	r_0	$r_{i-1}+1$	$i-1 \in J^{\max}(\sigma)$	s'_{i-1}
$p-2-r_i$	$p-1-r_0$	$p-2-r_{i-1}$	$i-1 \in J^{\max}(\sigma)$	s'_{i-1}
$p-3-r_i$	$p-2-r_0$	$p-3-r_{i-1}$	$i-1 \notin J^{\max}(\sigma)$	$p-1$

(ii) *Assume* $\overline{\rho}$ *is reducible (split). Then for any* $0 \leq i \leq f-1$

s_i	s'_{i-1}		$c^\sigma_{1,i-1}$
r_i	r_{i-1}	$i-1 \notin J^{\max}(\sigma)$	$p-1$
r_i+1	$r_{i-1}+1$	$i-1 \in J^{\max}(\sigma)$	$r_{i-1}+1$
$p-2-r_i$	$p-2-r_{i-1}$	$i-1 \in J^{\max}(\sigma)$	$p-2-r_{i-1}$
$p-3-r_i$	$p-3-r_{i-1}$	$i-1 \notin J^{\max}(\sigma)$	$p-1$

Proof. This is an easy exercise using the relation between J_σ and $J_{\delta(\sigma)}$. Note also that $i \notin J_\sigma$ if and only if $s_{i+1} \in \{r_{i+1}, p-2-r_{i+1}\}$. ☐

Starting from σ, we have defined $\underline{c_n^\sigma} \in \mathbb{Z}^f_{\geq 0}$ and $\underline{a_n^\sigma} \in \mathbb{Z}^f_{\geq 0}$ for $n \geq 1$ in §3.4. The following result determines $\underline{a_{d'}^\sigma}$ in terms of the s_i (recall $d' = df$). For $0 \leq i \leq f-1$, recall that $h_j = r_j + 1$ and define

$$h^{[i]} \overset{\text{def}}{=} h_i + p h_{i+1} + \cdots + p^{f-1-i} h_{f-1} \tag{113}$$

(thus $h^{[0]} = h$).

Lemma 3.9.2.

(i) *Assume $\overline{\rho}$ is irreducible. If $i=0$ then*

s_0	r_0	r_0-1	$p-2-r_0$	$p-1-r_0$
$\frac{a^{\sigma}_{d',0}}{1-p^{d'}}$	$-1+\frac{h}{1+q}$	-1	-1	$-\frac{h}{1+q}$

while if $1 \le i \le f-1$ then

s_i	r_i	r_i+1	$p-2-r_i$	$p-3-r_i$
$\frac{a^{\sigma}_{d',i}}{1-p^{d'}}$	-1	$h^{[i]}-\frac{hp^{f-i}}{1+q}$	$-1-h^{[i]}+\frac{hp^{f-i}}{1+q}$	-1

(ii) *Assume $\overline{\rho}$ is reducible (split). Then for any $0 \le i \le f-1$:*

s_i	r_i	r_i+1	$p-2-r_i$	$p-3-r_i$
$\frac{a^{\sigma}_{d',i}}{1-p^{d'}}$	-1	$h^{[i]}+\frac{hp^{f-i}}{1-q}$	$-1-h^{[i]}-\frac{hp^{f-i}}{1-q}$	-1

Proof. (i) Note that we always have $2|d$ (as $d \nmid f$ but $d|(2f)$) and so $(2f)|d'$. Thus it suffices to prove the formulas for $\frac{a^{\sigma}_{2f,i}}{1-p^{2f}}$; we choose to work with $2f$ because $d|(2f)$ by [Bre11, Lem. 5.2]. Using Lemma 3.9.1, we can inductively determine $c^{\sigma}_{n,i}$ for $1 \le n \le 2f$, and then compute $a^{\sigma}_{2f,i}$ using the formula $a^{\sigma}_{2f,i} = \sum_{k=0}^{2f-1} p^k c^{\sigma}_{2f-k,i+k}$, where $c^{\sigma}_{n,j}$ is understood to be $c^{\sigma}_{n,j \pmod f}$ if $j \ge f$.

We do this in the case $i=0$ and $s_0 = r_0$, and leave the other cases to the reader. In this case, we obtain using Lemma 3.9.1 that

$$c^{\sigma}_{1,f-1} = p-2-r_{f-1},\ \ldots,\ c^{\sigma}_{f-1,1} = p-2-r_1,\ c^{\sigma}_{f,0} = p-1-r_0,$$

$$c^{\sigma}_{f+1,f-1} = r_{f-1}+1,\ \ldots,\ c^{\sigma}_{2f-1,1} = r_1+1,\ c^{\sigma}_{2f,0} = r_0,$$

and so

$$\begin{aligned} a^{\sigma}_{2f,0} &= r_0 + p(r_1+1) + \cdots + p^{f-1}(r_{f-1}+1) \\ &\quad + p^f(p-1-r_0) + p^{f+1}(p-2-r_1) + \cdots + p^{2f-1}(p-2-r_{f-1}) \\ &= (h-1) + p^f(p^f-h) \\ &= (1-p^{2f})(-1+\tfrac{h}{1+p^f}), \end{aligned}$$

proving the result.

(ii) In this case it suffices to prove the formulas for $\frac{a^{\sigma}_{f,i}}{1-p^f}$. The computation is similar to (i) and is easier, and we leave it to the reader. □

3.10 The main theorem on $D_A(\pi)$

We prove Theorem 3.1.3.

Recall $\overline{\rho}$ is as at the end of §3.1. For $\sigma \in W(\overline{\rho})$ and our fixed $v_\sigma \in \sigma^{I_1} \setminus \{0\}$, define as in (105):

$$x_{\sigma,k} \overset{\mathrm{def}}{=} \lambda_\sigma^n \underline{Y}^{\underline{a}^\sigma_{nd} - \underline{k}} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{nd} (v_\sigma)$$

for $k \geq 0$, $n \gg_k 0$ and λ_σ as in Lemma 3.4.5 (recall that λ_σ depends only on the orbit of σ), so that the sequence $x_\sigma \overset{\mathrm{def}}{=} (x_{\sigma,k})_{k \geq 0}$ defines an element of $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cts}}(D_A(\pi), \mathbb{F})$ of degree 0.

We know by Theorem 3.7.1 that $x_\sigma \in \mathrm{Hom}_A(D_A(\pi), A)$ (via μ_*) and that the x_σ for $\sigma \in W(\overline{\rho})$ form an A-basis of $\mathrm{Hom}_A(D_A(\pi), A)$. By Lemma 3.3.6(ii) and (88) the action of φ on $\mathrm{Hom}_A(D_A(\pi), A)$ can be computed on sequences as follows: for any $k \geq 0$,

$$(\varphi(x_\sigma))_k = (-1)^{f-1} \underline{Y}^{\underline{p\ell} - \underline{k}} \begin{pmatrix} p & \\ & 1 \end{pmatrix} (x_{\sigma,\ell}), \tag{114}$$

where ℓ is chosen arbitrarily so that $p\ell \geq k$.

Let $\{x_\sigma^* : \sigma \in W(\overline{\rho})\}$ denote the A-basis of $D_A(\pi)$ that is dual to $\{x_\sigma : \sigma \in W(\overline{\rho})\}$.

Fix $\sigma \in W(\overline{\rho})$ and write $\delta(\sigma) = (s'_0, \ldots, s'_{f-1}) \otimes \eta'$. By Lemmas 3.4.1 and 3.4.5 there exists a constant $\mu_\sigma \in \mathbb{F}^\times$ such that

$$v_{\delta(\sigma)} = \mu_\sigma \cdot \prod_{j \in J^{\max}(\sigma)} Y_j^{s'_j} \prod_{j \notin J^{\max}(\sigma)} Y_j^{p-1} \begin{pmatrix} p & \\ & 1 \end{pmatrix} (v_\sigma) \tag{115}$$

and moreover

$$\prod_{i=0}^{d-1} \mu_{\delta^i(\sigma)} = \lambda_\sigma^{-1}, \tag{116}$$

where λ_σ is as in (98).

Lemma 3.10.1. *We have*

$$\varphi(x_\sigma^*) = (-1)^{f-1} \mu_\sigma \prod_{j \in J^{\max}(\sigma)} Y_j^{s'_j} \prod_{j \notin J^{\max}(\sigma)} Y_j^{p-1} x^*_{\delta(\sigma)}.$$

Proof. By Lemma 3.2.2 the natural pairing $\langle\ ,\ \rangle : \mathrm{Hom}_A(D_A(\pi), A) \times D_A(\pi) \to A$ satisfies

$$\langle \varphi(x), \varphi(y) \rangle = \varphi(\langle x, y \rangle), \quad \langle a(x), a(y) \rangle = a(\langle x, y \rangle)\ \forall\ a \in \mathcal{O}_K^\times,$$

which characterizes the $(\varphi, \mathcal{O}_K^\times)$-module structure of $D_A(\pi)$. Thus, we are reduced to check the relation

$$x_{\delta(\sigma)} = (-1)^{f-1} \mu_\sigma \prod_{j \in J^{\max}(\sigma)} Y_j^{s'_j} \prod_{j \notin J^{\max}(\sigma)} Y_j^{p-1} \varphi(x_\sigma),$$

namely

$$x_{\delta(\sigma),k} = (-1)^{f-1}\mu_\sigma \prod_{j\in J^{\max}(\sigma)} Y_j^{s'_j} \prod_{j\notin J^{\max}(\sigma)} Y_j^{p-1}(\varphi(x_\sigma))_k \tag{117}$$

for any $k \geq 0$.

Write $\underline{Y}^{\underline{c_1^\sigma}} = \prod_{j\in J^{\max}(\sigma)} Y_j^{s'_j} \prod_{j\notin J^{\max}(\sigma)} Y_j^{p-1}$ as in §3.4, so that

$$\underline{a^\sigma_{nd+1}} = \underline{c^\sigma_1} + p\delta(\underline{a^\sigma_{nd}}) = \underline{a^{\delta(\sigma)}_{nd}} + p^{nd}\delta^{nd}\underline{c^\sigma_1} \tag{118}$$

by (96). By definition and using (115), we have

$$\begin{array}{rcl} x_{\delta(\sigma),k} & = & \lambda^n_{\delta(\sigma)}\underline{Y}^{\underline{a^{\delta(\sigma)}_{nd}}-\underline{k}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd}(v_{\delta(\sigma)}) \\ & = & \lambda^n_{\delta(\sigma)}\underline{Y}^{\underline{a^{\delta(\sigma)}_{nd}}-\underline{k}} \cdot \mu_\sigma \underline{Y}^{p^{nd}\delta^{nd}\underline{c^\sigma_1}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd+1}(v_\sigma) \\ & = & \mu_\sigma\lambda^n_{\delta(\sigma)}\underline{Y}^{\underline{a^\sigma_{nd+1}}-\underline{k}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd+1}(v_\sigma), \end{array}$$

where we have applied (118). On the other hand, using (114) we have (for ℓ large enough)

$$\begin{array}{rcl} \underline{Y}^{\underline{c^\sigma_1}}(\varphi(x_\sigma))_k & = & (-1)^{f-1}\underline{Y}^{\underline{c^\sigma_1}}\underline{Y}^{\underline{p\ell}-\underline{k}}\begin{pmatrix} p & \\ & 1\end{pmatrix}(x_{\sigma,\ell}) \\ & = & (-1)^{f-1}\underline{Y}^{\underline{c^\sigma_1}}\underline{Y}^{\underline{p\ell}-\underline{k}} \cdot \lambda^n_\sigma\underline{Y}^{p\delta(\underline{a^\sigma_{nd}})-\underline{p\ell}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd+1}(v_\sigma) \\ & = & (-1)^{f-1}\lambda^n_\sigma\underline{Y}^{\underline{a^\sigma_{nd+1}}-\underline{k}}\begin{pmatrix} p & \\ & 1\end{pmatrix}^{nd+1}(v_\sigma). \end{array}$$

As $\lambda_\sigma = \lambda_{\delta(\sigma)}$, relation (117) is verified. $\square$

Theorem 3.10.2. *Assume $\overline{\rho}$ is absolutely semi-simple. Then Conjecture 3.1.2 is true.*

Proof. We write $D_{A,\sigma_0}(\overline{\rho}) = Ae_0 \oplus Ae_1$ with (e_0, e_1) as in Lemma 2.2.1 (for $d = 2$ and noting e_i instead of $1 \otimes e_i$) when $\overline{\rho}$ is absolutely irreducible and where

$$\begin{cases} \varphi_q(e_0) & = & \lambda_0\Big(\dfrac{Y_0}{\varphi(Y_0)}\Big)^h e_0 \\ \varphi_q(e_1) & = & \lambda_1 e_1 \\ a(e_0) & = & \Big(\dfrac{f_{a,0}}{\varphi(f_{a,0})}\Big)^{\frac{h}{1-q}} e_0 \\ a(e_1) & = & e_1. \end{cases} \tag{119}$$

when $\overline{\rho}$ is reducible split. Let $I \overset{\text{def}}{=} \{0,1\}^f$ and denote by $\underline{i} = (i_j)_j$ an element of I, by (68) and since $\varphi^{f-1-j}(e_{i_j}) \in D_{A,\sigma_{j+1}}(\overline{\rho})$ (see (66)) we have $D_A^{\otimes}(\overline{\rho}) = \bigoplus_{\underline{i}\in I} AE_{\underline{i}}$, where

$$E_{\underline{i}} \overset{\text{def}}{=} \bigotimes_{j=0}^{f-1} \varphi^{f-1-j}(e_{i_j}).$$

We will define an explicit A-linear isomorphism from $D_A^{\otimes}(\overline{\rho}^{\vee}(1))$ to $D_A(\pi)$ and check that it is actually a morphism of $(\varphi, \mathcal{O}_K^{\times})$-modules. Twisting $\overline{\rho}$ and π by the same unramified character and using Lemma 2.9.7 and Lemma 3.1.1, we can assume $\det(\overline{\rho})(p) = 1$, i.e. $\det(\overline{\rho}) = \omega_f^{\sum_{i=0}^{f-1} p^i(r_i+1)}$. Then

$$D_A^{\otimes}(\overline{\rho}^{\vee}(1)) \cong D_A^{\otimes}(\overline{\rho} \otimes \det(\overline{\rho})^{-1}\omega) = D_A^{\otimes}\Big(\overline{\rho} \otimes \omega_f^{-\sum_{i=0}^{f-1} p^i r_i}\Big),$$

and using Lemma 2.9.7 and Lemma 3.1.1 again, it is equivalent to define a morphism of $(\varphi, \mathcal{O}_K^{\times})$-modules

$$D_A^{\otimes}(\overline{\rho}) \longrightarrow D_A\Big(\pi \otimes \omega_f^{-\sum_{i=0}^{f-1} p^i r_i}\Big).$$

Below we write x_J^* instead of x_σ^* for convenience, where $J = J_\sigma$.

(i) Assume first $\overline{\rho}$ is absolutely irreducible. For $J \subseteq \{0, 1, \ldots, f-1\}$, with corresponding Serre weight $\sigma \in W(\overline{\rho})$, define

$$\vartheta : E_{\underline{i_J}} \longmapsto \alpha_J \underline{Y}^{\underline{b_J} - \underline{1}} x_J^*, \tag{120}$$

where $\alpha_J \in \mathbb{F}^{\times}$ are suitable constants, $\underline{i_J} \overset{\text{def}}{=} \mathbf{1}_J$ (i.e. $i_{J,j} = 1$ if $j \in J$ and $i_{J,j} = 0$ if $j \notin J$), and

- $b_{J,i} \overset{\text{def}}{=} 0$ if either $i = 0$ and $s_0 \in \{r_0, r_0 - 1\}$, or $i > 0$ and $s_i \in \{r_i, p - 3 - r_i\}$;
- $b_{J,0} \overset{\text{def}}{=} -h^{[0]} + 1$ if $s_0 = p - 1 - r_0$;
- $b_{J,i} \overset{\text{def}}{=} h^{[i]} + 1$ if $i > 0$ and $s_i = r_i + 1$;
- $b_{J,i} \overset{\text{def}}{=} -h^{[i]}$ if $s_i = p - 2 - r_i$.

Below we check that for well-chosen α_J, ϑ commutes with φ, i.e. $\vartheta\Big(\varphi(E_{\underline{i_J}})\Big) = \varphi(\alpha_J \underline{Y}^{\underline{b_J} - \underline{1}} x_J^*)$. Writing $J' = J_{\delta(\sigma)}$, Lemma 3.10.1 implies

$$\varphi(x_J^*) = (-1)^{f-1} \mu_J \underline{Y}^{\underline{c_{J'}}} x_{J'}^*, \tag{121}$$

where $\mu_J \overset{\text{def}}{=} \mu_{\sigma_J}$, and $\underline{c_{J'}}$ is defined as in §3.4 with respect to the pair $(\sigma, \delta(\sigma))$. Also, using Lemma 2.2.1 it is easy to check that

$$\varphi(E_{\underline{i_J}}) = \begin{cases} E_{\underline{i_{J'}}} & \text{if } i_{J,0} = 0, \\ -\big(\frac{Y_{\sigma_0}}{Y_{\sigma_{f-1}}^p}\big)^h E_{\underline{i_{J'}}} & \text{if } i_{J,0} = 1. \end{cases} \tag{122}$$

Thus, we are reduced to check:

- if $i_{J,0} = 0$ then
$$\begin{cases} \alpha_J \cdot \mu_J = (-1)^{f-1} \alpha_{J'} \\ p\delta(\underline{b_J}) + \underline{1 - p} + \underline{c_{J'}} = \underline{b_{J'}} \end{cases} \tag{123}$$

- if $i_{J,0} = 1$ then

$$\begin{cases} \alpha_J \cdot \mu_J = (-1)^f \alpha_{J'} \\ p\delta(\underline{b_J}) + \underline{1-p} + \underline{c_{J'}} = \underline{b_{J'}} + (h, 0, \ldots, -ph). \end{cases} \tag{124}$$

First assume $0 \notin J$, i.e. $s_0 \in \{r_0, r_0 - 1\}$; note that this implies $s_1 \in \{r_1, p-2-r_1\}$ by the property of $W(\overline{\rho})$. We need to check

$$pb_{J,i} + 1 - p + c_{J',i-1} = b_{J',i-1} \tag{125}$$

for any $0 \le i \le f-1$. It is a direct check using Lemma 3.9.1. We do it for $i = 0, 1$ and leave the other cases as an exercise. Recall that $c_{J',i-1} = s'_{i-1}$ if $i - 1 \in J^{\max}(\sigma_J)$ and $c_{J',i-1} = p - 1$ otherwise.

- If $i = 0$ and $s_0 = r_0$, then $b_{J,0} = 0$ by definition and $c_{J',f-1} = s'_{f-1} = p-2-r_{f-1}$ by Lemma 3.9.1, so we obtain

  $$p \cdot 0 + (1-p) + (p-2-r_{f-1}) = -h^{[f-1]}$$

  which is equal to $b_{J',f-1}$.

- If $i = 0$ and $s_0 = r_0 - 1$, then $b_{J,0} = 0$ by definition and $c_{J',f-1} = p - 1$ by Lemma 3.9.1, so we obtain

  $$p \cdot 0 + (1-p) + (p-1) = 0$$

  which is equal to $b_{J',f-1}$ (as $s'_{f-1} = p - 3 - r_{f-1}$).

- If $i = 1$ and $s_1 = r_1$, then $b_{J,1} = 0$ by definition and $c_{J',0} = p - 1$ by Lemma 3.9.1, so we obtain

  $$p \cdot 0 + (1-p) + (p-1) = 0$$

  which is equal to $b_{J',0}$ (as $s'_0 = r_0 - 1$).

- If $i = 1$ and $s_1 = p - 2 - r_1$, then $b_{J,1} = -h^{[1]}$ by definition and $c_{J',0} = s'_0 = p - 1 - r_0$, so we obtain

  $$p(-h^{[1]}) + (1-p) + (p-1-r_0) = -h^{[0]} + 1$$

  which is equal to $b_{J',0}$.

Assume $0 \in J$, i.e. $s_0 \in \{p-2-r_0, p-1-r_0\}$; note that this implies $s_1 \in \{r_1+1, p-3-r_1\}$. We check (124) for $i = 0$ and leave the other cases as an exercise.

- If $s_0 = p-2-r_0$, then $b_{J,0} = -h^{[0]}$ by definition and $c_{J',f-1} = p-1$ by Lemma 3.9.1, so we obtain

  $$p(-h^{[0]}) + (1-p) + (p-1) = -ph$$

  which equals to $b_{J',f-1} - ph$ (as $b_{J',f-1} = 0$, since $s'_{f-1} = r_{f-1}$).

- If $s_0 = p-1-r_0$, then $b_{J,0} = -h^{[0]}+1$ by definition and $c_{J',f-1} = s'_{f-1} = r_{f-1}+1$, so we obtain

  $$p(-h^{[0]}+1) + (1-p) + (r_{f-1}+1) = (r_{f-1}+1) + 1 - ph$$

  which is equal to $b_{J',f-1} - ph$ (as $b_{J',f-1} = h^{[f-1]} + 1$).

Now we show that the constants α_J can be compatibly chosen so that ϑ is φ-equivariant. Using (123) and (124) it suffices to check, for any J whose orbit has length d, that

$$(-1)^{d(f-1)} \prod_{j=0}^{d-1} \mu_{\delta^j(J)} = \prod_{j=0}^{d-1} (-1)^{i_{\delta^j(J),0}}.$$

As the left-hand side is equal to $(-1)^{-\frac{d}{2}} = (-1)^{\frac{d}{2}}$ by Remark 3.4.6 and (116) (and $\det(\overline{\rho})(p) = 1$), it suffices to show that

$$\sharp\Big\{0 \leq j \leq d-1, 0 \in \delta^j(J)\Big\} = \frac{d}{2}. \tag{126}$$

By the proof of [Bre11, Lemma 5.2], letting $J' = J \cup \{f+j, j \in \overline{J}\}$ (where $\overline{J}$ is the complement of J), then d is also the smallest positive integer such that $J' = J' - d$ as subsets of $\mathbb{Z}/2f\mathbb{Z}$, and in particular d divides $2f$. Since $|J'| = f$ and $J' \cap \{0, 1, \ldots, f-1\} = J$, it is easy to see that

$$\sharp\Big\{0 \leq j \leq 2f-1,\ 0 \in \delta^j(J)\Big\} = f$$

from which we deduce (126).

We now check that ϑ is $\mathcal{O}_K^\times$-equivariant. By Lemma 2.2.1 we know that

$$a(E_{\underline{i_J}}) = \prod_{i=0}^{f-1} \varphi^{f-1-i}\big(f_{a,0}^{h(1-\varphi)q^{i_{J,i}}/(1-q^2)}\big) E_{\underline{i_J}}$$

and by Lemma 3.8.3 we have

$$a(x_J^*) = N_{k/\mathbb{F}_p}(\overline{a}) \chi_\sigma(\overline{a})^{-1} \overline{a}^{\sum_{i=0}^{f-1} p^i r_i} \left(\prod_{i=0}^{f-1} f_{a,i}^{a^\sigma_{d',i}/(1-p^{d'})} \right) x_J^*,$$

where $d' = df$. Thus it suffices to show that

$$\begin{aligned} a(\underline{Y}^{\underline{b_J - 1}}) N_{k/\mathbb{F}_p}(\overline{a}) \chi_\sigma(\overline{a})^{-1} \overline{a}^{\sum_{i=0}^{f-1} p^i r_i} & \left(\prod_{i=0}^{f-1} f_{a,i}^{a^\sigma_{d',i}/(1-p^{d'})} \right) \\ & = \prod_{i=0}^{f-1} f_{a,i+1}^{p^{f-1-i} h q^{i_{J,i}}/(1-q^2)} f_{a,i}^{-p^{f-i} h q^{i_{J,i}}/(1-q^2)} \underline{Y}^{\underline{b_J - 1}}, \end{aligned}$$

which is implied by the following claims (where we use that $2f \mid d'$):

(a) $\chi_\sigma(\overline{a}) = \overline{a}^{\sum_{i=0}^{f-1} p^i b_{J,i} + \sum_{i=0}^{f-1} p^i r_i}$,

(b)

$$\frac{a^\sigma_{d',i}}{1-p^{d'}} + 1 - b_{J,i} = \begin{cases} \dfrac{hp^{f-i}}{1-q^2}[q^{\mathbf{1}_J(i-1)} - q^{\mathbf{1}_J(i)}] & \text{if } 1 \le i \le f-1, \\ \dfrac{h}{1-q^2}[q^{\mathbf{1}_J(f-1)} - q \cdot q^{\mathbf{1}_J(0)}] & \text{if } i = 0. \end{cases}$$

To verify the first claim, note from [Bre11, §2] that

$$\chi_\sigma(\overline{a}) = \overline{a}^{\frac{1}{2}\left(\sum_{i=0}^{f-1} p^i(r_i+s_i) + (q-1)\mathbf{1}_J(f-1)\right)}.$$

It then suffices to show that

$$\frac{1}{2}\Big(\sum_{i=0}^{f-1} p^i(s_i - r_i) + (q-1)\mathbf{1}_J(f-1)\Big) \equiv \sum_{i=0}^{f-1} p^i b_{J,i} \pmod{q-1}.$$

First assume $f-1 \notin J$ (so that $\mathbf{1}_J(f-1) = 0$), equivalently $s_0 \in \{r_0, p-2-r_0\}$. Then $(s_0, \ldots, s_{f-1})$ consists of subsequences of the form $p-2-r_j, p-3-r_{j+1}, \ldots, p-3-r_{j'-1}, r_{j'}+1$ for some $0 \le j < j' \le f-1$ (and r_i for $i \notin \{j, \ldots, j'\}$). Since $b_{J,i} = 0$ if $s_i = r_i$, we are reduced to prove that for $0 \le j < j' \le f-1$,

$$\frac{1}{2}\Big((p-2-2r_j) + \sum_{j<i<j'} p^i(p-3-2r_i) + p^{j'}\Big) \equiv \sum_{j \le i \le j'} p^i b_{J,i} \pmod{q-1}. \tag{127}$$

It is direct to check that the left-hand side of (127) is equal to

$$p^j(p-1-r_j) + \sum_{j<i<j'} p^i(p-2-r_i) = p^j(p-h_j) + \sum_{j<i<j'} p^i(p-1-h_i).$$

On the other hand, by the definition of $b_{J,i}$ the right-hand side of (127) is equal to

$$\begin{aligned} p^j(-h^{[j]}) + p^{j'}(h^{[j']}+1) &= p^{j'} - p^j h_j - \cdots p^{j'-1} h_{j'-1} \\ &= p^j(p-h_j) + \textstyle\sum_{j<i<j'} p^i(p-1-h_i), \end{aligned}$$

hence (127) is verified in this case (we actually have an equality). Now assume $f-1 \in J$ (so that $\mathbf{1}_J(f-1) = 1$), equivalently $s_0 \in \{r_0 - 1, p-1-r_0\}$.

- If $s_0=r_0-1$, then $(s_0,\dots,s_{f-1})$ contains a subsequence of the form $p-2-r_j, p-3-r_{j+1},\dots,r_0-1$ for some $0<j\leq f-1$ (note that the case $j=f-1$ is allowable), and one computes

$$\begin{aligned}&\tfrac{1}{2}\Big(p^j(p-2-2r_j)+\textstyle\sum_{j<i\leq f-1}p^i(p-3-2r_i)+(-1)+q-1\Big)\\ =\ &p^j(p-1-r_j)+\textstyle\sum_{j<i\leq f-1}p^i(p-2-r_i)+(-1)\\ \equiv\ &p^j(-h^{[j]}) \pmod{q-1}.\end{aligned}$$

- If $s_0=p-1-r_0$, then $(s_0,\dots,s_{f-1})$ contains a subsequence of the form $p-2-r_j, p-3-r_{j+1},\cdots,p-1-r_0,p-3-r_1,\dots,p-3-r_{j'-1},r_{j'}+1$ for some $0<j'<j\leq f-1$, and one checks the following congruence relation mod $q-1$:

$$\frac{1}{2}\Big((p-2-2r_j)+\sum_{j<i<j',i\neq 0}p^i(p-3-2r_i)+(p-1-2r_0)+p^{j'}+(q-1)\Big)\equiv\sum_{j\leq i\leq j'}p^ib_{J,i}$$

where $\sum_{j<i<j'}$ means $\sum_{j<i\leq f-1}+\sum_{0\leq i<j'}$ and similarly for $\sum_{j\leq i\leq j'}$.

Together with (127), the claim (a) is verified in this case.

Let's check the claim (b). Using Lemma 3.9.2 and the definition of $\underline{b_J}$ one checks that if $i=0$ we have

s_0	r_0	r_0-1	$p-2-r_0$	$p-1-r_0$
$\frac{a^{\sigma}_{d',0}}{1-p^{d'}}+1-b_{J,0}$	$\frac{h}{1+q}$	0	h	$\frac{hq}{1+q}$

while if $1\leq i\leq f-1$ we have

s_i	r_i	r_i+1	$p-2-r_i$	$p-3-r_i$
$\frac{a^{\sigma}_{d',i}}{1-p^{d'}}+1-b_{J,i}$	0	$-\frac{hp^{f-i}}{1+q}$	$\frac{hp^{f-i}}{1+q}$	0

Then (b) can easily be checked case by case.

(ii) Assume $\overline{\rho}$ is reducible (split). For $J\subseteq\{0,1,\dots,f-1\}$, with corresponding Serre weight $\sigma\in W(\overline{\rho})$, define

$$\vartheta:\ E_{\underline{i_J}}\longmapsto\alpha_J Y^{\underline{b_J}-\underline{1}}x_J^*,$$

where $\alpha_J\in\mathbb{F}^\times$ are suitable constants, $\underline{i_J}\overset{\text{def}}{=}\mathbf{1}_{J^c}$ (i.e. $i_{J,j}=1$ if $j\notin J$ and $i_{J,j}=0$ if $j\in J$), and

- $b_{J,i}=0$ if $s_i=r_i$;
- $b_{J,i}=-h^{[i]}$ if $s_i=p-2-r_i$;

- $b_{J,i} = h^{[i]} + 1$ if $s_i = r_i + 1$ and $i > 0$ (resp. $b_{J,0} = 1$ if $i = 0$);
- $b_{J,i} = 0$ if $s_i = p - 3 - r_i$ and $i > 0$ (resp. $b_{J,0} = -h^{[0]}$ if $i = 0$).

Write $J' = J_{\delta(\sigma)}$. Then (121) remains true, and it is easy to check that

$$\varphi(E_{\underline{i_J}}) = \begin{cases} \lambda_0(\frac{Y_{\sigma_0}}{Y^p_{\sigma_{f-1}}})^h E_{\underline{i_{J'}}} & \text{if } i_{J,0} = 0, \\ \lambda_1 E_{\underline{i_{J'}}} & \text{if } i_{J,0} = 1. \end{cases} \tag{128}$$

Thus, to check that ϑ is φ-equivariant it is equivalent to check

- if $i_{J,0} = 0$ then

$$\begin{cases} \alpha_J \cdot \mu_J = (-1)^{f-1}\lambda_0\alpha_{J'} \\ p\delta(\underline{b_J}) + \underline{1-p} + \underline{c_{J'}} = \underline{b_{J'}} + (h, 0, \dots, -ph) \end{cases} \tag{129}$$

- if $i_{J,0} = 1$ then

$$\begin{cases} \alpha_J \cdot \mu_J = (-1)^{f-1}\lambda_1\alpha_{J'} \\ p\delta(\underline{b_J}) + \underline{1-p} + \underline{c_{J'}} = \underline{b_{J'}}. \end{cases} \tag{130}$$

We leave it as an exercise to check the second equation of (129), resp. (130), using Lemma 3.9.1. Thus, to show that the constants α_J can be compatibly chosen so that ϑ is φ-equivariant, it suffices to check, for any J whose orbit has length d,

$$(-1)^{d(f-1)} \prod_{j=0}^{d-1} \mu_{\delta^j(J)} = \lambda_0^{|J|\frac{d}{f}} \lambda_0^{-|\overline{J}|\frac{d}{f}},$$

where we have used $\det(\overline{\rho})(p) = 1$ and the fact that

$$\sharp\Big\{0 \le j \le d-1, 0 \in \delta^j(J)\Big\} = |J|\frac{d}{f}, \quad \sharp\Big\{0 \le j \le d-1, 0 \notin \delta^j(J)\Big\} = |\overline{J}|\frac{d}{f}.$$

We conclude by Remark 3.4.6 and (116).

We now check that ϑ is $\mathcal{O}_K^\times$-equivariant. Using (119) we know that

$$a(E_{i_J}) = \prod_{i: i_{J,i}=0} \varphi^{f-1-i}(f_{a,0}^{h(1-\varphi)/(1-q)}) E_{i_J}$$

and by Lemma 3.8.3 we have

$$a(x_J^*) = N_{k/\mathbb{F}_p}(\overline{a})\chi_\sigma(\overline{a})^{-1}\overline{a}^{\sum_{i=0}^{f-1} p^i r_i} \left(\prod_{i=0}^{f-1} f_{a,i}^{a^\sigma_{d',i}/(1-p^{d'})} \right) x_J^*,$$

where $d' = df$. Thus it suffices to show that

$$a(\underline{Y}^{\underline{b_J}-\underline{1}})N_{k/\mathbb{F}_p}(\overline{a})\chi_\sigma(\overline{a})^{-1}\overline{a}^{\sum_{i=0}^{f-1}p^i r_i}\left(\prod_{i=0}^{f-1} f_{a,i}^{a^\sigma_{d',i}/(1-p^{d'})}\right)$$

$$= \prod_{\substack{0\le i\le f-1\\ i_{J,i}=0}} (f_{a,i+1}^{hp^{f-i-1}/(1-q)} f_{a,i}^{-hp^{f-i}/(1-q)})\underline{Y}^{\underline{b_J}-\underline{1}},$$

which is implied by the following claims (where we use that $f \mid d'$):

(a) $\chi_\sigma(\overline{a}) = \overline{a}^{\sum_{i=0}^{f-1}p^i b_{J,i}+\sum_{i=0}^{f-1}p^i r_i}$,

(b)

$$\frac{a^\sigma_{d',i}}{1-p^{d'}} + 1 - b_{J,i} = \begin{cases} \dfrac{hp^{f-i}}{1-q}[\mathbf{1}_J(i-1) - \mathbf{1}_J(i)] & \text{if } 1 \le i \le f-1, \\ \dfrac{h}{1-q}[\mathbf{1}_J(f-1) - q\mathbf{1}_J(0)] & \text{if } i = 0. \end{cases}$$

Both claims are checked as in the irreducible case (we omit the details). □

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