DENSITY OF AUTOMORPHIC POINTS IN DEFORMATION RINGS OF POLARIZED GLOBAL GALOIS REPRESENTATIONS

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Abstract. Conjecturally, the Galois representations that are attached to essentially selfdual regular algebraic cuspidal automorphic representations are Zariski-dense in a polarized Galois deformation ring. We prove new results in this direction in the context of automorphic forms on definite unitary groups over totally real fields. This generalizes the infinite fern argument of Gouvea-Mazur and Chenevier, and relies on the construction of non-classical p-adic automorphic forms, and the computation of the tangent space of the space of trianguline Galois representations. This boils down to a surprising statement about the linear envelope of intersections of Borel subalgebras.

Contents

1. Introduction

Let F be a number field, fix a positive integer $n \geqslant 1$ and a prime number p. The goal of this paper is to study some properties of deformation spaces of continuous representations

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_n(\mathbf{F})$$

where $\mathbf{F}$ is a finite extension of $\mathbf{F}_p$. Assume that $\overline{\rho}$ is absolutely irreducible and unramified outside a finite set of places S containing the set S_p of places dividing p. Mazur proved in [Maz89] that there exists a universal deformation of $\overline{\rho}$ unramified outside of S, that is, for $F_S \subset \overline{F}$ the maximal algebraic extension of F unramified outside of S, a complete local noetherian ring $R_{\overline{\rho},S}$ and a continuous representation

$$\rho_S^{\mathrm{univ}} : \mathrm{Gal}(F_S/F) \longrightarrow \mathrm{GL}_n(R_{\overline{\rho},S})$$

pro-representing the functor of deformations of $\overline{\rho}$ unramified outside S. The generic fiber $\mathcal{X}_{\overline{\rho},S}$ of the formal scheme $\mathrm{Spf}\, R_{\overline{\rho},S}$ is a rigid analytic space over $W(\mathbf{F})[\frac{1}{p}]$ whose closed points can be canonically identified with liftings of $\overline{\rho}$ to finite extensions of the p-adic field $W(\mathbf{F})[\frac{1}{p}]$.

When F is a totally real field or a CM field, it is known that we can attach to each regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ an n-dimensional p-adic continuous representation

$$\rho_\pi : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_n(\overline{\mathbf{Q}_p}).$$

This representation is characterized by some local compatibility with π at almost all finite places of F. As a consequence it is unramified outside a finite number of places. A very natural problem with regard to the rigid analytic spaces $\mathcal{X}_{\overline{\rho},S}$ concerns the distribution of automorphic points in $\mathcal{X}_{\overline{\rho},S}$, that is points corresponding to regular algebraic cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\overline{\rho}_\pi$ reduces to $\overline{\rho} \otimes_{\mathbf{F}} \overline{\mathbf{F}_p}$ modulo p.

Beyond the case $n = 1$ which is a consequence of class field theory, the case $n = 2$, $F = \mathbf{Q}$ and $\overline{\rho}$ attached to a modular form has been solved by the works of Gouvea-Mazur ([GM98]), Böckle ([Bö01]), Diamond-Flach-Guo ([DFG04]) and Khare-Wintenberger ([KW09a], [KW09b]), we refer to [Eme, §7.3] for more details. It follows from their results that the space $\mathcal{X}_{\overline{\rho},S}$ is equidimensional of dimension 3 and that the automorphic points are Zariski-dense inside $\mathcal{X}_{\overline{\rho},S}$.

For general values of n, the case of polarized representations has been studied by Chenevier in the paper [Che11]. Let $\varepsilon : \mathrm{Gal}(\overline{F}/F) \to \mathbf{Z}_p^\times$ be the cyclotomic character. We now introduce the global set up that we will use in the paper. Assume that E is a totally imaginary quadratic extension of a totally real number field F and let c be the non trivial element of $\mathrm{Gal}(E/F)$. We recall that an n-dimensional p-adic representation $\rho : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_n(\overline{\mathbf{Q}_p})$ is *polarized* if there exists an isomorphism

$$\rho^\vee \circ c \simeq \rho \otimes \varepsilon^{n-1}.$$

When the regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ is *conjugate self dual*, that is $\pi^{\vee,c} \simeq \pi$, the representation ρ_π is polarized. Moreover the representation ρ_π is crystalline at p if and only if the representations $\pi_{\tilde{v}}$ are unramified for $\tilde{v} \mid p$. In this situation, we have the following conjecture of Chenevier ([Che11, Conj. 1.15]):

Conjecture 1.1. *Assume that $\overline{\rho}$ is absolutely irreducible and polarized, then the set of points of the form $\rho_{\pi'}$ for π' a regular conjugate self dual algebraic cuspidal automorphic representation unramified outside of S is Zariski dense in $\mathcal{X}_{\overline{\rho},S}$.*

In this paper, we are proving some new cases of Conjecture 1.1 under the assumption that $\overline{\rho}$ is *automorphic*, i.e. that there exists a regular conjugate self dual algebraic cuspidal automorphic representation π, unramified at p and outside of S, such that $\overline{\rho} \otimes_{\mathbf{F}} \overline{\mathbf{F}_p} \simeq \overline{\rho}_\pi$.

When $n = 3$, $E_{\tilde{v}} = \mathbf{Q}_p$ for $\tilde{v} \mid p$ and the deformation functor of $\overline{\rho}$ is *unobstructed*, this conjecture has been proven by Chenevier for $\overline{\rho}$ automorphic in *loc. cit.*

The main result of this paper is the following.

Theorem 1.2. *Assume $p > 2$ and the following assumptions*

- *the extension E/F is unramified and E does not contain a p-th root of unity ζ_p;*
- $2 \mid [F : \mathbf{Q}]$ *if $n \equiv 2$ (mod. 4);*
- *S contains only places which are split in E ;*
- *the representation $\overline{\rho}$ is absolutely irreducible and the group* $\overline{\rho}(\mathrm{Gal}(\overline{E}/E(\zeta_p)))$ *is* adequate *in the sense of* [Tho12].

Assume moreover that there exists some regular conjugate self dual cuspidal automorphic representation π which is unramified outside of $S \setminus S_p$ and such that $\overline{\rho_\pi} \simeq \overline{\rho} \otimes_{\mathbf{F}} \overline{\mathbf{F}_p}$. Then the Zariski closure of automorphic points in $\mathcal{X}_{\overline{\rho},S}$ is a union of irreducible components.

In the paper [All19], Patrick Allen proved that, assuming standard automorphy lifting conjectures, it is true that all irreducible components of the space $\mathcal{X}_{\overline{\rho},S}$ contain some regular conjugate self dual cuspidal automorphic point. As such points are smooth by [All16], Theorem 1.2 covers substantial new cases of Conjecture 1.1 under the standard automorphy lifting conjectures.

Let us also mention that David Guiraud proved in [Gui] that, when the weight of π satisfies a strong condition of regularity, the set of places λ of the coefficient field of π such that the pair $(\overline{\rho}_{\pi,\lambda}, S \cup S_{\ell_\lambda})$ is unobstructed has density one (with S the set of ramification of π).

Following [Che11], our strategy to prove Theorem 1.2 is to use base change results between automorphic representations of unitary groups and conjugate self dual automorphic representations of linear groups to deduce this result from an analogous result concerning automorphic forms on some definite unitary group G.

For this definite unitary group we can rely on a well developed theory of families of p-adic automorphic forms, so called eigenvarieties: there exists a rigid analytic space called the eigenvariety $Y(U^p, \overline{\rho})$ parametrizing overconvergent p-adic eigenforms on G. This space is a generalization of the eigencurve of Coleman and Mazur, and was first introduced by Chenevier in the setting of definite unitary groups. The existence of a family of Galois representations on $Y(U^p, \overline{\rho})$ gives rise to a map $f : Y(U^p, \overline{\rho}) \to \mathcal{X}_{\overline{\rho},S}$. The image of this map is the so called “infinite fern”.

The main idea is to consider the Zariski-closure $\mathcal{X}^{\mathrm{aut}}_{\bar{\rho},S} \subset \mathcal{X}_{\overline{\rho},S}$ of all automorphic points and show that each of its irreducible components contains a smooth point ρ such that there is an equality of tangent spaces

$$T_\rho \mathcal{X}^{\mathrm{aut}}_{\bar{\rho},S} = T_\rho \mathcal{X}_{\overline{\rho},S}.$$

We are hence reduced to proving that the left hand side is large enough.

It is well known that (in our set up starting with a definite unitary group) automorphic points form a Zariski dense subset of the eigenvariety $Y(U^p, \bar{\rho})$ and hence the canonical map

$$\bigoplus_x T_x Y(U^p, \overline{\rho}) \longrightarrow T_\rho \mathcal{X}_{\overline{\rho},S} \tag{1.1}$$

factors through the tangent space $T_\rho \mathcal{X}^{\mathrm{aut}}_{\bar{\rho},S}$, where the direct sum is indexed by all the preimages $x \in Y(U^p, \overline{\rho})$ of ρ. Hence it would suffice to prove that (1.1) is surjective.

One of the main results of [BHS19] is the precise determination of the fiber $f^{-1}(\rho)$. In [Che11] it is shown that the map (1.1) is surjective, if the restriction of ρ to the local Galois groups at places dividing p satisfies some genericity assumption: roughly, the representations should be crystalline and the Hodge filtration in general position with respect to all possible Frobenius stable flags. The main problem is that in higher dimensions there is (for the time being) no way to guarantee that $\mathcal{X}_{\overline{\rho},S}$ contains any point satisfying this assumption. The point of our paper is the proof of the surjectivity of (1.1) without this genericity assumption.

Remark 1.3. For the sake of clarity let us point out that in fact we are not going to prove that (1.1) is a surjection. Instead we will prove the following slightly weaker statement that will be sufficient to conclude with the proof. As ρ is crystalline at p, a point $x \in Y(U^p, \overline{\rho})$ such that $f(x) = \rho$ determines a refinement of ρ, i.e. some additional data depending on the restriction of ρ at decomposition groups at p. The set $\{x \in Y(U^p, \overline{\rho})|\ f(x) = \rho\}$ has a partition $\{x \in Y(U^p, \overline{\rho})|\ f(x) = \rho\} = \bigcup_{\mathcal{R}} A_{\mathcal{R}}$ into subsets $A_{\mathcal{R}}$ indexed by the refinements $\mathcal{R}$ of ρ. Locally in a neighborhood V_x of $x \in Y(U^p, \overline{\rho})$ with $f(x) = \rho$ and a neighborhood U_ρ of ρ the map f turns out to be a closed immersion $V_x \hookrightarrow U_\rho$. After localizing further we can consider closed subspaces $Z_{\mathcal{R}} \subset U_\rho \subset \mathcal{X}_{\overline{\rho},S}$ defined as the union of the closed subspaces V_x, where x varies over $A_{\mathcal{R}}$. What we are really going to prove is that the map

$$\bigoplus_{\mathcal{R}} T_\rho Z_{\mathcal{R}} \longrightarrow T_\rho \mathcal{X}_{\overline{\rho},S}$$

is surjective when ρ is moreover supposed to be φ*-generic*. In the body of the paper we will rather work with deformation rings than with the open neighborhoods V_x and U_ρ and the notations we used here will not appear later on. We just introduced them here to make precise which statement we are going to prove.

As in [Che11] we do so by proving a similar surjectivity result for local avatars of the spaces $\mathcal{X}_{\overline{\rho},S}$ and $Y(U^p, \overline{\rho})$: the global deformation ring is replaced by a local deformation ring, and the eigenvariety is replaced by the so called space of trianguline Galois representations. The key construction of [BHS19] is a local model for the space of trianguline representations. This local model allows us to reduce the surjectivity of (1.1) to a problem in linear algebra. In this paper we will mainly work with deformation functors bypassing the definition of the trianguline variety.

The problem of linear algebra mentionned above is to determine the linear envelope of the intersection of a Borel algebra $\mathfrak{b}$ in the Lie-algebra $\mathfrak{gl}_n$ with the Weyl group translates of a fixed Borel $\mathfrak{b}_0$. This statement seems to be a very nice and interesting statement in its own right:

Theorem 1.4. *Let n be a positive integer, $\mathfrak{S}_n$ the symmetric group of order n, and $\mathfrak{gl}_n$ the algebra of $n \times n$ matrices with entries in a fixed field k. Let $\mathrm{GL}_n(k)$ be the group of the non-singular elements in $\mathfrak{gl}_n$ and $\mathfrak{b}_0 \subset \mathfrak{gl}_n$ the Borel subalgebra of upper triangular matrices. For any element $g \in \mathrm{GL}_n(k)$ let $\mathfrak{b}_g = g^{-1}\mathfrak{b}_0 g$ denote the Borel subalgebra conjugate to $\mathfrak{b}_0$ by g^{-1}.*

Any Borel subalgebra $\mathfrak{b}$ coincides with the linear envelope of its intersections with the conjugate of $\mathfrak{b}_0$ under $\mathfrak{S}_n$

$$\mathfrak{b} = \sum_{w \in \mathfrak{S}_n} \mathfrak{b} \cap \mathfrak{b}_w.$$

The plan of the paper is the following. In a first section, we prove Theorem 1.4 and prove as an application a surjectivity result for a map between tangent spaces of our local models. The second section is purely local and its purpose is to prove our main local result concerning the sum of tangent spaces of quasi-trianguline deformation spaces. Finally the last section is of global nature and contains the proof of our main global theorem.

Remark 1.5. After finishing the redaction of this paper, the authors discovered that the recent paper [EP20] of Emerton and Paškūnas can be used to show that some hypothesis in Theorem 1.2 could be relaxed. Namely we require the existence of some regular conjugate self dual cuspidal automorphic representation π unramified outside of $S\setminus S_p$ and such that $\overline{\rho_\pi}\simeq\overline{\rho}$. It should be sufficient to require that π is unramified outside of S (and can be ramified at p). Namely if such a point exists, then it follows from Theorem 5.1 in [EP20] that there exists a point in $\mathcal{X}_{\overline{\rho},S}$ which is automorphic, potentially crystalline with abelian descent data, hence of finite slope. This implies that there exists some tame level U^p unramified outside of S such that the corresponding eigenvariety $Y(U^p,\overline{\rho})$ is non empty. The irreducible components of $Y(U^p,\overline{\rho})$ are finite over some Fredholm hypersurfaces so that they have a "large image" in the weight space which contain regular algebraic weights. We can deduce from this fact that each irreducible component of $Y(U^p,\overline{\rho})$ contains classical points of regular dominant weight which are unramified at p whose images in $\mathcal{X}_{\overline{\rho},S}$ give rise to Galois representations associated to regular conjugate self dual cuspidal automorphic representation π which are unramified outside of $S\setminus S_p$.

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Notation : We fix a prime number p. Let K be a finite extension of $\mathbf{Q}_p$ and $\overline{K}$ an algebraic closure of K. We denote by v_K the unique valuation of K taking the value 1 on uniformizers of K. We use the notation $\mathcal{G}_K$ for the Galois group $\mathrm{Gal}(\overline{K}/K)$. Let $\chi_{\mathrm{cyc}}:\mathcal{G}_K\to\mathbf{Z}_p^\times$ be the cyclotomic character. We denote by $\mathcal{O}_K$ the ring of integers of K, by $\mathfrak{m}_K$ the unique maximal ideal of $\mathcal{O}_K$ and by $k_K\coloneqq\mathcal{O}_K/\mathfrak{m}_K$ its residue field. Let $|\cdot|_p$ be the unique norm on K inducing the p-adic norm on $\mathbf{Q}_p$. Let ε_K be the character $N_{K/\mathbf{Q}_p}|N_{K/\mathbf{Q}_p}|$ from $K^\times$ to $\mathbf{Z}_p^\times$. Let $\mathrm{rec}_K:K^\times\to W_K^{\mathrm{ab}}$ be the local reciprocity isomorphism sending a uniformizer of K onto a *geometric* Frobenius element so that $\varepsilon_K\circ\mathrm{rec}_K^{-1}$ is the cyclotomic character.

If X is some algebraic variety defined over K, we will use the notation $X_{K/\mathbf{Q}_p}$ for the Weil restriction of X from K to $\mathbf{Q}_p$. If L is a finite extension of $\mathbf{Q}_p$ and if Y is some algebraic variety defined over $\mathbf{Q}_p$, we will use the notation Y_L for the base change of Y from $\mathbf{Q}_p$ to L, so that $X_{K/\mathbf{Q}_p,L} = (X_{K/\mathbf{Q}_p}) \times_{\operatorname{Spec}\mathbf{Q}_p} \operatorname{Spec} L$. Let Σ be the set of $\mathbf{Q}_p$-algebra homomorphisms from K to L. If L is big enough so that $|\Sigma| = [K : \mathbf{Q}_p]$, or equivalently $L \otimes_{\mathbf{Q}_p} K \simeq L^{[K:\mathbf{Q}_p]}$, then we have an isomorphism of algebraic varieties over L

$$X_{K/\mathbf{Q}_p,L} \simeq \prod_{\tau\in\Sigma} X_\tau \tag{1.2}$$

where X_τ is the base change of X from K to L via the embedding τ. If x is some L-point of $X_{K/\mathbf{Q}_p}$ we will denote $(x_\tau) \in \prod_{\tau\in\Sigma} X_\tau$ its image by the isomorphism (1.2).

If $\mathbf{k} \in (\mathbf{Z}^n)^{[K:\mathbf{Q}_p]}$, we define the algebraic character $\delta_{\mathbf{k}} : (K^\times)^n \to L^\times$ by the formula

$$(a_1, \dots, a_n) \mapsto \prod_{i=1}^n \prod_{\tau\in\Sigma} \tau(a_i)^{k_{i,\tau}}$$

If X is a scheme, or a rigid analytic space and $x \in X$ is a point, we write T_xX for the tangent space of X at x. Similarly, if $\mathfrak{X}$ is a deformation functor defined on a category of local artinian L-algebras with fixed residue field L (or a formal scheme pro-representing such a functor), we write $T\mathfrak{X} := \mathfrak{X}(L[\varepsilon])$ for the tangent space of $\mathfrak{X}$, where $L[\varepsilon] = L[X]/X^2$.

2. On intersections of Borel algebras

2.1. **Envelopes of intersections of Borel subalgebras.** Let n be a positive integer and k a field. We denote by $\mathfrak{gl}_n$ the Lie algebra of $n\times n$-matrices with coefficients in k and by $\mathfrak{b}_0 \subset \mathfrak{gl}_n$ the Borel subalgebra of upper triangular matrices. Given an element $g \in \mathrm{GL}_n(k)$ we write $\mathfrak{b}_g = g^{-1}\mathfrak{b}_0 g$ for the Borel subalgebra that is conjugate to $\mathfrak{b}_0$ by g. We denote by B the subgroup of upper triangular matrices in $\mathrm{GL}_n(k)$ and by W the Weyl group N/T, where N denotes the subgroup of matrices with exactly one non-zero entry in each row and each column and T for the subgroup of diagonal matrices, $T = B \cap N$; the Weyl group W can be identified with the subgroup of $\mathrm{GL}_n(k)$ of $n \times n$ permutation matrices, and as such is isomorphic to the group $\mathfrak{S}_n$ of permutations over n elements. When speaking of elements of maximal length in W we refer to the generating set S of W whose elements are (the permutation matrices associated to) the transpositions $(j, j+1)$, $j \in [\![1, n-1]\!]$. The quadruple $(\mathrm{GL}_n(k), B, N, S)$ forms a Tits system ([Bou68, IV, 2.2]). From now on we freely identify an element w of W with its image in $\mathrm{GL}_n(k)$, the associated permutation matrix, and with its image in $\mathfrak{S}_n$, the underlying permutation. All scalars to be considered will be taken in k.

By its very definition the linear envelope $\sum_{w\in W} \mathfrak{b}\cap\mathfrak{b}_w$ of the intersection of any Borel subalgebra $\mathfrak{b}\subset\mathfrak{gl}_n$ with the conjugates of $\mathfrak{b}_0$ under the elements of the Weyl group, is contained in $\mathfrak{b}$; we discuss here the reverse inclusion and show the nice identity,

$$\mathfrak{b}=\sum_{w\in W}\mathfrak{b}\cap\mathfrak{b}_w,$$

that states the envelope does coincide with $\mathfrak{b}$.

Since any Borel subalgebra of $\mathfrak{gl}_n$ is a conjugate of the standard Borel subalgebra $\mathfrak{b}_0$, it will be enough to establish the identity for $\mathfrak{b}=\mathfrak{b}_g$, for an arbitrary element $g\in\mathrm{GL}_n(k)$. By the Bruhat decomposition, every such element g can be written as a product $g=u_1su$ of two (invertible) upper triangular matrices, u_1 and u and a permutation matrix s associated to a permutation $s\in\mathfrak{S}_n$, so that the identity to discuss reads

$$\mathfrak{b}_{su}=\sum_{w\in W}\mathfrak{b}_{su}\cap\mathfrak{b}_w.$$

This means that we only have to prove this identity for an arbitrary $n\times n$ permutation matrix s and an arbitrary upper triangular matrix $u\in B=\mathrm{GL}_n(k)\cap\mathfrak{b}_0$.

In a first part we settle this identity for $s=w_0$ *the* permutation of maximal length in $\mathfrak{S}_n$, i.e. the involution $w_0=(1,n)(2,n-1)...(\lfloor\frac{n}{2}\rfloor,n-\lfloor\frac{n}{2}\rfloor+1)$. Since conjugation by any element $g\in\mathrm{GL}_n(k)$, is a linear isomorphism of $\mathfrak{gl}_n$, the conjugate of a linear envelope coincides with the envelope of the conjugates, and since intersection and conjugation trivially commute, we find

$$\sum_{w\in W}\mathfrak{b}_{w_0u}\cap\mathfrak{b}_w=\sum_{w\in W}(\mathfrak{b}_{w_0}\cap\mathfrak{b}_{wu^{-1}})_u.$$

Hence the envelope $\sum_{w\in W}\mathfrak{b}_{w_0u}\cap\mathfrak{b}_w$ coincides with the Borel subalgebra $\mathfrak{b}_{w_0u}$ if and only if $\mathfrak{b}_{w_0}=\sum_{w\in W}\mathfrak{b}_{w_0}\cap\mathfrak{b}_{wu^{-1}}$ (the reader will notice that the Borel subalgebra $\mathfrak{b}_{w_0}$ coincides with the Borel algebra of lower triangular matrices).

The proof proceeds through an explicit "dévissage", which the following lemma will make clear; it does *not* rely on any induction on the dimension, nor does it require any further assumption on the fixed base field k: anyone will do. The elementary $n\times n$ matrix whose (l,m)-entry is given by $\delta_{i,l}\delta_{j,m}$ for $l,m\in[\![1,n]\!]$ will be denoted by $e^{i,j}$ and we will use the symbols $x_{i,j}$, $i,j\in[\![1,n]\!]$ to denote scalars in the base field k.

Lemma 2.1. *Let k be an arbitrary field. For any $u\in B$, and for any ordered pair (i,j) in $[\![1,n]\!]^2$, $i\geqslant j$, there is some permutation $s_{i,j}\in\mathfrak{S}_n$, and $i-j$ scalars $(x_{i,l})_{l=j+1}^{i}$ in k, such that the matrix $a^{i,j}:=e^{i,j}+\sum_{l=j+1}^{i}x_{i,l}e^{i,l}$ lies in the Borel subalgebra $\mathfrak{b}_{s_{i,j}u^{-1}}$.*

The matrices $a^{i,j}$ with (i,j) in $[\![1,n]\!]^2$, $i \geqslant j$, then form a basis of the Borel subalgebra $\mathfrak{b}_{w_0}$ of lower triangular matrices in $\mathfrak{gl}_n$ all elements of which lie in the envelope $\sum_{w \in \mathfrak{S}_n} \mathfrak{b}_{w_0} \cap \mathfrak{b}_{wu^{-1}}$.

The permutation $s_{i,j}$ may be chosen to be the (i,j)-transposition.

Proof of Lemma 2.1. Let us start by reformulating the claim[1]. Let $e_1, \ldots, e_n$ denote the standard basis of k^n, and let

$$V_\bullet = (0 = V_0 \subset V_1 \subset \cdots \subset V_n = k^n)$$

denote the standard flag $V_i = \langle e_1, \ldots, e_i \rangle$ of k^n. Let (i,j) in $[\![1,n]\!]^2$, $i \geqslant j$. Let $s^{i,j}$ be the permutation matrix corresponding to the transposition exchanging i and j. We are looking for a matrix $a^{i,j}$ which is the matrix in the standard basis of a morphism

$$\pi : k^n \to ke_i \hookrightarrow k^n$$

such that

(i) $e_l \in \ker(\pi)$ for $l < j$ or $l > i$.
(ii) $\pi(e_j) = e_i$.
(iii) the endomorphism $u^{-1}\pi u$ stabilizes the flag $s_{i,j}V_\bullet$.

A projection π like that can easily be constructed as follows: consider the basis

$$\mathcal{B} = \{u(e_1), \ldots, u(e_{j-1}), e_j, u(e_{j+1}), \ldots, u(e_i), e_{i+1}, \ldots, e_n\}$$

of k^n. As u is upper triangular this is indeed easily seen to be a basis of k^n. Define π by $\pi(x) = 0$ for $x \in \mathcal{B} \backslash \{e_j\}$ and $\pi(e_j) = e_i$, then (noting again that u is upper triangular) we easily see that π indeed satisfies (i) and (ii). For (iii) note that

$$u^{-1}\pi u(s_{i,j}V_l) = \begin{cases} 0 & l = 1, \ldots, i-1 \\ k \cdot u^{-1}e_i & l = i, \ldots, n. \end{cases}$$

In particular, as $k \cdot u^{-1}e_i \in \langle e_1, \ldots, e_i \rangle = s_{i,j}V_i$, we find that $u^{-1}\pi u(s_{i,j}V_l) \subset s_{i,j}V_l$ for all l, as claimed.

We finish the proof by showing that the matrices $a^{i,j}$ with (i,j) in $[\![1,n]\!]^2$, $i \geqslant j$ form a basis of $\mathfrak{b}_{w_0}$.

By construction one passes from the matrices $e^{i,j}$ to the matrices $a^{i,j}$, $i,j \in [\![1,n]\!]$, $i \geqslant j$, by some unipotent triangular matrix (with many zeros, since it is n-block diagonal), provided the order we choose on the set $\{(i,j), i,j \in [\![1,n]\!], i \geqslant j\}$ is compatible with the row order, i.e. such that for all $i \in [\![1,n]\!]$ $(i,j) < (i,k)$ if $j < k$ (the lexicographic order clearly has the property); the matrices $a^{i,j}$, $i,j \in [\![1,n]\!]$, $i \geqslant j$, then form a basis of the

[1]We thank one of the referees for pointing out this simplification of the rather involved computation in our original proof.

Borel subalgebra $\mathfrak{b}_{w_0}$, whith each one lying in exactly one of the generators $\mathfrak{b}_{w_0} \cap \mathfrak{b}_{s_{(i,j)}u^{-1}}$ of the envelope $\sum_{w\in\mathfrak{W}} \mathfrak{b}_{w_0} \cap \mathfrak{b}_{wu^{-1}}$. $\square$

The reader may notice that the above argument in fact proves the stronger statement that for all u, $u \in B$, $\mathfrak{b}_{w_0u} = \sum_{t\in\mathfrak{T}_n} \mathfrak{b}_{w_0u} \cap \mathfrak{b}_t$, where the sum is taken over the subset $\mathfrak{T}_n \subset W_0$ consisting of the identity and the i,j-transpositions, $n \geqslant i > j \geqslant 1$. This is a small subset of $\mathfrak{S}_n$ with only $(n^2 - n + 2)/2$ elements.

We now discuss the details of the reduction of the general statement to Lemma 2.1. The following lemma is a direct consequence of a refined version of the Bruhat decomposition established in [Jan87, II, 1.9].

Lemma 2.2. *Let k be an arbitrary field. Any invertible $n \times n$-matrix in $\mathrm{GL}_n(k)$ splits as the product of an upper triangular matrix by a lower triangular matrix by a permutation matrix: for all matrices $m \in \mathrm{GL}_n(k)$, there exist an upper triangular matrix u, a lower triangular matrix l, and a permutation matrix p, such that $m = ulp$.*

If useful, one may require the upper triangular u or the lower triangular l to be unipotent (but not both simultaneously of course).

Proof. For an algebraically closed field k this is precisely the statement of the displayed formula before just before (6) of [Jan87, II, 1.9]. As we deal with a split group (the group GL_n) the statement is actually true over an arbitrary field. Indeed in the proof of loc. cit. we only need to note that the groups U^+, $U(R')$ and $U(R'')$ as well as the isomorphism

$$U(R') \times U(R'') \xrightarrow{\cong} U^+$$

are defined over any field. $\square$

Let's turn back to our main object, realizing a Borel subalgebra as the envelope of its intersections with the conjugates of any fixed Borel subalgebra under the Weyl group. From Lemmas 2.1 and 2.2 we can deduce the following statement.

Theorem 2.3. *Let n be a positive integer, $\mathfrak{S}_n$ the symmetric group of order n, and $\mathfrak{gl}_n$ the algebra of $n \times n$ matrices with entries in a fixed field k. Let $\mathrm{GL}_n(k)$ be the group of the non-singular elements in $\mathfrak{gl}_n$ and $\mathfrak{b}_0 \subset \mathfrak{gl}_n$ the Borel subalgebra of upper triangular matrices. For any element $g \in \mathrm{GL}_n(k)$ let $\mathfrak{b}_g = g^{-1}\mathfrak{b}_0 g$ denote the Borel subalgebra conjugate to $\mathfrak{b}_0$ by g^{-1}.*

Any Borel subalgebra $\mathfrak{b}$ coincides with the linear envelope of its intersections with the conjugate of $\mathfrak{b}_0$ under $\mathfrak{S}_n$

$$\mathfrak{b} = \sum_{w\in\mathfrak{S}_n} \mathfrak{b} \cap \mathfrak{b}_w.$$

Proof. All Borel subalgebras are known to be conjugate, and it is enough to prove the identity in the theorem for $\mathfrak{b} = \mathfrak{b}_g = g^{-1}\mathfrak{b}_0 g$ for all $g \in \mathrm{GL}_n(k)$. By Lemma 2.2 the element g splits as a product of an upper triangular matrix, u, by a lower triangular matrix, l, by a permutation matrix p i.e. we can write $g = ulp$. The matrix l can be written as the conjugate $l = w_0 u_2 w_0^{-1}$ of an upper triangular matrix u_2 by the permutation matrix w_0 associated with the permutation of maximal length in $\mathfrak{S}_n$, that is

$$w_0 = (1, n)(2, n-1)\dots(\lfloor \tfrac{n}{2} \rfloor, n - \lfloor \tfrac{n}{2} \rfloor + 1).$$

Substituting for l in $g = ulp$ accordingly, and introducing the permutation matrix $q \coloneqq w_0^{-1}p$ we get $g = uw_0u_2w_0^{-1}p = uw_0u_2q$, and for the Borel subalgebra $\mathfrak{b}_g = \mathfrak{b}_{w_0u_2q}$.

Now, as observed above, conjugation trivially commutes with taking linear envelope and intersection so that $\sum_{w\in\mathfrak{S}_n} \mathfrak{b}_{w_0u_2} \cap \mathfrak{b}_w$ coincides with $(\sum_{w\in\mathfrak{S}_n} \mathfrak{b}_{w_0} \cap \mathfrak{b}_{wu_2^{-1}})_{u_2}$ and the identity $\mathfrak{b}_{w_0u_2} = \sum_{w\in\mathfrak{S}_n} \mathfrak{b}_{w_0u_2} \cap \mathfrak{b}_w$ is equivalent to $\mathfrak{b}_{w_0} = \sum_{w\in\mathfrak{S}_n} \mathfrak{b}_{w_0} \cap \mathfrak{b}_{wu_2^{-1}}$, which, in turn, is precisely the conclusion of Lemma 2.1.

Again, conjugation commutes with taking linear envelope and intersection, to the effect that the identity $\mathfrak{b}_{w_0u_2} = \sum_{w\in\mathfrak{S}_n} \mathfrak{b}_{w_0u_2} \cap \mathfrak{b}_w$ reads

$$\begin{aligned}\mathfrak{b}_g = \mathfrak{b}_{w_0u_2q} = (\sum_{w\in\mathfrak{S}_n} \mathfrak{b}_{w_0u_2} \cap \mathfrak{b}_w)_q &= \sum_{w\in\mathfrak{S}_n} (\mathfrak{b}_{w_0u_2} \cap \mathfrak{b}_w)_q \\ = \sum_{w\in\mathfrak{S}_n} \mathfrak{b}_{w_0u_2q} \cap \mathfrak{b}_{wq} &= \sum_{w\in\mathfrak{S}_n} \mathfrak{b}_g \cap \mathfrak{b}_{wq}.\end{aligned}$$

Since in any group (right-) translation by any element is a bijection, the latter sum can be rewritten $\sum_{w\in\mathfrak{S}_n} \mathfrak{b}_g \cap \mathfrak{b}_w$, which proves the claim that

$$\mathfrak{b}_g = \sum_{w\in\mathfrak{S}_n} \mathfrak{b}_g \cap \mathfrak{b}_w.$$

This finishes the proof. □

Remark 2.4. We expect that the analogue of Theorem 2.3 holds true for the Lie algebra of any split reductive group. Indeed, one would only need to prove a generalization of Lemma 2.1 to the case of a reductive group. The proof of Lemma 2.2 and the proof of Theorem 2.3 then will still work (with the obvious straightforward modifications).

One will note that the remark closing the proof of Lemma 2.1 implies, when inserted in the previous discussion, that in general it is enough to consider the envelope of the intersections of the Borel subalgebra $\mathfrak{b}_g$ with the Borel subalgebras $\mathfrak{b}_{tq}$, $t \in \mathfrak{T}_n$, where $q = w_0^{-1}p$, for p the permutation factor of the ulp decomposition of g: the sum in the above decomposition can be taken on a prescribed translate of $\mathfrak{T}_n$, a small subset (of cardinal $(n^2 - n + 2)/2$) of $\mathfrak{S}_n$.

2.2. **A surjectivity result.** Let k be a field and let $G := \mathrm{GL}_{n,k}$. Let B be a Borel subgroup in G and let $\mathfrak{g} := \operatorname{Lie} G$ and $\mathfrak{b} := \operatorname{Lie} B$. The quotient scheme G/B is identified to the projective scheme classifying the complete flag in k^n. As two complete flags in k^n are conjugate under $G(k)$, we have a natural isomorphism of sets $(G/B)(k) \simeq G(k)/B(k)$, so that we will identify k-points of G/B with right cosets $gB(k)$, for $g \in G(k)$. Let $\widetilde{\mathfrak{g}}$ be the Grothendieck simultaneous resolution of $\mathfrak{g}$: it coincides with the closed subscheme $\{(A, gB) \mid \operatorname{Ad}(g^{-1})A \in \mathfrak{b}\}$ of the product $\mathfrak{g} \times_k G/B$. Let π_1 and π_2 be the projections of $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ onto $\widetilde{\mathfrak{g}}$ with respect to the first and second factors.

Lemma 2.5. *Let $g \in G(k)$. The tangent space of $\widetilde{\mathfrak{g}}$ at the point $(0, gB(k))$ of $\mathfrak{g} \times G/B$ is the k-linear subspace $g\mathfrak{b}g^{-1} \oplus T_{gB(k)}G/B$ of $\mathfrak{g} \oplus T_{gB(k)}G/B$.*

Proof. Let B^- be the Borel subgroup of G, opposite to B and let U^- be the unipotent radical of B^-. There is an open embedding of U^- into G/B sending $u \in U^-$ to guB (see for example [Jan87, II.1.10]). This open embedding induces an isomorphism $T_{gB(k)}G/B \simeq T_{\mathrm{Id}}U^-$. This implies that the tangent space $T_{(0,gB(k))}\widetilde{\mathfrak{g}}$ can be identified with the set of pairs (A, C) in $\mathfrak{g} \times \operatorname{Lie} U^-$ such that $(\varepsilon A, g(\mathrm{Id} + \varepsilon C)) \in \widetilde{\mathfrak{g}}(L[\varepsilon])$, which means

$$(g(\mathrm{Id} + \varepsilon C))^{-1}\varepsilon A g(\mathrm{Id} + \varepsilon C) \in k[\varepsilon] \otimes_k \mathfrak{b}. \tag{2.1}$$

Using the fact that $\varepsilon^2 = 0$, (2.1) is equivalent to $g^{-1}Ag \in \mathfrak{b}$. □

The same kind of computation shows the following result.

Lemma 2.6. *The tangent space of $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ at the point $(gB(k), 0, hB(k)) \in G/B \times \mathfrak{g} \times G/B$ is the subspace*

$$T_{gB(k)}G/B \oplus (g\mathfrak{b}g^{-1} \cap h\mathfrak{b}h^{-1}) \oplus T_{hB(k)}G/B$$

of $T_{gB(k)}G/B \oplus \mathfrak{g} \oplus T_{hB(k)}G/B$.

Let T be a maximal split torus in G and let $\mathfrak{t}$ be its Lie algebra. The following result will turn out to be important for the computation of tangent spaces later. Let us write $(G/B)^T$ for the set of k-points of G/B which are fixed by the group $T(k)$. Note that this set is in bijection with the Weyl group W of (G, T).

Theorem 2.7. *Let $hB(k) \in (G/B)(k)$. We have*

$$\sum_{gB(k) \in (G/B)^T} d\pi_2(T_{(gB(k),0,hB(k))}(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}})) = T_{(0,hB(k))}\widetilde{\mathfrak{g}}$$

Proof. Let us remark that $gB(k) \in (G/B)^T$ if and only if $T \subset gB(k)g^{-1}$ which is equivalent to $\mathfrak{t} \subset g\mathfrak{b}g^{-1}$. Let $\mathcal{B}$ be the set of Borel sub-algebras of $\mathfrak{g}$ containing $\mathfrak{t}$. Using Lemmas 2.5 and 2.6, we see that the statement is equivalent to the following identity :

$$\sum_{\mathfrak{b}' \in \mathcal{B}} (\mathfrak{b}' \cap h\mathfrak{b}h^{-1}) = h\mathfrak{b}h^{-1}. \tag{2.2}$$

This, in turn, is a consequence of Theorem 2.3. □

3. Local deformation rings

Let K be a finite extension of $\mathbf{Q}_p$. We fix L a finite extension of $\mathbf{Q}_p$ such that $L \otimes_{\mathbf{Q}_p} K \simeq L^{[K:\mathbf{Q}_p]}$.

We will define and study several deformation problems related to Galois representations and (φ, Γ_K)-modules. Let $\mathcal{C}$ be the category of finite local L-algebras A with residue field isomorphic to L. If A is an object of $\mathcal{C}$ we denote by $\mathfrak{m}_A$ its unique maximal ideal.

3.1. (φ, Γ_K)**-modules.** Let K' be the maximal unramified extension of $\mathbf{Q}_p$ contained in $K(\mu_{p^\infty})$, it is a finite extension of $\mathbf{Q}_p$. Let $\mathcal{R}$ be the Robba ring of K defined as $\varinjlim_{r<1} \mathcal{R}^{]r,1[}$ where $\mathcal{R}^{]r,1[}$ is the ring of rigid analytic functions on the open annulus $\{r < |X| < 1\}$ over K'. This ring is a Bezout domain (see [Ber02, Prop. 4.12]). The ring $\mathcal{R}$ is endowed with a Frobenius endomorphism ϕ and a continuous action of the group $\Gamma_K \coloneqq \mathrm{Gal}(K(\zeta_{p^\infty})/K)$ commuting with ϕ (see [KPX14, Def. 2.2.2]). The ring $\mathcal{R}$ contains an element t which is the image in $\mathcal{R}$ of the rigid analytic function $x \mapsto \log(1+x)$ defined over the open unit disc over $\mathbf{Q}_p$. This element has the properties $\phi(t) = pt$ and $[\mathrm{rec}_K(a)] \cdot t = \chi_{\mathrm{cyc}}(\mathrm{rec}_K(a))t = at$, for $a \in K^\times$.

We will study the (φ, Γ_K)-modules over $\mathcal{R}$ and $\mathcal{R}[\frac{1}{t}]$, as well as their deformations to Artin rings. In this section we recall some definitions and first properties of these objects.

For an object A of $\mathcal{C}$ we let $\mathcal{R}_A \coloneqq A \otimes_{\mathbf{Q}_p} \mathcal{R}$ denote the scalar extension of $\mathcal{R}$ to A. We define a (φ, Γ_K)*-module over* $\mathcal{R}_A$ as a pair $(\mathcal{D}_A, \varphi)$ where $\mathcal{D}_A$ is a finite free $\mathcal{R}_A$-module, φ is a ϕ-semilinear endomorphism of $\mathcal{D}_A$ inducing an isomorphism $\mathcal{R}_A \otimes_{\phi, \mathcal{R}_A} \mathcal{D}_A \xrightarrow{\sim} \mathcal{D}_A$, and $\mathcal{D}_A$ is equipped with a continuous semilinear action of Γ_K commuting with φ (here $\mathcal{D}_A$ is a $\mathcal{R}$-module of finite type and has the canonical topology induced from the topology of $\mathcal{R}$). As A is a finite local $\mathbf{Q}_p$-algebra, this definition coincides with [KPX14, Def. 2.2.12].

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two (φ, Γ_K)-modules over $\mathcal{R}_A$. There is a (φ, Γ_K)-module $\mathrm{Hom}(\mathcal{D}_1, \mathcal{D}_2)$ defined over $\mathcal{R}_A$ whose underlying $\mathcal{R}_A$-module is the space of $\mathcal{R}_A$-linear maps from $\mathcal{D}_1$ to $\mathcal{D}_2$. The φ-structure on this module is defined as follows: first note that $\mathcal{R}$ is a flat $\mathcal{R}$-module via ϕ. Hence the canonical map $\mathcal{R} \otimes_{\phi, \mathcal{R}} \mathrm{Hom}_{\mathcal{R}}(M_1, M_2) \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{R} \otimes_{\phi, \mathcal{R}} M_1, \mathcal{R} \otimes_{\phi, \mathcal{R}} M_2)$ is an isomorphism. This isomorphism is used to define φ on $\mathrm{Hom}_{\mathcal{R}}(M_1, M_2)$.

For $i \geqslant 0$, the i-th cohomology group $H^i_{(\varphi,\Gamma_K)}(\mathcal{D})$ of a (φ,Γ_K)-module $\mathcal{D}$ is defined in [Liu07, §3.1]. If $\mathcal{D}_A$ is a (φ,Γ_K)-module over $\mathcal{R}_A$, it follows from [Liu07, Thm. 5.3] that $H^i_{(\varphi,\Gamma)}(\mathcal{D}_A)$ is of finite type over A and zero for $i > 2$.

For any continuous group homomorphism $\delta : K^\times \to L^\times$, we recall that we can construct a rank one (φ,Γ_K)-module $\mathcal{R}_L(\delta)$ over $\mathcal{R}_L$ such that the map $\delta \mapsto \mathcal{R}_L(\delta)$ induces a bijection between the set of continuous group homomorphisms $K^\times \to L^\times$ and the set of isomorphism classes of rank one (φ,Γ_K)-modules over $\mathcal{R}_L$ (see [KPX14, §6.1] for the precise construction of $\mathcal{R}_L(\delta)$).

By definition a (φ,Γ_K)*-module over* $\mathcal{R}[\frac{1}{t}]$ is a finite free $\mathcal{R}[\frac{1}{t}]$-module $\mathcal{M}$ with a ϕ-semilinear endomorphism φ and a semilinear action of Γ_K such that there exists a sub-$\mathcal{R}$-module $\mathcal{D}$ of $\mathcal{M}$ which is stable by φ and Γ_K, generates $\mathcal{M}$ as a $\mathcal{R}[\frac{1}{t}]$-module and is a (φ,Γ_K)-module over $\mathcal{R}$.

Lemma 3.1. *Let $\mathcal{M}$ be (φ,Γ_K)-module over $\mathcal{R}[\frac{1}{t}]$ and let $\mathcal{N}$ a sub-$\mathcal{R}[\frac{1}{t}]$-module of $\mathcal{M}$ which a direct factor as $\mathcal{R}[\frac{1}{t}]$-module and stable under φ and Γ_K. Then $\mathcal{N}$ is a (φ,Γ_K)-module over $\mathcal{R}[\frac{1}{t}]$.*

Proof. Let $\mathcal{D}$ be a sub-$\mathcal{R}$-module of $\mathcal{M}$, which is a (φ,Γ_K)-module and generates $\mathcal{M}$ as a $\mathcal{R}[\frac{1}{t}]$-module. It is sufficient to prove that $\mathcal{D}' \coloneqq \mathcal{D} \cap \mathcal{N}$ is a (φ,Γ_K)-module over $\mathcal{R}$. Let us first note that $\mathcal{D}'$ is finitely generated over $\mathcal{R}$. Indeed, as $\mathcal{N} \subset \mathcal{M}$ is a direct factor there exist $e_1,\dots,e_r \in \mathcal{D}\cap\mathcal{N} \subset \mathcal{M}$ such that $e_1,\dots,e_r$ generates $\mathcal{N}$ as an $\mathcal{R}[\frac{1}{t}]$-module. Let $\mathcal{D}'' \subset \mathcal{D}$ be the submodule generated by $e_1,\dots,e_r$. Then $\mathcal{D}/\mathcal{D}''$ is a finitely presented module over the Bezout domain $\mathcal{R}$. Its torsion sub-module $(\mathcal{D}/\mathcal{D}'')_{\mathrm{tors}}$ is a direct summand of $\mathcal{D}/\mathcal{D}''$ and hence finitely generated over $\mathcal{R}$. As $\mathcal{D}'$ coincides with the pre-image of $(\mathcal{D}/\mathcal{D}'')_{\mathrm{tors}}$ under the projection $\mathcal{D} \to \mathcal{D}/\mathcal{D}''$ it follows that $\mathcal{D}'$ is finitely generated as well.

It follows from [Ber02, Lem. 4.13] that $\mathcal{D}'$ is a finite free $\mathcal{R}$-module. Consequently we have to prove that the $\mathcal{R}$-linear map $\mathcal{R}\otimes_{\phi,\mathcal{R}}\mathcal{D}' \to \mathcal{D}'$ given by the restriction of φ is an isomorphism. The snake lemma applied to the following morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}\otimes_{\phi,\mathcal{R}}\mathcal{D}' \arrow[r] \arrow[d, "\varphi_{\mathcal{D}'}"] & \mathcal{R}\otimes_{\phi,\mathcal{R}}\mathcal{D} \arrow[r] \arrow[d, "\varphi_{\mathcal{D}}"] & \mathcal{R}\otimes_{\phi,\mathcal{R}}(\mathcal{D}/\mathcal{D}') \arrow[r] \arrow[d, "\varphi_{\mathcal{D}/\mathcal{D}'}"] & 0 \\
0 \arrow[r] & \mathcal{D}' \arrow[r] & \mathcal{D} \arrow[r] & \mathcal{D}/\mathcal{D}' \arrow[r] & 0,
\end{tikzcd}$$

where $\varphi_{\mathcal{D}}$ is bijective by the very definition of a (φ,Γ_K)-module, shows that the map $\varphi_{\mathcal{D}/\mathcal{D}'}$ is surjective and that it is enough to check it is also injective to get that $\varphi_{\mathcal{D}'}$ is an isomorphism. Since $\mathcal{D}/\mathcal{D}'$ is finitely presented and $\mathcal{R}$-torsion free it is free of finite rank (as $\mathcal{R}$ is a Bezout domain) and hence $\varphi_{\mathcal{D}/\mathcal{D}'}$ is a surjection between finite free $\mathcal{R}$-modules of the same rank. Hence it is an isomorphism. $\square$

If A is an object of $\mathcal{C}$ (i.e. A is a finite local L-algebra with residue field isomorphic to L) we define a (φ,Γ_K)-*module over* $\mathcal{R}_A[\frac{1}{t}]$ to be a (φ,Γ_K)-module $\mathcal{M}$ over $\mathcal{R}[\frac{1}{t}]$ together with a morphism of $\mathbf{Q}_p$-algebras from A into $\mathrm{End}_{\varphi,\Gamma_K}\mathcal{M}$ such that $\mathcal{M}$ is a finite free $\mathcal{R}_A[\frac{1}{t}]$-module.

3.2. **Filtered deformation functors.** We recall the notion of trianguline deformation of (φ,Γ_K)-module introduced by Bellaïche and Chenevier in [BC09, §2] and its non-saturated generalization studied in [BHS19, §3].

Let A be an object of $\mathcal{C}$ and let $\mathcal{D}_A$ be a (φ,Γ_K)-module over $\mathcal{R}_A$. We define a *filtration* $\mathcal{F}$ of $\mathcal{D}_A$ as a sequence

$$0=\mathcal{F}_0\subset\mathcal{F}_1\subset\cdots\subset\mathcal{F}_d=\mathcal{D}_A$$

of sub-(φ,Γ_K)-modules of $\mathcal{D}_A$ such that each $\mathcal{F}_i$ is a direct factor of $\mathcal{D}_A$ as an $\mathcal{R}_A$-module. When $d=\mathrm{rk}_{\mathcal{R}_A}\mathcal{D}_A$ and each quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is of rank 1 over $\mathcal{R}_A$, we say that $\mathcal{F}$ is a *triangulation* of $\mathcal{D}_A$.

Let $\mathcal{F}$ be a triangulation of a (φ,Γ_K)-module $\mathcal{D}$ over $\mathcal{R}_L$. For $1\leqslant i\leqslant d$, let δ_i the unique continuous morphism $K^\times\to L^\times$ such that $\mathcal{F}_i/\mathcal{F}_{i-1}\simeq\mathcal{R}_L(\delta_i)$. The character $\delta_1\otimes\cdots\otimes\delta_d$ from $(K^\times)^d$ to $L^\times$ depends only on $\mathcal{D}$ and $\mathcal{F}$ and is called the *parameter* of the triangulation $\mathcal{F}$.

If $\mathbf{k}=(k_\tau)_{\tau\in\Sigma}\in\mathbf{Z}^\Sigma$, we note $z^{\mathbf{k}}$ the character $z\mapsto\prod_{\tau\in\Sigma}\tau(z)^{k_\tau}$ from $K^\times$ into $L^\times$. A character $\delta_1\otimes\cdots\otimes\delta_d$ of $(K^\times)^d$ is called *regular* if, for all $i\neq j$, we have

$$\delta_i\delta_j^{-1}\notin\{z^{\mathbf{k}},z^{\mathbf{k}}\varepsilon_K;\ \mathbf{k}\in\mathbf{Z}^{[K:\mathbf{Q}_p]}\}. \tag{3.1}$$

From now on we fix $\mathcal{D}$ a (φ,Γ_K)-module over $\mathcal{R}_L$ and $\mathcal{F}$ a filtration of $\mathcal{D}$. If A is an object of $\mathcal{C}$, we define $\mathfrak{X}_{\mathcal{D},\mathcal{F}}(A)$ as the set of isomorphism classes of triples $(\mathcal{D}_A,\pi_A,\mathcal{F}_A)$ where $\mathcal{D}_A$ is a (φ,Γ_K)-module over $\mathcal{R}_A$, π_A is an $\mathcal{R}_A$-linear map from $\mathcal{D}_A$ to $\mathcal{D}$ commuting to φ and Γ_K, inducing an isomorphism $\mathcal{D}_A\otimes_A L\xrightarrow{\sim}\mathcal{D}$, and $\mathcal{F}_A=(\mathcal{F}_{A,i})_{0\leqslant i\leqslant m}$ is a filtration of $\mathcal{D}_A$ such that $\pi_A(\mathcal{F}_{A,i})=\mathcal{F}_i$ for all $0\leqslant i\leqslant m$. This construction can be promoted naturally into a functor from $\mathcal{C}$ to the category of sets. In the case $K=\mathbf{Q}_p$, the functor $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ was defined by Chenevier in [Che11]. Below we will make reference to the statements [Che11, Prop. 3.4] and [Che11, Prop. 3.6 (i) and (iii)] which concern only the case $K=\mathbf{Q}_p$. However the statements and proofs extend verbatim to the general case where K is a finite extension of $\mathbf{Q}_p$ so that we will apply them without more explanation to our situation. It follows from [Che11, Prop. 3.4] that the functor $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ admits a versal deformation L-algebra, i.e. a complete noetherian local L-algebra R such that

$$\mathrm{Hom}_{\mathrm{pro-}\mathcal{C}}(R,-)\simeq\mathfrak{X}_{\mathcal{D},\mathcal{F}}.$$

When $\mathcal{F}=(0\subset\mathcal{D})$, we simply write $\mathfrak{X}_{\mathcal{D}}$ for the functor $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$, which then coincides with the deformation functor of $\mathcal{D}$.

There is a natural map of functors $\mathfrak{X}_{\mathcal{D},\mathcal{F}} \to \mathfrak{X}_{\mathcal{D}}$ defined by

$$(\mathcal{D}_A, \pi_A, \mathcal{F}_A) \mapsto (\mathcal{D}_A, \pi_A).$$

If we assume in addition that $\mathrm{Hom}_{(\varphi,\Gamma_K)}(\mathrm{gr}_i(\mathcal{D}), \mathcal{D}/\mathcal{F}_i) = 0$ for all i, then [Che11, Prop. 3.6.(i)] shows that the map of functors $\mathfrak{X}_{\mathcal{D},\mathcal{F}} \to \mathfrak{X}_{\mathcal{D}}$ is injective and therefore we can identify $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ with a subfunctor of $\mathfrak{X}_{\mathcal{D}}$. In the particular case that $\mathcal{F}$ is a triangulation, the functor $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ was introduced in [BC09, Def. 2.3.2]; then the map of functors $\mathfrak{X}_{\mathcal{D},\mathcal{F}} \to \mathfrak{X}_{\mathcal{D}}$ is relatively representable if we assume $\mathrm{Hom}_{(\varphi,\Gamma_K)}(\mathrm{gr}_i(\mathcal{D}), \mathcal{D}/\mathcal{F}_i) = 0$ for all i (see [BC09, Prop. 2.3.9]), an assumption that is satisfied if $\mathcal{F}$ is a triangulation of $\mathcal{D}$ with regular parameter as follows from [Liu07, Prop. 3.10.(1)]. If $\mathcal{F}$ is a filtration of $\mathcal{D}$, let $\mathrm{End}_{\mathcal{F}} \mathcal{D}$ be the sub-$\mathcal{R}_L$-module of $\mathrm{End}\,\mathcal{D}$ whose elements are $\mathcal{R}_L$-linear maps respecting $\mathcal{F}$. It is a sub-(φ, Γ_K)-module of $\mathrm{End}\,\mathcal{D}$. It follows from [Che11, Prop. 3.6.(iii)] that if $H^2_{(\varphi,\Gamma_K)}(\mathrm{End}_{\mathcal{F}} \mathcal{D}) = 0$, the functor $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ is formally smooth. In particular if $H^2_{(\varphi,\Gamma_K)}(\mathrm{End}\,\mathcal{D}) = 0$, the functor $\mathfrak{X}_{\mathcal{D}}$ is formally smooth, which implies that a versal deformation ring for $\mathfrak{X}_{\mathcal{D}}$ is a formally smooth complete noetherian local L-algebra with residual field isomorphic to L, i.e. of the form $L[\![X_1, \dots, X_m]\!]$ for some non-negative integer m.

For the purpose of this paper we need an other kind of deformation problem that we are going to introduce now.

Let A be an object of $\mathcal{C}$ and $\mathcal{D}_A$ a (φ, Γ_K)-module over $\mathcal{R}_A$. The element $t \in \mathcal{R}$ satisfies $\phi(t) = pt$ and $\gamma \cdot t = \chi_{\mathrm{cyc}}(\gamma)t$, and hence the endomorphism ϕ and the action of Γ_K extends canonically to the ring $\mathcal{R}[\frac{1}{t}]$. The same observation applies to $\mathcal{R}_A[\frac{1}{t}]$ and, if $\mathcal{D}$ is a (φ, Γ_K)-module over $\mathcal{R}_A$, there are canonical semilinear extensions of φ and of the action of Γ_K to $\mathcal{D}_A[\frac{1}{t}]$. A *filtration* of $\mathcal{D}_A[\frac{1}{t}]$ is a sequence

$$\mathcal{M} = \left(0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_m = \mathcal{D}_A \left[\tfrac{1}{t}\right]\right)$$

by sub-$\mathcal{R}_A[\frac{1}{t}]$-modules which are direct factors and are stable under φ and Γ_K.

Remark 3.2. If $\mathcal{F} = (\mathcal{F}_i)_{0 \leqslant i \leqslant m}$ is a filtration of $\mathcal{D}_A$, the family $\mathcal{F}[\frac{1}{t}] \coloneqq (\mathcal{F}_i[\frac{1}{t}])$ is a filtration of $\mathcal{D}_A[\frac{1}{t}]$. However, if $(\mathcal{M}_i)_{0 \leqslant i \leqslant m}$ is a filtration of $\mathcal{D}_A[\frac{1}{t}]$, the family $(\mathcal{M}_i \cap \mathcal{D}_A)_{0 \leqslant i \leqslant m}$ need not be a filtration of $\mathcal{D}_A$ since the $\mathcal{R}_A$-modules $\mathcal{M}_i \cap \mathcal{D}_A$ may fail to be projective. A family of the form $(\mathcal{M}_i \cap \mathcal{D}_A)_{0 \leqslant i \leqslant m}$, for a given filtration $\mathcal{M}_i$ of $\mathcal{D}_A[\frac{1}{t}]$, is what we call an *unsaturated filtration* of $\mathcal{D}_A$. When $A = L$ the family $\mathcal{M} \cap \mathcal{D}_A \coloneqq (\mathcal{M}_i \cap \mathcal{D}_A)_{0 \leqslant i \leqslant m}$ is actually a filtration of $\mathcal{D}_A$ and the map $\mathcal{M} \mapsto \mathcal{M} \cap \mathcal{D}_A$ is a bijection from the set of filtrations of $\mathcal{D}_A[\frac{1}{t}]$ onto the set of filtrations of $\mathcal{D}_A$ whose inverse is $\mathcal{F} \mapsto \mathcal{F}[\frac{1}{t}]$.
For $K = \mathbf{Q}_p$ this is explained in [BC09, 2.4.2], and the argument works verbatim in the general case: the key step is to prove that the saturation

$$\mathcal{D}'_{\mathrm{sat}} = \{m \in \mathcal{D}_L |\ \exists\ 0 \neq f \in \mathcal{R}_L \text{ such that } fm \in \mathcal{D}'\}$$

of a sub-(φ, Γ_K)-module $\mathcal{D}' \subset \mathcal{D}_L$ can be identified with $\mathcal{D}'[1/t] \cap \mathcal{D}$, as in [BC09, Prop. 2.2.2 (i)]. The same proof as in loc. cit. works after noting that the product of the elementary divisors of $\mathcal{D}'$ divides a power of t. Indeed by the argument of loc. cit. we are reduced to show that if $\mathcal{D}' \subset \mathcal{D}$ is an embedding of (φ, Γ_K)-modules of rank one over $\mathcal{R}_L$, then $\mathcal{D}/\mathcal{D}'$ is killed by a power of t, wich in turn follows from the computation of the (φ, Γ_K)-invariants of rank one objects [KPX14, Prop. 6.2.8].

Let $\mathcal{D}$ be a (φ, Γ_K)-module over $\mathcal{R}_L$ and let $\mathcal{M}$ be a filtration of $\mathcal{D}[\frac{1}{t}]$. If A is an object of $\mathcal{C}$, we define $\mathfrak{X}_{\mathcal{D},\mathcal{M}}(A)$ as the set of isomorphism classes of triples $(\mathcal{D}_A, \pi_A, \mathcal{M}_A)$ where $\mathcal{D}_A$ is a (φ, Γ_K)-module over $\mathcal{R}_A$, π_A is a (φ, Γ_K)-module morphism $\mathcal{D}_A \to \mathcal{D}$ inducing an isomorphism $L \otimes_A \mathcal{D}_A \xrightarrow{\sim} \mathcal{D}$ and $\mathcal{M}_A$ is a filtration of $\mathcal{D}_A[\frac{1}{t}]$ such that $\pi_A(\mathcal{M}_{A,i}) = \mathcal{M}_i$. The construction $A \mapsto \mathfrak{X}_{\mathcal{D},\mathcal{M}}(A)$ can be promoted into a functor from $\mathcal{C}$ to the category of sets. When $\mathcal{F} := \mathcal{M} \cap \mathcal{D}$ we can check that the map $(\mathcal{D}_A, \pi_A, \mathcal{F}_A) \mapsto (\mathcal{D}_A, \pi_A, \mathcal{F}_A[\frac{1}{t}])$ induces an injection of functors $\mathfrak{X}_{\mathcal{D},\mathcal{F}} \hookrightarrow \mathfrak{X}_{\mathcal{D},\mathcal{M}}$ and we use it to identify $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ with a subfunctor of $\mathfrak{X}_{\mathcal{D},\mathcal{M}}$.

Remark 3.3. When $\mathcal{M}$ is a triangulation of $\mathcal{D}$, the functor $\mathfrak{X}_{\mathcal{D},\mathcal{M}}$ coincides with the functor of isomorphism classes of the groupoid $X_{\mathcal{D},\mathcal{M}}$ introduced in [BHS19, §3.5].

Remark 3.4. In the proof of density results of automorphic points (in the global case) respectively crystalline representations (in the local case) one main point is to control the tangent space $T\mathfrak{X}_{\mathcal{D}}$, where $\mathcal{D}$ is the (φ, Γ_K)-module associated to a given crystalline Galois representation. The strategy of [Che11] is to analyze the map $\mathfrak{X}_{\mathcal{D},\mathcal{F}} \to \mathfrak{X}_{\mathcal{D}}$ on tangent spaces in order to show that

$$\bigoplus_{\mathcal{F}} T\mathfrak{X}_{\mathcal{D},\mathcal{F}} \longrightarrow T\mathfrak{X}_{\mathcal{D}} \tag{3.2}$$

is surjective if the crystalline representation is φ-generic and all its triangulations are non-critical (in the language introduced in the sections below). Here the direct sum runs over all triangulations $\mathcal{F}$ of $\mathcal{D}$. (We caution the reader that we are using a slightly different terminology than Chenevier: in Chenevier's article these representations are called generic. This notion of being generic should not be confused with the notion of being φ-generic that we use here! In the language introduced below generic enough means that the crystalline representation is φ-generic and that all triangulations $\mathcal{F}$ are non-critical.) We need to generalize this theorem in order to allow crystalline representations that have critical triangulations. This can be achieved by studying the functor $\mathfrak{X}_{\mathcal{D},\mathcal{M}}$ just introduced instead of $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$. Indeed the main local theorem Theorem 3.14 will be the generalization of (3.2) to the case of critical triangulations.

The following statement is a direct consequence of the definitions; we state it for the sake of completeness and comfort of reading.

Scholium 3.5. *Let $\pi : \mathfrak{X} \to \mathfrak{Y}$ be a relatively representable morphism between functors from $\mathcal{C}$ to the category of sets. If $\mathfrak{Y}$ admits a versal deformation L-algebra, then $\mathfrak{X}$ admits a versal deformation L-algebra. More precisely if* $\operatorname{Spf} R \to \mathfrak{Y}$ *is a hull for $\mathfrak{Y}$ then the functor* $\operatorname{Spf} R \times_{\mathfrak{Y}} \mathfrak{X}$ *is pro-representable by a local complete noetherian L-algebra S and* $\operatorname{Spf} S \to \mathfrak{X}$ *is a hull for $\mathfrak{X}$.*

We will also need the following fact which is a direct consequence of [BHS19, Prop. 3.4.6].

Proposition 3.6. *Let $\mathcal{D}$ be a (φ, Γ_K)-module and let $\mathcal{F}$ be a triangulation of $\mathcal{D}$ whose parameter is regular in the sense of* (3.1)*. Let $\mathcal{M} := \mathcal{F}[\frac{1}{t}]$. Then the forgetful map $\mathfrak{X}_{\mathcal{D},\mathcal{M}} \to \mathfrak{X}_{\mathcal{D}}$ is injective and relatively representable. This implies that $\mathfrak{X}_{\mathcal{D},\mathcal{M}}$ admits a versal deformation L-algebra.*

This means that the map $\mathfrak{X}_{\mathcal{D},\mathcal{M}} \to \mathfrak{X}_{\mathcal{D}}$ is injective and that for all objects A in $\mathcal{C}$ and all $x \in \mathfrak{X}_{\mathcal{D}}(A)$, there is a unique quotient A_x of A such that for any map $A \to B$ in $\mathcal{C}$, the image of x in $\mathfrak{X}_{\mathcal{D}}(B)$ is in $\mathfrak{X}_{\mathcal{D},\mathcal{M}}(B)$ if and only if the map $A \to B$ factors through A_x.

3.3. **Crystalline (φ, Γ_K)-modules.** We study the deformation functors defined in the previous section in the case of crystalline (φ, Γ_K)-modules.

Let $K_0 = W(k_K)[\frac{1}{p}]$, let σ be the absolute Frobenius automorphism of K_0 and $f = [k_K : \mathbf{F}_p] = [K_0 : \mathbf{Q}_p]$. If A is an object of $\mathcal{C}$, an *isocrystal* over k_K with coefficients in A is a pair (V, φ) where V is a finite projective $A \otimes_{\mathbf{Q}_p} K_0$-module and φ is an $\mathrm{Id}_A \otimes \sigma$-semilinear automorphism of V. Actually these conditions automatically imply that V is a finite free $A \otimes_{\mathbf{Q}_p} K_0$-module. Its rank is by definition its rank as an $A \otimes_{\mathbf{Q}_p} K_0$-module. If (V, φ) is an isocrystal over k_K with coefficients in A, we define $\chi(V, \varphi)$ as the characteristic polynomial of the $A \otimes_{\mathbf{Q}_p} K_0$-linear endomorphism φ^f. This polynomial is invariant under $\mathrm{Id}_A \otimes \sigma$, hence $\chi(V, \varphi)$ lies in $A[X]$. Assume now that $A = L$. If $\chi(V, \varphi) = PQ$ where P and Q are coprime elements in $L[X]$, then there exists a unique φ-stable $L \otimes_{\mathbf{Q}_p} K_0$-submodule $W \subset V$ such that $\chi(W, \varphi|_W) = P$. Actually we have explicitely $W = \ker P(\varphi^f)$.

Recall from the work of Berger [Ber02] and [Ber08a] (we follow the exposition in [BC09, 2.2.7.] here) that there exists a left exact functor D_{cris} from the category of (φ, Γ_K)-modules over $\mathcal{R}_L$ to the category of isocrystals over k_K with coefficients in L defined by $D_{\mathrm{cris}}(\mathcal{D}) := \mathcal{D}[\frac{1}{t}]^{\Gamma_K}$ (see [BC09, 2.2.7.] for the case where $K = \mathbf{Q}_p$). We say that a (φ, Γ_K)-module $\mathcal{D}$ over $\mathcal{R}_L$ is *crystalline* if

$$\dim_{K_0} D_{\mathrm{cris}}(\mathcal{D}) = \mathrm{rk}_{\mathcal{R}}\, \mathcal{D}.$$

A *refinement* of a rank d isocrystal (D, φ) over k_K with coefficients in L is a filtration $F = (F_i)_{0 \leqslant i \leqslant d}$ of D

$$F_0 = 0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = D$$

such that each F_i is a $L\otimes_{\mathbf{Q}_p}K_0$-submodule stable under φ. Note that each F_i is necessarily free over $L\otimes_{\mathbf{Q}_p}K_0$ and consequently of rank i. The following lemma relates the notions of refinements and triangulations for crystalline (φ,Γ_K)-modules.

Lemma 3.7. *Let $\mathcal{D}$ be a crystalline (φ,Γ_K)-module over $\mathcal{R}_L$, there is a bijection*

$$\mathcal{F}=(\mathcal{F}_i)_{0\leqslant i\leqslant d}\mapsto D_{\mathrm{cris}}(\mathcal{F}):=(D_{\mathrm{cris}}(\mathcal{F}_i))_{0\leqslant i\leqslant d}$$

between the set of triangulations of $\mathcal{D}$ and the set of refinements of $D_{\mathrm{cris}}(\mathcal{D})$.

Proof. In the case $K=\mathbf{Q}_p$ this is [BC09, 2.4.2]. The case of general K works verbatim. □

Remark 3.8. Slightly more generally the same argument also shows the following: Let $\mathcal{D}$ be a crystalline (φ,Γ)-module over $\mathcal{R}_L$. Then D_{cris} induces a bijection between sub-(φ,Γ_K)-modules of $\mathcal{D}$ which are direct summands as $\mathcal{R}_L$-module and φ-stable sub-$L\otimes_{\mathbf{Q}_p}K_0$-modules of $D_{\mathrm{cris}}(\mathcal{D})$.

A refinement F of a k_K-isocrystal (D,φ) with coefficients in K gives rise to a decomposition of $\chi(V,\varphi)$ as a product of polynomials of degree one

$$\chi(D,\varphi)=\prod_{i=1}^{n}\chi(F_i/F_{i-1},\varphi).$$

In particular $\chi(D,\varphi)$ is split over L and the refinement defines an ordering $(\phi_1,\dots,\phi_d)$ of the roots of $\chi(D,\varphi)$ such that $\chi(F_i,\varphi)=\prod_{j=1}^{i}(X-\phi_j)$. We define δ_F to be the unramified character $(K^\times)^d\to L^\times$ given by the formula

$$(a_1,\dots,a_d)\longmapsto\prod_{i=1}^{d}\phi_i^{v_K(a_i)}.$$

If $(D,\varphi)=D_{\mathrm{cris}}(\mathcal{D})$ for a crystalline (φ,Γ_K)-module $\mathcal{D}$ over $\mathcal{R}_L$ and if $\mathcal{F}$ is a triangulation of $\mathcal{D}$, we define $\delta_{\mathcal{F}}:=\delta_{D_{\mathrm{cris}}(\mathcal{F})}$. It follows from the classification of sub-(φ,Γ_K)-modules of rank one (φ,Γ_K)-modules ([KPX14, Prop. 6.2.8.(1)]) that the parameter of the triangulation $\mathcal{F}$ is the product of $\delta_{\mathcal{F}}$ with an algebraic character of $(K^\times)^d$.

Conversely if the polynomial $\chi(D,\varphi)$ is separable and split in $L[X]$, each ordering $(\phi_1,\dots,\phi_d)$ of its roots comes from a unique refinement of D. In this case, the character δ_F completely determines the refinement F.

We say that a crystalline (φ,Γ_K)-module $\mathcal{D}$ over $\mathcal{R}_L$ is *φ-generic* if the polynomial $\chi(D_{\mathrm{cris}}(\mathcal{D}))$ is separable split over L with pairwise distinct roots $(\phi_1,\dots,\phi_d)$ such that $\phi_i\phi_j^{-1}\neq p^f$ for $i\neq j$. This property in particular implies that for each triangulation $\mathcal{F}$ of $\mathcal{D}$, the parameter of $\mathcal{F}$ is regular so that the assumption on the triangulation $\mathcal{F}$ in Proposition 3.6 is satisfied. As a consequence we have the following relatively representable inclusions

$$\mathfrak{X}_{\mathcal{D},\mathcal{F}}\subset\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]}\subset\mathfrak{X}_{\mathcal{D}},$$

where both $\mathfrak{X}_{\mathcal{D}}$ and $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ are formally smooth. The functor $\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]}$ is not formally smooth in general. In section 3.7 we will see that (after choosing some framings/rigidifications to make the functor pro-representable) the ring pro-representing this functor can have several minimal prime ideals.

Let $\mathcal{D}$ be a crystalline (φ,Γ_K)-module over $\mathcal{R}_L$. For an object A of $\mathcal{C}$, let $\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}(A)$ the subset of $\mathfrak{X}_{\mathcal{D}}(A)$ of isomorphism classes of pairs $(\mathcal{D}_A,\pi_A)$ with $\mathcal{D}_A$ a crystalline (φ,Γ_K)-module. The subfunctor $A\mapsto \mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}(A)$ of $\mathfrak{X}_{\mathcal{D}}$ is simply denoted $\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}$. If $\mathcal{D}_A$ is crystalline, the $A\otimes_{\mathbf{Q}_p}K_0$-module $D_{\mathrm{cris}}(\mathcal{D}_A)$ is finite free of rank $\mathrm{rk}_{\mathcal{R}_L}\mathcal{D}$.

Lemma 3.9. *Let $\mathcal{D}$ be a (φ,Γ_K)-module over $\mathcal{R}_L$ that is φ-generic crystalline. Let $\mathcal{F}$ be a triangulation of $\mathcal{D}$ with associated refinement $F = D_{\mathrm{cris}}(\mathcal{F})$. Let A be an object of $\mathcal{C}$ and let $(\mathcal{D}_A,\pi_A)\in\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}(A)$. There exists a unique complete flag F_A of $A\otimes_{\mathbf{Q}_p}K_0$-submodules of $D_{\mathrm{cris}}(\mathcal{D}_A)$ which is stable under φ and reduces to F modulo $\mathfrak{m}_A$.*

Proof. By assumption, the polynomial $\chi(D_{\mathrm{cris}}(\mathcal{D}))$ is separable split in $L[X]$ so that we can write

$$\chi(D_{\mathrm{cris}}(\mathcal{D}))=\prod_{i=1}^{n}(X-x_i)$$

and assume that the filtration F is given by $F_i=\ker\prod_{j=1}^{i}(\varphi^f-x_j)$.

Let $\chi_A(D_{\mathrm{cris}}(\mathcal{D}_A))\in A[X]$ be the characteristic polynomial of the $A\otimes_{\mathbf{Q}_p}K_0$-linear endomorphism φ^f of $D_{\mathrm{cris}}(\mathcal{D}_A)$. The reduction modulo $\mathfrak{m}_A$ of $\chi_A(D_{\mathrm{cris}}(\mathcal{D}_A))$ is the polynomial $\chi(D_{\mathrm{cris}}(\mathcal{D}))\in L[X]$ which is separable split in $L[X]$. Thus there exists a unique $(\tilde{x}_1,\dots,\tilde{x}_n)\in A^n$ such that $\chi_A(D_{\mathrm{cris}}(\mathcal{D}_A))=\prod_{i=1}^{n}(X-\tilde{x}_i)$ and, for $1\leqslant i\leqslant n$, $\tilde{x}_i\equiv x_i \mod \mathfrak{m}_A$. Considering the characteristic polynomials of the $\varphi^f|_{F_{A,i}}$ we can check that $F_{A,i}=\ker\prod_{j=1}^{i}(\varphi^f-\tilde{x}_j)$ defines the desired filtration. On the other hand any complete flag with the desired properties must fulfill this condition. $\square$

Let F_A denote the filtration whose existence is proved in Lemma 3.9, and write

$$\mathcal{M}_A\coloneqq\left(\mathcal{R}_A\left[\tfrac{1}{t}\right]\otimes_{K_0\otimes_{\mathbf{Q}_p}A}F_{A,1}\subsetneq\cdots\subsetneq\mathcal{R}_A\left[\tfrac{1}{t}\right]\otimes_{K_0\otimes_{\mathbf{Q}_p}A}F_{A,n}=\mathcal{D}_A\left[\tfrac{1}{t}\right]\right)$$

where we used the canonical isomorphism ([Ber02, Thm. 0.2])

$$\mathcal{R}_A\left[\tfrac{1}{t}\right]\otimes_{K_0\otimes_{\mathbf{Q}_p}A}D_{\mathrm{cris}}(\mathcal{D}_A)\simeq\mathcal{D}_A\left[\tfrac{1}{t}\right].$$

Then $(\mathcal{D}_A,\pi_A,\mathcal{M}_A)$ is an element of $\mathfrak{X}_{\mathcal{D},\mathcal{M}}(A)$. This implies that we have a sequence of inclusions

$$\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}\subset\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]}\subset\mathfrak{X}_{\mathcal{D}}. \tag{3.3}$$

Remark 3.10. We point out that in general $\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}$ does not embed into $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$. This is only true if we impose some conditions on the relative position of $\mathcal{F}$ with respect to the Hodge filtration (see below).

3.4. **$\mathbf{B}_{\mathrm{dR}}^+$-representations.** We compare deformation functors of (φ,Γ_K)-modules with deformation functors of $\mathbf{B}_{\mathrm{dR}}^+$-representations and recall some results of [BHS19] which will be useful later.

Recall that we write $\mathcal{G}_K = \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group of our fixed field K. Further recall that a $\mathbf{B}_{\mathrm{dR}}^+$-representation (resp. a $\mathbf{B}_{\mathrm{dR}}$-representation) of $\mathcal{G}_K$ is a finite free $\mathbf{B}_{\mathrm{dR}}^+$-module (resp. a finite dimensional $\mathbf{B}_{\mathrm{dR}}$-vector space) equipped with a continuous action of $\mathcal{G}_K$ that is semi-linear with respect to the $\mathcal{G}_K$-action on $\mathbf{B}_{\mathrm{dR}}^+$ (resp. $\mathbf{B}_{\mathrm{dR}}$). In [Ber08b, Prop. 2.2.6.], Berger constructs an exact functor W_{dR}^+ from the category of (φ,Γ_K)-modules over $\mathcal{R}$ to the category of $\mathbf{B}_{\mathrm{dR}}^+$-representations of $\mathcal{G}_K$. Moreover, this functor preserves the rank.

Let $\mathcal{M}$ be a (φ,Γ_K)-module over $\mathcal{R}[\frac{1}{t}]$. By definition there exists a (φ,Γ_K)-module D over $\mathcal{R}$ with $\mathcal{M} = \mathcal{D}[\frac{1}{t}]$. It follows from the construction of W_{dR}^+ that the $\mathbf{B}_{\mathrm{dR}}$-representation

$$W_{\mathrm{dR}}(\mathcal{M}) = W_{\mathrm{dR}}^+(\mathcal{D}) \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbf{B}_{\mathrm{dR}}.$$

is independent of the choice of $\mathcal{D}$. Hence this procedure defines an exact functor W_{dR} from the category of (φ,Γ_K)-modules over $\mathcal{R}[1/t]$ to the category of $\mathbf{B}_{\mathrm{dR}}$-representations of $\mathcal{G}_K$.

We consider variants of these notions for deformations of (φ,Γ_K)-modules and $\mathbf{B}_{\mathrm{dR}}^+$-representations to Artin rings.

Let A be an object of $\mathcal{C}$, an $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ is a finite free module over $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$ with a continuous semilinear action of $\mathcal{G}_K$. If $\mathcal{D}_A$ is a (φ,Γ_K)-module over $\mathcal{R}_A$, the $L \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation $W_{\mathrm{dR}}^+(\mathcal{D}_A)$ is actually an $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-module with a continuous semilinear action of $\mathcal{G}_K$. It follows from [BHS19, Lem. 3.3.5.(i)] that $W_{\mathrm{dR}}^+(\mathcal{D}_A)$ is a finite free $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-module and consequently an $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$. If W is an $L \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ we define $\mathfrak{X}_W$ to be the functor from $\mathcal{C}$ to the category of sets such that $\mathfrak{X}_W(A)$ is the set of equivalence classes of pairs (W_A,π_A) where W_A is a $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ and π_A is an $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-linear and $\mathcal{G}_K$-equivariant morphism from W_A to W inducing an isomorphism $L \otimes_A W_A \xrightarrow{\sim} W$. If $\mathcal{D}$ is a (φ,Γ_K)-module over $\mathcal{R}_L$, the functor W_{dR}^+ induces a map from $\mathfrak{X}_{\mathcal{D}}$ to $\mathfrak{X}_{W_{\mathrm{dR}}^+(\mathcal{D})}$.

Let $\mathcal{D}$ be a (φ,Γ_K)-module over $\mathcal{R}$. Let $D_{\mathrm{dR}}(\mathcal{D}) \coloneqq (W_{\mathrm{dR}}^+(\mathcal{D}) \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbf{B}_{\mathrm{dR}})^{\mathcal{G}_K}$. The *de Rham filtration* on $D_{\mathrm{dR}}(\mathcal{F})$ is defined by

$$\mathrm{Fil}_{\mathrm{dR}}^i(D_{\mathrm{dR}}(\mathcal{D})) \coloneqq (t^i W_{\mathrm{dR}}^+(\mathcal{D}))^{\mathcal{G}_K} \subset D_{\mathrm{dR}}(\mathcal{D})$$

We say that $\mathcal{D}$ is *de Rham* if $\dim_K D_{\mathrm{dR}}(\mathcal{D}) = \mathrm{rk}_{\mathcal{R}}\, \mathcal{D}$. If A is an object of $\mathcal{C}$ and if $\mathcal{D}_A$ is a (φ, Γ_K)-module over $\mathcal{R}_A$, then $D_{\mathrm{dR}}(\mathcal{D}_A)$ is a $A \otimes_{\mathbf{Q}_p} K$-module. If we assume that $\mathcal{D}_A$ is de Rham, then it is finite free over $A \otimes_{\mathbf{Q}_p} K$, and each $\mathrm{Fil}^i_{\mathrm{dR}}\, D_{\mathrm{dR}}(\mathcal{D}_A)$ is a sub-$A \otimes_{\mathbf{Q}_p} K$-module (though these submodules are not necessarily free over $A \otimes_{\mathbf{Q}_p} K$).

A *filtered* $L \otimes_{\mathbf{Q}_p} K$-module is a finite free $L \otimes_{\mathbf{Q}_p} K$-module with a separated and exhaustive filtration by sub-$L \otimes_{\mathbf{Q}_p} K$-modules. The functor D_{dR} is a left exact functor from the category of (φ, Γ_K)-modules over L to the category of filtered $L \otimes_{\mathbf{Q}_p} K$-modules. The restriction of the functor D_{dR} to the subcategory of de Rham (φ, Γ_K)-modules is exact and a crystalline (φ, Γ_K)-module over $\mathcal{R}_L$ is de Rham. Moreover there is a canonical isomorphism of $L \otimes_{\mathbf{Q}_p} K$-modules $D_{\mathrm{cris}}(\mathcal{D}) \otimes_{K_0} K \simeq D_{\mathrm{dR}}(\mathcal{D})$ (see for example [Ber08b, Prop. 2.3.3]).

Let $\mathcal{D}$ be a crystalline (φ, Γ_K)-module over $\mathcal{R}_L$. For all $\tau \in \Sigma$, we define

$$D_{\mathrm{dR},\tau}(\mathcal{D}) \coloneqq L \otimes_{K \otimes_{\mathbf{Q}_p} L, \tau} D_{\mathrm{dR}}(\mathcal{D})$$

It is a direct factor of $D_{\mathrm{dR}}(\mathcal{D})$ and we define a separated and exhaustive filtration on $D_{\mathrm{dR},\tau}(\mathcal{D})$ by

$$\mathrm{Fil}^i_{\mathrm{dR},\tau}\, D_{\mathrm{dR},\tau}(\mathcal{D}) \coloneqq L \otimes_{K \otimes_{\mathbf{Q}_p} L, \tau} (\mathrm{Fil}^i_{\mathrm{dR}}\, D_{\mathrm{dR}}(\mathcal{D}))$$

A *Hodge-Tate type* is an element $\mathbf{k} = (\mathbf{k}_\tau)_{\tau \in \Sigma} \in (\mathbf{Z}^n)^{[K:\mathbf{Q}_p]}$ where each $\mathbf{k}_\tau$ is an increasing sequence of integers. We say that the Hodge-Tate type is *regular* if all these sequences of integers are *strictly* increasing. If $\mathcal{D}$ is a de Rham (φ, Γ_K)-module, *its Hodge-Tate type* is by definition $(k_{1,\tau} \leqslant \cdots \leqslant k_{n,\tau})_{\tau \in \Sigma}$ where the $k_{i,\tau}$ are the integers m such that $\mathrm{gr}^{-m}\, D_{\mathrm{dR},\tau}(\mathcal{D}) \neq 0$, counted with multiplicity, where the multiplicity of m is defined as the dimension $\dim_L \mathrm{gr}^{-m}\, D_{\mathrm{dR},\tau}(\mathcal{D})$.

Let $\mathcal{D}$ be a crystalline (φ, Γ_K)-module over $\mathcal{R}_L$ and let $\mathcal{F}$ be a triangulation of $\mathcal{D}$. We say that $\mathcal{F}$ is *non critical*, if for all $1 \leqslant i \leqslant \mathrm{rk}_{\mathcal{R}_L}\, \mathcal{D}$ and for all $\tau \in \Sigma$, there exists some $m \in \mathbf{Z}$ such that

$$(L \otimes_{K_0 \otimes_{\mathbf{Q}_p} L, \tau|_{K_0}} D_{\mathrm{cris}}(\mathcal{F}_i)) \oplus \mathrm{Fil}^m_{\mathrm{dR},\tau}\, D_{\mathrm{dR},\tau}(\mathcal{D}) = D_{\mathrm{dR},\tau}(\mathcal{D}).$$

In this case we obviously have

$$\dim_L \mathrm{Fil}^m_{\mathrm{dR},\tau}\, D_{\mathrm{dR},\tau}(\mathcal{D}) + i = \mathrm{rk}\ \mathcal{D}.$$

If A is an object of $\mathcal{C}$, the image under W_{dR} of a (φ, Γ_K)-module over $\mathcal{R}_A[\frac{1}{t}]$ is finite free as an $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}$-module and consequently an $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}$-representation of $\mathcal{G}_K$ (see [BHS19, Lem. 3.3.5.(ii)]).

If A is an object of $\mathcal{C}$ and W_A is an $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}$-representation of $\mathcal{G}_K$ of rank n, we define a *complete flag* of W_A to be a filtration $(F_i)_{0 \leqslant i \leqslant n}$ of W_A by sub-$A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}$-modules stable under $\mathcal{G}_K$ such that F_i is a free $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}$-module of rank i.

Let W be an $L\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ and let F be a complete flag of $W\otimes_{\mathbf{B}_{\mathrm{dR}}^+}\mathbf{B}_{\mathrm{dR}}$ stable under the action of $\mathcal{G}_K$. Let A be an object of the category $\mathcal{C}$. A *deformation of the pair* (W,F) over A is an element (W_A,π_A,F_A) where W_A is a $A\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$, π_A a $\mathcal{G}_K$-equivariant isomorphism from $W_A\otimes_A L$ to W and F_A a complete flag of $W_A\otimes_{\mathbf{B}_{\mathrm{dR}}^+}\mathbf{B}_{\mathrm{dR}}$ such that $F=(\pi_A\otimes\mathrm{Id}_{\mathbf{B}_{\mathrm{dR}}})(F_A)$. We denote by $\mathfrak{X}_{W,F}$ the functor from the category $\mathcal{C}$ to the category of sets, that maps an object A of $\mathcal{C}$ to the isomorphism class of deformations of (W,F).

Let $\mathcal{D}_A$ be a (φ,Γ_K)-module over $\mathcal{R}_A$ and $\mathcal{M}_A$ a triangulation of $\mathcal{D}_A[\frac{1}{t}]$. It follows from Lemma 3.1 that each $\mathcal{M}_{A,i}$ is a (φ,Γ_K)-module over $\mathcal{R}_A[\frac{1}{t}]$. Thus

$$W_{\mathrm{dR}}(\mathcal{M}_A)\coloneqq\left(W_{\mathrm{dR}}(\mathcal{M}_{A,0})\subset\cdots\subset W_{\mathrm{dR}}(\mathcal{M}_{A,n})=W_{\mathrm{dR}}\left(\mathcal{D}_A\left[\tfrac{1}{t}\right]\right)\right)$$

is a complete flag of $W_{\mathrm{dR}}^+(\mathcal{D}_A)\otimes_{\mathbf{B}_{\mathrm{dR}}^+}\mathbf{B}_{\mathrm{dR}}$. For $\mathcal{D}$ a (φ,Γ_K)-module over $\mathcal{R}_L$ and $\mathcal{F}$ a triangulation of $\mathcal{D}$, we deduce from this fact that the functor W_{dR}^+ extends to a map of functors

$$\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]}\longrightarrow\mathfrak{X}_{W_{\mathrm{dR}}^+(\mathcal{D}),W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])}. \tag{3.4}$$

The following proposition is part of [BHS19, Cor. 3.5.6.] (except that in loc. cit. we chose to talk about deformation groupoids rather than deformation functors).

Proposition 3.11. *If $\mathcal{D}$ is a φ-generic crystalline (φ,Γ_K)-module over $\mathcal{R}_L$ with regular Hodge-Tate type and $\mathcal{F}$ is a triangulation of $\mathcal{D}$, then the map* (3.4) *is formally smooth.*

If W is an $L\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ and A is an object of $\mathcal{C}$, the *trivial* deformation of W is the isomorphism class of the pair $(A\otimes_L W,\pi_A)$ where π_A is the reduction mod $\mathfrak{m}_A$. Similarly if F is a filtration of $W\otimes_{\mathbf{B}_{\mathrm{dR}}^+}\mathbf{B}_{\mathrm{dR}}$, the *trivial* deformation of (W,F) is the isomorphism class of the triple $(A\otimes_L W,\pi_A,A\otimes_L F)$ with π_A as above.

Proposition 3.12. *Let $\mathcal{D}$ be a φ-generic crystalline (φ,Γ_K)-module over $\mathcal{R}_L$ of regular Hodge-Tate type. Let A be an object of $\mathcal{C}$ and $\mathcal{D}_A\in\mathfrak{X}_{\mathcal{D}}(A)$. If the image of $\mathcal{D}_A$ in $\mathfrak{X}_{W_{\mathrm{dR}}^+}(\mathcal{D})(A)$ is a trivial deformation of $W_{\mathrm{dR}}^+(\mathcal{D})$, then $\mathcal{D}_A\in\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}(A)$.*

Proof. If the image of $(\mathcal{D}_A,\pi_A)$ under W_{dR}^+ is the trivial deformation $W_{\mathrm{dR}}^+(\mathcal{D})\otimes_L A$, then $\mathcal{D}_A$ is a de Rham (φ,Γ_K)-module, as $\mathcal{D}$ is. We can conclude as in the proof of [HS16, Cor. 2.7.(i)]. Namely it follows from the p-adic monodromy theorem ([Ber08b, Thm. 2.3.5.(1)]) that $\mathcal{D}_A$ is a potentially semistable (φ,Γ_K)-module. Being an extension of finitely many crystalline (φ,Γ_K)-modules, it is actually semistable. It follows from the φ-genericity assumption on $\mathcal{D}$ that no quotient of eigenvalues of φ^f on $D_{\mathrm{st}}(\mathcal{D}_A)$ can be equal to p^f, so that the monodromy operator of $D_{\mathrm{st}}(\mathcal{D}_A)$ is trivial. Hence $\mathcal{D}_A$ is crystalline. $\square$

As a consequence of Proposition 3.12 with $A = L[\varepsilon]$, the kernel of the induced map on tangent spaces $T\mathfrak{X}_{\mathcal{D}} \to T\mathfrak{X}_{W^+_{\mathrm{dR}}(\mathcal{D})}$ is contained in $T\mathfrak{X}^{\mathrm{cris}}_{\mathcal{D}}$.

Corollary 3.13. *Let $\mathcal{D}$ be a φ-generic crystalline (φ, Γ_K)-module over $\mathcal{R}_L$ of regular Hodge-Tate type and $\mathcal{F}$ a triangulation of $\mathcal{D}$.*

(i) For all objects A of $\mathcal{C}$, the preimage of the trivial deformation of $(W^+_{\mathrm{dR}}(\mathcal{D}), W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}]))$ under the map (3.4) *is contained in $\mathfrak{X}^{\mathrm{cris}}_{\mathcal{D}}(A)$. Here we use the inclusion*

$$\mathfrak{X}^{\mathrm{cris}}_{\mathcal{D}} \subset \mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]}$$

constructed in (3.3)

(ii) Let $U \subset T\mathfrak{X}^{\mathrm{cris}}_{\mathcal{D}}$ be the kernel of the map $T\mathfrak{X}_{\mathcal{D}} \to T\mathfrak{X}_{W^+_{\mathrm{dR}}(\mathcal{D})}$. Then the following sequence is exact:

$$0 \longrightarrow U \longrightarrow T\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]} \longrightarrow T\mathfrak{X}_{W^+_{\mathrm{dR}}(\mathcal{D}), W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} \longrightarrow 0.$$

Proof. (i) This is a direct consequence of Proposition 3.12.

(ii) After evaluating on $L[\varepsilon]$ Proposition 3.11 shows that the map

$$T\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]} \longrightarrow T\mathfrak{X}_{W^+_{\mathrm{dR}}(\mathcal{D}), W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])}$$

is surjective and by (i) its kernel is contained in $T\mathfrak{X}^{\mathrm{cris}}_{\mathcal{D}}$. As $\mathcal{D}$ is φ-generic, we have an inclusion $T\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]} \subset T\mathfrak{X}_{\mathcal{D}}$ and we need to show that we can identify this kernel with U. As $\mathfrak{X}^{\mathrm{cris}}_{\mathcal{D}} \subset \mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]}$ is a subfunctor we have an inclusion

$$T\mathfrak{X}^{\mathrm{cris}}_{\mathcal{D}} \subset T\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]}.$$

The claim follows from the observation that, given a deformation $(\mathcal{D}_{L[\varepsilon]}, \mathcal{M}_{L[\varepsilon]})$ of $(\mathcal{D}, \mathcal{F}[\frac{1}{t}])$, the pair

$$(W^+_{\mathrm{dR}}(\mathcal{D}_{L[\varepsilon]}), W_{\mathrm{dR}}(\mathcal{M}_{L[\varepsilon]}))$$

is the trivial deformation of

$$(W^+_{\mathrm{dR}}(\mathcal{D}), W_{\mathrm{dR}}(\mathcal{F}[\tfrac{1}{t}]))$$

if and only if $W^+_{\mathrm{dR}}(\mathcal{D}_{L[\varepsilon]})$ is the trivial deformation of $W^+_{\mathrm{dR}}(\mathcal{D})$. The only if part is trivial. The if part follows from the fact that $\mathcal{D}_{L[\varepsilon]}$ must be crystalline by (i). Hence the φ-genericity assumption implies that the deformation $\mathcal{M}_{L[\varepsilon]}$ of $\mathcal{F}[\frac{1}{t}]$ is uniquely determined by $\mathcal{D}_{L[\varepsilon]}$ as follows from Lemma 3.9. $\square$

3.5. **Main theorem: the local version.** Let $\mathcal{D}$ be a φ-generic crystalline (φ, Γ_K)-module over $\mathcal{R}_L$. We write $\mathrm{Tri}(\mathcal{D})$ for the set of triangulations of $\mathcal{D}$, which is in bijection with the set of refinements of $D_{\mathrm{cris}}(\mathcal{D})$. Our main result about local deformation spaces is the following theorem.

Theorem 3.14. *Let $\mathcal{D}$ be a φ-generic crystalline (φ, Γ_K)-module over $\mathcal{R}_L$ of regular Hodge-Tate type. Let $\mathrm{Tri}(\mathcal{D})$ be the set of triangulations of $\mathcal{D}$. Then the L-linear map*

$$\bigoplus_{\mathcal{F} \in \mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{\mathcal{D}, \mathcal{F}[\frac{1}{t}]} \longrightarrow T\mathfrak{X}_{\mathcal{D}}$$

is surjective.

Remark 3.15. The special case of the result where all refinements of D are assumed to be non critical is a theorem due to G. Chenevier for $K = \mathbf{Q}_p$ ([Che11, Thm. 3.19]) and to K. Nakamura for an arbitrary extension K of $\mathbf{Q}_p$ ([Nak13, Thm. 2.62.]).

Before giving the proof of Theorem 3.14, let us recall some constructions and results from [BHS19], to which we refer the reader for relevent definitions if needed. The construction in [BHS19] relates the deformation functor $\mathfrak{X}_{\mathcal{D}, \mathcal{F}[\frac{1}{t}]}$ to complete local rings of the variety X introduced in section 2.2.

Let W be an almost de Rham $L \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ (see [BHS19, 3.1]). Let i be a $L \otimes_{\mathbf{Q}_p} K$-linear isomorphism $(L \otimes_{\mathbf{Q}_p} K)^n \xrightarrow{\sim} D_{\mathrm{pdR}}(W)$. Let $\mathfrak{X}_W^{\square}$ be the functor from $\mathcal{C}$ to the category of sets such that $\mathfrak{X}_W^{\square}(A)$ is the set of isomorphism classes of triples (W_A, π_A, i_A) where W_A is some $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$, π_A is map from W_A to W inducing an isomorphism from $L \otimes_A W_A$ to W and i_A is an isomorphism between $(A \otimes_{\mathbf{Q}_p} K)^n$ and $D_{\mathrm{pdR}}(W_A)$ compatible with π_A and i. Let $\mathfrak{g}$ be the Lie algebra of the algebraic group $\mathrm{GL}_{n,K}$ and let $\tilde{\mathfrak{g}} \to \mathfrak{g}$ be Grothendieck's simultaneous resolution of singularities. As in section 2.2 we consider the scheme $X \coloneqq \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$.

We assume in addition that the almost de Rham $L \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation W is regular, see [BHS19, Def. 3.2.4]. Under this assumption the functor $\mathfrak{X}_W^{\square}$ is pro-representable by the completion of $\widetilde{\mathfrak{g}}_{K/\mathbf{Q}_p, L}$ at the point $x = (0, i^{-1}(\mathrm{Fil}_{\mathrm{dR}}))$, by [BHS19, Thm. 3.2.5].

Let F be a complete flag of $W \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbf{B}_{\mathrm{dR}}$ stable under $\mathcal{G}_K$. We can define $\mathfrak{X}_{W,F}$ as in section 3.4 and $\mathfrak{X}_{W,F}^{\square}$ by adding a framing of $D_{\mathrm{pdR}}(W_A)$, that is an isomorphism

$$D_{\mathrm{pdR}}(W_A) \cong (A \otimes_{\mathbf{Q}_p} K)^n,$$

for $(W_A, \pi_A, F_A) \in \mathfrak{X}_{W,F}(A)$. The forgetful map $\mathfrak{X}_{W,F}^{\square} \to \mathfrak{X}_{W,F}$ is then formally smooth and by [BHS19, Cor. 3.5.8.(i)] the functor $\mathfrak{X}_{W,F}^{\square}$ is pro-representable by the completion of $X_{K/\mathbf{Q}_p, L}$ at the point $x_F = (F_1, 0, F_2)$

where $F_1 = i^{-1}(D_{\mathrm{pdR}}(F))$ and $F_2 = i^{-1}(\mathrm{Fil}_{\mathrm{dR}})$. More precisely we obtain the following commutative diagram:

$$\begin{tikzcd} \mathfrak{X}^{\square}_{W,F} \arrow[r, "\text{forget}"] \arrow[d, "\wr"] & \mathfrak{X}^{\square}_{W} \arrow[d, "\wr"] \\ \widehat{X_{K/\mathbf{Q}_p,L}}_{x_F} \arrow[r, "\pi_2"] & \widehat{\widetilde{\mathfrak{g}}_{K/\mathbf{Q}_p,L}}_{x} \end{tikzcd} \tag{3.5}$$

where the upper horizontal map is the forgetful map and the lower horizontal map is induced by the second projection of X on $\widetilde{\mathfrak{g}}$. If $W = W^+_{\mathrm{dR}}(\mathcal{D})$ for a (φ, Γ_K)-module $\mathcal{D}$ and $F = W_{\mathrm{dR}}(\mathcal{M})$ for $\mathcal{M}$ a triangulation of $\mathcal{D}[\frac{1}{t}]$ we will use the shorter notation $x_{\mathcal{M}}$ in place of $x_{W_{\mathrm{dR}}(\mathcal{M})}$.

Proof of Theorem 3.14. Let $W := W^+_{\mathrm{dR}}(\mathcal{D})$. In a first step we prove that the L-linear map

$$\bigoplus_{\mathcal{F}\in\mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{W,W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} \longrightarrow T\mathfrak{X}_W$$

is surjective. Let's consider the commutative diagram

$$\begin{tikzcd} \bigoplus_{\mathcal{F}\in\mathrm{Tri}(\mathcal{D})} T\mathfrak{X}^{\square}_{W,W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} \arrow[r] \arrow[d] & T\mathfrak{X}^{\square}_W \arrow[d] \\ \bigoplus_{\mathcal{F}\in\mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{W,W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} \arrow[r] & T\mathfrak{X}_W. \end{tikzcd}$$

As the forgetful map $\mathfrak{X}^{\square}_W \to \mathfrak{X}_W$ is formally smooth, it induces a surjection on the tangent spaces. Consequently it is sufficient to prove that the upper horizontal map is surjective.

Because of the commutative diagram (3.5) this is equivalent to the surjectivity of the map

$$\pi_2 : \sum_{\mathcal{F}\in\mathrm{Tri}(\mathcal{D})} T_{x_{\mathcal{F}[\frac{1}{t}]}} X_{K/\mathbf{Q}_p,L} \longrightarrow T_x \widetilde{\mathfrak{g}}_{K/\mathbf{Q}_p,L}$$

induced by the second projection. Let α be the morphism of K-schemes $X = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ given by the second projection. Let $\alpha_{K/\mathbf{Q}_p}$ be its image under the Weil restriction functor from K to $\mathbf{Q}_p$ and let $\alpha_{K/\mathbf{Q}_p,L}$ be the base change of $\alpha_{K/\mathbf{Q}_p}$ to L. For each $\tau \in \Sigma$ we write α_τ for the base change of α along $\tau : K \to L$. Then we have the following decompositions

$$\begin{aligned} X_{K/\mathbf{Q}_p} \times_{\mathbf{Q}_p} L &\simeq \prod_{\tau\in\Sigma} X_\tau, \\ \widetilde{\mathfrak{g}}_{K/\mathbf{Q}_p,L} &\simeq \prod_{\tau\in\Sigma} \widetilde{\mathfrak{g}}_\tau, \\ \alpha_{K/\mathbf{Q}_p,L} &= (\alpha_\tau)_{\tau\in\Sigma}. \end{aligned}$$

Therefore it only remains to prove that for each $\tau \in \Sigma$, the L-linear map

$$d\alpha_\tau : \bigoplus_{\mathcal{F}\in \mathrm{Tri}(\mathcal{D})} T_{x_{\mathcal{F}[\frac{1}{t}],\tau}} X_\tau \longrightarrow T_{x_\tau}\widetilde{\mathfrak{g}}_\tau \tag{3.6}$$

is surjective.

The $L \otimes_{\mathbf{Q}_p} K_0$-linear endomorphism $\Phi := \varphi^f$ of $D_{\mathrm{cris}}(\mathcal{D})$ induces an L-linear endomorphism Φ_τ of $D_{\mathrm{dR},\tau}(\mathcal{D}) = D_{\mathrm{cris},\tau|_{K_0}}(\mathcal{D})$ for all $\tau \in \Sigma$. This endomorphism is killed by the polynomial $\chi(D_{\mathrm{cris}}(\mathcal{D}), \varphi) \in L[X]$ which, by assumption, is separable and split. This implies that Φ_τ is contained in a unique maximal split torus T_τ of $\mathrm{GL}(D_{\mathrm{dR},\tau}(\mathcal{D}))$ or equivalently that the centralizer T_τ of Φ_τ is a maximal split torus. If $\mathcal{F}$ is a triangulation of $\mathcal{D}$, the complete flag $D_{\mathrm{cris}}(\mathcal{F})$ of $D_{\mathrm{cris}}(\mathcal{D})$ is stable under φ, as is the complete flag $D_{\mathrm{dR},\tau}(\mathcal{F}[\frac{1}{t}])$ under Φ_τ, and thus also T_τ. However the maximal split torus T_τ fixes exactly $n!$ complete flags of $D_{\mathrm{dR},\tau}(\mathcal{D})$. As $\mathcal{D}$ has exactly $n!$ triangulations we conclude that the set

$$\{x_{\mathcal{F}[\frac{1}{t}],\tau},\ \mathcal{F} \in \mathrm{Tri}(\mathcal{D})\}$$

is exactly the set of points $(F, 0, i^{-1}(\mathrm{Fil}_{\mathrm{dR},\tau})) \in X_\tau(L)$ such that F is fixed by the maximal split torus T_τ.

The surjectivity of the map (3.6) is thus a direct consequence of Theorem 2.7. This concludes the first step of the proof.

As in Corollary 3.13 we write $U \subset T\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}$ for the kernel of the canonical map

$$T\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}} \longrightarrow T\mathfrak{X}_{W_{\mathrm{dR}}^+(\mathcal{D})}.$$

Now we consider the commutative diagram with exact lines and columns

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & \\
\bigoplus_{\mathcal{F}\in\mathrm{Tri}(\mathcal{D})} U \arrow[rr, "\sum"] \arrow[d] & & U \arrow[r] \arrow[d] & 0 \\
\bigoplus_{\mathcal{F}\in\mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]} \arrow[rr, "\beta"] \arrow[d] & & T\mathfrak{X}_{\mathcal{D}} \arrow[d, "W_{\mathrm{dR}}^+"] & \\
\bigoplus_{\mathcal{F}\in\mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{W_{\mathrm{dR}}^+(\mathcal{D}),W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} \arrow[rr] \arrow[d] & & T\mathfrak{X}_{W_{\mathrm{dR}}^+(\mathcal{D})} \arrow[r] & 0 \\
0 & & &
\end{tikzcd}$$

The exactness of the vertical lines are a consequences of Proposition 3.12 and Corollary 3.13. The surjectivity of $\sum$ is trivial and the surjectivity of the lower horizontal map is what we proved as a first step. We can deduce from this that the map $W_{\mathrm{dR}}^+ \circ \beta$ is surjective and call upon the “five” Lemma to conclude that the map β itself is surjective. This finishes the proof of Theorem 3.14. □

3.6. **The case of Galois representations.** We specialize the constructions and results to the (φ, Γ_K)-modules which correspond to Galois representations.

We fix a continuous representation (ρ, V) of the group $\mathcal{G}_K$ on some finite dimensional L-vector space V. Let us write $\mathfrak{X}_{(\rho,V)}$ for the deformation functor of (ρ, V) over $\mathcal{C}$. According to [Ber02] there exists a functor D_{rig} from the category of continuous representation of the group $\mathcal{G}_K$ on finite dimensional L-vector spaces to the category of (φ, Γ_K)-modules over $\mathcal{R}_L$, which is fully faithful by [Col08, Cor. 1.5]. Then, [BC09, Lem. 2.2.7] shows that if A is an object of $\mathcal{C}$ and (ρ, V) is a continuous representation of $\mathcal{G}_K$ with some L-algebras morphism $A \to \mathrm{End}_{\mathcal{G}_K} V$, then V is a finite free A-module if and only if $D_{\mathrm{rig}}(V)$ is a finite free $\mathcal{R}_A$-module. The essential image of the functor D_{rig} is moreover stable under extensions. From these facts we conclude that if (ρ, V) is a continuous representation of $\mathcal{G}_K$ on some finite dimensional L-vector space, then the functor D_{rig} induces an isomorphism of functors

$$\mathfrak{X}_{(\rho,V)} \xrightarrow{D_{\mathrm{rig}}} \mathfrak{X}_{D_{\mathrm{rig}}(V)}. \tag{3.7}$$

If moreover $\mathcal{F}$ is a triangulation of $D_{\mathrm{rig}}(V)$ we define $\mathfrak{X}_{(\rho,V),\mathcal{F}[\frac{1}{t}]}$ as the functor from $\mathcal{C}$ to the category of sets sending an object A to the set of isomorphism classes of tuples $(\rho_A, V_A, \pi_A, \mathcal{M}_A)$ where (ρ_A, V_A) is a continuous representation of $\mathcal{G}_K$ on some finite free A-module V_A, π_A is $\mathcal{G}_K$ equivariant A-linear map $V_A \to V$ inducing an isomorphism $L \otimes_A V_A \xrightarrow{\sim} V$ and $\mathcal{M}_A$ is a triangulation of $D_{\mathrm{rig}}(V_A)[\frac{1}{t}]$ such that $D_{\mathrm{rig}}(\pi_A)(\mathcal{M}_A) = \mathcal{F}[\frac{1}{t}]$.

It follows from [Ber02, Thm. 0.2] that (ρ, V) is crystalline if and only if the (φ, Γ_K)-module $D_{\mathrm{rig}}(V)$ is and that (ρ, V) and $D_{\mathrm{rig}}(V)$ have the same Hodge-Tate type. We say that a crystalline representation (ρ, V) is *φ-generic* if $D_{\mathrm{rig}}(V)$ is.

The functor D_{rig} gives rise to a commutative diagram that relates the surjection of Theorem 3.14 to the linear map

$$\bigoplus_{\mathcal{F} \in \mathrm{Tri}(D_{\mathrm{rig}}(V))} T\mathfrak{X}_{(\rho,V),\mathcal{F}[\frac{1}{t}]} \longrightarrow T\mathfrak{X}_{(\rho,V)}$$

induced by the forgetful functors and allows us to derive the following rephrasing of Theorem 3.14.

Corollary 3.16. *Let (ρ, V) be an L-linear φ-generic crystalline representation of the group $\mathcal{G}_K$ with regular Hodge-Tate type. The forgetful functors induce a surjective L-linear map*

$$\bigoplus_{\mathcal{F} \in \mathrm{Tri}(D_{\mathrm{rig}}(V))} T\mathfrak{X}_{(\rho,V),\mathcal{F}[\frac{1}{t}]} \longrightarrow T\mathfrak{X}_{(\rho,V)}.$$

3.7. **Components of the non saturated deformation ring.** This section contains some complements about the geometry of the deformation space $\mathfrak{X}_{(\rho,V),\mathcal{F}[\frac{1}{t}]}$ that will be useful in the next chapter. One of the most important results here is that several deformation rings and their irreducible components are Cohen-Macaulay. In order to ensure representability of the deformation functors introduced here, we will choose some framings. Let us begin by fixing the set up.

Let (ρ, V) be some n-dimensional L-linear φ-generic crystalline representation of $\mathcal{G}_K$ with regular Hodge-Tate type. Let $\mathcal{D} = D_{\mathrm{rig}}(V)$ and let $\mathcal{F}$ be a triangulation of $\mathcal{D}$. Moreover, write $\mathcal{M} := \mathcal{F}[\frac{1}{t}]$ and $W_{\mathrm{dR}}(\mathcal{D}) := \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{dR}}^+} W_{\mathrm{dR}}^+(\mathcal{D})$. We fix a basis of V, i.e. an L-linear isomorphism $\iota : L^n \simeq V$ so that ρ can be identified with a group homomorphism $\rho : \mathcal{G}_K \to \mathrm{GL}_n(L)$. Let $\mathfrak{X}_\rho$ be the deformation functor of the pair $((\rho, V), \iota)$ which comes with a map $\mathfrak{X}_\rho \to \mathfrak{X}_{(\rho,V)}$ that forgets ι. Note that it is isomorphic to the deformation functor of the *morphism* $\rho : \mathcal{G}_K \to \mathrm{GL}_n(L)$. Using the identification (3.7) and pulling back along this map we define the deformation functors

$$\mathfrak{X}_{\rho,\mathcal{F}}, \mathfrak{X}_{\rho,\mathcal{M}}, \mathfrak{X}_\rho^{\mathrm{cris}} \subset \mathfrak{X}_\rho.$$

These are the obvious variants of the above functors with the corresponding decorations in the context of (φ, Γ_K)-modules.

The study of the deformation functor $\mathfrak{X}_{\rho,\mathcal{M}}$ in [BHS19] is very much inspired by its relation to complete local rings on the trianguline variety. Let us briefly review the definition of the trianguline variety. We refer the reader to [BHS17b, 2.2] for details.

Recall that K is a finite extension of $\mathbf{Q}_p$ and write $\mathcal{G}_K = \mathrm{Gal}(\overline{K}/K)$ for its absolute Galois group. We fix a continuous residual representation

$$\overline{\rho} : \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k_L).$$

Let $R_{\overline{\rho}}$ denote the universal deformation ring of $\overline{\rho}$. Moreover, we assume that the representation ρ fixed above is a lift of $\overline{\rho}$. We write $\mathcal{X}_{\overline{\rho}} = (\mathrm{Spf}\, R_{\overline{\rho}})^{\mathrm{rig}}$ for the rigid-analytic generic fiber of $\mathrm{Spf}\, R_{\overline{\rho}}$ and write $\mathcal{T} = \mathrm{Hom}_{\mathrm{cont}}(K^\times, \mathbb{G}_m(-))$ for the space of continuous characters of $K^\times$. Then by definition $X_{\mathrm{tri}}(\overline{\rho}) \subset \mathcal{X}_{\overline{\rho}} \times \mathcal{T}^n$ is the Zariski-closure of the subset

$$\left\{ (\rho', \delta_1, \ldots, \delta_n) \,\middle|\, \begin{array}{c} \delta_1 \otimes \cdots \otimes \delta_n \text{ is regular and } D_{\mathrm{rig}}(\rho') \text{ has} \\ \text{a triangulation with parameter } \delta_1 \otimes \cdots \otimes \delta_n. \end{array} \right\} \tag{3.8}$$

The closure process adds several new points to the set (3.8) and if a point $(\rho, \delta_1, \ldots, \delta_n) \in X_{\mathrm{tri}}(\overline{\rho})$ maps to ρ under the canonical projection $f : X_{\mathrm{tri}}(\overline{\rho}) \to \mathcal{X}_{\overline{\rho}}$, then $\delta_1 \otimes \cdots \otimes \delta_n$ is not necessarily a parameter of a triangulation of ρ. In the case of crystalline representations, the results of [BHS19, 4.2] allow us to characterize the pre-image $f^{-1}(\rho)$ of our fixed φ-generic crystalline representation ρ. Moreover [BHS19, Cor. 3.7.8] allows us to identify the complete local rings of $X_{\mathrm{tri}}(\overline{\rho})$ at points in $f^{-1}(\rho)$ with the irreducible components of the rings pro-representing $\mathfrak{X}_{\rho,\mathcal{M}}$ (for the various possible $\mathcal{M}$) and [BHS19, Cor. 3.7.10] determines the local geometry of these complete local rings.

In this subsection we review these results that will prove to be important in section 4. Recall that we have fixed a triangulation $\mathcal{F}$ of $\mathcal{D}$ and that $\mathcal{M} = \mathcal{F}[\frac{1}{t}]$. We start by investigating $\mathfrak{X}_{\rho,\mathcal{M}}$.

Proposition 3.17. *The deformation functor $\mathfrak{X}_{\rho,\mathcal{M}}$ is pro-representable by a complete local noetherian L-algebra $R_{\rho,\mathcal{M}}$. This L-algebra is reduced, a complete intersection and equidimensional of dimension*

$$n^2 + [K : \mathbf{Q}_p]\frac{n(n+1)}{2}.$$

In particular $R_{\rho,\mathcal{M}}$ is Cohen-Macaulay.

Proof. The representability, as well as the calculation of the dimension is [BHS19, Thm. 3.6.2 (i)]. The fact that the ring is a complete intersection follows from the smoothness of the map $X_{D,\mathcal{M}} \to X_{W^+,\mathcal{F}}$ in [BHS19, Cor. 3.5.6] and the fact that the target is pro-represented by the complete local ring on an explicit scheme (more precisely: the scheme $X_{K/\mathbf{Q}_p,L}$) by [BHS19, Cor. 3.5.8]. This scheme is a complete intersection by [BHS19, Prop. 2.2.5]. The fact that $R_{\rho,\mathcal{M}}$ is Cohen-Macaulay is a direct consequence from the fact that it is a complete intersection □

In general the space $\operatorname{Spec} R_{\rho,\mathcal{M}}$ can have several irreducible components, i.e. the ring $R_{\rho,\mathcal{M}}$ can have several minimal prime ideals. It turns out that its minimal prime ideals can be described in terms of combinatorial data (Weyl group elements) attached to the triangulation. We recall this description of the minimal primes from [BHS19].

Let $F \coloneqq D_{\mathrm{cris}}(\mathcal{F})$ be complete flag of $D_{\mathrm{cris}}(\mathcal{D})$ associated to the triangulation $\mathcal{F}$. For each $\tau \in \Sigma$, let F_τ be the complete flag of $D_{\mathrm{dR},\tau}(D)$ image of F under the functor $(-) \otimes_{L\otimes_{\mathbf{Q}_p} K_0,\tau} L$. The stabilizer $B_{F,\tau}$ of F_τ is a Borel subgroup of $\mathrm{GL}(D_{\mathrm{dR},\tau}(\mathcal{D}))$ and there exists a unique $w_{F,\tau} \in \mathfrak{S}_n$ such that $\mathrm{Fil}_{\mathrm{dR},\tau} \in B_{F,\tau} w_{F,\tau}(F_\tau)$. We define $w_{\mathcal{F}} \coloneqq (w_{F,\tau})_{\tau\in\Sigma} \in \mathcal{W} = (\mathfrak{S}_n)^{[K:\mathbf{Q}_p]}$. We write $\preceq$ for the Bruhat order in the Weyl group $\mathcal{W}$ (in this subsection we use calligraphic letters for the Weyl group in order to distinguish the Weyl group from the $\mathbf{B}_{\mathrm{dR}}$-representations. In the following sections $\mathbf{B}_{\mathrm{dR}}$-representations won't appear and we will return to the usual roman letters for the Weyl groups).

Proposition 3.18. *There is a bijection of the set of minimal primes of $R_{\rho,\mathcal{M}}$ with the set $\{w \in \mathcal{W} | w_{\mathcal{F}} \preceq w\}$. The corresponding quotient $R^w_{\rho,\mathcal{M}}$ of $R_{\rho,\mathcal{M}}$ is normal and Cohen-Macaulay.*

In order to prove this proposition we recall some details from the construction in [BHS19]. Recall that we defined the scheme $X = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ below

Theorem 3.14. Let $\mathfrak{t}$ be the diagonal torus of $\mathfrak{g} = \mathfrak{gl}_{n,K}$. There is a canonical map $\tilde{\mathfrak{g}} \to \mathfrak{t}$ mapping $(A, gB) \in \tilde{\mathfrak{g}}$ to the class of $\mathrm{Ad}(g^{-1})A$ in $\mathfrak{b}/\mathfrak{u}$. Here $\mathfrak{u} \subset \mathfrak{b}$ is the sub-Lie-algebra of nilpotent upper triangular matrices, and the quotient $\mathfrak{b}/\mathfrak{u}$ is canonically identified with $\mathfrak{t}$. Thus we obtain a map

$$\Theta : X_{K/\mathbf{Q}_p,L} \longrightarrow \mathfrak{t}_{K/\mathbf{Q}_p,L} \times_{\mathfrak{t}_{K/\mathbf{Q}_p,L}/\mathcal{W}} \mathfrak{t}_{K/\mathbf{Q}_p,L}.$$

The source of this map decomposes into irreducible components X_w indexed by the elements w of the Weyl group $\mathcal{W}$ of $\mathrm{Res}_{K/\mathbf{Q}_p} \mathrm{GL}_n$. This follows from the decomposition $X_{K/\mathbf{Q}_p,L} \cong \prod_{\tau\in\Sigma} X_L$ and the description of the irreducible components of X, see [BHS19, Prop. 2.2.5] which can be applied to each factor of this product. Similarly, the target decomposes into irreducible components $\mathfrak{t}_w$ indexed by the elements of $\mathcal{W}$, see [BHS19, Lem. 2.5.1]. The map Θ is called (κ_1, κ_2) in *loc. cit.* and using Lemma 2.5.1 of *loc. cit.* again we deduce that $X_w \subset X_{K/\mathbf{Q}_p,L}$ is the unique irreducible component such that $\Theta(X_w) = \mathfrak{t}_w$.

Now we return to the deformation problem $\mathfrak{X}_{\rho,\mathcal{M}}$ and relate it to a complete local ring on $X_{K/\mathbf{Q}_p,L}$. We use the notations introduced just before the beginning of the proof of Theorem 3.14 and write

$$(W, M) = (W^+_{\mathrm{dR}}(\mathcal{D}), W_{\mathrm{dR}}(\mathcal{M})).$$

Consider the following diagram:

$$\begin{tikzcd}
 & \mathfrak{X}^{\square}_{\rho,\mathcal{M}} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & \\
\mathfrak{X}_{\rho,\mathcal{M}} & & \mathfrak{X}^{\square}_{W,M} \arrow[r, "\gamma"] & (\widehat{X_{K/\mathbb{Q}_p,L}})_{x_M} \arrow[d] \\
 & & & (\mathfrak{t}_{K/\mathbf{Q}_p,L} \times_{\mathfrak{t}_{K/\mathbf{Q}_p,L}/\mathcal{W}} \mathfrak{t}_{K/\mathbf{Q}_p,L})\widehat{}_{(0,0)}.
\end{tikzcd}$$

Here α is the formally smooth morphism that adds a framing of $W^+_{\mathrm{dR}}(-)$, β is the formally smooth map (3.4) and γ is left vertical isomorphism from (3.5). Moreover we note that $\Theta(x_M) = (0, 0)$, as ρ is crystalline (and hence $W \otimes_{\mathbf{B}^+_{\mathrm{dR}}} \mathbf{B}_{\mathrm{dR}}$ is the trivial $\mathbf{B}_{\mathrm{dR}}$-representation) and hence the right vertical arrow is well defined. By the argument just before [BHS19, Cor. 3.5.12] the composition

$$\mathfrak{X}^{\square}_{\rho,\mathcal{M}} \longrightarrow (\mathfrak{t}_{K/\mathbf{Q}_p,L} \times_{\mathfrak{t}_{K/\mathbf{Q}_p,L}/\mathcal{W}} \mathfrak{t}_{K/\mathbf{Q}_p,L})\widehat{}_{(0,0)}$$

descends to a map

$$\mathfrak{X}_{\rho,\mathcal{M}} \longrightarrow (\mathfrak{t}_{K/\mathbf{Q}_p,L} \times_{\mathfrak{t}_{K/\mathbf{Q}_p,L}/\mathcal{W}} \mathfrak{t}_{K/\mathbf{Q}_p,L})\widehat{}_{(0,0)}$$

that we still call Θ by abuse of notation. Similarly, for $w \in \mathcal{W}$, the preimage $\mathfrak{X}^{\square,w}_{\rho,\mathcal{M}}$ of the closed formal subscheme $(\widehat{X_w})_{x_M} \subset (\widehat{X_{K/\mathbb{Q}_p,L}})_{x_M}$ in $\mathfrak{X}^{\square}_{\rho,\mathcal{M}}$

descends to a closed formal subscheme
$$\mathfrak{X}^w_{\rho,\mathcal{M}} \subset \mathfrak{X}_{\rho,\mathcal{M}}.$$
Note that this formal subscheme is empty if and only if $x_M \notin X_w \subset X_{K/\mathbb{Q}_p,L}$. Moreover, we note that the discussion implies that the canonical maps

$$\begin{tikzcd} & \mathfrak{X}^{\square,w}_{\rho,\mathcal{M}} \arrow[dl] \arrow[dr] & \\ \mathfrak{X}^w_{\rho,\mathcal{M}} & & (\widehat{X_w})_{x_M} \end{tikzcd} \tag{3.9}$$

are both formally smooth (by the very definition of the spaces involved).

Proof of Proposition 3.18. By [BHS19, Thm. 3.6.2 (ii)] the minimal primes $\mathfrak{p}_w$ of $R_{\rho,\mathcal{M}}$ are in bijection with a subset $\mathcal{S}(\mathcal{D},\mathcal{F}) \subset \mathcal{W}$ defined just before Corollary 3.5.11 of *loc. cit.*, and the quotient $R^w_{\rho,\mathcal{M}} = R_{\rho,\mathcal{M}}/\mathfrak{p}_w$ satisfies $\operatorname{Spf} R^w_{\rho,\mathcal{M}} = \mathfrak{X}^w_{\rho,\mathcal{M}}$. By definition $w \in \mathcal{S}(\mathcal{D},\mathcal{F})$ if and only if $x_M \in X_{K/\mathbb{Q}_p,L,w}$, i.e. if and only if $(\widehat{X_w})_{x_M} \neq \emptyset$. By [BHS19, Prop. 3.6.4] we have
$$\mathcal{S}(\mathcal{D},\mathcal{F}) \subset \{w \in \mathcal{W} | w_{\mathcal{F}} \preceq w\}.$$
We claim that $\mathcal{S}(\mathcal{D},\mathcal{F}) = \{w \in \mathcal{W} | w_{\mathcal{F}} \preceq w\}$. Let $w \in \mathcal{W}$ such that $w_{\mathcal{F}} \preceq w$. By [BHS19, Thm. 4.2.3] the trianguline variety $X_{\mathrm{tri}}(\overline{\rho})$ contains a specific point x_w depending on $\rho, \mathcal{F}$ and w (see loc.cit for the precise description). By [BHS19, Cor. 3.7.8] the completion of the trianguline variety at this point is given by
$$\widehat{X_{\mathrm{tri}}(\overline{\rho})}_{x_w} \xrightarrow{\cong} \mathfrak{X}^w_{\rho,\mathcal{M}}. \tag{3.10}$$
In particular the target of the isomorphism is non empty and we deduce that $w \in \mathcal{S}(\mathcal{D},\mathcal{F})$. Finally it follows from
$$\operatorname{Spf} R^w_{\rho,\mathcal{M}} = \mathfrak{X}^w_{\rho,\mathcal{M}}$$
together with the formally smooth maps in (3.9) and [BHS19, Prop. 2.3.3, Thm. 2.3.6] that $R^w_{\rho,\mathcal{M}}$ is normal and Cohen-Macaulay (compare also [BHS19, Cor. 3.7.10] and (3.10)). $\square$

Remark 3.19. In this paper the Cohen-Macaulayness of several local rings is an extremely important tool: it is a crucial ingredient in the arguments in section 4. In [BHS19] we deduce this Cohen-Macaulayness of the local rings $R^w_{\rho,\mathcal{M}}$ from the Cohen-Macaulayness of the irreducible components of the scheme X. This result in turn was proven by Bezrukavnikov and Riche [BR12], so the argument here crucially relies on their paper.

We recall one further consequence of the above constructions from [BHS19]. For $w \in \mathcal{W}$ we write $\widehat{\mathfrak{t}_w}$ its completion at $(0,0)$ and recall the ensuing characterization of $\mathfrak{X}^w_{\rho,\mathcal{M}}$ which follows from [BHS19, Cor. 3.5.12].

Proposition 3.20. *Let w_1 and w_2 two elements of $\{w \in \mathcal{W}, w_{\mathcal{F}} \preceq w\}$. Then $\Theta(\mathfrak{X}_{\rho,\mathcal{M}}^{w_1}) \subset \widehat{\mathfrak{t}_{w_2}}$ if and only if $w_1 = w_2$.*

We finally recall some facts about crystalline deformations.

Proposition 3.21. *(i) The functor $\mathfrak{X}_\rho^{\mathrm{cris}}$ is pro-representable by a formally smooth L-algebra of dimension $n^2 + [K : \mathbf{Q}_p]\frac{n(n-1)}{2}$.*

(ii) The deformation functor $\mathfrak{X}_\rho^{\mathrm{cris}} \subset \mathfrak{X}_{\rho,\mathcal{M}}$ is a subfunctor of $\mathfrak{X}_{\rho,\mathcal{M}}^{w_0}$, where $w_0 \in \mathcal{W}$ denotes the longest Weyl group element.

Proof. (i) This is a direct consequence of the main result of [Kis08].

(ii) This follows from the proof of [BHS19, Thm. 4.2.3], or, slightly more precisely, from the embedding (4.2) in *loc.cit.* □

4. Global deformation rings

Let F be a totally real field and E a totally imaginary quadratic extension that we assume to be unramified over F, such that all places v dividing p are split in E and such that E doesn't contain primitive p-th roots of 1. Let G be a unitary group in n variables defined over F such that $G \times_F E$ is an inner form of $\mathrm{GL}_{n,E}$. We assume moreover that $G(F \otimes_{\mathbf{Q}} \mathbf{R})$ is compact and that the group G is quasi-split over all finite places of F. This implies that n is odd or that $4|n[F : \mathbf{Q}]$. Moreover, if v is a place of F which splits in E, the group G splits at v. In this case we fix a place $\tilde{v}$ of E dividing v and an isomorphism $G \times_F E_{\tilde{v}} \cong \mathrm{GL}_{n,E_{\tilde{v}}}$ which induces an isomorphism $G \times_F F_v \simeq \mathrm{GL}_{n,F_v}$. Let $B_v \subset G(F_v)$ be the subgroup corresponding to the Borel subgroup of upper triangular matrices of $\mathrm{GL}_n(F_v)$ under this isomorphism and $T_v \subset B_v$ the subgroup corresponding to the subgroup of diagonal matrices in $\mathrm{GL}_n(F_v)$ that we identify with $(F_v^\times)^n \simeq (E_{\tilde{v}}^\times)^n$ using the diagonal elementary matrices. We write $T_p = \prod_{v|p} T_v$ and $B_p = \prod_{v|p} B_v$. Moreover we define $U_v \subset G(F_v)$ the maximal compact subgroup of $G(F_v)$ corresponding to $\mathrm{GL}_n(\mathcal{O}_{F_v})$ under this isomorphism.

Let U^p be a compact open subgroup of $G(\mathbb{A}_F^{p,\infty})$ of the form $\prod_{v\nmid p} U_v$ with U_v a compact open subgroup of $G(F_v)$. We assume that U_v is hyperspecial for all places v of F which are inert in E. Let S_p denote the set of places of F that divide p and let S be a finite set of places of F containing S_p and the finite set of places of F for which U_v is not hyperspecial. Finally we write $U = U^p \times U_p$, where $U_p = \prod_{v|p} U_v$ is a maximal compact subgroup of $G(F \otimes_{\mathbf{Q}} \mathbf{Q}_p)$.

We write E_S for the maximal extension of E that is unramified outside all places of E above the places in S and denote by $\mathcal{G}_{E,S} = \mathrm{Gal}(E_S/E)$ the corresponding Galois group. Let A be a $\mathbf{Z}_p$-algebra and ρ_A a representation of $\mathcal{G}_{E,S}$ on some finite free A-module V_A of rank n. We write ρ_A^c for the representation $g \mapsto \rho_A(cgc)$, where $c \in \mathrm{Gal}(\overline{F}/F)$ is a complex conjugation.

We recall that the representation (ρ_A, V_A) is called *polarizable*, if there exists an isomorphism

$$(\rho^{\vee,c}, V_A^{\vee}) \cong (\rho \otimes \varepsilon^{n-1}, V_A),$$

where ε is the cyclotomic character. Such an isomorphism is called a *polarization*.

We fix L a finite extension of $\mathbf{Q}_p$ and $(\overline{\rho}, \overline{V})$ a continuous polarized representation $\mathcal{G}_{E,S} \to \mathrm{GL}_n(k_L)$ which is absolutely irreducible so that it has a unique polarization up to scalar multiplication. We denote by $R_{\overline{\rho},S}$ the universal polarized deformation $\mathcal{O}_L$-algebra of $\overline{\rho}$. That is, the complete local $\mathcal{O}_L$-algebra pro-representing the functor of isomorphism classes of triples (ρ_A, V_A, ι_A), with V_A a finite free A-module with a continuous polarized action ρ_A of $\mathcal{G}_{E,S}$ and an isomorphism $\iota_A : V_A/\mathfrak{m}_A V_A \cong \overline{V}$ of $\mathcal{G}_{E,S}$-representations, on the category of local Artinian $\mathcal{O}_L$-algebras A with residue field k_L. The existence of the $\mathcal{O}_L$-algebra $R_{\overline{\rho},S}$ follows from [Che11, §1.1]. We assume that L is large enough in the sense that $L \subset \overline{\mathbf{Q}_p}$ contains all the $\tau(E_{\tilde{v}})$ for all places $v|p$ and all embeddings $\tau : F_v = E_{\tilde{v}} \hookrightarrow \overline{\mathbf{Q}_p}$.

Let $\mathcal{X}_{\overline{\rho},S} = (\mathrm{Spf}\, R_{\overline{\rho},S})^{\mathrm{rig}}$ be the rigid analytic generic fiber of the formal scheme $\mathrm{Spf}\, R_{\overline{\rho},S}$. As $(\overline{\rho}, \overline{V})$ is absolutely irreducible, the L-points of $\mathcal{X}_{\overline{\rho},S}$ are in bijection with the set of isomorphism classes of continuous representations (ρ, V) of $\mathcal{G}_{E,S}$ on L-vector spaces such that $(\rho^{\vee,c}, V^{\vee}) \simeq (\rho \otimes \varepsilon^{n-1}, V)$ and such that there exists a $\mathcal{G}_{E,S}$-stable $\mathcal{O}_L$-lattice $V^\circ \subset V$ and a $\mathcal{G}_{E,S}$-equivariant isomorphism $V^\circ/\varpi_L V^\circ \simeq \overline{V}$. Given a point $x \in \mathcal{X}_{\overline{\rho}}$, we denote by (ρ_x, V_x) the associated representation of $\mathcal{G}_{E,S}$.

Fix an isomorphism $\iota : \overline{\mathbf{Q}_p} \simeq \mathbf{C}$. Recall that, if π is a cuspidal automorphic representation of G, there exists a unique semisimple polarized n-dimensional $\overline{\mathbf{Q}_p}$-representation (ρ_π, V_π) of $\mathrm{Gal}(\overline{E}/E)$ associated to π. If $(\pi^{p,\infty})^{U^p} \neq 0$ then this representation factors through $\mathcal{G}_{E,S}$. The existence of this Galois representation is a consequence of base change ([Lab11, Cor. 5.3]) and of the construction of Galois representations associated to some automorphic representation of $\mathrm{GL}_{n,E}$ (see [CH13]). We say that a point $x \in \mathcal{X}_{\overline{\rho},S}(L)$ is (G, U^p)-automorphic (resp. (G, U)-automorphic) if there exists a cuspidal automorphic representation π of G such that $(\pi^{p,\infty})^{U^p} \neq 0$ (resp. such that $(\pi^\infty)^U \neq 0$) and such that there is an isomorphism $(\rho_x, V_x \otimes_L \overline{\mathbf{Q}_p}) \simeq (\rho_\pi, V_\pi)$. Moreover we say that $(\overline{\rho}, \overline{V})$ is (G, U)-automorphic over L if there exists a (G, U)-automorphic point $x \in \mathcal{X}_{\overline{\rho},S}(L)$. Let $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$ be the Zariski closure of the set of (G, U)-automorphic points in $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$.

The aim of this section is to prove the following theorem:

Theorem 4.1. *Assume that $p > 2$, that all places of S are split in E and that the group $\overline{\rho}(\mathcal{G}_{E(\zeta_p)})$ is adequate in the sense of* [Tho12, Def. 2.3]. *Then the inclusion $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S} \subset \mathcal{X}_{\overline{\rho},S}$ is the inclusion of a union of irreducible components (possibly empty if $(\overline{\rho}, \overline{V})$ is not (G, U)-automorphic).*

>From now on we assume $p > 2$, that the places of S split in E, that $\overline{\rho}(\mathcal{G}_{E(\zeta_p)})$ is adequate and that $(\overline{\rho}, \overline{V})$ is (G, U)-automorphic.

Recall that, for a place $v \in S$, we fix a place $\tilde{v}$ of E dividing v. We write $\mathcal{G}_{E_{\tilde{v}}}$ for the choice of a decomposition group at $\tilde{v}$. Given a representation ρ of $\mathcal{G}_{E,S}$ we write $\rho_{\tilde{v}}$ for the restriction of ρ to $\mathcal{G}_{E_{\tilde{v}}}$.

In subsections 4.1 to 4.2, we assume that U^p is sufficiently small so that the compact open subgroup $U := \prod_v U_v$ is such that

$$\forall g \in G(\mathbb{A}_F^\infty),\ G(F) \cap gUg^{-1} = \{1\}. \tag{4.1}$$

4.1. **Recollections about eigenvarieties and patching.** Attached to the data G, U^p and $\overline{\rho}$ there is a so-called *eigenvariety*. We briefly recall the main notations and objects and refer to [BHS17a, 3.1] for details of this construction.

For a place v dividing p let us write $\hat{T}_v$ for the rigid analytic space of continuous characters of T_v and similarly $\hat{T}_v^0$ for the space of continuous characters of the maximal compact subgroup $T_v^0 \subset T_v$. Further let

$$\hat{T}_p = \prod_{v|p} \hat{T}_v \qquad \text{and} \qquad \hat{T}_p^0 = \prod_{v|p} \hat{T}_v^0.$$

We write $\hat{S}(U^p, L)$ for the space of p-adic automorphic forms of tame level U^p on G. This is a Banach space over L which comes with an action of $G(F \otimes_{\mathbf{Q}} \mathbf{Q}_p) = \prod_{v|p} G(F_v)$ and a commuting action of a Hecke-algebra. Let $\mathfrak{m}$ denote a fixed maximal ideal of this Hecke-algebra corresponding to the residual Galois representation $\overline{\rho}$, and write $\hat{S}(U^p, L)_{\mathfrak{m}}$ for the localization of $\hat{S}(U^p, L)$ with respect to this maximal ideal of the Hecke-algebra. The eigenvariety associated to G, U^p and $\overline{\rho}$ is by definition the Zariski-closed rigid analytic subspace $Y(U^p, \overline{\rho}) \subset \mathcal{X}_{\overline{\rho},S} \times \hat{T}_{p,L}$ that is the (scheme-theoretic) support of dual of the locally analytic Jacquet-module $J_{B_p}(\hat{S}(U^p, L)_{\mathfrak{m}}^{\mathrm{an}})$ of the locally analytic representation underlying the $G(F \otimes_{\mathbf{Q}} \mathbf{Q}_p)$-representation on $\hat{S}(U^p, L)_{\mathfrak{m}}$.

We recall the notion of a *classical point* on $Y(U^p, \overline{\rho})$: We write $X^*(T)$ for the space of algebraic characters of the product of the diagonal tori in

$$(\mathrm{Res}_{F/\mathbf{Q}}\, G)_{\mathbf{C}} \cong \prod_{\tau: F \hookrightarrow \mathbf{C}} \mathrm{GL}_{n,\mathbf{C}}\,. \tag{4.2}$$

This space comes equipped with an action of the Weyl group W of $(\mathrm{Res}_{F/\mathbf{Q}}\, G)_{\mathbf{C}}$. As usual we write $w \cdot \lambda$ for the shifted *dot*-action of W on $X^*(T)$. We write w_0 for the longest element of W.

The fixed isomorphism $\overline{\mathbf{Q}_p} \cong \mathbf{C}$ identifies $\lambda \in X^*(T)$ with a character $T_p \to \overline{\mathbf{Q}_p}^\times$ that we denote by z^λ. If L is a finite extension of $\mathbf{Q}_p$ such that $\mathrm{Res}_{F/\mathbf{Q}}\, G$ splits over L, then z^λ takes values in L and we may view it as an L-valued point of $\hat{T}_p$.

Given a representation π_∞ of $G(F \otimes_{\mathbf{Q}} \mathbf{R})$ we say that π_∞ is of weight $\lambda \in X^*(T)$ if it is the restriction to $G(F \otimes_{\mathbf{Q}} \mathbf{R})$ of the irreducible algebraic representation of $(\mathrm{Res}_{F/\mathbf{Q}} G)_{\mathbf{C}}$ of highest weight λ.

Let $\pi = \pi_\infty \otimes_{\mathbf{C}} \pi^{p,\infty} \otimes_{\mathbf{C}} \pi_p$ be an automorphic representation of G such that $(\pi^{p,\infty})^{U^p} \neq 0$ and such that π_∞ is of weight λ. Moreover we assume that, for all $v \in S_p$, the representation π_v is an unramified quotient of the smooth induced representation $(\mathrm{Ind}_{B_v}^{G(F_v)} \delta_{\mathrm{sm},v}\delta_v)^{\mathrm{sm}}$ for some unramified character $\delta_{\mathrm{sm},v}$ of T_v with values in $L^\times$ and where δ_v is the smooth character

$$\delta_v = (1 \otimes |\cdot|_v \otimes \cdots \otimes |\cdot|_v^{n-1}).$$

Let $\delta_{\mathrm{sm}} := \bigotimes_{v \in S_p} \delta_{\mathrm{sm},v}$. The Galois representation ρ_π associated to π is (G,U)-automorphic by definition and we have

$$(\rho_\pi, \delta_{\mathrm{sm}} z^\lambda) \in Y(U^p, \overline{\rho})(L) \subset \mathcal{X}_{\overline{\rho},S}(L) \times \hat{T}_p(L),$$

see [BHS17a, Prop. 3.4] for example. The point $x = (\rho_\pi, \delta_{\mathrm{sm}} z^\lambda)$ is called the classical point associated with $(\pi, \delta_{\mathrm{sm}})$.

It follows from [CH13] that, for $v \in S_p$, the representation $\rho_{\tilde{v}}$ is crystalline and that the character $\delta_{\mathrm{sm},v}$ is of the form $\delta_{\mathcal{F}_{\tilde{v}}}$ for $\mathcal{F}_{\tilde{v}}$ a triangulation of $\rho_{\tilde{v}}$ (see §3.3 for the definition of $\delta_{\mathcal{F}_{\tilde{v}}}$ and use the identification $T_v = (F_v^\times)^n = (E_{\tilde{v}}^\times)^n$). We say that ρ is crystalline φ-generic if $\rho_{\tilde{v}}$ is crystalline φ-generic for all places v dividing p.

Assuming that ρ is crystalline φ-generic, the map

$$\mathcal{F}_{\tilde{v}} \longmapsto \delta_{\mathcal{F}_{\tilde{v}}}$$

induces a bijection between the set of smooth characters δ_{sm} such that $\pi_v \cong (\mathrm{Ind}_{B_v}^{G(F_v)} \delta_{\mathrm{sm}}\delta_v)^{\mathrm{sm}}$ and the triangulations of $\rho_{\tilde{v}}$. This follows from the classification of unramified representations of $G(F_v)$ (see for example [BC09, §6.4.3] and [BC09, Prop. 6.4.8]) and from the classification of triangulations of a crystalline φ-generic representation discussed in section 3.3.

Similarly, given a tuple $\underline{\mathcal{F}} = (\mathcal{F}_{\tilde{v}})_{v \in S_p}$ of triangulations we write $\delta_{\underline{\mathcal{F}}} = (\delta_{\mathcal{F}_{\tilde{v}}})_v$ for the corresponding unramified character of T_p. By construction, in this case $x_{\underline{\mathcal{F}}} := (\rho, z^\lambda \delta_{\underline{\mathcal{F}}})$ is a classical point of $Y(U^p, \overline{\rho})$ associated to the pair $(\pi, \delta_{\underline{\mathcal{F}}})$.

Fix an embedding $\tau : E_{\tilde{v}} = F_v \hookrightarrow \overline{\mathbf{Q}_p}$. Via the identification $\overline{\mathbf{Q}_p} \cong \mathbf{C}$ this embedding restricts to an embedding $F \hookrightarrow \mathbf{C}$ and we write W_τ for the factor of the Weyl group W corresponding to this embedding via the decomposition (4.2).

The relative position of the τ-part of the Hodge filtration

$$\begin{aligned}\mathrm{Fil}_{\mathrm{dR},\tau} := \mathrm{Fil}_{\mathrm{dR}} \otimes_{E_{\tilde{v}} \otimes_{\mathbf{Q}_p} L, \tau \otimes \mathrm{Id}} \overline{\mathbf{Q}_p} &\subset D_{\mathrm{dR}}(\rho_{\tilde{v}}) \otimes_{E_{\tilde{v}} \otimes_{\mathbf{Q}_p} L, \tau \otimes \mathrm{Id}} \overline{\mathbf{Q}_p} \\ &= D_{\mathrm{cris}}(\rho_{\tilde{v}}) \otimes_{E_{\tilde{v},0} \otimes_{\mathbf{Q}_p} L, \tau|_{E_{\tilde{v},0}} \otimes \mathrm{Id}} \overline{\mathbf{Q}_p}\end{aligned}$$

with respect to $\mathcal{F}_{\tilde{v}} \otimes_{E_{\tilde{v},0}\otimes_{\mathbf{Q}_p} L, \tau\otimes \mathrm{Id}} \overline{\mathbf{Q}_p}$ defines an element of the Weyl group $w_{\mathcal{F}_\tau} \in W_\tau$. We write $w_{\tilde{v}} = (w_{\mathcal{F}_\tau})_\tau \in \prod_{\tau: E_{\tilde{v}} \hookrightarrow \overline{\mathbf{Q}_p}} W_\tau = W_{\tilde{v}}$ and $w_{\underline{\mathcal{F}}} = (w_{\tilde{v}})_v \in W = \prod_v W_{\tilde{v}}$ for the Weyl group element defined by the tuple $\underline{\mathcal{F}}$.

The following proposition summarizes the properties of the eigenvariety needed for the proof of the main theorem:

Proposition 4.2. *(i) The eigenvariety $Y(U^p, \overline{\rho})$ is reduced and equi-dimensional of dimension*

$$\dim Y(U^p, \overline{\rho}) = \dim \hat{T}_p^0 = n[F : \mathbf{Q}].$$

(ii) The set of classical points as defined above is Zariski-dense and has the accumulation property, i.e. for every classical point x and every open connected neighborhood U of x the classical, crystalline φ-generic points are Zariski-dense in U.

(iii) Let $x = x_{\underline{\mathcal{F}}}$ be a classical, crystalline φ-generic point associated to $(\pi, \delta_{\underline{\mathcal{F}}})$ as above. For a weight $\mu \in X^*(T)$, one has

$$(\rho, z^\mu \, \delta_{\underline{\mathcal{F}}}) \in Y(U^p, \overline{\rho}) \Longleftrightarrow \mu = ww_0 \cdot \lambda \text{ with } w \in W,\ w_{\underline{\mathcal{F}}} \preceq w.$$

For $w_{\underline{\mathcal{F}}} \preceq w$, we define

$$x_{\underline{\mathcal{F}},w} := (\rho, z^{ww_0\cdot\lambda} \delta_{\underline{\mathcal{F}}}).$$

(iv) Let x be as in (iii). Then the projection $\omega : Y(U^p, \overline{\rho}) \to \hat{T}_p^0$ is flat at the points $x_{\underline{\mathcal{F}},w}$.

Proof. Points (i) and (ii) are contained in [Che05, 3.8]. The statements can be obtained as well along the lines of Corollaire 3.12, Théorème 3.19 and Corollaire 3.20 of [BHS17b]. See Definition 3.2 and Proposition 3.4 of [BHS17a] for a comparison of the (a priori different) notions of classical points.
(iii) This is a direct consequence of [BHS19, Thm. 5.3.3].
(iv) This is contained in [BHS19, Thm. 5.4.2]. □

We further recall the *patched eigenvariety* $X_p(\overline{\rho})$ and its relation to the global object $Y(U^p, \overline{\rho})$. In [BHS17b, 3.2] we have carried out the following construction: Let us write

$$R_{\overline{\rho}_p} = \widehat{\bigotimes_{v\in S_p}} R_{\overline{\rho}_{\tilde{v}}} \text{ and } R_{\overline{\rho}^p} = \widehat{\bigotimes_{v\in S\setminus S_p}} R_{\overline{\rho}_{\tilde{v}}}$$

for the completed tensor products over $\mathcal{O}_L$ of the maximal reduced and $\mathbf{Z}_p$-flat quotients $R_{\overline{\rho}_{\tilde{v}}}$ of the universal framed deformation rings $R'_{\overline{\rho}_{\tilde{v}}}$ of $\overline{\rho}_{\tilde{v}}$. Let

$$R_{\overline{\rho},\mathcal{S}} := R_{\overline{\rho},S} \otimes_{\left(\widehat{\bigotimes}_{v\in S} R'_{\overline{\rho}_{\tilde{v}}}\right)} \left(\widehat{\bigotimes_{v\in S}} R_{\overline{\rho}_{\tilde{v}}}\right).$$

As in loc.cit. (see also the Erratum in [BHS17a, 6]), there exists an integer $g \geqslant 1$ and a commutative diagram with maps of local $\mathcal{O}_L$-algebras

$$\begin{tikzcd} S_\infty \coloneqq \mathcal{O}_L[\![\mathbf{Z}_p^q]\!] \arrow[r] \arrow[d] & R_\infty \coloneqq (R_{\overline{\rho}_p} \widehat{\otimes}_{\mathcal{O}_L} R_{\overline{\rho}^p})[\![y_1, \dots, y_g]\!] \arrow[d, two heads] \\ R_\infty \otimes_{S_\infty} \mathcal{O}_L \arrow[r, two heads] & R_{\overline{\rho},\mathcal{S}} \end{tikzcd} \tag{4.3}$$

where the left vertical map is induced by the augmentation map $S_\infty \to \mathcal{O}_L$ and where $q = g + [F : \mathbf{Q}]\frac{n(n-1)}{2} + n^2|S|$. Let $\mathcal{X}_{\overline{\rho},\mathcal{S}}$ be the rigid analytic generic fiber of $\operatorname{Spf} R_{\overline{\rho},\mathcal{S}}$. The surjective map $R_{\overline{\rho},S} \twoheadrightarrow R_{\overline{\rho},\mathcal{S}}$ induces a closed immersion of rigid analytic spaces ([dJ95, Prop. 7.2.1.d)])

$$\mathcal{X}_{\overline{\rho},\mathcal{S}} \hookrightarrow \mathcal{X}_{\overline{\rho},S}. \tag{4.4}$$

As a continuous morphism of $R_{\overline{\rho},S}$ to a finite extension of L factors necessarily through $R_{\overline{\rho},\mathcal{S}}$, it follows that the injection (4.4) induces an isomorphism at the level of reduced rigid analytic subspaces. In particular the closed immersion $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S} \hookrightarrow \mathcal{X}_{\overline{\rho},S}$ factors through $\mathcal{X}_{\overline{\rho},\mathcal{S}}$ since the left hand side is reduced by definition.

We write $\mathcal{X}_\infty$, $\mathcal{X}_{\overline{\rho}_p}$ and $\mathcal{X}_{\overline{\rho}^p}$ for the rigid analytic generic fibers of $\operatorname{Spf} R_\infty$, $\operatorname{Spf} R_{\overline{\rho}_p}$ and $\operatorname{Spf} R_{\overline{\rho}^p}$. Moreover we denote by $X_p(\overline{\rho}) \subset \mathcal{X}_\infty \times \hat{T}_p$ the patched eigenvariety constructed in [BHS17b, Def. 3.6]. Then there are canonical embeddings

$$\begin{aligned} Y(U^p, \overline{\rho}) \hookrightarrow X_p(\overline{\rho}) \times_{\mathcal{X}_\infty} \mathcal{X}_{\overline{\rho},\mathcal{S}} &\hookrightarrow X_p(\overline{\rho}) \times_{(\operatorname{Spf} S_\infty)^{\mathrm{rig}}} \operatorname{Sp} L \\ &\subset (\mathcal{X}_\infty \times_{(\operatorname{Spf} S_\infty)^{\mathrm{rig}}} \operatorname{Sp} L) \times \hat{T}_p \subset \mathcal{X}_\infty \times \hat{T}_p, \end{aligned} \tag{4.5}$$

see [BHS17b, 4.1] or [BHS19, (5.38)]. Let us abbreviate $X_p(\overline{\rho}) \times_{(\operatorname{Spf} S_\infty)^{\mathrm{rig}}} \operatorname{Sp} L$ by $Y_p(\overline{\rho})$ for the moment. The precise relation of the local geometry of the patched eigenvariety and the (global) eigenvariety is given by the following proposition:

Proposition 4.3. *Assume that $x = x_{\underline{\mathcal{F}}}$ is a classical, crystalline φ-generic point. For each $w \in W$ such that $x_{\underline{\mathcal{F}},w} \in Y(U^p, \overline{\rho})$ the morphism* (4.5) *induces an isomorphism of complete local rings*

$$\hat{\mathcal{O}}_{Y_p(\overline{\rho}), x_{\underline{\mathcal{F}},w}} \cong \hat{\mathcal{O}}_{Y(U^p,\overline{\rho}), x_{\underline{\mathcal{F}},w}}.$$

Proof. This is essentially [BHS19, Prop. 5.4.1]. We note that *loc. cit.* concerns only the case $w = w_0$. However the same proof applies *mutatis mutandis* for any choice of w. The point which has to be checked is the Cohen-Macaulay property of $X_p(\overline{\rho})$ at $x_{\underline{\mathcal{F}},w}$ which follows from [BHS19, Cor. 3.7.10] exactly as in the proof of [BHS19, Prop. 5.4.1]. $\square$

Finally we recall the relation of the patched eigenvariety with the space of trianguline representations, see [BHS17b, Thm. 3.21].

Let $X_{\mathrm{tri}}(\overline{\rho}) = \prod_{v\in S_p} X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}}) \subset \mathcal{X}_{\overline{\rho}_p} \times \hat{T}_p$. Then there is a commutative diagram

$$\begin{tikzcd} X_p(\overline{\rho}) \arrow[r, "\iota"] \arrow[d] & X_{\mathrm{tri}}(\overline{\rho}) \times \mathcal{X}_{\overline{\rho}^p} \times \mathbb{U}^g \arrow[d] \\ \hat{T}_p^0 \arrow[r, "\iota_{T_p^0}"] & \hat{T}_p^0, \end{tikzcd} \tag{4.6}$$

where ι is a closed embedding that identifies $X_p(\overline{\rho})$ with a union of irreducible components of the target. Here $\mathbb{U}^g = (\mathrm{Spf}\, \mathcal{O}_L[\![y_1,\dots,y_g]\!])^{\mathrm{rig}}$ and the $\iota_{T_p^0}$ is an isomorphism induced by shifting the weight, see [BHS17b, Thm. 3.21].

4.2. **A characterization of the tangent space.** We fix a (G,U)-automorphic representation $\rho \in \mathcal{X}_{\overline{\rho},S} \subset \mathcal{X}_\infty$ that is crystalline φ-generic. For the remainder of this subsection we introduce the following notation:

Let R be the complete local ring of $\mathcal{X}_\infty$ at ρ so that $\widehat{\mathcal{X}_{\infty}}_\rho = \mathrm{Spf}\, R$. For a given refinement $\underline{\mathcal{F}} = (\mathcal{F}_{\tilde{v}})_{v\in S_p}$ of ρ and $w \in W$ such that $w_{\underline{\mathcal{F}}} \preceq w$ (hence $x_{\underline{\mathcal{F}},w} \in Y(U^p,\bar{\rho}) \subset X_p(\bar{\rho})$ by Proposition 4.2 (iii)) let $R_{\underline{\mathcal{F}},w}$ be the complete local ring of $X_p(\overline{\rho})$ at the point $x_{\underline{\mathcal{F}},w}$, so that $\widehat{X_p(\overline{\rho})}_{x_{\underline{\mathcal{F}},w}} = \mathrm{Spf}\, R_{\underline{\mathcal{F}},w}$. By [BHS19, Lem. 4.3.3], the canonical map $R \to R_{\underline{\mathcal{F}},w}$ is a surjection. Similarly we define S as the complete local ring of $\mathcal{X}_{\overline{\rho},\mathcal{S}}$ at ρ, $S_{\underline{\mathcal{F}},w}$ the complete local ring of $Y_p(\overline{\rho})$ at $x_{\underline{\mathcal{F}},w}$ and $\widehat{S}_\infty$ the complete local ring of $\mathrm{Spf}(S_\infty)^{\mathrm{rig}}$ at the augmentation ideal of $S_\infty[\frac{1}{p}]$. Then we have a canonical surjection $R \twoheadrightarrow S$ and the sequence of embeddings (4.5) gives rise to maps

$$\widehat{\mathcal{O}}_{Y_p(\overline{\rho}),x_{\underline{\mathcal{F}},w}} = R_{\underline{\mathcal{F}},w} \otimes_{\widehat{S}_\infty} L \twoheadrightarrow R_{\underline{\mathcal{F}},w} \otimes_R S \twoheadrightarrow S_{\underline{\mathcal{F}},w} = \widehat{\mathcal{O}}_{Y(U^p,\overline{\rho}),x_{\underline{\mathcal{F}},w}}. \tag{4.7}$$

Proposition 4.3 shows that the composite map is an isomorphism so that all the maps of this sequence are actually isomorphisms. In particular the map

$$S = \widehat{\mathcal{O}}_{\mathcal{X}_{\overline{\rho},\mathcal{S}},\rho} \longrightarrow \widehat{\mathcal{O}}_{Y(U^p,\overline{\rho}),x_{\underline{\mathcal{F}},w}} = S_{\underline{\mathcal{F}},w} \tag{4.8}$$

is surjective.

Let ρ^p be the image of ρ in $\mathcal{X}_{\overline{\rho}^p} \times \mathbb{U}^g$, and for $v|p$ let $x_{\mathcal{F}_{\tilde{v}},w_{\tilde{v}}} = (\rho_{\tilde{v}}, z^{w_{\tilde{v}}w_0\cdot\lambda_v}\delta_{\mathcal{F}_{\tilde{v}}})$ which is a point of $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$. As ρ is automorphic the space $\mathcal{X}_{\overline{\rho}^p}$ is smooth of dimension $n^2|S\backslash S_p|$ at ρ^p, by [Car12, Thm. 1.2] and [BLGGT14, Lem. 1.3.2 (1)]. Moreover [BHS19, Thm. 1.4] implies that the complete local ring $R_{\mathcal{F}_{\tilde{v}},w_{\tilde{v}}}$ of $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ at $x_{\mathcal{F}_{\tilde{v}},w_{\tilde{v}}}$ is irreducible. Hence the diagram (4.6) implies that the ring $R_{\underline{\mathcal{F}},w}$ decomposes as a tensor product

$$R_{\underline{\mathcal{F}},w} = \widehat{\bigotimes_{v|p}} R_{\mathcal{F}_{\tilde{v}},w_{\tilde{v}}} \widehat{\otimes}_L \widehat{\mathcal{O}}_{\mathcal{X}_{\overline{\rho}^p}\times\mathbb{U}^g,\rho^p}.$$

By [BHS19, Cor. 3.7.8] and [BHS19, Thm. 3.6.2.(ii)] the quotient $R_{\mathcal{F}_{\tilde{v}},w_{\tilde{v}}}$ of $R_{\rho_{\tilde{v}}}$ coincides with the quotient $R^{w_v}_{\rho_{\tilde{v}},\mathcal{F}_{\tilde{v}}[\frac{1}{t}]}$ of $R_{\rho_{\tilde{v}},\mathcal{F}_{\tilde{v}}[\frac{1}{t}]}$ defined in section

3.7 and is therefore normal and Cohen-Macaulay by Proposition 3.18. As $\widehat{\mathcal{O}}_{\mathcal{X}_{\overline{\rho}^p}\times\mathbb{U}^g,\rho^p}$ is isomorphic to a ring of power series with coefficients in L ([Gro64, Cor. 19.6.5]), the topological ring $R_{\underline{\mathcal{F}},w}$ is isomorphic to a power series ring with coefficients in $\widehat{\bigotimes}_{v|p}R_{\mathcal{F}_{\tilde{v}},w_{\tilde{v}}}$. Similarly R is a power series ring over $\widehat{\bigotimes}_{v|p}R_{\rho_{\tilde{v}}}$ and using (4.6) we conclude that the map $R\to R_{\underline{\mathcal{F}},w}$ induces an isomorphism

$$R\otimes_{\left(\widehat{\bigotimes}_{v|p}R_{\rho_{\tilde{v}}}\right)}\left(\widehat{\bigotimes_{v|p}}R^{w_{\tilde{v}}}_{\rho_{\tilde{v}},\mathcal{F}_{\tilde{v}}[\frac{1}{t}]}\right)\xrightarrow{\sim}R_{\underline{\mathcal{F}},w}. \tag{4.9}$$

Let us write $\operatorname{Spec}R_{\underline{\mathcal{F}}}$ for the scheme theoretic image of the canonical morphism

$$\coprod_{w_{\underline{\mathcal{F}}}\preceq w}\operatorname{Spec}R_{\underline{\mathcal{F}},w}\longrightarrow\operatorname{Spec}R$$

and $\operatorname{Spec}S_{\underline{\mathcal{F}}}$ for the scheme theoretic image of the canonical morphism

$$\coprod_{w_{\underline{\mathcal{F}}}\preceq w}\operatorname{Spec}S_{\underline{\mathcal{F}},w}\longrightarrow\operatorname{Spec}S.$$

We point out that $R_{\underline{\mathcal{F}}}$ is reduced and that

$$R_{\underline{\mathcal{F}}}=\widehat{\bigotimes_{v|p}}R_{\rho_{\tilde{v}},\mathcal{F}_{\tilde{v}}[\frac{1}{t}]}\widehat{\otimes}_L\widehat{\mathcal{O}}_{\mathcal{X}_{\overline{\rho}^p}\times\mathbb{U}^g,\rho^p}$$

using the notation from section 3.7 again: indeed both sides are reduced quotients of R and have the same minimal prime ideals, the right hand side being reduced by [Gro65, Cor. 7.5.7] and the fact that all the $R_{\rho_{\tilde{v}},\mathcal{F}_{\tilde{v}}[\frac{1}{t}]}$ are reduced because normal.

Lemma 4.4. *(i) The schemes* $\operatorname{Spec}R_{\underline{\mathcal{F}}}$ *and* $\operatorname{Spec}R_{\underline{\mathcal{F}},w}$ *are reduced and Cohen-Macaulay of dimension* $g+[F:\mathbf{Q}]\frac{n(n+1)}{2}+n^2|S|=q+n[F:\mathbf{Q}]$.

(ii) The scheme $\operatorname{Spec}S_{\underline{\mathcal{F}},w}$ *is reduced and equi-dimensional of dimension* $n[F:\mathbf{Q}]$*. The same holds true for* $\operatorname{Spec}S_{\underline{\mathcal{F}}}$.

(iii) Let $\eta\in\operatorname{Spec}S_{\underline{\mathcal{F}},w}$ *be a generic point, then* $\eta\notin\operatorname{Spec}R_{\underline{\mathcal{F}},w'}$ *for* $w'\neq w$.

Proof. (i) As explained above the ring $\widehat{\mathcal{O}}_{\mathcal{X}_{\overline{\rho}^p}\times\mathbb{U}^g,\rho^p}$ is isomorphic to a power series ring with coefficients in L. The claim now follows from the fact that $R^{w_{\tilde{v}}}_{\rho_{\tilde{v}},\mathcal{F}_{\tilde{v}}}$ and $R_{\rho_{\tilde{v}},\mathcal{F}_{\tilde{v}}}$ are reduced and Cohen-Macaulay of dimension $n^2+[F_v:\mathbf{Q}_p]\frac{n(n+1)}{2}$ (see Proposition 3.17 and Proposition 3.18) and the fact that the completed tensor product over L of two reduced Cohen-Macaulay complete local noetherian L-algebras with residue field L is still a reduced Cohen-Macaulay complete local noetherian L-algebra with residue field L (see [Gro65, Lem. 7.5.5], [Gro65, Cor. 7.5.7] and [Tab, Thm. 3.1]).

(ii) As $S_{\underline{\mathcal{F}},w}$ is the complete local ring at some point of the eigenvariety $Y(U^p,\overline{\rho})$, the claims follow from the corresponding statements on $Y(U^p,\overline{\rho})$ in Proposition 4.2. The claim on $\operatorname{Spec}S_{\underline{\mathcal{F}}}$ then is a direct consequence.

(iii) Let us write $\operatorname{Spf} A$ for the formal completion of $\hat{T}_p^0$ at $\omega(\iota(x))$. Thus A is just the completed tensor-product of power series rings of dimension $n[F_v : \mathbf{Q}_p]$ indexed by $v \in S_p$. We identify $\operatorname{Spf} A$ with the formal completion of $\prod_{v|p} \mathfrak{t}_{E_{\tilde{v}}/\mathbf{Q}_p,L}$ at the origin via the weight map $(\delta_v)_{v|p} \mapsto (\mathrm{wt}(\delta_v) - \mathrm{wt}(\omega(\iota(x)_v))_{v|p}$ (see [BHS19, (3.16)]).

Let $\underline{\Theta} : \operatorname{Spf} R_{\underline{\mathcal{F}}} \to \operatorname{Spf} A \times \operatorname{Spf} A$ be the composition of the projection on $\operatorname{Spf}\left(\widehat{\bigotimes}_{v|p} R_{\rho_{\tilde{v}},\mathcal{F}_{\tilde{v}}}[\frac{1}{t}]\right)$ with the product of all the maps Θ for all $v|p$, as defined before the proof of Proposition 3.18. It follows from Proposition 3.20 that its restriction to $\operatorname{Spf} R_{\underline{\mathcal{F}},w}$ factors through the weight map $\operatorname{Spf} R_{\underline{\mathcal{F}},w} \to \operatorname{Spf} A$ and the quotient map $\psi_w : \operatorname{Spf} A \to \operatorname{Spf} A \times \operatorname{Spf} A$ factors through the closed formal subscheme $\operatorname{Spf} A_w \subset \operatorname{Spf} A \times \operatorname{Spf} A$ image of $(1, \mathrm{Ad}(w)^{-1}) : \operatorname{Spf} A \hookrightarrow \operatorname{Spf} A \times \operatorname{Spf} A$ (we recall that we have identified $\operatorname{Spf} A$ with the completion of $\prod_{v|p} \mathfrak{t}_{E_{\tilde{v}}/\mathbf{Q}_p,L}$ at the origin). We obtain a commutative diagram (compare [BHS19, 2.5]):

$$\begin{tikzcd}
\operatorname{Spf} S_{\underline{\mathcal{F}},w} \arrow[r, hook] \arrow[dr, "\omega"'] & \operatorname{Spf} R_{\underline{\mathcal{F}},w} \arrow[rr, hook] \arrow[d] & & \operatorname{Spf} R_{\underline{\mathcal{F}}} \arrow[d, "\underline{\Theta}"] \\
& \operatorname{Spf} A \arrow[r, "\psi_w"] & \operatorname{Spf} A_w \arrow[r, hook] & \operatorname{Spf} A \times \operatorname{Spf} A.
\end{tikzcd}$$

Passing to rings and applying $\operatorname{Spec}(-)$ the above diagram becomes:

$$\begin{tikzcd}
\operatorname{Spec} S_{\underline{\mathcal{F}},w} \arrow[r, hook] \arrow[dr, "\omega"'] & \operatorname{Spec} R_{\underline{\mathcal{F}},w} \arrow[rr, hook] \arrow[d] & & \operatorname{Spec} R_{\underline{\mathcal{F}}} \arrow[d, "\underline{\Theta}"] \\
& \operatorname{Spec} A \arrow[r, "\psi_w"] & \operatorname{Spec} A_w \arrow[r, hook] & \operatorname{Spec} A \times \operatorname{Spec} A,
\end{tikzcd}$$

where we use the same letters for the maps by abuse of notation. Now, by Proposition 4.2, the map ω is flat and hence dominant when restricted to each irreducible component of $\operatorname{Spec} S_{\underline{\mathcal{F}},w}$.

On the other hand $\underline{\Theta}(\operatorname{Spec} R_{\underline{\mathcal{F}},w'}) = \operatorname{Spec} A_{w'}$ by [BHS19, Cor. 3.5.12] (compare also Proposition 3.20), and $\operatorname{Spec} A_{w'}$ does not contain the generic point of $\operatorname{Spec} A_w$ for $w \neq w'$. The claim follows from this. $\square$

Lemma 4.5. *The canonical maps $R_{\underline{\mathcal{F}}} \otimes_{\widehat{S}_\infty} L \to R_{\underline{\mathcal{F}}} \otimes_R S$ and $R_{\underline{\mathcal{F}}} \otimes_R S \to S_{\underline{\mathcal{F}}}$ are isomorphisms, i.e.*

$$\operatorname{Spec} S_{\underline{\mathcal{F}}} = \operatorname{Spec} R_{\underline{\mathcal{F}}} \cap \operatorname{Spec} S = \operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} R} \operatorname{Spec} S \tag{4.10}$$

as subschemes of $\operatorname{Spec} R$. *In particular* $\operatorname{Spec} S_{\underline{\mathcal{F}}} \subset \operatorname{Spec} R_{\underline{\mathcal{F}}}$ *is a closed subscheme that is cut out by q equations.*

Proof. From (4.7), we deduce a sequence of surjective maps

$$R_{\underline{\mathcal{F}}} \otimes_{\widehat{S}_\infty} L \twoheadrightarrow R_{\underline{\mathcal{F}}} \otimes_R S \twoheadrightarrow S_{\mathcal{F}}.$$

First note that $\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} \widehat{S}_\infty} \operatorname{Spec} L$ is cut out by q equations in $\operatorname{Spec} R_{\underline{\mathcal{F}}}$.

By definition $\operatorname{Spec} R_{\underline{\mathcal{F}}} = \bigcup_{w \geqslant w_{\underline{F}}} \operatorname{Spec} R_{\underline{\mathcal{F}},w}$ and $\operatorname{Spec} S_{\underline{\mathcal{F}}} = \bigcup_{w \geqslant w_{\underline{F}}} \operatorname{Spec} S_{\underline{\mathcal{F}},w}$ as topological spaces. Consequently we have an equality of *sets*

$$\begin{aligned}\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} \widehat{S}_\infty} \operatorname{Spec} L &= \bigcup_{w \geqslant w_{\underline{\mathcal{F}}}} (\operatorname{Spec} R_{\underline{\mathcal{F}},w} \times_{\operatorname{Spec} \widehat{S}_\infty} \operatorname{Spec} L)\\ &= \bigcup_{w \geqslant w_{\underline{\mathcal{F}}}} (\operatorname{Spec} R_{\underline{\mathcal{F}},w} \times_{\operatorname{Spec} R} \operatorname{Spec} S)\\ &= \operatorname{Spec} S_{\underline{\mathcal{F}}}\end{aligned}$$

Indeed $\operatorname{Spec} S_{\underline{\mathcal{F}},w} = \operatorname{Spec} R_{\underline{\mathcal{F}},w} \times_{\operatorname{Spec} R} \operatorname{Spec} S = \operatorname{Spec} R_{\underline{\mathcal{F}},w} \times_{\operatorname{Spec} \widehat{S}_\infty} \operatorname{Spec} L$ by Proposition 4.3. Hence (4.10) is true on the level of topological spaces. As $\operatorname{Spec} S_{\underline{\mathcal{F}}}$ is reduced it remains to show that $\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} \widehat{S}_\infty} \operatorname{Spec} L$ is reduced as well.

We have

$$\dim(\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} \widehat{S}_\infty} \operatorname{Spec} L) = \dim \operatorname{Spec} S_{\underline{\mathcal{F}}} = \dim \operatorname{Spec} R_{\underline{\mathcal{F}}} - q$$

and consequently $\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} \widehat{S}_\infty} \operatorname{Spec} L$ is Cohen-Macaulay as $\operatorname{Spec} R_{\underline{\mathcal{F}}}$ is (see e.g. [Gro64, Cor. 16.5.6]).

By [Gro65, Prop. 5.8.5], it remains to prove that $\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} \widehat{S}_\infty} \operatorname{Spec} L$ is generically reduced. Let $\eta \in \operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} \widehat{S}_\infty} \operatorname{Spec} L$ be the generic point of some irreducible component and write $\mathfrak{q}$ for the corresponding prime ideal of R. Then η is the generic point of an irreducible component in $\operatorname{Spec} S_{\underline{\mathcal{F}},w}$ for some w and it follows that

$$(L \otimes_{\widehat{S}_\infty} R_{\underline{\mathcal{F}}})_{\mathfrak{q}} = L \otimes_{\widehat{S}_\infty} (R_{\underline{\mathcal{F}}})_{\mathfrak{q}} = L \otimes_{\widehat{S}_\infty} (R_{\underline{\mathcal{F}},w})_{\mathfrak{q}} = (S_{\underline{\mathcal{F}},w})_{\mathfrak{q}},$$

where the second equality is a consequence of Lemma 4.4 (iii) and the last equality is Proposition 4.3. As $(S_{\underline{\mathcal{F}},w})_{\mathfrak{q}}$ is reduced so is $(L \otimes_{\widehat{S}_\infty} R_{\underline{\mathcal{F}}})_{\mathfrak{q}}$ and the claim follows. $\square$

Let us write R_{cris} for the quotient of R such that for all $v \in S_p$ and any morphism $R \to A$ (for some finite dimensional L-algebra A) the $\mathcal{G}_{E_{\tilde{v}}}$ representation on A^n induced by $R_{\rho_{\tilde{v}}} \to R \to A$ is crystalline if and only if the morphism $R \to A$ factors through R_{cris}. This quotient exists by the main result of [Kis08] and is isomorphic to $\widehat{\bigotimes}_{v \in S_p} R^{\mathrm{cris}}_{\rho_{\tilde{v}}} \widehat{\otimes}_L \widehat{\mathcal{O}}_{\mathcal{X}_{\bar{\rho}^p} \times \mathbb{U}^g, \rho^p}$ where $R^{\mathrm{cris}}_{\rho_{\tilde{v}}}$ is the ring representing the functor $\mathfrak{X}^{\mathrm{cris}}_{\rho_{\tilde{v}}}$, see Proposition 3.21.

If A is a complete noetherian local ring, we write t_A for the tangent space of the functor $\operatorname{Spf} A$, ie $t_A := T \operatorname{Spf} A$.

Lemma 4.6. *(i) The local ring R_{cris} is formally smooth of dimension q.*
(ii) The tangent spaces of R_{cris} and S intersect trivially inside t_R, i.e.

$$t_{R_{\mathrm{cris}}} \cap t_S = 0.$$

(iii) There is an inclusion $t_{R_{\mathrm{cris}}} \subset t_{R_{\underline{\mathcal{F}}}}$ *of subspaces of* t_R*.*

Proof. (i) This follows from the smoothness of $\mathcal{X}_{\overline{\rho}^p}$ at the image of ρ (see above) and the fact that the generic fiber of a crystalline deformation rings is smooth of dimension $n^2 + [F_v : \mathbf{Q}_p]\frac{n(n-1)}{2}$ by [Kis08] (see also Proposition 3.21) and the definition of q.
(ii) With the notation introduced here this is the statement of [All16, Thm. A.1].
(iii) This is a direct consequence of $\operatorname{Spec} R_{\mathrm{cris}} \subset \operatorname{Spec} R_{\underline{\mathcal{F}},w_0} \subset \operatorname{Spec} R_{\underline{\mathcal{F}}}$, which follows from Proposition 3.21. □

Corollary 4.7. *There is a direct sum decomposition*

$$t_{R_{\mathrm{cris}}} \oplus t_{S_{\underline{\mathcal{F}}}} = t_{R_{\underline{\mathcal{F}}}}$$

of subspaces of t_R*.*

Proof. As $\operatorname{Spec} S_{\underline{\mathcal{F}}} \subset \operatorname{Spec} S$ we have $t_{S_{\underline{\mathcal{F}}}} \subset t_S$ and hence $t_{S_{\underline{\mathcal{F}}}} \cap t_{R_{\mathrm{cris}}} = 0$ by Lemma 4.6 (ii). Moreover $t_{R_{\mathrm{cris}}} \subset t_{R_{\underline{\mathcal{F}}}}$ and $t_{R_{\mathrm{cris}}}$ has dimension q. The claim now follows from the fact that $\operatorname{Spec} \overline{S}_{\underline{\mathcal{F}}} \subset \operatorname{Spec} R_{\underline{\mathcal{F}}}$ is cut out by q equations by Lemma 4.5 and hence

$$\operatorname{codim}(t_{S_{\underline{\mathcal{F}}}}, t_{R_{\underline{\mathcal{F}}}}) \leqslant q.$$ □

Corollary 4.8. *The canonical map of tangent spaces*

$$\bigoplus_{\underline{\mathcal{F}}} t_{S_{\underline{\mathcal{F}}}} \longrightarrow t_S$$

is a surjection. Here the sum is taken over all tuples $\underline{\mathcal{F}} = (\mathcal{F}_{\tilde{v}})_{v \in S_p}$ *of Frobenius stable flags* $\mathcal{F}_{\tilde{v}}$ *of* $D_{\mathrm{cris}}(\rho_{\tilde{v}})$*.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\bigoplus_{\underline{\mathcal{F}}} t_{R_{\underline{\mathcal{F}}}} \arrow[r, two heads] \arrow[d, two heads] & t_R \arrow[d, two heads] \\
\bigoplus_{\underline{\mathcal{F}}} t_{R_{\underline{\mathcal{F}}}}/t_{R_{\mathrm{cris}}} \arrow[r] & t_R/t_{R_{\mathrm{cris}}} \\
\bigoplus_{\underline{\mathcal{F}}} t_{S_{\underline{\mathcal{F}}}} \arrow[r, "\gamma"] \arrow[u, "\alpha"] & t_S. \arrow[u, "\beta"']
\end{tikzcd}$$

It follows from Corollary 4.7 that α is an isomorphism, and from Lemma 4.6 (ii) that β is injective. Moreover the upper horizontal arrow is surjective by Corollary 3.16. It follows from an obvious diagram chaise that γ is a surjection. □

Remark 4.9. We point out that it is a direct consequence of the proof of Corollary 4.8 that the map

$$t_S \longrightarrow t_R/t_{R_{\mathrm{cris}}}$$

is an isomorphism.

4.3. **Proof of Theorem 4.1.** We now prove the main result, Theorem 4.1. With the above preparation, the final argument just follows the original method of Gouvea-Mazur [Maz97] and Chenevier [Che11] in the case of modular forms (i.e. in the case $n = 2$), resp. in the case $n = 3$. The main result of this section is the computation of the tangent space of $\mathcal{X}^{\rm aut}_{\bar\rho,S}$ at an automorphic point.

We start with a computation of the tangent space of $\mathcal{X}^{\rm aut}_{\bar\rho,S}$ at certain (G, U)-automorphic crystalline points.

Proposition 4.10. *Assume that the group U^p is sufficiently small to satisfy* (4.1) *and let $\rho \in \mathcal{X}^{\rm aut}_{\bar\rho,S}$ be a (G, U)-automorphic and crystalline φ-generic point. Then there is an equality of tangent spaces*

$$T_\rho \mathcal{X}^{\rm aut}_{\bar\rho,S} = T_\rho \mathcal{X}_{\bar\rho,\mathcal{S}}.$$

Proof. The inclusion $T_\rho \mathcal{X}^{\rm aut}_{\bar\rho,S} \subset T_\rho \mathcal{X}_{\bar\rho,\mathcal{S}}$ is obvious and we need to prove the converse inclusion.

After enlarging L if necessary, we may assume that the point ρ is an L-valued point of $\mathcal{X}_{\bar\rho,S}$ and that L contains all eigenvalues of the crystalline Frobenius on the Weil-Deligne representation $\mathrm{WD}(D_{\rm cris}(\rho_{\tilde v}))$ associated to $D_{\rm cris}(\rho_{\tilde v})$ for all $v \in S_p$.

For each choice of a tuple $\underline{\mathcal{F}}$ of complete Frobenius stable flags $\mathcal{F}_{\tilde v}$ in $D_{\rm cris}(\rho_{\tilde v})$ and each Weyl group element $w_{\underline{\mathcal{F}}} \preceq w$ we have constructed points $x_{\underline{\mathcal{F}},w} \in Y(U^p, \bar\rho)$ that map to ρ under the canonical projection $f : Y(U^p, \bar\rho) \to \mathcal{X}_{\bar\rho,\mathcal{S}}$.

We recall (cf. (4.8)) that the induced map

$$\hat{\mathcal{O}}_{\mathcal{X}_{\bar\rho,\mathcal{S}},\rho} \longrightarrow \hat{\mathcal{O}}_{Y(U^p,\bar\rho),x_{\underline{\mathcal{F}},w}}$$

is a surjection. It follows that this map is unramified and by the discussion in the beginning of [dJvdP96, 3.1] the morphism $\mathcal{O}_{\mathcal{X}_{\bar\rho,\mathcal{S}},\rho} \to \mathcal{O}_{Y(U^p,\bar\rho),x_{\underline{\mathcal{F}},w}}$ is unramified as well. In particular we can find neighborhoods $V_{\underline{\mathcal{F}},w}$ of $x_{\underline{\mathcal{F}},w}$ and U of ρ such that the restriction of f is an unramified morphism $V_{\underline{\mathcal{F}},w} \to U$. By [Hub96, Prop. 1.6.8] we may shrink $V_{\underline{\mathcal{F}},w}$ and U such that f can be written as the composition of a closed embedding $V_{\underline{\mathcal{F}},w} \hookrightarrow V'_{\underline{\mathcal{F}},w}$ with an étale morphism $f' : V'_{\underline{\mathcal{F}},w} \to U$. Using [dJvdP96, Lem. 3.1.5] we can further shrink $V'_{\underline{\mathcal{F}},w}$ and U such that $f' : V'_{\underline{\mathcal{F}},w} \to U$ is an isomorphism. In particular after shrinking $V_{\underline{\mathcal{F}},w}$ and U we may assume that $f : V_{\underline{\mathcal{F}},w} \to U$ is a closed embedding. Finally we may choose U and $V_{\underline{\mathcal{F}},w}$ such that this assertion holds true for all the points $x_{\underline{\mathcal{F}},w}$ at the same time.

As the classical points are Zariski-dense in $V_{\underline{\mathcal{F}},w}$ by Proposition 4.2 (ii), we find that $\bigcup_w V_{\underline{\mathcal{F}},w} \subset U \cap \mathcal{X}^{\rm aut}_{\bar\rho,S}$.

The formation of scheme-theoretic images commutes with flat base change, hence in particular with passing to the complete local ring at ρ. It follows (using the notation from subsection 4.2) that

$$\operatorname{Spec} S_{\underline{\mathcal{F}}} \subset \operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S},\rho} \subset \operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X}_{\overline{\rho},\mathcal{S}},\rho} = \operatorname{Spec} S.$$

We deduce that

$$t_{S_{\underline{\mathcal{F}}}} \subset t_{\hat{\mathcal{O}}_{\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S},\rho}} = T_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}.$$

As this conclusion holds true for each choice of $\underline{\mathcal{F}}$, Corollary 4.8 implies the claimed inclusion $T_\rho \mathcal{X}_{\overline{\rho},\mathcal{S}} = t_S \subset T_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$. $\square$

Proof of Theorem 4.1. Given a rigid analytic space Z and a point $z \in Z$ we write $\dim_z Z$ for the dimension of Z at the point z, i.e. for the dimension of the local ring $\mathcal{O}_{Z,z}$ of Z at z.

Let us write $\mathcal{X}^{\mathrm{aut,sm}}_{\overline{\rho},S} \subset \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$ for the smooth locus which is Zariski-open and dense in $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$.

Let us fix an irreducible component C of $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$. We need to show that C is an irreducible component of $\mathcal{X}_{\overline{\rho},S}$. By slight abuse of notations we write $C^{\mathrm{sm}} = C \cap \mathcal{X}^{\mathrm{aut,sm}}_{\overline{\rho},S}$. As by definition $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$ is the Zariski-closure of the (G,U)-automorphic points in $\mathcal{X}_{\overline{\rho},S}$, it follows that C^{sm} contains a (G,U)-automorphic point ρ.

Assume in the first place that the group U^p is sufficiently small to satisfy (4.1). By the construction preceding Proposition 4.2 there is a classical point $y = (\rho,\delta) \in Y(U^p,\overline{\rho})$ and by Proposition 4.2 (ii) the classical, crystalline φ-generic points accumulate at y. It follows that there is a classical, crystalline φ-generic point $y' = (\rho',\delta') \in Y(U^p,\overline{\rho})$ such that $\rho' \in C^{\mathrm{sm}}$. We may replace ρ by ρ' (and y by y') and hence assume that ρ is (G,U)-automorphic and crystalline φ-generic.

We then have a chain of inequalities

$$\dim_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S} = \dim_\rho C \leqslant \dim_\rho \mathcal{X}_{\overline{\rho},S} \leqslant \dim T_\rho \mathcal{X}_{\overline{\rho},\mathcal{S}} = \dim T_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S} = \dim_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S},$$

as ρ is (by assumption) a smooth point of $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$. Here the equality $\dim T_\rho \mathcal{X}_{\overline{\rho},\mathcal{S}} = \dim T_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$ is Proposition 4.10. It follows that equality holds and hence the (necessarily unique) irreducible component C of $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$ containing ρ is an irreducible component of $\mathcal{X}_{\overline{\rho},S}$. Note that the smoothness of $\mathcal{X}_{\overline{\rho},\mathcal{S}}$ at ρ (see [All16, Thm. C]) is also a direct consequence of the equality of tangent spaces, as ρ by assumption lies in the smooth locus of $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$.

Now assume that U^p does not necessary satisfy (4.1). Then we can find a place $v_1 \notin S$ of F such that v_1 is split in E. Let $V_{v_1} \subset G(F_{v_1})$ sufficiently small so that the group $V^p = \prod_v V_v := V_{v_1} \times \prod_{v \nmid p, v \neq v_1} U_v$ satisfies (4.1) and let $S' = S \cup \{v_1\}$. We have a closed immersion $\mathcal{X}_{\overline{\rho},S} \subset \mathcal{X}_{\overline{\rho},S'}$ and it follows from local-global compatibility theorems that a point of $\mathcal{X}_{\overline{\rho},S}$ is (G,V)-automorphic if and only if it is (G,U)-automorphic: indeed assume that

$\rho \in \mathcal{X}_{\overline{\rho},S}$ is (G,V)-automorphic, i.e. associated to an automorphic representation $\pi = \bigotimes \pi_v$ such that $\pi_v^{V_v} \neq 0$. By [Car12, Thm. 1.1] the representation π_v is associated to $\mathrm{WD}(\rho_{\tilde{v}})^{\mathrm{F-ss}}$ under the local Langlands correspondence, and hence $\pi_v^{U_v} \neq 0$, as $\rho \in \mathcal{X}_{\overline{\rho},S} \subset \mathcal{X}_{\overline{\rho},S'}$.

Let x be a (G,U)-automorphic point and let Z be an irreducible component of $\mathcal{X}_{\overline{\rho},S}$ containing x. Let Z' be some irreducible component of $\mathcal{X}_{\overline{\rho},S'}$ containing x. As x is a (G,V)-automorphic point, it is a smooth point of Z' and $\dim Z' = [F:\mathbf{Q}]\frac{n(n+1)}{2}$ (see [All16, Thm. C], note that this result is true for $\mathcal{X}_{\overline{\rho},S}$, not only for $\mathcal{X}_{\overline{\rho},S}$). On the other hand, we have $\dim Z \geqslant [F:\mathbf{Q}]\frac{n(n+1)}{2}$ so that we have $Z = Z'$. We have proved that (G,V)-automorphic points are Zariski-dense in $Z' = Z$. As these points are also (G,U)-automorphic we can conclude that (G,U)-automorphic points are Zariski-dense in Z. $\square$

4.4. **Proof of Theorem 1.2.** We finally turn to the proof of the main theorem as stated in the introduction. This result follows from Theorem 4.1 using base change results for unitary groups.

Proof of Theorem 1.2. Let G be a unitary group over F which is an outer form of $\mathrm{GL}_{n,\overline{F}}$, which is quasi-split at every finite place and such that $G(F \otimes_{\mathbf{Q}} \mathbf{R})$ is compact. The existence of such a unitary group follows for example from the results of [Clo91, §2]. By assumption there exists some regular cohomological cuspidal automorphic representation π such that $\overline{\rho} \otimes_{\mathbf{F}} \overline{\mathbf{F}_p} \simeq \overline{\rho}_\pi$. It follows from [Lab11, Thm. 5.4] that the representation π is the weak base change of some automorphic representation σ of G (which is automatically cuspidal) so that $\overline{\rho} \otimes_{\mathbf{F}} \overline{\mathbf{F}_p} \simeq \overline{\rho}_\sigma$. Let U^p be some compact open subgroup of $G(\mathbb{A}_F^{p,\infty})$ such that $\sigma^{U^p} \neq 0$. The representation π is unramified at finite places $v \notin S$. Consequently it follows from [Lab11, Thm. 5.9] that we can choose the group U^p spherical at places not in S and that U_v is spherical for all $v|p$. Then the representation $\overline{\rho}$ is (G,U)-automorphic. Consequently we can apply Theorem 4.1 to conclude that the Zariski closure of the (G,U)-automorphic points in $\mathcal{X}_{\overline{\rho},S}$ is a union of irreducible components. However it follows from Cor. 5.3, Thm. 5.4 and Thm. 5.9 in [Lab11] that a point of $\mathcal{X}_{\overline{\rho},S}$ is automorphic if and only if it is (G,U^p)-automorphic for some U^p as above. This concludes the proof. $\square$

4.5. **Remarks on the existence of enough automorphic points.** We end by discussing that the main theorem conjecturally should imply density of automorphic points in $\mathcal{X}_{\overline{\rho},S}$. Let us write $\mathcal{X}_{\overline{\rho},S} = \bigcup C_i$ for the decomposition into finitely many irreducible components. Then, obviously Theorem 1.2 implies that the automorphic points are Zariski-dense in $\mathcal{X}_{\overline{\rho},S}$, if one could prove that $C_i \backslash \bigcup_{j\neq i} C_j$ contains an automorphic point for each i.

Assuming standard automorphy lifting conjectures a result like this is a consequence of the main result of Allen [All19, Thm. 5.3.1].

Theorem 4.11. *Assume that*

(a) $p \nmid 2n$, *the group* $\overline{\rho}(\mathrm{Gal}(\overline{E}/E(\zeta_p))$ *is adequate and* $\zeta_p \notin E$;
(b) *the group* $H^0(\mathcal{G}_{E_{\tilde{v}}}, \mathrm{ad}^0(\overline{\rho})(1))$ *vanishes for each* $v \in S_p$;
(c) *there exists a cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_E)$ *such that* $\overline{\rho}$ *is isomorphic to* $\overline{\rho}_\pi$ *and* $\rho_{\pi,\tilde{v}}$ *is potentially diagonalizable for each* $v \in S_p$.

Then each irreducible component of $\mathcal{X}_{\overline{\rho},S}$ *contains a point of the form* ρ_π *where* π *is a regular conjugate self dual algebraic cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_E)$.

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