Practice exercises

Exercise 1 1) Let $x \in F$. We have

$$\mathcal{G}_x = \varinjlim_{x \in U} \mathcal{G}(U) = \varinjlim_{x \in U} \mathcal{F}(U \cap F).$$

By definition of the induced topology on F, the set of open subsets of F containing x is exactly the set of $F \cap U$ where U is an open subset of X containing x, therefore

$$\mathcal{G}_x \simeq \mathcal{F}_x.$$

Let $x \notin F$. As F is closed in X, there exists some open subset V of X containing x and such that $V \cap F = \emptyset$. Then

$$\mathcal{G}(V) = \mathcal{F}(V \cap F) = \mathcal{F}(\emptyset) = 0.$$

Therefore $\mathcal{G}_x = 0$.

2) Let $U_1 \subset U_2$ be open subsets of F. Let res_{U_2,U_1} be the map

$$\prod_{x \in U_2} \mathcal{G}_x \longrightarrow \prod_{x \in U_1} \mathcal{G}_x$$

defined by $(s_x)_{x \in U_2} \mapsto (s_x)_{x \in U_1}$ (the projection). This is of course a morphism of abelian groups. This map sends $\mathcal{H}(U_2)$ into $\mathcal{H}(U_1)$. Namely if $s = (s_x) \in \mathcal{H}(U_2)$. Let V_2 open subset of X such that $U_2 = V_2 \cap F$ and $s' \in \mathcal{G}(V_2)$ such that $s_x = s'_x$ for all $x \in U_2$. Choose $V_1 \subset V_2$ open such that $V_1 \cap F = U_1$ and let s'' be the image of s' in $\mathcal{G}(V_1)$ (as $\mathcal{G}$ is a sheaf). Then $s''_x = s_x$ for all $x \in U_1$ showing that $(s_x)_{x \in U_1} \in \mathcal{H}(U_1)$. Then $\mathcal{H}$ is a presheaf as $\mathrm{res}_{U_2,U_1} \circ \mathrm{res}_{U_3,U_2} = \mathrm{res}_{U_3,U_1}$ for $U_1 \subset U_2 \subset U_3$.

Let's compute $\mathcal{H}_x$ for $x \in F$. If $U \subset F$ is an open subset containing x we can define a map

$$\alpha_U : \mathcal{H}(U) \longrightarrow \mathcal{G}_x$$

sending $(s_x)_{x \in U}$ on s_x. If $U_1 \subset U_2$ we have $\alpha_{U_2} = \alpha_{U_1} \circ \mathrm{res}_{U_2,U_1}$ therefore these maps give rise to a group homomorphism

$$\alpha : \mathcal{H}_x \longrightarrow \mathcal{G}_x.$$

Let's prove that α is an isomorphism. Let $t \in \mathcal{H}_x$ such that $\alpha(t) = 0$. Then t is the image of some $\widetilde{t} = (\widetilde{t}_y)_{y \in U} \in \mathcal{H}(U)$ for some open $U \subset F$ and we have $\widetilde{t}_x = \alpha(t) = 0$. Let $V \subset X$ open such that $U = V \cap F$ and $s' \in \mathcal{G}(V)$ such that $s'_y = \widetilde{t}_y$ for $y \in U$. We have $s'_x = \widetilde{t}_x = 0$. As $\mathcal{G}$ is a sheaf, there exists some open subset $V' \subset V$ containing x such that $s'_{|V'} = 0$ in $\mathcal{G}(V')$. This implies that $\widetilde{t}_y = 0$ for $y \in V' \cap F$ and then that the restriction of $\widetilde{t}$ to $V' \cap F$ is zero, thus that $t = 0$.

We have proved that α is injective. Let's prove that it is surjective. Let $t \in \mathcal{G}_x$. By definition there exists some open subset $V \subset X$ containing x and some $\widetilde{t} \in \mathcal{G}(V)$ such that $t = \widetilde{t}_x$. Then $s = (\widetilde{t}_y)_{y \in V \cap F} \in \mathcal{H}(V \cap F)$ and clearly $\alpha(s) = t$. Thus α is surjective.

The sheaf $\mathcal{G}_{|F}$ associated to $\mathcal{H}$ has the same stalks as $\mathcal{H}$, therefore

$$\mathcal{H}_x \simeq \mathcal{G}_{|F,x} \simeq \mathcal{G}_x.$$

3) By 1), if $\mathcal{G}$ is isomorphic to $i_*(\mathcal{G}_{|F})$ we have $\mathcal{G}_x = 0$ for $x \notin F$. Assume that $\mathcal{G}_x = 0$ for $x \notin F$. Let's construct a morphism of presheaves $\mathcal{G} \to i_*(\mathcal{H})$. For $V \subset X$ open and $s \in \mathcal{G}(V)$ we can send s to $(s_x)_{x \in V \cap F}$ which is an element of $i_*(\mathcal{H})(V)$. This map commutes with restriction so that we have a well defined morphism of presheaves $\mathcal{G} \to i_*(\mathcal{H})$. By composition with the natural map $\mathcal{H} \to \mathcal{G}_{|F}$ and the functoriality of i_* we obtain a morphism of abelian sheaves $\mathcal{G} \to i_*(\mathcal{G}_{|F})$. Now it follows from 2) that this map is an isomorphism at the stalk of $x \in F$ and from our assumption (and 1)) that it is also an isomorphism on the stalks of $x \notin F$. Therefore this is an isomorphism on the stalks of all points of X, thus it is an isomorphism of sheaves.

4) Clearly $\mathcal{I}(U)$ is an ideal of $\mathcal{O}_X(U)$. It is easy to check that it is a sheaf.

5) By exactness of the stalks, for all $x \in X$, we have an exact sequence

$$0 \longrightarrow \mathcal{I}_x \longrightarrow \mathcal{O}_{X,x} \longrightarrow (\mathcal{O}_X/\mathcal{I})_x \longrightarrow 0.$$

If $x \notin F$, there is an open subset $U \subset X$ such that $x \in U$ and $U \cap F = \emptyset$. By definition, we have $\mathcal{I}(U) = \mathcal{O}_X(U)$ so that $\mathcal{I}_x = \mathcal{O}_{X,x}$. Assume that $x \in F$. Let $U = \mathrm{Spec}(A)$ be an affine open neighborhood of x in X. As F is closed, $F \cap U$ is closed in U and $F \cap U = V(I)$ for $I \subset A$ some ideal. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x, we have $I \subset \mathfrak{p}$. In particular if $f \in \mathcal{I}(U)$, the condition $f(x) = 0$ implies $f \in \mathfrak{p} \subsetneq A = \mathcal{O}_X(U)$. Therefore the image of f in $\mathcal{O}_{X,x} \simeq A_{\mathfrak{p}}$ is contained in $\mathfrak{p}A_{\mathfrak{p}}$ (the same argument applies to every open neighborhood of x, so that all the germs of $\mathcal{I}$ at x lie in $\mathfrak{p}A_{\mathfrak{p}}$). This shows that $\mathcal{I}_x \subseteq \mathfrak{p}A_{\mathfrak{p}} \subsetneq \mathcal{O}_{X,x}$.

6) We need to define a structure of scheme on F. Take $\mathcal{O}_Z = (\mathcal{O}_X/\mathcal{I})_{|F}$. We need to check that the ringed space $(F, \mathcal{O}_Z)$ is locally isomorphic to an affine scheme. By 3) and 5), we have $i_*(\mathcal{O}_Z) \simeq \mathcal{O}_X/\mathcal{I}$. Let $x \in F$ and $U = \mathrm{Spec}(A)$ an affine open subset of X containing x. Then $F \cap U = V(I)$ for some ideal $I \subset A$ and we have

$$\mathcal{I}(U) = \{f \in A \mid f(x) = 0 \text{ for all } x \in V(I)\} = \bigcap_{\mathfrak{q} \in V(I)} \mathfrak{q} = \sqrt{I}.$$

We deduce that, for $f \in A$, $\mathcal{I}(D(f)) = \sqrt{IA_f} = (\sqrt{I})_f$ so that the ringed space $(F \cap U, (\mathcal{O}_Z)_{|F \cap U})$ is isomorphic to the scheme $\mathrm{Spec}(A/\sqrt{I})$. Therefore $Z = (F, \mathcal{O}_Z)$ is a closed subscheme of X with $|Z| = F$ and $i_*(\mathcal{O}_Z) \simeq \mathcal{O}_X/\mathcal{I}$. Finally the scheme Z is reduced: it is locally isomorphic to $\mathrm{Spec}(A/\sqrt{I})$ and the ring $A/\sqrt{I}$ is reduced.

Exercise 2 1) We have

$$I = (Y, Y^2 - XZ) = (Y, XZ) = (Y, X) \cap (Y, Z).$$

For the last equality, we can remark that (Y, XZ) is the inverse image of (XZ) by the map $k[X, Y, Z] \to k[X, Z]$ defined by $Y \mapsto 0$ and that $(XZ) = (X) \cap (Z)$ in $k[X, Z]$ as $k[X, Z]$ is a factorial ring.

Then

$$V(I) = V(X, Y) \cup V(Y, Z).$$

The closed subschemes $V(X, Y)$ and $V(Y, Z)$ are irreducible as $k[X, Y, Z]/(X, Y) \simeq k[Z]$ and $k[X, Y, Z]/(Y, Z) \simeq k[X]$ are domains. Their generic points are the prime ideals (X, Y) and (Y, Z), and they are the two irreducible components of S, as they are not contained in each other. To check if $V(I)$ is reduced we need to check if the quotient ring $k[X, Y, Z]/I$ is reduced. The equality $(Y, XZ) = (Y, X) \cap (Y, Z)$ shows that this quotient ring is isomorphic to a subring of the reduced ring

$$k[X, Y, Z]/(Y, X) \oplus k[X, Y, Z]/(Y, Z)$$

so it is reduced.

2) Here we have

$$I = (Y - X^2 + 1) \cdot (X, Y)$$

so that

$$V(I) = V(Y - X^2 + 1) \cup V(X, Y).$$

Note that $1 \in (Y - X^2 + 1) + (X, Y)$ so that $(Y - X^2 + 1) + (X, Y) = k[X, Y, Z]$ and $V(Y - X^2 + 1) \cap V(X, Y) = \emptyset$. Therefore $V(Y - X^2 + 1)$ and $V(X, Y)$ are the two irreducible components. As the quotient rings $k[X, Y, Z]/(Y - X^2 + 1) \simeq k[X, Z]$ and $k[X, Y, Z]/(X, Y) \simeq k[Z]$ are domains, the ideals $(Y - X^2 + 1)$ and (X, Y) are prime and are the generic points of $V(Y - X^2 + 1)$ and $V(X, Y)$. The equality $(Y - X^2 + 1) + (X, Y) = k[X, Y, Z]$ implies $I = (Y - X^2 + 1) \cap (X, Y)$ and an isomorphism

$$k[X, Y, Z]/I \simeq k[X, Y, Z]/(Y - X^2 + 1) \times k[X, Y, Z]/(X, Y) \simeq k[X, Z] \times k[Z].$$

Therefore the scheme $V(I)$ is reduced.

3) Here we have

$$(X^2 - YZ, X) \cdot (X^2 - YZ, Z - 1) \subset I \subset (X^2 - YZ, X) \cap (X^2 - YZ, Z - 1).$$

As

$$\begin{aligned} V((X^2 - YZ, X) \cap (X^2 - YZ, Z - 1)) &= V(X^2 - YZ, X) \cup V(X^2 - YZ, Z - 1) \\ &= V((X^2 - YZ, X) \cdot (X^2 - YZ, Z - 1)) \end{aligned}$$

we have

$$V(I) = V(X^2 - YZ, X) \cup V(X^2 - YZ, Z - 1).$$

Moreover $(X^2 - YZ, X) = (X, YZ) = (X, Y) \cap (X, Z)$ (as in 1)) and $(X^2 - YZ, Z - 1) = (X^2 - Y, Z - 1)$. Therefore

$$V(I) = V(X, Y) \cup V(X, Z) \cup V(X^2 - Y, Z - 1).$$

The ideals (X, Y), (X, Z) and $(X^2 - Y, Z - 1)$ are primes:

$$k[X, Y, Z]/(X, Y) \simeq k[Z], \quad k[X, Y, Z]/(X, Z) \simeq k[Y], \quad k[X, Y, Z]/(X^2 - Y, Z - 1) \simeq k[X].$$

Therefore the subsets $V(X, Y)$, $V(X, Z)$ and $V(X^2 - Y, Z - 1)$ are irreducible of generic points (X, Y), (X, Z) and $(X^2 - Y, Z - 1)$. To prove that they are the irreducible components of $V(I)$ it is sufficient to prove that they are not contained in each other. This can be checked by remarking that $Z - 1 \notin (X, Y), (X, Z)$, $Y \notin (X, Z), (X^2 - Y, Z - 1)$ and $Z \notin (X, Y), (X^2 - Y, Z - 1)$.

We want to check if $V(I)$ is reduced. As $V(I) = (V(I) \cap D_Z) \cup (V(I) \cap D_{Z-1})$ with $(V(I) \cap D_Z)$ and $(V(I) \cap D_{Z-1})$ open, it is sufficient to prove that $(V(I) \cap D_Z)$ and $(V(I) \cap D_{Z-1})$ are reduced. They are affine schemes and we have

$$\begin{aligned} \mathcal{O}(V(I) \cap D_Z) \simeq (k[X, Y, Z]/I)_Z &\simeq k[X, Y, Z, Z^{-1}]/(X^2 - YZ, X(Z - 1)) \\ &\simeq k[X, Z, Z^{-1}]/(X(Z - 1)). \end{aligned}$$

As $k[X, Z]/(X(Z - 1))$ is reduced (as in 1) for instance), its localization along Z, $k[X, Z, Z^{-1}]/(X(Z - 1))$ is reduced and so is $\mathcal{O}(V(I) \cap D_Z)$. Similarly we have

$$\begin{aligned} \mathcal{O}(V(I) \cap D_{Z-1}) \simeq (k[X, Y, Z]/I)_{Z-1} &\simeq k[X, Y, Z, (Z-1)^{-1}]/(X^2 - YZ, X(Z-1)) \\ &\simeq k[Y, Z, (Z - 1)^{-1}]/(YZ) \end{aligned}$$

which is reduced. Therefore $k[X, Y, Z]/I$ is reduced. This wasn't asked, but it implies that

$$I = (X^2 - YZ, Z - 1) \cap (X, Y) \cap (X, Z).$$

Exercise 3 Let's consider first the generic point x of $\mathrm{Spec}(\mathbb{Z})$ corresponding to the prime ideal (0). Then $\pi^{-1}(x)$ is the affine scheme $\mathrm{Spec}(\mathbb{Q}[T_1, T_2]/(T_1^2 + T_2^2 - 10))$. The ring $\mathbb{Q}[T_1, T_2]/(T_1^2 + T_2^2 - 10)$ is a domain. Namely the polynomial, $10 - T_2^2$ is not a square in $\mathbb{Q}(T_2)$ so that $T_1^2 + T_2^2 - 10$ is irreducible in $\mathbb{Q}[T_1, T_2]$, and as $\mathbb{Q}[T_1, T_2]$ is factorial, the ideal $(T_1^2 + T_2^2 - 10)$ is prime. Therefore $\pi^{-1}(x)$ is an integral scheme: irreducible and reduced.

Consider the case where x corresponds to a prime p. Then $\pi^{-1}(x)$ is isomorphic to the affine scheme $\mathrm{Spec}(A)$ with $A = \mathbb{F}_p[T_1, T_2]/(T_1^2 + T_2^2 - 10)$. Assume first that p is different from 2 and 5. Then, as $10 - T_2^2$ is not a square in $\mathbb{F}_p(T_2)$, the polynomial

$T_1^2+T_2^2-10$ is irreducible in $\mathbb{F}_p[T_1,T_2]$. As $\mathbb{F}_p[T_1,T_2]$ is factorial, the ideal $(T_1^2+T_2^2-10)$ is prime and A is a domain. Then $\pi^{-1}(x)$ is an integral scheme.

Now consider the case where $p=5$. In $\mathbb{F}_5[T_1,T_2]$ we have $T_1^2+T_2^2=(T_1+2T_2)(T_1+3T_2)$ and the ideals (T_1+2T_2) and (T_1+3T_2) are prime (the quotient rings are isomorphic to $\mathbb{F}_5[T]$) and non-associate. Therefore $V(T_1+2T_2)$ and $V(T_1+3T_2)$ are two irreducible components of $\pi^{-1}(x)$. The change of variables $U=T_1+2T_2$, $V=T_1+3T_2$ gives

$$A\simeq \mathbb{F}_5[U,V]/(UV),$$

a ring which embeds into the reduced ring $\mathbb{F}_5[U]\oplus\mathbb{F}_5[V]$; hence A is reduced.

Finally let's consider the case $p=2$. Then, in $\mathbb{F}_2[T_1,T_2]$, we have $T_1^2+T_2^2-10=(T_1+T_2)^2$ so that, setting $T=T_2$ and $U=T_1+T_2$ (an invertible change of variables),

$$A\simeq \mathbb{F}_2[T_1,T_2]/(T_1+T_2)^2\simeq \mathbb{F}_2[T][U]/(U^2).$$

This time $\pi^{-1}(x)$ is irreducible but non reduced: its unique irreducible component is $V(T_1+T_2)$, with generic point (T_1+T_2), and U is a nonzero nilpotent element.

In conclusion, the fibre $\pi^{-1}(x)$ is irreducible if and only if x is the generic point (0) of $\mathrm{Spec}(\mathbb{Z})$ or a closed point (p) with $p\neq 5$, and it is reduced if and only if $x\neq(2)$.

Exercise 4 1) The three generators $AY-BX$, $AZ-CX$ and $BZ-CY$ are homogeneous of degree 2. Moreover an ideal generated by homogeneous elements is homogeneous. Hence I is homogeneous.

2) Let $S=V_+(I)\subset\mathbb{P}^5_{\mathbb{Z}}$. Show that $|S|$ is irreducible.

Let's prove that $S\cap D_+(A)$ is irreducible. We have $S\cap D_+(A)\simeq\mathrm{Spec}(\mathbb{Z}[X,Y,Z,B,C]/(Y-BX,Z-CX,BZ-CY))$. However we have

$$\mathbb{Z}[X,Y,Z,B,C]/(Y-BX,Z-CX,BZ-CY)\simeq\mathbb{Z}[X,B,C]/(BCX-BCX)\simeq\mathbb{Z}[X,B,C].$$

Therefore $S\cap D_+(A)$ is irreducible. A similar comptuation shows that the intersection of S with $D_+(B)$, $D_+(C)$, $D_+(X)$, $D_+(Y)$ and $D_+(Z)$ is irreducible. Then for $x\in\{A,B,C,X,Y,Z\}$ we have $S\cap D_+(x)$ is irreducible. Note that

$$U=S\cap D_+(A)\cap D_+(B)\cap D_+(C)\cap D_+(X)\cap D_+(Y)\cap D_+(Z)$$

is non empty, it contains for instance the point of homogeneous coordinates $(1:1:\cdots:1)$ and is thus a non empty open subset in each of the $D_+(x)\cap S$ and thus irreducible. Therefore the closure of U in S contains all the $S\cap D_+(x)$ and thus contains S as S is the union of the $D_+(x)\cap S$. As S is the closure of an irreducible subset, it is irreducible.

3) To be reduced can be checked locally on S and we shown that each $S\cap D_+(x)$ is reduced for $x\in\{A,B,C,X,Y,Z\}$. Hence S is reduced.

4) The intersection is the closed subscheme

$$V_+(I) \cap V_+(C) = V_+(I + (C)) = V_+(J), \qquad J = (C,\ AY - BX,\ AZ,\ BZ),$$

where we have used that, modulo C, the generators $AZ - CX$ and $BZ - CY$ of I become AZ and BZ. We have

$$(C, AY-BX, A, B)\cdot(C, AY-BX, Z) \subset J \subset (C, AY-BX, A, B)\cap(C, AY-BX, Z)$$

so that

$$V_+(J) = V_+(C, AY - BX, A, B) \cup V_+(C, AY - BX, Z)$$

and as $(C, AY - BX, A, B) = (A, B, C)$ we have

$$V_+(J) = V_+(C, AY-BX, A, B)\cup V_+(C, AY-BX, A, Z)\cup V_+(C, AY-BX, Z, B) \cup V_+(C, AY - BX, Z).$$

We claim that $V_+(J)$ has exactly two irreducible components, namely

$$V_+(\mathfrak{p}_1), \quad \mathfrak{p}_1 = (C, Z, AY - BX), \qquad \text{and} \qquad V_+(\mathfrak{p}_2), \quad \mathfrak{p}_2 = (C, A, B).$$

Both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are homogeneous prime ideals not containing (A, B, C, X, Y, Z): for $\mathfrak{p}_2$ this is clear, as $R/\mathfrak{p}_2 \simeq \mathbb{Z}[X, Y, Z]$ is a domain; for $\mathfrak{p}_1$ we note that $R/\mathfrak{p}_1 \simeq \mathbb{Z}[X, Y, A, B]/(AY - BX)$ and that the polynomial $AY - BX$ is irreducible in the factorial ring $\mathbb{Z}[X, Y, A, B]$ (it is of degree one in A, and a factor independent of A would divide both coefficients Y and $-BX$, whose gcd is 1, hence would be a unit), and in a factorial ring an irreducible element generates a prime ideal. Moreover we have $V_+(\mathfrak{p}_1) \not\subset V_+(\mathfrak{p}_2)$ and $V_+(\mathfrak{p}_2) \not\subset V_+(\mathfrak{p}_1)$.

5) Let $R = \mathbb{Z}[X, Y, Z, A, B, C]$; the zero section is the closed subscheme $\{0\} = V(X, Y, Z, A, B, C)$ of $\mathbb{A}^6_{\mathbb{Z}} = \operatorname{Spec} R$, and $\mathbb{A}^6_{\mathbb{Z}} \smallsetminus \{0\}$ is covered by the six distinguished open subsets $D(T)$, $T \in \{X, \dots, C\}$. Note that $I \subseteq (X, Y, Z, A, B, C)$, so that $\{0\} \subseteq V(I)$. The morphism π restricted to $D(T)$ is the morphism $\operatorname{Spec} R_T \to \operatorname{Spec} R_{(T)}$ induced by the inclusion $R_{(T)} \subseteq R_T$. As $V_+(I)\cap D_+(T) = \operatorname{Spec}((R/IR)_{(T)})$, the preimage of $V_+(I)\cap D_+(T)$ in $D(T)$ is $V(IR_T)$, that is $V(I)\cap D(T)$. Therefore

$$\pi^{-1}(V_+(I)) = \bigcup_T V(I) \cap D(T) = V(I) \smallsetminus \{0\}.$$

Finally let us determine the closure of $V(I) \smallsetminus \{0\}$ in $\mathbb{A}^6_{\mathbb{Z}}$. It is contained in $V(I)$, which is closed. Conversely, consider the diagonal closed subscheme

$$L = V(Y - X,\ Z - X,\ A - X,\ B - X,\ C - X) \simeq \operatorname{Spec} \mathbb{Z}[X] = \mathbb{A}^1_{\mathbb{Z}}$$

($L \smallsetminus \{0\}$ is the preimage of the $\mathbb{Z}$-point $(1 : \cdots : 1)$). The generators of I vanish identically on L, so $L \subseteq V(I)$; moreover $\{0\} = V(X, \dots, C) \subseteq L$. The intersection

$L \cap D(X)$ is a nonempty open subset of the irreducible space L, hence dense in L; as $L \cap D(X) \subseteq V(I) \smallsetminus \{0\}$, we get

$$L \subseteq \overline{L \cap D(X)} \subseteq \overline{V(I) \smallsetminus \{0\}},$$

and therefore

$$V(I) = (V(I) \smallsetminus \{0\}) \cup \{0\} \subseteq (V(I) \smallsetminus \{0\}) \cup L \subseteq \overline{V(I) \smallsetminus \{0\}} \subseteq V(I).$$

Each $V(I) \cap D(T)$ is irreducible. For instance, in R_X/IR_X the relations in I give $B = AY/X$ and $C = AZ/X$, the third relation being then automatic, whence

$$R_X/IR_X \simeq \mathbb{Z}[X^{\pm 1}, Y, Z, A],$$

and is a domain. The other five charts are treated by symmetry. Note that the $V(I) \cap D(T)$ intersect pairwise. Then $V(I) \smallsetminus \{0\}$ is a union of six irreducible open subsets which intersect pairwise thus it is irreducible.

Finally the closure $V(I)$ of $\pi^{-1}(V_+(I)) = V(I) \smallsetminus \{0\}$ in $\mathbb{A}^6_{\mathbb{Z}}$ is irreducible.

6) The ideal of $\mathbb{Z}[A, B, C, X, Y, Z]$ generated by the 2×2-minors of the matrix

$$\begin{pmatrix} A & X \\ B & Y \\ C & Z \end{pmatrix}$$

is the ideal I. It follows that $\mathcal{X}$ is represented by the scheme $V(I)$. Namely for any ring R, we have a functorial bijection

$$\mathrm{Hom}(\mathbb{Z}[A, B, C, X, Y, Z]/I, R) \xrightarrow{\sim} \mathcal{X}(R)$$

sending φ on the matrix

$$\begin{pmatrix} \varphi(A) & \varphi(X) \\ \varphi(B) & \varphi(Y) \\ \varphi(C) & \varphi(Z) \end{pmatrix}.$$

The $\mathcal{X}$ is irreducible as $V(I)$ is irreducible by 5).

To check that $\mathcal{X}$ is reduced was more complicated than I expected. Here is direct proof. We can actually check directly that the ring $\mathbb{Z}[A, B, C, X, Y, Z]/I$ is a domain by considering the map

$$\begin{array}{ccc} R/I & \longrightarrow & \mathbb{Z}[U, V, W, T] \\ (A, B, C, X, Y, Z) & \longmapsto & (U, V, W, TU, TV, TW). \end{array}$$

It is possible to show that this map is injective so that R/I is a domain. Let's do it. Using the relation $AY = BX$, $AZ = CX$ and $BZ = CY$ we can show that the family of monomials

$$A^k B^\ell C^m X^n Y^p Z^q$$

satisfying the relations

$$\begin{cases} k > 0 & \Rightarrow p = q = 0 \\ \ell > 0 & \Rightarrow q = 0 \end{cases}$$

is a family of generators of R/I as a $\mathbb{Z}$-module. The image of this family in $\mathbb{Z}[U,V,W,T]$ is $\mathbb{Z}$-free. Namely, $A^k B^\ell C^m X^n Y^p Z^q$ is sent on $U^{k+n}V^{\ell+p}W^{m+q}T^{n+p+q}$ and, as the family of monomials $(U^\alpha V^\beta W^\gamma T^\delta)$ is $\mathbb{Z}$-free in $\mathbb{Z}[U,V,W,T]$, it is sufficient to prove that the relation

$$U^{k+n}V^{\ell+p}W^{m+q}T^{n+p+q} = U^{k'+n'}V^{\ell'+p'}W^{m'+q'}T^{n'+p'+q'}$$

implies $(k,\ell,m,n,p,q) = (k',\ell',m',n',p',q')$. Namely we have

$$\begin{cases} k+n = k'+n' \\ \ell+p = \ell'+p' \\ m+q = m'+q' \\ n+p+q = n'+p'+q'. \end{cases}$$

Assume that we have $k > k'$. Then $k > 0$ so that $p = q = 0$ and thus $n \geqslant n'$ so that $k' \geqslant k$. Contradiction. Thus $k \leqslant k'$. By Symmetry $k' \leqslant k$ and thus $k = k'$ and $n = n'$. Similarly we have $\ell = \ell'$ and $p = p'$. We deduce plainly the other equalities.

Exercise 5 Let A be a graded ring.

1) If every element of A_+ is nilpotent, then A_+ is contained in every prime ideal of A (the nilpotent elements form the nilradical, which is the intersection of all the prime ideals), in particular in every homogeneous prime ideal. As a point of $\mathrm{Proj}(A)$ is a homogeneous prime ideal which does not contain A_+, this shows that $\mathrm{Proj}(A) = \emptyset$.

   Conversely, assume that some element $x \in A_+$ is not nilpotent. We can assume that x is homogeneous. Namely if the elements of A_i are nilpotent for all $i > 0$, then all elements of A_+ are nilpotent. Namely a sum of commuting nilpotent elements is nilpotent. We can assume that $x \in A_k$ for some $k > 0$. As x is not nilpotent, the ring A_x is nonzero, i.e. we have $1 \neq 0$ in A_x. Therefore the ring $A_{(x)}$ is nonzero and thus $\mathrm{Spec}(A_{(x)}) \neq \emptyset$. We conclude that $D_+(x) \simeq \mathrm{Spec}(A_{(x)}) \neq \emptyset$ and thus that $\mathrm{Proj}(A) \neq \emptyset$.

2) Assume first that $A_n = 0$ for some $n > 0$. As A is generated by A_1 as an A_0-algebra, every element of A_{m+1} is a sum of products of $m+1$ elements of A_1, that is $A_{m+1} = A_1 A_m$ for $m \geqslant 1$; by induction $A_m = 0$ for all $m \geqslant n$. Now let $x \in A_+$ and write its homogeneous decomposition $x = x_{d_1} + \cdots + x_{d_r}$ with $d_i \geqslant 1$. In the

expansion of x^n, every product involves n homogeneous elements of degrees $\geqslant 1$, hence has degree $\geqslant n$, so

$$x^n \in \bigoplus_{d \geqslant n} A_d = 0.$$

Thus every element of A_+ is nilpotent and $\mathrm{Proj}(A) = \emptyset$ by question 1).

Conversely, assume $\mathrm{Proj}(A) = \emptyset$. By question 1), every element of A_+ is nilpotent. Let $x_1, \ldots, x_r$ be generators of A_1 as an A_0-module and let $n_i \geqslant 1$ be such that $x_i^{n_i} = 0$; put $N = n_1 + \cdots + n_r$. For $m \geqslant N$, the A_0-module A_m is generated by the products $x_{i_1} \cdots x_{i_m}$ of m elements of A_1, since A is generated by A_1 as an A_0-algebra. In such a product with $m \geqslant N$ factors, some index i occurs at least n_i times: otherwise the number of factors would be at most $\sum_i (n_i - 1) = N - r < N \leqslant m$. The product is then divisible by $x_i^{n_i} = 0$, hence vanishes. So $A_m = 0$ for all $m \geqslant N$, and we may take $n = \max(N, 1)$.

Exercise 6 It is sufficient to check that $V_+(I(\mathfrak{p}))$ satisfies the universal hypothesis of the fiber product

$$\pi^{-1}_{V_+(I)}(\mathfrak{p}) = \mathrm{Spec}(k(\mathfrak{p})) \times_{\mathrm{Spec}(A)} V_+(I).$$

We need to construct a map $V_+(I(\mathfrak{p})) \to V_+(I)$. There is a morphism of graded rings $\psi : R/I \to (R \otimes_A k(\mathfrak{p}))/I(\mathfrak{p})$. Moreover if $\mathfrak{p} \subset (R \otimes_A k(\mathfrak{p}))/I(\mathfrak{p})$ is a graded prime ideal not containing $(X_0, \ldots, X_n)$ its inverse image doesn't contain $(X_0, \ldots, X_n)$ so that ψ gives rise to a morphism of schemes $\eta : V_+(I(\mathfrak{p})) \to V_+(I)$. Let x_i be the image of X_i in R/IR. The affine scheme $V_+(I) \cap D_+(X_i)$ is canonically isomorphic to $\mathrm{Spec}((R/IR)_{(x_i)})$. Similarly the affine scheme $V_+(I(\mathfrak{p})) \cap D_+(X_i)$ is canonically isomorphic to $\mathrm{Spec}(((k(\mathfrak{p}) \otimes_A R)/I(\mathfrak{p}))_{(x_i)})$ (denoting abusively x_i the image of x_i in $(k(\mathfrak{p}) \otimes_A R)/I(\mathfrak{p})$). By construction for any $0 \leqslant i \leqslant n$, η induces a morphism $V_+(I(\mathfrak{p})) \cap D_+(X_i) \to V_+(I) \cap D_+(X_i)$ such that the following diagram commutes

$$\begin{array}{ccc} V_+(I(\mathfrak{p})) \cap D_+(X_i) & \xrightarrow{\eta} & V_+(I) \cap D_+(X_i) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Spec}(((k(\mathfrak{p}) \otimes_A R)/I(\mathfrak{p}))_{(x_i)}) & \longrightarrow & \mathrm{Spec}((R/IR)_{(x_i)}) \end{array}$$

where the bottom map is deduced from $R/IR \to (k(\mathfrak{p}) \otimes_A R)/I(\mathfrak{p})$ by localization along x_i.

We deduce from these observations that we have a commutative diagram

$$\begin{array}{ccc} V_+(I(\mathfrak{p})) & \xrightarrow{\eta} & V_+(I) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k(\mathfrak{p})) & \longrightarrow & \mathrm{Spec}(A). \end{array}$$

Now let T be a scheme and a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{f} & V_+(I) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k(\mathfrak{p})) & \longrightarrow & \mathrm{Spec}(A). \end{array}$$

We need to check that there exists a unique map of $k(\mathfrak{p})$-schemes $g : T \to V_+(I(\mathfrak{p}))$ such that $f = \eta \circ g$. For $0 \leqslant i \leqslant n$ let T_i be the inverse image of the open susbet $D_+(X_i)$ so that we have a commutative diagram

$$\begin{array}{ccc} T_i & \xrightarrow{f} & V_+(I) \cap D_+(X_i) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k(\mathfrak{p})) & \longrightarrow & \mathrm{Spec}(A). \end{array}$$

Let x_i be the image of X_i in R/IR. The affine scheme $V_+(I) \cap D_+(X_i)$ is canonically isomorphic to $\mathrm{Spec}((R/IR)_{(x_i)})$. By definition we have

$$\mathrm{Spec}(k(\mathfrak{p})) \times_{\mathrm{Spec}(A)} \mathrm{Spec}((R/IR)_{(x_i)}) = \mathrm{Spec}(k(\mathfrak{p}) \otimes_A ((R/IR)_{(x_i)})).$$

Moreover we have an isomorphism of graded rings

$$k(\mathfrak{p}) \otimes_A (R/IR) \simeq (k(\mathfrak{p}) \otimes_A R)/I(\mathfrak{p})$$

which induces an isomorphism of rings

$$k(\mathfrak{p}) \otimes_A (R/I)_{(x_i)} \simeq ((k(\mathfrak{p}) \otimes_A R)/I(\mathfrak{p}))_{(x_i)}.$$

We obtain an isomorphism over $\mathrm{Spec}(k(\mathfrak{p}))$:

$$\mathrm{Spec}(k(\mathfrak{p})) \times_{\mathrm{Spec}(A)} (V_+(I) \cap D_+(X_i)) \simeq V_+(I(\mathfrak{p})) \cap D_+(X_i).$$

and a unique morphism $g_i : T_i \to V_+(I(\mathfrak{p}))$ such that $\eta \circ g_i = f_{|T_i}$.

For $0 \leqslant i, j \leqslant n$ the following diagram is commutative

$$\begin{array}{ccc} k(\mathfrak{p}) \otimes_A (R/I)_{(x_i)} & \xrightarrow{\sim} & ((k(\mathfrak{p}) \otimes_A R)/I(\mathfrak{p}))_{(x_i)} \\ \downarrow & & \downarrow \\ k(\mathfrak{p}) \otimes_A (R/I)_{(x_i x_j)} & \xrightarrow{\sim} & ((k(\mathfrak{p}) \otimes_A R)/I(\mathfrak{p}))_{(x_i x_j)}. \end{array}$$

showing that $g_{i|T_i \cap T_j} = g_{j|T_i \cap T_j}$. Therefore the morphisms g_i glue to a morphism of $k(\mathfrak{p})$-schemes $g : T \to V_+(I(\mathfrak{p}))$ such that the composition of $\eta \circ g = f$.