

Practice exercises

Exercise 1 Let X be a topological space and let F be a closed subset of X . Let i denote the inclusion of F into X .

- 1) Let $\mathcal{F}$ be a sheaf of abelian groups on F and let $\mathcal{G} = i_*\mathcal{F}$. Show that, for every $x \in X$, we have $\mathcal{F}_x \simeq \mathcal{G}_x$ if $x \in F$ and $\mathcal{G}_x = 0$ if $x \notin F$.
- 2) Let $\mathcal{G}$ be a sheaf of abelian groups on X . For each open subset U of F , let $\mathcal{H}(U)$ be the set of families

$$(s_x) \in \prod_{x \in U} \mathcal{G}_x$$

for which there exist an open subset $V \subset X$ and a section $s' \in \mathcal{G}(V)$ such that $V \cap F = U$ and $s_x = s'_x$ in $\mathcal{G}_x$ for every $x \in U$. Show that $\mathcal{H}$ is a presheaf on F . Let $\mathcal{G}|_F$ be its associated sheaf. Show that, for every $x \in F$, we have $\mathcal{G}|_{F,x} \simeq \mathcal{G}_x$.

- 3) Let $\mathcal{G}$ be a sheaf of abelian groups on X . Show that $\mathcal{G}$ is isomorphic to $i_*(\mathcal{G}|_F)$ if and only if $\mathcal{G}_x = 0$ for every $x \notin F$.

Now let X be a scheme and let F be a closed subset of $|X|$. For each open subset $U \subset X$, set

$$\mathcal{I}(U) = \{f \in \mathcal{O}_X(U) \mid \forall x \in U \cap F, f(x) = 0 \text{ in } k(x)\}.$$

- 4) Show that $\mathcal{I}$ is a sheaf of ideals of $\mathcal{O}_X$.
- 5) Consider the sheaf of rings $\mathcal{O}_X/\mathcal{I}$. Show that $(\mathcal{O}_X/\mathcal{I})_x = 0$ if and only if $x \notin F$.
- 6) Show that there exists a closed subscheme Z of X such that $|Z| = F$ and $i_*(\mathcal{O}_Z) \simeq \mathcal{O}_X/\mathcal{I}$. Is Z reduced?

Exercise 2 Let k be a field. For each of the following closed subschemes $S = V(I)$ of $\mathbb{A}_k^3 = \text{Spec}(k[X, Y, Z])$, determine the irreducible components and their generic points. Also determine whether S is reduced in each case.

- 1) $S = V(I)$ with $I = (Y, Y^2 - XZ)$;
- 2) $S = V(I)$ with $I = (X(Y - X^2 + 1), Y(Y - X^2 + 1))$;
- 3) $S = V(I)$ with $I = (X^2 - YZ, XZ - X)$.

Exercise 3 Let $X = \text{Spec}(\mathbb{Z}[T_1, T_2]/(T_1^2 + T_2^2 - 10))$ and let $\pi : X \rightarrow \text{Spec}(\mathbb{Z})$ be the structure morphism. For which points $x \in \text{Spec}(\mathbb{Z})$ is the fibre $\pi^{-1}(x)$ irreducible? For which points is it reduced?

Exercise 4 Let I be the ideal of $\mathbb{Z}[X, Y, Z, A, B, C]$ generated by $AY - BX$, $AZ - CX$ and $BZ - CY$.

- 1) Show that I is a homogeneous ideal.
- 2) Let $S = V_+(I) \subset \mathbb{P}_{\mathbb{Z}}^5$. Show that $|S|$ is irreducible.
- 3) Show that the scheme S is reduced.
- 4) Determine the irreducible components of the intersection $V_+(I) \cap V_+(C)$.
- 5) Consider the morphism $\pi : \mathbb{A}_{\mathbb{Z}}^6 \setminus \{0\} \rightarrow \mathbb{P}_{\mathbb{Z}}^5$, where $\{0\}$ denotes the zero section. Determine the closure of $\pi^{-1}(V_+(I))$ in $\mathbb{A}_{\mathbb{Z}}^6$ and show that this closure is irreducible.
- 6) Consider the functor from the category of commutative rings to the category of sets given by

$$R \mapsto \mathcal{X}(R) = \{M \in \mathcal{M}_{3,2}(R) \mid \text{rg}(M) \leq 1\},$$

where $\text{rg}(M) \leq 1$ means that all 2×2 minors of M vanish. Show that $\mathcal{X}$ is represented by an irreducible reduced affine scheme.

Exercise 5 Let A be a graded ring.

- 1) Show that $\text{Proj}(A) = \emptyset$ if and only if every element of A_+ is nilpotent.
- 2) Assume in addition that A is generated by A_1 as an A_0 -algebra and that A_1 is a finitely generated A_0 -module. Show that $\text{Proj}(A) = \emptyset$ if and only if there exists $n > 0$ such that $A_n = 0$.

Exercise 6 Let A be a ring, let $R = A[X_0, \dots, X_n]$ and let $I \subset R$ be a homogeneous ideal. Write $\pi : \mathbb{P}_A^n \rightarrow \text{Spec}(A)$ for the structure morphism and $\pi_{V_+(I)} : V_+(I) \rightarrow \text{Spec}(A)$ for its restriction to $V_+(I)$. Let $\mathfrak{p} \in \text{Spec}(A)$. Show that $\pi_{V_+(I)}^{-1}(\mathfrak{p})$ is isomorphic to $V_+(I(\mathfrak{p}))$, where $I(\mathfrak{p})$ is the ideal generated by the image of I in $k(\mathfrak{p}) \otimes_A R \simeq k(\mathfrak{p})[X_0, \dots, X_n]$.