

INFINITESIMAL CHARACTERS IN ARITHMETIC FAMILIES

GABRIEL DOSPINESCU, VYTAUTAS PAŠKŪNAS, AND BENJAMIN SCHRAEN

ABSTRACT. We associate infinitesimal characters to (twisted) families of L -parameters and C -parameters of p -adic reductive groups. We use the construction to study the action of the centre of the universal enveloping algebra on the locally analytic vectors in the Hecke eigenspaces in the completed cohomology.

CONTENTS

1. Introduction	2
1.1. Quick overview	2
1.2. Problems in the p -adic world	3
1.3. Analytic continuation and infinitesimal characters	5
1.4. Infinitesimal characters and local Galois representations	7
1.5. Global input and applications	10
1.6. p -adic Jacquet–Langlands and infinitesimal characters	12
1.7. p -adic Langlands and infinitesimal characters	13
1.8. Beyond $l_0 = 0$ case	13
1.9. Overview of sections	14
1.10. Acknowledgements	14
2. The L -group	15
2.1. C -groups	16
2.2. Admissible representations	17
2.3. Twisted families of L - and C -parameters	17
3. Sen theory	20
4. Families of Galois representations and infinitesimal characters	23
4.1. Topology on the coefficients	23
4.2. The Sen operator in the affinoid case	24
4.3. The Sen operator and twisted families of L -parameters	27
4.4. Chevalley’s restriction theorem	28
4.5. Harish-Chandra homomorphism.	28
4.6. Definition of infinitesimal character	29
4.7. Modification for C -groups	31
4.8. The archimedean case	32
5. Hodge–Tate representations	33
5.1. Galois representations valued in L -groups	34
5.2. Galois representations valued in C -groups	35
5.3. Algebraic representations and regular Hodge–Tate weights	37
5.4. Automorphic representations	38

Date: December 2, 2020.

6. Fibres of Cohen–Macaulay modules	39
7. Density of algebraic vectors	40
8. Families of Banach space representations	43
9. Global applications	47
9.1. Locally symmetric spaces	48
9.2. Completed cohomology	50
9.3. Hecke algebra	50
9.4. Weakly non-Eisenstein ideals	51
9.5. Automorphic forms	52
9.6. Main result	54
9.7. Modular curves	57
9.8. Shimura curves	57
9.9. Definite unitary groups	58
9.10. Compact unitary Shimura varieties	59
9.11. Patched module	60
9.12. The p -adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$	60
9.13. A conjectural picture	61
References	63

1. INTRODUCTION

Let G be a connected reductive group over \mathbb{R} and let $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))$. The center $Z(\mathfrak{g})$ of the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ has a strong influence on the representation theory of G , as we will briefly recall. Let $\widehat{G(\mathbb{R})}$ be the unitary dual of $G(\mathbb{R})$. By Segal’s theorem¹ there is an “infinitesimal character map”

$$\widehat{G(\mathbb{R})} \rightarrow \mathrm{Hom}_{\mathbb{C}\text{-alg}}(Z(\mathfrak{g}), \mathbb{C}), \pi \mapsto \chi_{\pi},$$

and by Harish-Chandra’s finiteness theorem this map has finite fibres. Moreover, the study of the differential equations hidden in the existence of χ_{π} yields important information about the asymptotic behaviour of the matrix coefficients of π , and this can be used to prove Casselman’s subrepresentation theorem and the Langlands classification. Going somewhat in the opposite direction, one can use the existence of an infinitesimal character to deduce finiteness results: another classical result of Harish-Chandra ensures that any admissible Banach representation of $G(\mathbb{R})$ with an infinitesimal character has finite length.

It is both natural and tempting to investigate whether similar phenomena happen in the p -adic world, but so far the situation is far less rosy due to our rather poor understanding of these representations. The results of this paper and its sequel [29] point to some striking similarities with the above picture, as well as significant differences. Our results are most definite in the cases when a connection to Galois representations can be made.

1.1. Quick overview. There are essentially three main results in this paper.

The first is a very general construction (see sections 4.6 and 4.7) attaching infinitesimal characters to (twisted) families of L -parameters and C -parameters of

¹This crucially uses the unitarity hypothesis, and the result fails for irreducible Banach representations, making it impossible to adapt the proof in the p -adic world.

p -adic reductive groups. This is a p -adic analogue of a classical construction for L -parameters of real reductive groups (see section 4.8 for a review). For an L -parameter $\rho : \mathrm{Gal}_F \rightarrow {}^L G(\overline{\mathbb{Q}}_p)$ the associated infinitesimal character ζ_ρ is obtained from the conjugacy class of the semisimple part of the Sen operator attached to ρ .

The second is an interpolation result (theorems 8.5 and 8.6), stating that if we can produce sufficiently many² locally analytic vectors in sufficiently many members of a family of admissible Banach representations of a p -adic reductive group G , with the property that the center of the universal enveloping algebra of $\mathrm{Lie}(G)$ acts on them by characters in a compatible way, then these characters glue and all Banach representations of the family have an infinitesimal character obtained by specialisation.

The third exploits these two results to show (theorems 9.20 and 9.24) that many locally analytic representations arising “in nature” have infinitesimal characters, which can be explicitly computed from p -adic Hodge theoretic data. Informally, if we can attach Galois representations to Hecke eigenspaces in completed cohomology, so that at classical points the Hodge–Tate weights of the Galois representation match the infinitesimal character of the corresponding classical automorphic form (i.e. a weak form of local-global compatibility at p), then this property propagates by analytic continuation to all Hecke eigenspaces. Thus in favorable settings if the Hecke eigenspace is non-zero, then its locally analytic vectors have an infinitesimal character, which can be related to the generalized Hodge–Tate weights of the associated Galois representation.

Examples of situations covered by our results are given by (sufficiently non-degenerate) Hecke eigenspaces in the completed cohomology of modular curves, or more generally Shimura curves over totally real fields, or of compact unitary Shimura varieties studied by Caraiani–Scholze, or of definite unitary groups over totally real fields (see sections 9.7, 9.8, 9.9 and 9.10). The formalism developed in this paper also applies to “patched” modules obtained by patching these completed cohomology groups as in [15]. In particular, we show (see theorem 9.27) that the candidates for the p -adic local Langlands correspondence for p -adic $\mathrm{GL}_n(F)$ constructed in [15] have infinitesimal characters depending only on the local Galois representation one starts with, giving further nontrivial evidence that these candidates are independent of the choices made in their construction.

1.2. Problems in the p -adic world. Fix a prime number p and a connected reductive group G over \mathbb{Q}_p . Let L be the coefficient field of our representations, a sufficiently large finite extension of \mathbb{Q}_p . We assume that G is split over L . Let \mathfrak{g} be the Lie algebra of the p -adic Lie group $G(\mathbb{Q}_p)$ (or equivalently of G).

Before discussing the problems arising for p -adic representations, it is convenient to introduce some notation and recall a certain number of basic results. Let H be a p -adic Lie group and let $\mathfrak{h} = \mathrm{Lie}(H)$. Let $\mathrm{Ban}_L(H)$ be the category of admissible L -Banach space representations of H and let \widehat{H}_L be the set of isomorphism classes of absolutely irreducible objects of this category. Contrary to the world of real groups, admissibility of irreducible unitary representations does not come for free, thus it is better to impose it from the very beginning. Let $\mathrm{Ban}_L(H)^{\mathrm{unit}}$ be the full subcategory of $\mathrm{Ban}_L(H)$ consisting of unitary representations, i.e. those having an H -invariant norm defining the Banach space topology, and let $\widehat{H}_L^{\mathrm{unit}}$ be the

²In a sense which can and will be made precise later on.

set of isomorphism classes of absolutely irreducible objects of $\text{Ban}_L(H)^{\text{unit}}$. If $\Pi \in \text{Ban}_L(H)$ then we let Π^{la} be the subspace of locally analytic vectors in Π , i.e. vectors whose orbit map $g \mapsto g.v$ is a locally analytic map from H to Π . Then Π^{la} is a dense subspace of Π by [73], on which $U(\mathfrak{h})_L := U(\mathfrak{h}) \otimes_{\mathbb{Q}_p} L$ naturally acts ($U(\mathfrak{h})$ is the universal enveloping algebra of \mathfrak{h}). Let $Z(\mathfrak{h})_L$ be the center of $U(\mathfrak{h})_L$ and let $\chi : Z(\mathfrak{h})_L \rightarrow L$ be an L -algebra homomorphism. If $\Pi \in \text{Ban}_L(H)$, we say that Π^{la} has infinitesimal character χ if $Xv = \chi(X)v$ for all $X \in Z(\mathfrak{h})_L, v \in \Pi^{\text{la}}$. We will also abuse language and say that Π has infinitesimal character χ when Π^{la} does so.

A first major problem, which is unfortunately unsolved (to our knowledge) for any G beyond tori, is the existence of an infinitesimal character map

$$(1) \quad \widehat{G(\mathbb{Q}_p)}_L \rightarrow \text{Hom}_{L\text{-alg}}(Z(\mathfrak{g})_L, L),$$

i.e. whether Π^{la} has an infinitesimal character for each $\Pi \in \widehat{G(\mathbb{Q}_p)}_L$. It is conjectured in [30] that this is the case, and it is proved there (based on deep results of Ardakov and Wadsley [1]) that this is so if we further³ assume that Π^{la} is an absolutely irreducible locally analytic representation of $G(\mathbb{Q}_p)$. If we replace $\widehat{G(\mathbb{Q}_p)}$ by $\widehat{G(\mathbb{Q}_p)}^{\text{unit}}$, then essentially the only group for which the existence of this map is known is $G = \text{GL}_2$, and the argument in [28] fully uses the p -adic local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$. We will give a different proof in this paper, which still uses this input, as well as global results, but sheds some more light on what could happen for other groups. Contrary to the case of real groups, already for $\text{GL}_2(\mathbb{Q}_p)$ all fibres of the map (1) are uncountable. Still, the fibres of the restriction to $\widehat{\text{GL}_2(\mathbb{Q}_p)}^{\text{unit}}$ have nice geometric structures: they are the set of L -points of some (non quasi-compact) rigid analytic spaces.

Inspired by the situation for real groups, one may ask whether every admissible L -Banach representation Π of $G(\mathbb{Q}_p)$ with an infinitesimal character has finite length. One cannot expect such a result to hold in great generality (i.e. with $G(\mathbb{Q}_p)$ replaced by any p -adic Lie group), since there are admissible L -Banach representations Π of $\text{GL}_2(\mathbb{Z}_p)$ with an infinitesimal character and of infinite length. However, we will show in [29] that this holds for unitary representations of $\text{GL}_2(\mathbb{Q}_p)$ (the proof does *not* use the p -adic Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$), and that a similar result holds for the group of units of a quaternion division algebra D over \mathbb{Q}_p , if we further assume (in the case of D^\times) that the infinitesimal character is not the one of an irreducible algebraic representation. When this hypothesis on the infinitesimal character is no longer satisfied, it is not clear to us whether the $\text{GL}_2(\mathbb{Z}_p)$ or the $\text{GL}_2(\mathbb{Q}_p)$ phenomenon prevails, since D^\times is compact modulo the centre. This in turn has interesting local and global applications, for instance to the study of eigenspaces in the completed cohomology of certain Shimura curves or to the (still hypothetical, despite a lot of recent progress [75, 67, 49, 55]) p -adic Jacquet–Langlands correspondence for the pair $(\text{GL}_2(\mathbb{Q}_p), D^\times)$. We leave these applications for [29], but they should suggest that the existence of an infinitesimal character on a Banach representation has rather strong consequences on its structure, and it is precisely with this existence problem that we are dealing in this paper.

³This is not automatic; actually it is not even known if Π^{la} always has finite length, and it is not even clear that this is to be expected.

1.3. Analytic continuation and infinitesimal characters. The basic idea is very simple: suppose that a representation Π lives in a family of Banach representations Π_X , parameterized by some rigid analytic space X , and suppose that we can produce sufficiently “many” locally analytic vectors in sufficiently “many” members of a family of admissible Banach representations, with the property that $Z(\mathfrak{g})_L$ acts on them by characters in a compatible way, so that these characters interpolate to a character $\chi : Z(\mathfrak{g})_L \rightarrow \mathcal{O}_X(X)$. A density argument then shows that $Z(\mathfrak{g})_L$ acts by χ on Π_X^{la} , and thus all members of the family have infinitesimal characters, obtained as specialisations of χ .

In order to maximize its flexibility, it is convenient to state the analytic continuation argument in a rather general and abstract setting. Let \mathcal{O} be the ring of integers of L and let (R, \mathfrak{m}) be a complete local noetherian \mathcal{O} -algebra with the same residue field as L . Let $\mathfrak{X}^{\text{rig}}$ be the generic fibre of the formal scheme $\text{Spf } R$ and let $R^{\text{rig}} = H^0(\mathfrak{X}^{\text{rig}}, \mathcal{O}_{\mathfrak{X}^{\text{rig}}})$.

Let G be a connected reductive group over \mathbb{Q}_p and let K be a compact open subgroup of $G(\mathbb{Q}_p)$. Finally, let M be a finitely generated $R[[K]]$ -module. There is a canonical topology on M making it a compact topological $R[[K]]$ -module. Endowed with the supremum norm, the (not necessarily admissible) unitary Banach space representation of K

$$\Pi := \text{Hom}_{\mathcal{O}}^{\text{cont}}(M, L)$$

can be thought of as a family of objects of $\text{Ban}_L(K)^{\text{unit}}$, parameterized by the maximal spectrum $\mathfrak{m}\text{-Spec}(R[1/p])$: for any $x \in \mathfrak{m}\text{-Spec}(R[1/p])$ the subspace $\Pi[\mathfrak{m}_x]$ of Π consisting of elements killed by the maximal ideal \mathfrak{m}_x is an object of $\text{Ban}_{\kappa(x)}(K)^{\text{unit}}$ (and also an object of $\text{Ban}_L(K)^{\text{unit}}$ since the residue field $\kappa(x)$ of x is finite over L).

Choose the coefficient field L large enough so that G splits over L . Let $\text{Irr}_G(L)$ be the set of isomorphism classes of irreducible algebraic representations of G_L . For any $V \in \text{Irr}_G(L)$, by evaluating at L we get an action of $G(\mathbb{Q}_p) \subset G(L)$ on V , and V is absolutely irreducible as a representation of $G(\mathbb{Q}_p)$ (since G splits over L), thus it has an infinitesimal character. For such V consider the $R[1/p]$ -module

$$M(V) = \text{Hom}_K^{\text{cont}}(V, \Pi)',$$

where W' is the topological L -dual of the topological L -vector space W . It is not difficult to see⁴ that $M(V)$ is a finitely generated $R[1/p]$ -module. Let R_V be the quotient of R that acts faithfully on $M(V)$.

We refer the reader to theorems 8.5 and 8.6 for variations on the next basic theorem.

Theorem 1.1. *Let R, M be as above and suppose that there is an L -algebra homomorphism $\chi : Z(\mathfrak{g})_L \rightarrow R^{\text{rig}}$ such that the following hold*

- (1) *there is an M -regular sequence $y_1, \dots, y_h \in \mathfrak{m}$ such that $M/(y_1, \dots, y_h)M$ is a finitely generated projective $\mathcal{O}[[K]]$ -module.*
- (2) *for all $V \in \text{Irr}_G(L)$ the ring R_V is reduced.*
- (3) *for all $V \in \text{Irr}_G(L)$ and all $x \in \mathfrak{m}\text{-Spec}(R_V[1/p])$ the infinitesimal character of V is the specialisation $\chi_x : Z(\mathfrak{g})_L \rightarrow \kappa(x)$ of χ at x .*

⁴One can also present $M(V) = V \otimes_{\mathcal{O}[[K]]} M$, for the right $\mathcal{O}[[K]]$ -module structure on V induced by the anti-automorphism of $\mathcal{O}[[K]]$ sending $k \in K$ to k^{-1} .

Then for all $y \in \mathfrak{m}\text{-Spec}(R[1/p])$ the representation $\Pi[\mathfrak{m}_y]$ has infinitesimal character χ_y .

We end this paragraph with a detailed explanation of the key ingredients in the proof of this result, since this also explains the origin of the somewhat exotic hypotheses in the statement of the theorem. Let $D(K, L)$ be the algebra of L -valued distributions on K . The representation Π gives rise to an $R^{\text{rig}} \widehat{\otimes}_L D(K, L)$ -module $\Pi^{R\text{-la}}$, roughly speaking the space of vectors $v \in \Pi$ that are locally analytic both for the action of K and that of R . More precisely, we have an isomorphism

$$(\Pi^{R\text{-la}})' \simeq (R^{\text{rig}} \widehat{\otimes}_L D(K, L)) \otimes_{R[[K]]} M$$

describing $\Pi^{R\text{-la}}$ as the topological L -dual of the right-hand side. Moreover, the inclusion $\Pi^{R\text{-la}}[\mathfrak{m}_x] \rightarrow \Pi[\mathfrak{m}_x]^{\text{la}}$ is an isomorphism for all $x \in \mathfrak{m}\text{-Spec}(R[1/p])$.

It suffices therefore to prove that $1 \otimes D - \chi(D) \otimes 1 \in R^{\text{rig}} \widehat{\otimes}_L D(K, L)$ kills $\Pi^{R\text{-la}}$ for all $D \in Z(\mathfrak{g})_L$. By continuity, it suffices to prove this for a dense subspace of $\Pi^{R\text{-la}}$. We will explain how to find such a subspace under the hypotheses of the theorem. There are natural embeddings

$$(2) \quad \bigoplus_{[V] \in \text{Irr}_G(L)} \text{Hom}_K(V, \Pi^{R\text{-la}}) \otimes_L V \rightarrow \Pi^{R\text{-la}}$$

and

$$(3) \quad \bigoplus_{x \in \mathfrak{m}\text{-Spec}(R_V[1/p])} \text{Hom}_K(V, \Pi[\mathfrak{m}_x]^{\text{la}}) \otimes_L V \rightarrow \text{Hom}_K(V, \Pi^{R\text{-la}}) \otimes V,$$

the first one identifying the left-hand side with the space of K -algebraic vectors in $\Pi^{R\text{-la}}$. We prove that under the assumptions of the theorem both embeddings have dense image. Since the third hypothesis implies that $1 \otimes D - \chi(D) \otimes 1$ kills the image of the second embedding for all $[V] \in \text{Irr}_G(L)$ and $D \in Z(\mathfrak{g})_L$, this will finish the proof.

The proof that (2) and (3) have dense image crucially uses the first hypothesis, which allows us to replace R first by $S = \mathcal{O}[[x_1, \dots, x_h]]$ and then (this requires changing the groups K and G) by \mathcal{O} , while simultaneously reducing the proof to the case $M = \mathcal{O}[[K]]$. Sending x_i to y_i yields an action of S on M , and a key remark is that the first hypothesis forces M to be a finite projective module over $S[[K]]$.

Let us explain the proof that (2) has dense image. It is proved in [11] that $\Pi(M)^{R\text{-la}} = \Pi(M)^{S\text{-la}}$, thus we may forget about R . But $S[[K]]$ is identified with $\mathcal{O}[[K']]$, where $K' = (1 + 2p\mathbb{Z}_p)^h \times K$ is a compact open subgroup of $G' = \mathbb{G}_m^h \times G$. The transparent link between $\text{Irr}_G(L)$ and $\text{Irr}_{G'}(L)$ reduces the proof to the case $R = \mathcal{O}$ and M is finite projective over $\mathcal{O}[[K]]$ (up to replacing G and K by G' and K'). This further reduces to the case $M = \mathcal{O}[[K]]$. In this case the result we want to prove is equivalent to the density of the image of the natural map $\bigoplus_V V^* \otimes V \rightarrow \mathcal{C}^{\text{la}}(K, L)$. Since $\bigoplus_V V^* \otimes V \simeq \mathcal{O}_G(G)$, the required density is a consequence of the following more general result, a simple application of Amice's theorem: for any smooth affine scheme X over $\text{Spec}(\mathbb{Q}_p)$ and for any open and closed subset U of $X(\mathbb{Q}_p)$, the natural map $L \otimes_{\mathbb{Q}_p} \mathcal{O}_X(X) \rightarrow \mathcal{C}^{\text{la}}(U, L)$, obtained by restricting regular functions on X to U , has dense image.

Finally, let us explain the proof of the density of the map (3). We argue by duality, using the Hahn–Banach theorem. Thus we want to show that the dual of the right-hand side in (3) embeds in the dual of the left-hand side. The choice of

a K -stable lattice Θ in V induces an integral structure $M(\Theta)$ on $M(V)$. Simple arguments show that

$$\mathrm{Hom}_K(V, \Pi^{R\text{-la}})' \simeq M(\Theta) \otimes_{R_V} R_V^{\mathrm{rig}}$$

and for all $x \in \mathrm{m}\text{-Spec}(R_V[1/p])$

$$\mathrm{Hom}_K(V, \Pi[\mathfrak{m}_x]^{1\mathrm{a}})' \simeq M(\Theta) \otimes_{R_V} \kappa(x).$$

One can prove, using again the first hypothesis, that $M(\Theta)$ is a finite free S -module and deduce that it is a Cohen–Macaulay R_V -module. The result follows now from the second hypothesis and the following commutative algebra result: if R is a complete local noetherian \mathcal{O} -algebra, which is reduced and \mathcal{O} -torsion free, then for any faithful Cohen–Macaulay R -module M the natural map

$$M \otimes_R R^{\mathrm{rig}} \rightarrow \prod_{x \in (\mathrm{Spf} R)^{\mathrm{rig}}} M \otimes_R \kappa(x)$$

is injective. Note that this is equivalent to the density of the image of (3) in the case when G is trivial. The proof is a simple application of the Noether normalisation lemma and basic properties of faithful Cohen–Macaulay modules.

1.4. Infinitesimal characters and local Galois representations. The previous paragraph gives a systematic way of proving the existence of an infinitesimal character for a Banach representation, but one needs to be able to find the objects R, M and χ as in theorem 1.1. This is not at all a trivial problem. As we will explain in the next paragraph, the existence of R and M can be proved in many cases using global methods and the existence of Galois representations attached to p -adic automorphic forms, combined with suitable local-global compatibility results. In this paragraph we want to focus on a key result of this paper, namely the construction of a character χ starting from a family of local Galois representations. The construction uses Sen theory in families (as discussed in [3]), Tannakian formalism, Chevalley’s restriction theorem and the Harish-Chandra isomorphism.

We find it useful to consider first the simplest possible case. Let $\rho : \mathrm{Gal}_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_n(L)$ be a continuous Galois representation, where $\mathrm{Gal}_{\mathbb{Q}_p}$ is the absolute Galois group of \mathbb{Q}_p . We would like to attach to ρ a character $\zeta_\rho : Z(\mathfrak{g})_L \rightarrow L$, where \mathfrak{g} is the Lie algebra of $\mathrm{GL}_n(\mathbb{Q}_p)$. By the Harish-Chandra isomorphism $Z(\mathfrak{g})_L$ is a polynomial ring in n variables, thus giving ζ_ρ is equivalent to giving n elements of L . These are simply the coefficients of the characteristic polynomial of the Sen operator of ρ . In other words ζ_ρ encodes the generalized Hodge–Tate weights of ρ . When ρ is de Rham with regular Hodge–Tate weights, one can attach to ρ an algebraic representation of $\mathrm{GL}_n/\mathbb{Q}_p$ over L , and ζ_ρ encodes the highest weight of this representation.

Let us discuss now the general case. Let F be a finite extension of \mathbb{Q}_p and let G be a connected reductive group over F with Lie algebra \mathfrak{g} . Let \widehat{G} be the dual group of G (defined over \mathbb{Z}). This group comes with an action $\mu_G : \mathrm{Gal}_F \rightarrow \widehat{G}$ of the absolute Galois group of F , and we let ${}^L G = \widehat{G} \rtimes \mathrm{Gal}_F$ be the Langlands dual group of G . The map $g \mapsto (1, g)$ identifies $\mathrm{Ker}(\mu_G)$ with a normal subgroup of ${}^L G$, and we let ${}^L G_f = {}^L G / \mathrm{Ker}(\mu_G) = \widehat{G} \rtimes \Gamma$, where $\Gamma = \mathrm{Gal}_F / \mathrm{Ker}(\mu_G)$.

Next, let X be a rigid analytic variety and let P be a \widehat{G}_X -torsor, locally trivial for the étale topology⁵. Let P^{ad} be the sheaf of \widehat{G}_X -equivariant automorphisms of P , which leave the base X fixed. We define in this generality the L -group ${}^L P^{\text{ad}}(X)$ of G twisted by P , as well as a notion of admissible representations $\rho : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(X)$, see definitions 2.4 and 2.7. For instance, if $P = \widehat{G}_X$ is the trivial \widehat{G}_X -torsor, then ${}^L P^{\text{ad}}(X) = {}^L G_f(\mathcal{O}_X(X)) = \widehat{G}(\mathcal{O}_X(X)) \rtimes \Gamma$ and admissibility of a continuous representation $\rho : \text{Gal}_F \rightarrow \widehat{G}(\mathcal{O}_X(X)) \rtimes \Gamma$ is the usual notion, i.e. we ask that the induced map $\text{Gal}_F \rightarrow \Gamma$ is the natural projection. Our main construction associates to any admissible representation $\rho : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(X)$ an L -algebra homomorphism

$$(4) \quad \zeta_\rho : Z(\text{Res}_{F/\mathbb{Q}_p} \mathfrak{g}) \otimes_{\mathbb{Q}_p} L \rightarrow \mathcal{O}_X(X).$$

One may naturally wonder if working in the previous generality really has some interest. A motivating example appears in Chenevier's thesis [18], where he constructs an eigenvariety X for a unitary group compact at ∞ and split at p associated to a division algebra over a quadratic extension E of \mathbb{Q} , and a pseudo-representation $t : \text{Gal}_E \rightarrow \mathcal{O}_X(X)$, which at points corresponding to classical automorphic forms interpolates the traces of corresponding Galois representations. The locus X_{irr} , where t is absolutely irreducible, is an open rigid subvariety of X and Chenevier shows that over this locus, t gives rise to a representation $\rho : \text{Gal}_E \rightarrow \mathcal{A}^*$, where \mathcal{A} is an Azumaya algebra over X_{irr} , see [18, Thm. E]. A similar type of example arising in the deformation theory of pseudo-representations appears in [19, §4.2]. Although, in these cases $\widehat{G} = {}^L G_f = \text{GL}_n$, the Galois representation takes values not in $\text{GL}_n(\mathcal{O}_X(X_{\text{irr}}))$, but in an inner form of it. We would like our setup to cover this example and also the situations where the action of Gal_F on \widehat{G} is non-trivial.

All previous constructions can be adapted if we work with the C -group ${}^C G$ of G instead of the L -group. Recall that ${}^C G$ is related to the L -group by an exact sequence

$$(5) \quad 1 \rightarrow {}^L G \rightarrow {}^C G \xrightarrow{d} \mathbb{G}_m \rightarrow 1$$

and that ${}^C G$ is the L -group of a canonical central extension G^T of G by \mathbb{G}_m over F . In particular, one can define (see remark 2.12) a twisted C -group ${}^C P^{\text{ad}}(X)$ attached to a \widehat{G}_X -torsor P , as well as a notion of admissibility for a continuous representation $\rho : \text{Gal}_F \rightarrow {}^C P^{\text{ad}}(X)$. And to each such admissible representation ρ we can associate (see definition 4.23) a character

$$\zeta_\rho^C : Z(\text{Res}_{F/\mathbb{Q}_p} \mathfrak{g}) \otimes_{\mathbb{Q}_p} L \rightarrow \mathcal{O}_X(X).$$

If we consider ${}^C G$ as the L -group of G^T then ζ_ρ and ζ_ρ^C are related by a twisting construction: taking square roots at the Lie algebra level is just dividing by 2 and we can always do that in characteristic zero. We will only consider L -groups in the introduction, to avoid too many technicalities, but we emphasize⁶ that it is better to consider representations valued in C -groups instead of L -groups. The first issue is that one expects that the Galois representations attached to C -algebraic automorphic forms take values in the C -group and not the L -group, see [12, Conj. 5.3.4]. Secondly, if we take $L = \mathbb{Q}_p$, $G = \text{GL}_2$, so that ${}^L G_f = \text{GL}_2$, and

$$\rho : \text{Gal}_{\mathbb{Q}_p} \rightarrow \text{GL}_2(\mathbb{Q}_p), \quad g \mapsto \begin{pmatrix} \chi_{\text{cyc}}(g)^a & 0 \\ 0 & \chi_{\text{cyc}}(g)^b \end{pmatrix},$$

⁵One could allow other Grothendieck topologies as well.

⁶We thank Peter Scholze for pointing out this.

where $a > b$ are integers, then the Sen operator is the matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, but the character $\zeta_\rho : Z(\mathfrak{g}) \rightarrow \mathbb{Q}_p$ is not an infinitesimal character of an algebraic representation of GL_2 . The problem is caused by the shift by the half sum of positive roots appearing in the Harish-Chandra isomorphism. If we choose a twisting element $\tilde{\delta}(t) := \begin{pmatrix} t^{n+1} & 0 \\ 0 & t^n \end{pmatrix}$ for some $n \in \mathbb{Z}$ then to ρ one may attach a Galois representation with values in ${}^C\mathrm{GL}_2(\mathbb{Q}_p)$, which we denote by ρ^C . In this case, ζ_{ρ^C} is equal to the infinitesimal character of $\mathrm{Sym}^{a-b-1} \otimes \det^{b-n}$. See section 4.7 for more details.

The construction and the study of the characters ζ_ρ and ζ_ρ^C occupies the first four chapters of the paper and is a bit subtle since one needs to pay special attention to the action of the group $\Gamma = \mathrm{Gal}_F / \mathrm{Ker}(\mu_G)$. One important result we prove is the compatibility of our construction with the Buzzard–Gee conjecture, cf. proposition 5.15. This roughly says that for C -algebraic automorphic forms π , if ρ_π is “the” Galois representation conjecturally attached to π as in conjecture 5.3.4 in [12], then one can relate the infinitesimal characters attached to the restriction of ρ_π at places above p and the infinitesimal characters of the archimedean components of π . This plays a crucial role in global applications, and it was also a major motivation for our construction of ζ_ρ and ζ_ρ^C .

We end this long paragraph by sketching the construction of ζ_ρ in the case when X is an affinoid over L , P is the trivial \widehat{G}_X -torsor and $F = \mathbb{Q}_p$. This case already covers the key difficulties. Thus we start with an admissible representation $\rho : \mathrm{Gal}_{\mathbb{Q}_p} \rightarrow {}^L G_f(A)$, where A is an L -affinoid algebra, and we want to construct a character $\zeta_\rho : Z(\mathfrak{g})_L \rightarrow A$.

Pick a finite Galois extension E/\mathbb{Q}_p splitting G and pick an embedding $\tau : E \rightarrow L$. The map ζ_ρ will be a composition of two maps, each depending on the choice of τ , but whose composition is independent of any choice

$$Z(\mathfrak{g})_L \xrightarrow{\kappa_\tau} S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \xrightarrow{\theta_\tau} A.$$

We write $S(W)$ for the symmetric algebra of an L -vector space W , thought of as the ring of polynomial functions on the dual of W .

The map κ_τ is an isomorphism, and is obtained by combining the Harish-Chandra isomorphism and the Chevalley restriction theorem. More precisely, let T be a maximal torus in G_E and let \widehat{T} be the dual torus in \widehat{G} . Let \mathfrak{t} and $\widehat{\mathfrak{t}}$ be the associated Lie algebras of T and \widehat{T} , and let W be the Weyl group of the root system of (G, T) , which is the same as the Weyl group of the dual root system. Then κ_τ is the composition

$$Z(\mathfrak{g})_L \simeq S(\widehat{\mathfrak{t}}^*)^W \simeq S(\widehat{\mathfrak{g}}^*)^{\widehat{G}},$$

the first isomorphism being obtained by base change along $\tau : E \rightarrow L$ of the (normalised) Harish-Chandra isomorphism $Z(\mathfrak{g}_E) \simeq S(\mathfrak{t})^W$, where we also use τ to identify $\mathfrak{t} \otimes_{E, \tau} L \simeq \widehat{\mathfrak{t}}^*$, and the second map being the inverse of the isomorphism $S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \simeq S(\widehat{\mathfrak{t}}^*)^W$ induced by restriction (this is Chevalley’s restriction theorem).

On the other hand, the map θ_τ is the composition

$$S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \rightarrow E \otimes A \rightarrow A,$$

the second map being simply $x \otimes a \rightarrow \tau(x)a$ and the first map being the evaluation of \widehat{G} -invariant polynomial functions at a special element

$$\Theta_{\mathrm{Sen}, \rho} \in (\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}}.$$

A priori the evaluation map lands in $\mathbb{C}_p \widehat{\otimes} A$, but using the Ax–Sen–Tate theorem and special Gal_F -equivariance properties of $\Theta_{\text{Sen},\rho}$ and of the Harish-Chandra and Chevalley isomorphisms, we show that it lands in $E \otimes A$, and moreover that $\theta_\tau \circ \kappa_\tau$ is independent of τ . The element $\Theta_{\text{Sen},\rho}$ is obtained using the Tannakian formalism and the results of Berger–Colmez [3] on Sen theory in families of Galois representations.

Remark 1.2. Let R be a complete local noetherian \mathcal{O} -algebra with residue field k . We expect that even if ρ takes values in ${}^C G_f(R)$ the character ζ_ρ will take values in R^{rig} and not in $R[\frac{1}{p}]$, in general.

For example, if $G = \text{GL}_n$ and ζ_ρ takes values in $R[\frac{1}{p}]$ then this would imply that the Sen polynomial of ρ has coefficients in $R[\frac{1}{p}]$ and this would impose restrictions on the Hodge–Tate weights of the Galois representations obtained by specialising ρ at the maximal ideals of $R[\frac{1}{p}]$.

Specializing the example further, if $p \nmid 2n$ and $\rho^{\text{univ}} : \text{Gal}_{\mathbb{Q}_p} \rightarrow \text{GL}_n(R_{\bar{\rho}})$ is the universal deformation of an absolutely irreducible representation $\bar{\rho} : \text{Gal}_{\mathbb{Q}_p} \rightarrow \text{GL}_n(k)$, which is not isomorphic to its twist by the cyclotomic character, then the density results of [36] or [46] imply that the locus in $\text{m-Spec } R_{\bar{\rho}}[\frac{1}{p}]$ corresponding to de Rham representations with some fixed Hodge–Tate weights is dense in $\text{Spec } R_{\bar{\rho}}$. Thus if $\zeta_{\rho^{\text{univ}}}$ takes values in $R_{\bar{\rho}}[\frac{1}{p}]$ then this would imply that all the specialisations of ρ^{univ} have the same Hodge–Tate weights, which would contradict [42, Thm. D].

1.5. Global input and applications. We use Theorem 1.1 to study Hecke eigenspaces in the completed cohomology. Our work is motivated by Emerton’s ICM talk [34] and his paper [32]. The theorem 1.4 below requires a number of hypotheses to be satisfied, most serious of which is the existence of Galois representations attached to automorphic forms and satisfying a weak form of local-global compatibility at p . We explain in sections 9.7 to 9.10 that these hypotheses are satisfied for modular curves, or more generally Shimura curves over totally real fields, as well as in the setting of definite unitary groups over totally real fields and of compact unitary Shimura varieties studied by Caraiani–Scholze [16].

Let G be a connected reductive group over \mathbb{Q} and let A be the maximal split torus in the centre Z of G . Choose a maximal compact subgroup K_∞ of $G(\mathbb{R})$, as well as a sufficiently small⁷ compact open subgroup $K_f = \prod_{\ell \neq p} K_\ell$ of $G(\mathbb{A}_f^p)$, where K_ℓ is a compact open subgroup of $G(\mathbb{Q}_\ell)$. Let S be a finite set of prime numbers, containing p and such that G is unramified over \mathbb{Q}_ℓ and K_ℓ is hyperspecial for $\ell \notin S$, and consider the universal spherical Hecke algebra outside of S (a polynomial ring over \mathcal{O} in infinitely many variables)

$$\mathbb{T}^{\text{univ}} = \bigotimes_{\ell \notin S} \mathcal{H}_\ell,$$

where $\mathcal{H}_\ell = \mathcal{O}[K_\ell \backslash G(\mathbb{Q}_\ell) / K_\ell]$ and the tensor product is taken over \mathcal{O} .

If K_f is a compact open subgroup of $G(\mathbb{A}_f)$ let

$$\tilde{Y}(K_f) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / Z(\mathbb{R})^\circ K_\infty^\circ K_f,$$

⁷See the discussion preceding lemma 9.4 for the precise hypotheses.

where H° denotes the neutral connected component of the Lie group H . Let

$$\tilde{H}^i = \varprojlim_s \varinjlim_{K_p} H^i(\tilde{Y}(K_f^p K_p), \mathcal{O}/\varpi^s)$$

be the associated completed cohomology groups, the inductive limit being taken over compact open subgroups K_p of $G(\mathbb{Q}_p)$.

The algebra \mathbb{T}^{univ} acts on the various \tilde{H}^i by spherical Hecke operators, and following [34] one can define a suitable completion \mathbb{T} of a quotient of \mathbb{T}^{univ} , which still acts on \tilde{H}^i for all $i \geq 0$ (see the discussion preceding lemma 9.10 for the precise definition). The algebra \mathbb{T} is profinite and has only finitely many open maximal ideals, but it is not known in this level of generality whether it is Noetherian.

Let $x : \mathbb{T}[1/p] \rightarrow \overline{\mathbb{Q}}_p$ be a continuous homomorphism of \mathcal{O} -algebras with kernel \mathfrak{m}_x , such that the image of x is a finite extension of \mathbb{Q}_p (this condition is satisfied if \mathbb{T} is Noetherian). We conjecture that for all $n \geq 0$ the $G(\mathbb{Q}_p)$ -representation

$$(\tilde{H}^n \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]^{\text{la}} \otimes_{\mathbb{T}, x} \overline{\mathbb{Q}}_p$$

has an infinitesimal character, and we give a conjectural recipe for this character. More precisely, motivated by [12, Conj. 5.3.4] we conjecture that one can associate to x an admissible Galois representation

$$\rho : \text{Gal}_{\mathbb{Q}} \longrightarrow {}^C G_f(\overline{\mathbb{Q}}_p)$$

which is unramified outside of S and such that for $\ell \notin S$ the semisimplification of $\rho(\text{Frob}_{\ell})$ matches the homomorphism $x_{\ell} : \mathcal{H}_{\ell} \rightarrow \mathbb{T} \xrightarrow{x} \overline{\mathbb{Q}}_p$ via a suitable form of Satake isomorphism for the C -group, defined in [88], see section 9.13 for the precise definitions. Of course, there is no reason for ρ to be unique, but we prove that all such representations ρ (if they exist) have the same associated (by the recipe described in the previous section) infinitesimal character ζ_{ρ}^C . Let us note that if we fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ and if we suppose that $\iota \circ x : \mathbb{T} \rightarrow \mathbb{C}$ is associated to a C -algebraic automorphic form, our conjecture on the existence of Galois representations becomes [12, Conj. 5.3.4].

Conjecture 1.3. *Let ρ be an admissible representation associated to $x : \mathbb{T}[1/p] \rightarrow \overline{\mathbb{Q}}_p$ as in Conjecture 9.31, and let n be a non-negative integer. Then $Z(\mathfrak{g})$ acts on $(\tilde{H}^n \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]^{\text{la}} \otimes_{\mathbb{T}, x} \overline{\mathbb{Q}}_p$ via ζ_{ρ}^C .*

As it stands, the conjecture seems out of reach, but we prove it in a certain number of cases, cf. theorem 1.4 below. We now explain the extra assumptions one needs to make in our theorem. First, we assume for simplicity of the exposition that $Z(\mathbb{R})/A(\mathbb{R})$ is compact, cf. theorem 9.24 for a version with a fixed central character which allows one to suppress this hypothesis (this is useful in practice, for instance in the case of Shimura curves over totally real fields). Next, we choose an open maximal ideal \mathfrak{m} and we make the crucial hypothesis that \mathfrak{m} is weakly non-Eisenstein, i.e. that there is an integer q_0 such that

$$H^i(\tilde{Y}(K_f^p K_p), \mathcal{O}/\varpi)_{\mathfrak{m}} = 0$$

for all $i \neq q_0$ and all sufficiently small compact open subgroups K_p of $G(\mathbb{Q}_p)$. This assumption is automatically satisfied if $G(\mathbb{R})/Z(\mathbb{R})$ is compact. When this quotient is not compact, the assumption is rather strong, since one expects it to be satisfied only when the rank of $G(\mathbb{R})$ is the same as the rank of $Z(\mathbb{R})K_{\infty}$, i.e. the defect of the derived group of $G(\mathbb{R})$ is 0, in which case q_0 is half the dimension of the symmetric

space $G(\mathbb{R})/Z(\mathbb{R})^\circ K_\infty^\circ$. As mentioned in [34] (see also lemma 9.16) the hypothesis on \mathfrak{m} has very strong consequences, for instance the continuous \mathcal{O} -dual of \tilde{H}^n is finite projective over $\mathcal{O}[[K_p]]$ for all sufficiently small compact open subgroups K_p of $G(\mathbb{Q}_p)$, and $\mathbb{T}_{\mathfrak{m}}$ acts faithfully on this dual.

Fix a supercuspidal type τ , i.e. a smooth absolutely irreducible representation of K_p on an L -vector space such that for any smooth irreducible non supercuspidal \bar{L} -representation π_p of $G(\mathbb{Q}_p)$ we have $\mathrm{Hom}_{K_p}(\tau, \pi_p) = 0$. If V is an irreducible algebraic representation of G over L , let $V(\tau) = V \otimes_L \tau$ and let $\mathbb{T}_{\mathfrak{m}, V(\tau)}$ be the quotient of the localization $\mathbb{T}_{\mathfrak{m}}$ acting faithfully on $\mathrm{Hom}_{K_p}(V(\tau), \tilde{H}_{\mathfrak{m}}^{q_0}[1/p])$. As in [34], one can (thanks to Franke's theorem [39]) associate to any L -algebra homomorphism $x : \mathbb{T}_{\mathfrak{m}, V(\tau)} \rightarrow \bar{\mathbb{Q}}_p$ a cuspidal automorphic representation $\pi_x = \otimes'_v \pi_{x,v}$ of $G(\mathbb{A})$ such that (among other things, cf. lemma 9.18) $\mathbb{T}_{\mathfrak{m}}$ acts on $\pi_x^{K_f^p}$ via x and $\pi_{x,\infty}$ has the same infinitesimal character as V , thus π_x is cohomological and in particular C -algebraic. Conjecture 5.3.4 of [12] asserts that there should be an admissible Galois representation $\rho_x : \mathrm{Gal}_{\mathbb{Q}} \rightarrow {}^C G(\bar{\mathbb{Q}}_p)$ attached to π_x . This representation is not necessarily unique (for instance because π_x is not necessarily unique), but one of the properties imposed on ρ_x is a relation between the Hodge–Tate cocharacter of ρ_x at places above p and the infinitesimal character of π_x at the archimedean places, as discussed in [12, Rem. 5.3.5]. In particular $\zeta_{\rho,x}^C$ is well-defined, i.e. independent of the choice of π_x or ρ_x . Our main result is (we keep the previous hypotheses):

Theorem 1.4. *We assume that the following hold:*

- (i) $\mathbb{T}_{\mathfrak{m}}$ is noetherian and \mathfrak{m} is weakly non-Eisenstein;
- (ii) there is an admissible representation $\rho : \mathrm{Gal}_{\mathbb{Q}} \rightarrow {}^C G_f(\mathbb{T}_{\mathfrak{m}}^{\mathrm{rig}})$, such that for all $V \in \mathrm{Irr}_G(L)$ and all $x : \mathbb{T}_{\mathfrak{m}, V(\tau)}[1/p] \rightarrow \bar{\mathbb{Q}}_p$, the specialisation of ρ at x matches π_x according [12, Conj. 5.3.4];
- (iii) the composition $d \circ \rho$ is equal to the p -adic cyclotomic character.

Then for all $y \in \mathrm{m}\text{-Spec } \mathbb{T}_{\mathfrak{m}}[1/p]$ the centre $Z(\mathfrak{g})$ acts on $(\tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{m}_y]^{\mathrm{la}}$ by $\zeta_{\rho,y}^C$.

We apply this theorem in the settings of modular curves, definite unitary groups over totally real fields, compact unitary Shimura varieties, studied by Caraiani–Scholze, and we apply a version of the theorem with a fixed central character in the setting of Shimura curves. We note that Lue Pan has proved a version of the theorem above in the setting of modular curves in [61, prop. 6.1.5] using the geometry of the Hodge–Tate period map and relative Sen theory (as opposed to the classical one, used in this paper).

1.6. p -adic Jacquet–Langlands and infinitesimal characters. With a view towards the p -adic Jacquet–Langlands correspondence, one particularly interesting example is that of a quaternion algebra D over \mathbb{Q} split at ∞ , and the first completed cohomology group of the tower of Shimura curves associated to D (of a fixed tame level), localised at a maximal ideal corresponding to an absolutely irreducible 2-dimensional mod p Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. If the ideal \mathfrak{m}_x in the theorem corresponds to a p -adic Galois representation which is not Hodge–Tate at p then $\zeta_{\rho,x}^C$ cannot be an infinitesimal character of an irreducible finite dimensional $U(\mathfrak{sl}_2)_L$ -module. This implies that if K is a compact open subgroup of $(D \otimes \mathbb{Q}_p)^\times$ then $(\tilde{H}_{\mathfrak{m}}^1 \otimes L)[\mathfrak{m}_x]$ does not have a finite dimensional K -invariant subquotient. By a different argument (see [67] when locally at p the residual Galois representation

is reducible and generic and [29] for the general case) we can bound the Gelfand–Kirilov dimension of this Banach space representation by 1 and, putting the two ingredients together, conclude that $(\widetilde{H}_m^1 \otimes L)[\mathfrak{m}_x]$ is of finite length as a Banach space representation of K . This is interesting because there should be a p -adic Jacquet–Langlands correspondence realised by the completed cohomology, and the result suggests that the objects on the division algebra side should be (some) admissible unitary L -Banach space representations of $(D \otimes \mathbb{Q}_p)^\times$ of finite length and of Gelfand–Kirilov dimension 1. The case when the p -adic Galois representation is Hodge–Tate at p seems quite a bit more involved, in particular we do not know how to prove that the corresponding Banach representation has finite length.

1.7. p -adic Langlands and infinitesimal characters. In the applications discussed so far we applied theorem 1.1 with $h = 0$ in part (i), since the completed cohomology is admissible. However, we may also apply it to the modules coming out of Taylor–Wiles patching as done in [15], as we will now explain. Let F be a finite extension of \mathbb{Q}_p and let $n \geq 1$ be such that $p \nmid 2n$. Consider a mod p Galois representation $\bar{\rho} : \text{Gal}_F \rightarrow \text{GL}_n(k)$, with universal framed deformation $\rho^\square : \text{Gal}_F \rightarrow \text{GL}_n(R_\rho^\square)$. Let R_∞ be the patched deformation ring, which is a flat R_ρ^\square -algebra, and let $\rho : \text{Gal}_F \rightarrow \text{GL}_n(R_\infty)$ be the Galois representation obtained from ρ^\square by extending scalars to R_∞ .

Let $K = \text{GL}_n(\mathcal{O}_F)$ and let M_∞ be the compact $R_\infty[[K]]$ -module with a compatible R_∞ -linear action of G constructed in [15] by patching automorphic forms on definite unitary groups. Let $\Pi_\infty := \text{Hom}_{\mathcal{O}}^{\text{cont}}(M_\infty, L)$. Any $y \in \mathfrak{m}\text{-Spec } R_\infty[1/p]$ gives rise to a Galois representation $\rho_x : \text{Gal}_F \rightarrow \text{GL}_n(\kappa(x))$, where x is the image of y in $\mathfrak{m}\text{-Spec } R_\rho^\square[1/p]$, as well as to an admissible unitary $\kappa(y)$ -Banach space representation $\Pi_\infty[\mathfrak{m}_y]$ of $G := \text{GL}_n(F)$. It is expected but not known beyond the case $n = 2$ and $F = \mathbb{Q}_p$, that $\Pi_\infty[\mathfrak{m}_y]$ depends only on ρ_x , more precisely that $\Pi_\infty[\mathfrak{m}_y]$ and ρ_x should be related by the hypothetical p -adic Langlands correspondence, see [15, §6]. Theorem 1.5 below shows that the infinitesimal character of $\Pi_\infty[\mathfrak{m}_y]^{\text{la}}$ depends only on ρ_x , thus adding nontrivial evidence that the expectation should be true. Using a twisting element we may associate to ρ an admissible Galois representation ρ^C with values in the C -group of GL_n , see lemma 5.12 for more details.

Theorem 1.5. *Let \mathfrak{g} be the \mathbb{Q}_p -linear Lie algebra of G . The algebra $Z(\mathfrak{g})$ acts on $\Pi_\infty[\mathfrak{m}_y]^{\text{la}}$ through the character $\zeta_{\rho_x^C}^C$.*

If $F = \mathbb{Q}_p$ and $n = 2$ then the p -adic Langlands correspondence $\rho \mapsto \Pi(\rho)$, envisioned by Breuil, has been established by local means in [23], [62], building on the monumental work of Colmez [22], and the infinitesimal character of $\Pi(\rho)^{\text{la}}$ has been calculated by one of us (G.D.) in [28] in terms of the characteristic polynomial of the Sen operator of ρ . We give a new proof of this result in section 9.12, using the patched module and recent results of Tung [85], [86].

1.8. Beyond $l_0 = 0$ case. We end this long introduction by noting that one cannot hope to prove the conjecture 1.3 by applying theorems 8.5 and 8.6 to the completed cohomology directly, since the localisation of completed homology \widetilde{H}_n at a maximal ideal of the Hecke algebra is not expected to be projective over $\mathcal{O}[[K_p]]$. However, one might hope to be able to apply our results to the patched homology groups obtained via the patching method of Calegry–Geraghty [14]. The most accessible

case, when weakly non-Eisenstein maximal ideal are not expected to exist, is when $G = \mathrm{PGL}_2$ over a quadratic imaginary field F , such that p splits completely in F , studied by Gee–Newton in [43]. It follows from [43, Prop. 5.3.1] and its proof that under the assumptions made there the patched homology satisfies the conditions of Theorem 8.5. We do not pursue the matter here, since in this specific setting instead of applying Theorem 8.5 it might be easier to use local–global compatibility at p and appeal to the results on the infinitesimal character in the p -adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, see Theorem 9.29, but hope to come back to these questions in future work.

1.9. Overview of sections. We will review the content of each section.

In section 2 we review L -groups, C -groups, admissible Galois representations and introduce their variants twisted by a torsor in subsection 2.3.

In section 3 we review Sen theory for Galois representations valued in GL_n of a Banach algebra.

In section 4 we define the characters ζ_ρ and ζ_ρ^C . The main difficulty in this section is to come up with the right definition, when G is not split over F . At first reading one should ignore that and assume that G is split over F and the torsor P is trivial.

In section 5 we relate ζ_ρ and ζ_ρ^C to the Hodge–Tate cocharacter, when ρ is Hodge–Tate. We also explain how to attach algebraic representations to ρ , when the Hodge–Tate weights are regular in a setting, when G is (possibly) not split over F . In subsection 5.4 we check that if ρ is a Galois representation associated to a cohomological automorphic representation π then ζ_ρ^C is essentially the infinitesimal character of the archimedean component of π .

In section 6 we prove a commutative algebra result, which is used in section 8.

In section 7 we prove a density theorem, which is used in section 8.

In section 8 we prove the existence of infinitesimal character in an abstract setting, modelled on the representations that appear in completed cohomology, when the mod p Hecke eigenvalues contribute to only one cohomological degree, and its patched version.

In section 9 we first review completed cohomology following [34]. However, we do not make the assumption that the maximal \mathbb{Q} -split torus and the maximal \mathbb{R} -split torus in the centre of G coincide, since we want to handle Shimura curves. The main result is proved in subsection 9.6 by showing that we may apply the results of section 8 if certain conditions are satisfied. We then check that these conditions are satisfied in the settings of modular curves, Shimura curves, definite unitary groups, compact unitary Shimura varieties by appealing to the results already available in the literature. In subsection 9.11 we show that we may apply the results of section 8 in the setting of the patched module defined in [15]. In subsection 9.12 we reprove results on the infinitesimal character in the p -adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. In subsection 9.13 we formulate a general conjecture regarding the action of the center of the universal enveloping algebra on the locally analytic vectors in the Hecke eigenspaces in completed cohomology.

1.10. Acknowledgements. We would like to acknowledge the influence of the ideas of Michael Harris, regarding the connections between the representation theory of real Lie groups and the p -adic representation theory of p -adic groups. In particular, the question about the infinitesimal character was raised by him in 2003

in Luminy, where VP was a participant. We would like to thank Ana Caraiani, Matthew Emerton, Toby Gee, Florian Herzig, David Loeffler and Timo Richarz for the stimulating correspondence regarding various aspects of this paper. VP would like to thank especially his colleague Jochen Heinloth for several illuminating discussions regarding geometric group theory. Parts of the paper were written, when VP visited the Morning Side Center for Mathematics in Beijing in March 2019. VP would like to thank Yongquan Hu for the invitation and stimulating discussions and the Morning Side Center for Mathematics for providing excellent working conditions. GD and BS would like to thank Christophe Breuil, Laurent Clozel, Pierre Colmez and Olivier Taïbi for useful discussions on the topics of this paper. BS would like to acknowledge the support of the A.N.R. through the project CLap-CLap ANR-18-CE40-0026. We would like to thank Toby Gee and James Newton for their comments on the earlier version of the paper.

2. THE L -GROUP

We fix algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p and a finite extension F of \mathbb{Q}_p contained in $\overline{\mathbb{Q}_p}$. Let G be a connected reductive group defined over F . Let $E \subset \overline{\mathbb{Q}_p}$ be a finite Galois extension of F such that G splits over E . We follow [4, §I.2], [12, §2.1] and [88] to define the dual group \widehat{G} , the L -group ${}^L G$ and the C -group ${}^C G$. We discuss admissible representations of $\text{Gal}_F := \text{Gal}(\overline{\mathbb{Q}_p}/F)$ with values in these groups. In the last subsection we consider versions of these constructions twisted along a torsor.

We fix a maximal split torus T of G defined over E . Let $X = X^*(T)$ be the group of characters of T , Φ the set of roots, $X^\vee = X_*(T)$ the group of cocharacters of T and Φ^\vee the set of coroots. We denote the natural pairing $X \times X^\vee \rightarrow \mathbb{Z}$ by $\langle \cdot, \cdot \rangle$. The 4-tuple $R(G, T) = (X, \Phi, X^\vee, \Phi^\vee)$ together with the pairing $\langle \cdot, \cdot \rangle$ is a reduced *root datum* in the sense of [24, Def. 1.3.3] by [24, Prop. 5.1.6].

We further fix a Borel subgroup B of G defined over E and containing T . To the triple (G_E, T, B) one may attach a *based root datum*, which is a 6-tuple $(X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$, where Δ is the set of positive simple roots corresponding to B , Δ^\vee is the set of coroots a^\vee for $a \in \Delta$, see [24, §1.5].

Let $\text{Aut}(G)$ be the group of automorphisms of G as an algebraic group over E . Then $\text{Aut}(G)$ acts on the set of such pairs (B, T) as above. If $\phi \in \text{Aut}(G)$ then the pairs $(\phi(B), \phi(T))$ and (B, T) are conjugate by an element $g_\phi \in G(E)$, which is uniquely determined by ϕ up to an element of $T(E)$, see [24, Prop. 6.2.11 (2)] and its proof. This induces a group homomorphism $\text{Aut}(G) \rightarrow \text{Aut}(R(G, T), \Delta)$, defined by $\phi \mapsto \text{Ad}(g_\phi) \circ \phi$, where $\text{Aut}(R(G, T), \Delta)$ is the subgroup of the group of automorphisms of the root data $R(G, T)$, consisting of those automorphisms, which map Δ to itself. It follows from [24, Prop. 7.1.6] that this induces an exact sequence:

$$(6) \quad 1 \rightarrow (G/Z_G)(E) \rightarrow \text{Aut}(G) \rightarrow \text{Aut}(R(G, T), \Delta) \rightarrow 1,$$

where Z_G is the centre of G and the first non-trivial arrow is given by conjugation. Let $\text{Aut}(G, T, B)$ be the set of $\phi \in \text{Aut}(G)$ such that $\phi(T) = T$ and $\phi(B) = B$. If $\phi \in \text{Aut}(G, T, B)$ then $\phi(a) := a \circ \phi^{-1} \in \Delta$ for all $a \in \Delta$ and ϕ induces an isomorphism $\phi : U_a \rightarrow U_{\phi(a)}$ for all $a \in \Delta$, where U_a is the root subgroup corresponding to $a \in \Delta$. For each $a \in \Delta$ we fix an isomorphism $p_a : \mathbb{G}_a \xrightarrow{\cong} U_a$ over L . The data $\{p_a\}_{a \in \Delta}$ is called a *pinning* of (G, T, B) . The group $(T/Z_G)(E)$ acts

simply transitively on the set of pinning of (G, T, B) . Let $\text{Aut}(G, T, B, \{p_a\}_{a \in \Delta})$ be the set of $\phi \in \text{Aut}(G, T, B)$ such that $p_{\phi(a)} = \phi \circ p_a$ for all $a \in \Delta$. It is shown in [24, Prop. 7.1.6] that (6) induces an isomorphism $\text{Aut}(G, T, B, \{p_a\}_{a \in \Delta}) \xrightarrow{\cong} \text{Aut}(R(G, T), \Delta)$. Thus if we fix a pinning then there is a natural isomorphism:

$$(7) \quad \text{Aut}(G) \cong (G/Z_G)(E) \rtimes \text{Aut}(R(G, T), \Delta).$$

By [24, Thm. 6.1.17] there is a unique (up to unique \mathbb{Z} -isomorphism) pinned split reductive group $(\widehat{G}, \widehat{T}, \widehat{B}, \{p'_{a^\vee}\}_{a^\vee \in \Delta^\vee})$ over \mathbb{Z} such that its based root datum $(R(\widehat{G}, \widehat{T}), \Delta^\vee)$ is isomorphic to $(X^\vee, \Phi^\vee, \Delta^\vee, X, \Phi, \Delta)$, which is the based root datum dual to $(R(G, T), \Delta)$. We will refer to \widehat{G} as the dual group.

We will now give a linear algebra definition of the action

$$(8) \quad \mu_G : \text{Gal}_F \rightarrow \text{Aut}(R(G, T), \Delta),$$

defined in [4, §I.1.3]. Let \mathfrak{g} be the Lie algebra of G as an algebraic group over F . Then $\mathfrak{g}_E := \mathfrak{g} \otimes_F E$ is the Lie algebra of G_E . For $\gamma \in \text{Gal}_F$ let $\gamma : \mathfrak{g}_E \rightarrow \mathfrak{g}_E$ be the map $x \otimes \lambda \mapsto x \otimes \gamma(\lambda)$ for $x \in \mathfrak{g}$ and $\lambda \in E$. Let $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}_E$ be the Lie algebras of T and B respectively and let Ad denote the action of G on \mathfrak{g} by conjugation. Then $\gamma(\mathfrak{t})$ is a Cartan subalgebra of \mathfrak{g}_E and $\gamma(\mathfrak{b})$ is a Borel subalgebra of \mathfrak{g}_E . Thus there is $g \in (G/Z_G)(E)$ such that $\text{Ad}(g)(\gamma(\mathfrak{t})) = \mathfrak{t}$ and $\text{Ad}(g)(\gamma(\mathfrak{b})) = \mathfrak{b}$. The map $\text{Ad}(g) \circ \gamma$ maps the set of \mathfrak{t} -eigenspaces for the adjoint action on \mathfrak{b} to itself, thus there is a unique $\sigma \in \text{Aut}(R(G, T), \Delta)$ such that $\text{Ad}(g)(\gamma(\mathfrak{g}_{E, \alpha})) = \mathfrak{g}_{E, \sigma(\alpha)}$ for all $\alpha \in \Delta$. To show that the map $\gamma \mapsto \sigma$ is a group homomorphism it is enough to check that σ does not depend on the choice of g . This is the case, because any two choices differ by an element of $T(E)$, which does not change the eigenspaces. It follows from the construction that (8) factors through $\text{Gal}(E/F)$. By using the splitting in (7) we obtain a group homomorphism $\text{Gal}_F \rightarrow \text{Aut}(G)$, which induces an E -linear action of Gal_F on \mathfrak{g}_E .

The automorphism groups of $(R(G, T), \Delta)$ and $(R(\widehat{G}, \widehat{T}), \Delta^\vee)$ are canonically isomorphic. Using the pinning and [24, Thm. 7.1.9 (3)] we obtain a group homomorphism $\text{Gal}_F \rightarrow \text{Aut}(\widehat{G})$. We note that the resulting action of Gal_F on \widehat{G} is defined over \mathbb{Z} . The $\widehat{G}/Z_{\widehat{G}}$ -conjugacy class of this homomorphism is canonical and depends only on G . We define the L -group of G as a semidirect product:

$${}^L G := \widehat{G} \rtimes \text{Gal}_F.$$

In particular, ${}^L G$ is a split reductive group over \mathbb{Z} with identity component \widehat{G} and the component group Gal_F . The map $g \mapsto (1, g)$ identifies $\text{Ker } \mu_G$ with a normal subgroup ${}^L G_f$. We let ${}^L G_f := {}^L G / \text{Ker } \mu_G$.

2.1. C -groups. We follow the exposition in [88]. Let δ be a half sum of positive roots, thus $2\delta \in X^*(T) = X_*(\widehat{T})$. Since $\widehat{G}/Z_{\widehat{G}}$ is of adjoint type there is a unique $\delta_{\text{ad}} \in X_*(\widehat{T}/Z_{\widehat{G}})$, such that the image of 2δ in $X_*(\widehat{T}/Z_{\widehat{G}})$ is equal to $2\delta_{\text{ad}}$. We define an action of \mathbb{G}_m on \widehat{G} by

$$\text{Ad } \delta_{\text{ad}} : \mathbb{G}_m \xrightarrow{\delta_{\text{ad}}} \widehat{T}/Z_{\widehat{G}} \xrightarrow{\text{Ad}} \text{Aut}(\widehat{G}).$$

Since δ_{ad} is Gal_F -invariant, the action of \mathbb{G}_m commutes with the action of Gal_F . We let

$$\widehat{G}^T := \widehat{G} \rtimes \mathbb{G}_m, \quad {}^C G := \widehat{G}^T \rtimes \text{Gal}_F \cong \widehat{G} \rtimes (\mathbb{G}_m \times \text{Gal}_F).$$

Let ${}^C G_f := {}^C G / \text{Ker } \mu_G$ and let $d : {}^C G \rightarrow \mathbb{G}_m$ denote the projection map. Then we have an exact sequence of group schemes over \mathbb{Z} :

$$(9) \quad 1 \rightarrow {}^L G \rightarrow {}^C G \xrightarrow{d} \mathbb{G}_m \rightarrow 1$$

and a similar sequence with ${}^L G_f$ and ${}^C G_f$.

Following Remark 1 in [88], we may regard \widehat{G}^T as the dual group of a reductive group G^T , which is a central extension of G by \mathbb{G}_m over F , and then regard ${}^C G = {}^L G^T$ as the usual Langlands dual group of G^T . This is the definition given in [12, Def. 5.3.2].

2.2. Admissible representations. Let N be a topological group and let $\text{Aut}(N)$ be the group of continuous automorphisms of N , which we equip with discrete topology. Let $\theta : \text{Gal}_F \rightarrow \text{Aut}(N)$ be a continuous group homomorphism. We may then form the semi-direct product $N \rtimes \text{Gal}_F$ and let $\pi : N \rtimes \text{Gal}_F \rightarrow \text{Gal}_F$ denote the projection.

Definition 2.1. *A continuous representation $\rho : \text{Gal}_F \rightarrow N \rtimes \text{Gal}_F$ is admissible if $\pi \circ \rho$ induces the identity on Gal_F . Two admissible representations ρ, ρ' are equivalent if there is $n \in N$ such that $\rho'(\gamma) = n\rho(\gamma)n^{-1}$, for all $\gamma \in \text{Gal}_F$.*

Let $Z_{\text{cont}}^1(\text{Gal}_F, N)$ be the set of continuous functions $c : \text{Gal}_F \rightarrow N$, $\gamma \mapsto c_\gamma$ satisfying the cocycle condition $c_{\gamma\beta} = c_\gamma(\gamma \cdot c_\beta)$, where \cdot denotes the action of Gal_F on N . Two cocycles c and c' are cohomologous if there is $n \in N$ such that $c'_\gamma = nc_\gamma(\gamma \cdot n^{-1})$ for all $\gamma \in \text{Gal}_F$. This relation is an equivalence relation and we denote the set of equivalence classes by $H_{\text{cont}}^1(\text{Gal}_F, N)$.

Lemma 2.2. *The rule $\rho(\gamma) = (c_\gamma, \gamma)$ defines a bijection between the set of admissible representations $\rho : \text{Gal}_F \rightarrow N \rtimes \text{Gal}_F$ and $Z_{\text{cont}}^1(\text{Gal}_F, N)$, which induces a bijection between the respective equivalence classes.*

Proof. Let $\rho : \text{Gal}_F \rightarrow N \rtimes \text{Gal}_F$ be an admissible representation. Then we may write $\rho(\gamma) = (c_\gamma, \gamma)$ with $c_\gamma \in N$ for all $\gamma \in \text{Gal}_F$. The condition $\rho(\gamma\beta) = \rho(\gamma)\rho(\beta)$ translates into the cocycle condition. Conversely, if $c \in Z_{\text{cont}}^1(\text{Gal}_F, N)$ then $\rho(\gamma) := (c_\gamma, \gamma)$ defines an admissible representation $\text{Gal}_F \rightarrow N \rtimes \text{Gal}_F$. If $n \in N$ then $(n, 1)(c_\gamma, \gamma)(1, n)^{-1} = (nc_\gamma(\gamma \cdot n^{-1}), g)$. Thus two admissible representations are equivalent if and only if the corresponding cocycles are cohomologous. \square

If A is a topological algebra then the topology on A induces a topology on $\widehat{G}(A)$ and ${}^L G(A) = \widehat{G}(A) \rtimes \text{Gal}_F$, where the topology on Gal_F is the Krull topology. Thus we may apply the above discussion to $N = \widehat{G}(A)$. The description of admissible representations in terms of cocycles shows that admissible representations with values in ${}^L G_f(A)$ coincide with admissible representations with values in ${}^L G(A)$. We will exploit that ${}^L G_f$ is of finite type over \mathbb{Z} . The same discussion applies to ${}^C G(A)$ and ${}^C G_f(A)$.

2.3. Twisted families of L - and C -parameters. We want to allow more general families of Galois representations than considered in the previous section. A motivating example appears in Chenevier's thesis [18], where he constructs an eigenvariety X for a unitary group compact at ∞ and split at p associated to a division algebra over a quadratic extension E of \mathbb{Q} , and a pseudo-representation $t : \text{Gal}_E \rightarrow \mathcal{O}_X(X)$, which at points corresponding to classical automorphic forms

interpolates the traces of corresponding Galois representations. The locus X_{irr} , where t is absolutely irreducible, is an open rigid subvariety of X and Chenevier shows that over this locus, t gives rise to a representation $\rho : \text{Gal}_E \rightarrow \mathcal{A}^*$, where \mathcal{A} is an Azumaya algebra over X_{irr} , see [18, Thm. E]. A similar type of example arising in the deformation theory of pseudo-representations appears in [19, §4.2]. Although, in these cases $\widehat{G} = {}^L G_f = \text{GL}_n$, the Galois representation takes values not in $\text{GL}_n(\mathcal{O}_X(X_{\text{irr}}))$, but in an inner form of it. We would like our setup to cover this example and also the situations where the action of Gal_F on \widehat{G} is non-trivial. In this more general setting it is not obvious what *admissibility* for a Galois representation should mean.

We keep the discussion deliberately very general, since it applies in different contexts. Let X be a space with a sheaf of topological rings \mathcal{O}_X for some Grothendieck topology on X . (In the application, "space" will mean a rigid analytic space over \mathbb{Q}_p , but it could also be, for example, schemes over \mathbb{F}_ℓ , rigid spaces over \mathbb{Q}_ℓ , or differentiable manifolds.) Let H be a group over X and let P be an H -torsor, by which we mean a space $P \rightarrow X$, together with the left H -action $H \times_X P \rightarrow P$, such that the map

$$H \times_X P \rightarrow P \times_X P, \quad (g, p) \mapsto (gp, p)$$

is an isomorphism. We require in addition that there is a family of local sections $s_i : U_i \rightarrow P$ for some open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X . We note that this implies that the map $H \times U_i \rightarrow P \times_X U_i$ is an isomorphism, so that the restriction of P to U_i is a trivial H -torsor. Following [10], we let P^{ad} be the sheaf of H -equivariant automorphisms of P , which leave the base X fixed.

Recall that ${}^L G_f = \widehat{G} \rtimes \Gamma$, where $\Gamma = \text{Gal}_F / \text{Ker } \mu_G$. Since \widehat{G} and the action $\mu_G : \text{Gal}_F \rightarrow \text{Aut}(\widehat{G})$ are defined over \mathbb{Z} , we may view \widehat{G} , ${}^L G_f$, Γ as groups over X , so that $\widehat{G}_X(U) = \widehat{G}(\mathcal{O}_X(U))$. We have an isomorphism of groups over X :

$$(10) \quad {}^L G_{f,X} \cong \widehat{G}_X \rtimes \Gamma_X.$$

Let P be a \widehat{G} -torsor and let $P_f := {}^L G_{f,X} \times^{\widehat{G}_X} P$. It follows from (10) that the quotient $\widehat{G}_X \backslash P_f$ is a Γ_X -torsor on X together with a global trivialisation $t : \widehat{G}_X \backslash P_f \cong \Gamma_X$.

Lemma 2.3. *We have an exact sequence of sheaves of groups on X :*

$$1 \rightarrow P^{\text{ad}} \rightarrow P_f^{\text{ad}} \rightarrow (\widehat{G}_X \backslash P_f)^{\text{ad}} \rightarrow 1,$$

where we consider P as a \widehat{G}_X -torsor, P_f as an ${}^L G_{f,X}$ -torsor, $\widehat{G}_X \backslash P_f$ as a Γ_X -torsor.

Proof. If P is a trivial torsor then the assertion follows from (10). In general, one may reduce to this case by choosing a covering which trivialises P . \square

By taking global sections we get an exact sequence of pointed sets

$$(11) \quad 1 \rightarrow P^{\text{ad}}(X) \rightarrow P_f^{\text{ad}}(X) \xrightarrow{\pi} \Gamma_X(X) \rightarrow H^1(X, P^{\text{ad}}),$$

where we used the trivialisation t to identify $\Gamma_X(X)$ with $(\widehat{G}_X \backslash P_f)^{\text{ad}}(X)$. There is an injective group homomorphism $\Gamma \hookrightarrow \Gamma_X(X)$, $\gamma \mapsto r_\gamma$, where $r_\gamma((\beta, x)) = (\beta\gamma^{-1}, x)$.

Definition 2.4. Let P be a \widehat{G}_X -torsor over X . We define the L -group of G twisted by P to be

$${}^L P^{\text{ad}}(X) := \{\varphi \in P_f^{\text{ad}}(X) : \pi(\varphi) \in \Gamma \subset \Gamma_X(X)\},$$

where π is the projection in (11), equipped with the topology as explained below.

We have an exact sequence of groups

$$(12) \quad 1 \rightarrow P^{\text{ad}}(X) \rightarrow {}^L P^{\text{ad}}(X) \xrightarrow{\pi} \Gamma.$$

Since Γ is finite it is enough to define a topology on $P^{\text{ad}}(X)$. If P is a trivial \widehat{G}_X -torsor over X then we topologize $P^{\text{ad}}(X)$ by identifying it with $\widehat{G}(\mathcal{O}_X(X))$. Any two such identifications differ by a conjugation by an element of $\widehat{G}(\mathcal{O}_X(X))$, hence the topology on $P^{\text{ad}}(X)$ does not depend on a choice of section. In general, we choose a covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X trivialising P , topologize $P^{\text{ad}}(U_i)$ as above and use the sheaf property to put a subspace topology on $P^{\text{ad}}(X)$. In Remark 4.5 we show that if X is a rigid analytic space over \mathbb{Q}_p and P is a torsor on X for the étale or the analytic topology then the topology on $P^{\text{ad}}(X)$ does not depend on a choice of a covering.

Example 2.5. If $P = \widehat{G}_X$ then ${}^L P^{\text{ad}}(X) = {}^L G_f(\mathcal{O}_X(X)) = \widehat{G}(\mathcal{O}_X(X)) \rtimes \Gamma$.

Remark 2.6. One may also describe ${}^L P^{\text{ad}}(X)$ as the set of pairs $(\varphi, \gamma) \in P_f^{\text{ad}}(X) \times \Gamma$, such that the diagram

$$\begin{array}{ccc} P_f & \longrightarrow & \widehat{G}_X \backslash P_f \\ \varphi \downarrow & & \downarrow r_\gamma \\ P_f & \longrightarrow & \widehat{G}_X \backslash P_f \end{array}$$

commutes.

Definition 2.7. Let P be \widehat{G}_X -torsor over X . A continuous representation

$$\rho : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(X)$$

is admissible, if $\pi(\rho(\gamma)) = \gamma(\text{Ker } \mu_G)$ for all $\gamma \in \text{Gal}_F$.

Two admissible representations ρ and ρ' are equivalent if there is $\varphi \in P^{\text{ad}}(X)$ such that $\rho'(\gamma) = \varphi \circ \rho(\gamma) \circ \varphi^{-1}$ for all $\gamma \in \Gamma$.

Remark 2.8. If $P = \widehat{G}_X$ then we recover the definition of admissible representations and their equivalence classes for Galois representations valued in ${}^L G_f(\mathcal{O}_X(X))$.

Remark 2.9. For a general P the map $\pi : {}^L P^{\text{ad}}(X) \rightarrow \Gamma$ might not be surjective; the obstruction will lie in $H^1(X, P^{\text{ad}})$. In this case, there will be no admissible representations of Gal_F with values in ${}^L P^{\text{ad}}(X)$. However, we expect that as the theory develops examples of admissible representations with values in ${}^L P^{\text{ad}}(X)$ will arise in the context of eigenvarieties, similar to [18, Thm. E].

Definition 2.10. If $Y \rightarrow X$ is a map of spaces and $\rho : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(X)$ is an admissible representation then we define an admissible representation

$$\rho_Y : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(Y)$$

by composing ρ with the natural map $P_f^{\text{ad}}(X) \rightarrow (P_f \times_X Y)^{\text{ad}}(Y)$.

Lemma 2.11. *Let $\mathcal{U} = \{U_i \rightarrow X\}_{i \in I}$ be a cover of X together with local sections $\{s_i : U_i \rightarrow P\}_{i \in I}$. Let $g_{ij} \in \widehat{G}(\mathcal{O}_X(U_{ij}))$ be such that $s_i = g_{ij}s_j$ for all $i, j \in I$. Then to give an admissible representation $\rho : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(X)$ is equivalent to giving a family of admissible representations*

$$\rho_{U_i} : \text{Gal}_F \rightarrow {}^L G(\mathcal{O}_X(U_i)), \quad \forall i \in I,$$

such that

$$(13) \quad \rho_{U_i}(\gamma) = g_{ij} \rho_{U_j}(\gamma) g_{ij}^{-1}, \quad \forall \gamma \in \text{Gal}_F, \quad \forall i, j \in I,$$

where the equality takes place in ${}^L G(\mathcal{O}_X(U_{ij}))$.

Proof. We note that the cover \mathcal{U} also trivialises P_f and the gluing data for P_f is also given by $\{g_{ij}\}_{i, j \in I}$. The assertion follows from the description of $P_f^{\text{ad}}(X)$ in terms of the gluing data in [10, §1, (1.1.4)]. Let us just indicate how to obtain ρ_{U_i} from ρ and vice versa, leaving the rest of the details for the interested reader.

If $u_i \in P_f^{\text{ad}}(U_i)$ then there is a unique $h_i \in {}^L G_f(\mathcal{O}_X(U_i))$ such that $u_i(s_i) = h_i^{-1} s_i$. The map $P_f^{\text{ad}}(U_i) \rightarrow {}^L G_f(\mathcal{O}_X(U_i))$, $u_i \mapsto h_i$ is a group isomorphism, which is in fact a homeomorphism (by construction). We let ρ_{U_i} be the composition

$$\rho_{U_i} : \text{Gal}_F \xrightarrow{\rho} {}^L P^{\text{ad}}(X) \subset P_f^{\text{ad}}(X) \rightarrow P_f^{\text{ad}}(U_i) \xrightarrow{\cong} {}^L G_f(\mathcal{O}_X(U_i)).$$

Conversely, if we start with the family $\{\rho_{U_i}\}_{i \in I}$ then for each $\gamma \in \text{Gal}_F$, (13) implies that $\{\rho_{U_i}(\gamma)\}_{i \in I}$ glue to $\rho(\gamma) \in P_f^{\text{ad}}(X)$. Since the gluing data for P_f is given by elements of $\widehat{G}(\mathcal{O}_X(U_{ij}))$, we get that $\rho(\gamma)$ lies in ${}^L P^{\text{ad}}(X)$. The admissibility of ρ_{U_i} then implies that $\rho : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(X)$, $\gamma \mapsto \rho(\gamma)$ is admissible. \square

Remark 2.12. We make the analogous definitions for C -groups as follows. If P is a \widehat{G}_X -torsor on X then $P^T := \widehat{G}_X^T \times^{\widehat{G}_X} P$ is \widehat{G}_X^T -torsor on X . By interpreting the C -group ${}^C G_f$ as an L -group ${}^L G_f^T$ and using P^T instead of P we may make the same definitions for C -groups, so that ${}^C P^{\text{ad}}(X) = {}^L P^{T, \text{ad}}(X)$, and an admissible representation $\rho : \text{Gal}_F \rightarrow {}^C P^{\text{ad}}(X)$ is just a representation $\rho : \text{Gal}_F \rightarrow {}^L P^{T, \text{ad}}(X)$, which is admissible in the sense of Definition 2.7. Moreover, the exact sequence (9) induces an exact sequence of groups

$$(14) \quad 1 \rightarrow {}^L P^{\text{ad}}(X) \rightarrow {}^C P^{\text{ad}}(X) \xrightarrow{d} \mathcal{O}_X(X)^*.$$

3. SEN THEORY

Let F be a finite extension of \mathbb{Q}_p and let $F_\infty = F(\mu_{p^\infty}) = \cup_n F_n$, where $F_n = F(\mu_{p^n})$. Let $H_F = \text{Gal}(\bar{F}/F_\infty)$ and $\Gamma_F = \text{Gal}(F_\infty/F) \simeq \text{Gal}_F/H_F$.

Let A be a \mathbb{Q}_p -Banach algebra and let V be a finite free A -module with a continuous A -linear action of Gal_F (one could as well assume that V is locally free over A , but in applications V will actually be free). Define

$$\widetilde{D}_{\text{Sen}}(V) := (\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} V)^{H_F}$$

and define $D_{\text{Sen}}(V)$ as the set of Γ_F -finite vectors in $\widetilde{D}_{\text{Sen}}(V)$, i.e. those $v \in \widetilde{D}_{\text{Sen}}(V)$ for which $A[\Gamma_F]v$ is a finitely generated A -module. Observe that $\widetilde{D}_{\text{Sen}}(V)$ is an $\widehat{F}_\infty \widehat{\otimes}_{\mathbb{Q}_p} A$ -module and that $D_{\text{Sen}}(V)$ is an $F_\infty \otimes_{\mathbb{Q}_p} A$ -submodule of $\widetilde{D}_{\text{Sen}}(V)$. The following result is classical when $A = L$, but somewhat curiously it is not easy to find it in the literature in the form below:

Theorem 3.1. *With the above notations the $F_\infty \otimes_{\mathbb{Q}_p} A$ -module $D_{\text{Sen}}(V)$ is free and the natural map*

$$(15) \quad (\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} A) \otimes_{F_\infty \otimes_{\mathbb{Q}_p} A} D_{\text{Sen}}(V) \rightarrow \mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} V$$

is an isomorphism. Moreover, for all $x \in D_{\text{Sen}}(V)$ the limit

$$(16) \quad \theta(x) = \lim_{\gamma \rightarrow 1} \frac{\gamma \cdot x - x}{\chi_{\text{cyc}}(\gamma) - 1}$$

exists in $D_{\text{Sen}}(V)$ and $\theta \in \text{End}_{F_\infty \otimes_{\mathbb{Q}_p} A}(D_{\text{Sen}}(V))$.

Proof. We will deduce this by descent from the results of Berger–Colmez [3, Prop. 4.1.2]. For this we need the following lemma:

Lemma 3.2. *If E is a finite Galois extension of F and V_E is the restriction of V to Gal_E , then the natural map $(E_\infty \otimes_{\mathbb{Q}_p} A) \otimes_{F_\infty \otimes_{\mathbb{Q}_p} A} D_{\text{Sen}}(V) \rightarrow D_{\text{Sen}}(V_E)$ is an isomorphism.*

Proof. Consider the finite group $G = \text{Gal}(E_\infty/F_\infty)$. We clearly have $\widetilde{D}_{\text{Sen}}(V) = \widetilde{D}_{\text{Sen}}(V_E)^G$. We claim that a vector $v \in \widetilde{D}_{\text{Sen}}(V)$ is Γ_F -finite if and only if it is Γ_E -finite, or equivalently v is a G -invariant Γ_E -finite vector of $\widetilde{D}_{\text{Sen}}(V_E)$. In other words $D_{\text{Sen}}(V) \simeq D_{\text{Sen}}(V_E)^G$. Thus given the claim, the result follows then by Galois descent for the finite Galois extension E_∞/F_∞ .

To prove the claim, we observe that one direction is trivial and the other follows from Cayley–Hamilton. Namely, let e_1, \dots, e_n be some generators of $A[\Gamma_F]v$ as an A -module, let γ be a topological generator of Γ_F and $M = (a_{ij})$ be a matrix such that $\gamma e_i = \sum_{j=1}^n a_{ij} e_j$ for $1 \leq i \leq n$. It follows from Cayley–Hamilton that, for all $k \geq 0$, M^{mk} is an A -linear combination of M^{mi} for $0 \leq i \leq n-1$. If γ^m is a topological generator of Γ_E , we deduce that $\gamma^{mi} v$ for $0 \leq i \leq n-1$ generate $A[\Gamma_E]v$ as an A -module. \square

Coming back to the proof of the Theorem, let A^+ be the unit ball in A . By continuity, there is a finite Galois extension E of F and a free A^+ -module $T \subset V$ such that $A \otimes_{A^+} T \simeq V$, T is stable under Gal_E and Gal_E acts trivially on $T/(12pT)$. By a result of Berger–Colmez [3, Prop. 4.1.2], for n sufficiently large we can find a free $E_n \otimes_{\mathbb{Q}_p} A$ -module $D_n \subset (\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} A) \otimes_A V$ such that

- D_n is stable under Gal_E , fixed by H_E and has a basis \mathcal{B} over $E_n \otimes_{\mathbb{Q}_p} A$ in which the matrix M_γ of each $\gamma \in \Gamma_E$ satisfies $\|M_\gamma - 1\| < 1$, i.e. the entries of $M_\gamma - 1$ have positive valuations.
- the natural map $(\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} A) \otimes_{E_n \otimes_{\mathbb{Q}_p} A} D_n \rightarrow (\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} A) \otimes_A V$ is an isomorphism.

We claim that $D_{\text{Sen}}(V_E) = (E_\infty \otimes_{\mathbb{Q}_p} A) \otimes_{E_n \otimes_{\mathbb{Q}_p} A} D_n$. This immediately implies that the theorem holds with F replaced by E . But then the previous lemma combined with the fact that $E_\infty \otimes_{\mathbb{Q}_p} A$ is faithfully flat over $F_\infty \otimes_{\mathbb{Q}_p} A$ allows one to conclude.

To prove the claim, first take H_E -invariants in the isomorphism

$$(\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} A) \otimes_{E_n \otimes_{\mathbb{Q}_p} A} D_n \simeq (\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} A) \otimes_A V$$

to deduce that $\widetilde{D}_{\text{Sen}}(V_E) \simeq (\widehat{E}_\infty \widehat{\otimes}_{\mathbb{Q}_p} A) \otimes_{E_n \otimes_{\mathbb{Q}_p} A} D_n$, where \widehat{E}_∞ is the closure of E_∞ in \mathbb{C}_p . This implicitly uses that $(\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} A) \otimes_A V \simeq \mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} V$ (since V is finite over A) and $(\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} A)^{H_E} \simeq \widehat{E}_\infty \widehat{\otimes}_{\mathbb{Q}_p} A$ (pick an orthonormal basis of A over \mathbb{Q}_p and

use Ax-Sen-Tate). Now [2, Lem. 4.2.7] implies that Γ_E -finite vectors in $\widetilde{D}_{\text{Sen}}(V_E)$ coincide with $(E_\infty \otimes_{\mathbb{Q}_p} A) \otimes_{E_n \otimes_{\mathbb{Q}_p} A} D_n = E_\infty \otimes_{E_n} D_n$.

We will now show that the operator θ is well defined. Since

$$D_{\text{Sen}}(V) = D_{\text{Sen}}(V_E)^{\text{Gal}(E_\infty/F_\infty)}$$

by Lemma 3.2, it is enough to compute the limit inside $D_{\text{Sen}}(V_E)$. Since

$$D_{\text{Sen}}(V_E) = E_\infty \otimes_{E_n} D_n = \varinjlim_{m \geq n} E_m \otimes_{E_n} D_n$$

it is enough to compute the limit inside D_n . If $\gamma \in \Gamma_{E_n}$ then the series

$$\log(M_\gamma) = - \sum_{m=1}^{\infty} \frac{(1 - M_\gamma)^m}{m}$$

converges in $\text{End}_{E_n \otimes A}(D_n)$. Since $\log(M_{\gamma^k}) = \log(M_\gamma^k) = k \log(M_\gamma)$ and Γ_E is pro-cyclic the operator $\theta' := \log(M_\gamma)(\log_p(\chi_{\text{cyc}}(\gamma)))^{-1}$ is independent of γ . Moreover, the Γ_{E_n} -action on D_n is given by $\gamma \mapsto \exp(\log_p(\chi_{\text{cyc}}(\gamma))\theta') = M_\gamma$. Thus the limit exists and $\theta = \theta'$. \square

Definition 3.3. *The Sen operator of V is the $\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} A$ -linear endomorphism $\Theta_{\text{Sen},V}$ of $\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} V$ obtained from the endomorphism θ of $D_{\text{Sen}}(V)$ by linearity using (15).*

Remark 3.4. It follows from the proof of Theorem 3.1 that the Sen operator does not change if we restrict the Galois representation to the Galois group of a finite extension.

Lemma 3.5. *Let V and W be free A -modules of finite rank with continuous Gal_F -action. Then*

$$\Theta_{\text{Sen},V \oplus W} = \Theta_{\text{Sen},V} \oplus \Theta_{\text{Sen},W}$$

and

$$\Theta_{\text{Sen},V \otimes_A W} = \Theta_{\text{Sen},V} \otimes \text{id}_W + \text{id}_V \otimes \Theta_{\text{Sen},W}.$$

Proof. The first assertion is trivial. To prove the second assertion we note that the natural map $D_{\text{Sen}}(V) \otimes_{F_\infty \otimes_{\mathbb{Q}_p} A} D_{\text{Sen}}(W) \rightarrow D_{\text{Sen}}(V \otimes_A W)$ becomes an isomorphism after extending scalars to $\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} A$. Since $\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} A$ is faithfully flat over $F_\infty \otimes_{\mathbb{Q}_p} A$ we conclude that the map is an isomorphism. The assertion then follows from (16) applied to an element of the form $x \otimes y$ with $x \in D_{\text{Sen}}(V)$ and $y \in D_{\text{Sen}}(W)$. \square

Lemma 3.6. *Let $A \rightarrow B$ be a continuous map of \mathbb{Q}_p -Banach algebras. There is a natural isomorphism $D_{\text{Sen}}(B \otimes_A V) \simeq D_{\text{Sen}}(V) \otimes_{F_\infty \otimes A} (F_\infty \otimes B)$ inducing an identification*

$$\Theta_{\text{Sen},V \otimes_A B} = \Theta_{\text{Sen},V} \otimes \text{id}.$$

Proof. This is proved in the same way as the previous lemma. \square

We may identify $\text{End}_{\mathbb{C}_p \widehat{\otimes} A}(\mathbb{C}_p \widehat{\otimes} V) = \mathbb{C}_p \widehat{\otimes} \text{End}_A(V)$ and define two commuting, A -linear actions of Gal_F on it via

$$\gamma * (z \otimes X) = \gamma(z) \otimes X, \quad \gamma \cdot (z \otimes X) = z \otimes \text{Ad}(\rho(\gamma))(X),$$

for all $z \in \mathbb{C}_p$ and $X \in \text{End}_A(V)$, where ρ denotes the action of Gal_F on V .

Lemma 3.7. *We have $\gamma \cdot (\gamma * \Theta_{\text{Sen},V}) = \Theta_{\text{Sen},V}$, for all $\gamma \in \text{Gal}_F$.*

Proof. We observe that the $*$ -action maybe described as follows: every $\gamma \in \text{Gal}_F$ induces a semi-linear map $\gamma \otimes \text{id} : \mathbb{C}_p \widehat{\otimes} V \rightarrow \mathbb{C}_p \widehat{\otimes} V$, $z \widehat{\otimes} v \mapsto \gamma(z) \widehat{\otimes} v$, and

$$\gamma * \varphi = (\gamma \otimes \text{id}) \circ \varphi \circ (\gamma \otimes \text{id})^{-1}$$

for all $\varphi \in \text{End}_{\mathbb{C}_p \widehat{\otimes} A}(\mathbb{C}_p \widehat{\otimes} V)$. Thus the assertion of the lemma is equivalent to the equality

$$(\gamma \otimes \rho(\gamma)) \circ \Theta_{\text{Sen}, V} \circ (\gamma \otimes \rho(\gamma))^{-1} = \Theta_{\text{Sen}, V}.$$

Note that $\gamma \mapsto \gamma \otimes \rho(\gamma)$ is just the diagonal action of Gal_F on $\mathbb{C}_p \widehat{\otimes} V$. Since both operators are $\mathbb{C}_p \widehat{\otimes} A$ -linear, it is enough to show that they agree on $D_{\text{Sen}}(V)$. Since the action of Gal_F on $D_{\text{Sen}}(V)$ commutes with the limit in (16), we obtain the assertion. \square

4. FAMILIES OF GALOIS REPRESENTATIONS AND INFINITESIMAL CHARACTERS

Recall that we fix algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p and a finite extension F of \mathbb{Q}_p contained in $\overline{\mathbb{Q}_p}$. Let G be a connected reductive group defined over F . Let $E \subset \overline{\mathbb{Q}_p}$ be a finite Galois extension of F such that G_E is split and let $\Gamma = \text{Gal}(E/F)$. Let L be a further finite extension of \mathbb{Q}_p , let \mathcal{O} be the ring of integers of L with residue field k and a uniformizer ϖ . We assume that L is large enough so that there are $[E : \mathbb{Q}_p]$ field embeddings of E into L . In particular, $G \times_{F, \sigma} L$ is split for any embedding $\sigma : F \hookrightarrow L$.

4.1. Topology on the coefficients. Let \mathcal{T} be a Grothendieck topology on the category of rigid analytic spaces over L , such that the following hold:

- the functor $X \mapsto \mathcal{O}_X(X)$ is a sheaf on \mathcal{T} ;
- every X admits a covering in \mathcal{T} by affinoids;
- every covering $\{U_i \rightarrow X\}_{i \in I}$ with U_i and X affinoid admits a finite sub-cover.

An example of such topology is the analytic topology, [8, §9.3.1], or the étale topology, [41, §8.2]. We want to topologize the rings $\mathcal{O}_X(X)$. If $X = \text{Sp}(A)$ then $\mathcal{O}_X(X) = A$, which is naturally an L -Banach space. Moreover, a map between affinoids induces a continuous map on the ring of functions.

Lemma 4.1. *Let $\{U_i\}_{i \in I}$ be a covering of X in \mathcal{T} . If X and U_i for $i \in I$ are affinoid then the map*

$$\mathcal{O}_X(X) \rightarrow \prod_{i \in I} \mathcal{O}_{U_i}(U_i)$$

is a homeomorphism onto its image for the product topology on the target and Banach space topology for the rings of global sections.

Proof. For each $i, j \in I$ we choose a covering by affinoids $\{V_{ijk} \rightarrow U_i \times_X U_j\}_k$. The maps in the equalizer diagram

$$(17) \quad \mathcal{O}_X(X) \longrightarrow \prod_i \mathcal{O}_{U_i}(U_i) \rightrightarrows \prod_{i,j,k} \mathcal{O}_{V_{ijk}}(V_{ijk})$$

are continuous and hence $\mathcal{O}_X(X)$ is closed in $\prod_{i \in I} \mathcal{O}_{U_i}(U_i)$ for the subspace topology, which we denote by σ . Let $F \subset I$ be finite such that $\{U_i\}_{i \in F}$ is a cover of X . Then by the same argument $\mathcal{O}_X(X)$ is closed in $\prod_{i \in F} \mathcal{O}_{U_i}(U_i)$ for the subspace topology, which we denote by σ' . Since F is finite, σ' is a Banach space topology. Since by the Open Mapping theorem every continuous bijection

between Banach spaces is a homeomorphism, β and σ' coincide. The projection map $\prod_{i \in I} \mathcal{O}_{U_i}(U_i) \rightarrow \prod_{i \in F} \mathcal{O}_{U_i}(U_i)$ induces continuous maps $(\mathcal{O}_X(X), \beta) \xrightarrow{\text{id}} (\mathcal{O}_X(X), \sigma) \xrightarrow{\text{id}} (\mathcal{O}_X(X), \sigma')$, thus σ and β coincide. \square

For an arbitrary X we define the topology τ on $\mathcal{O}_X(X)$ to be the coarsest topology such that for all coverings $\{U_i \rightarrow X\}_i$ by affinoids the map $\mathcal{O}_X(X) \rightarrow \prod_i \mathcal{O}_{U_i}(U_i)$ is continuous, where we put the Banach space topology on each $\mathcal{O}_{U_i}(U_i)$ and the product topology on the target.

Lemma 4.2. *If $\{U_i \rightarrow X\}_{i \in I}$ is any covering of X in \mathcal{T} by affinoids then the subspace topology on $\mathcal{O}_X(X)$ induced by $\mathcal{O}_X(X) \rightarrow \prod_{i \in I} \mathcal{O}_{U_i}(U_i)$ coincides with τ , for the for the Banach space topology on $\mathcal{O}_{U_i}(U_i)$ and the product topology on the target.*

Proof. Let $\{U'_j \rightarrow X\}_{j \in J}$ be another covering in \mathcal{T} of X by affinoids. Then $\{U_i \times_X U'_j \rightarrow U_i\}_i$ and $\{U_i \times_X U'_j \rightarrow U'_j\}_j$ are both coverings in \mathcal{T} . For each i, j we let $\{V_{ijk}\}_k$ be a cover of $U_i \times_X U'_j$ by affinoids. It follows from Lemma 4.1 that the maps

$$\prod_i \mathcal{O}_{U_i}(U_i) \rightarrow \prod_{i,j,k} \mathcal{O}_{V_{ijk}}(V_{ijk}), \quad \prod_j \mathcal{O}_{U'_j}(U'_j) \rightarrow \prod_{i,j,k} \mathcal{O}_{V_{ijk}}(V_{ijk})$$

are homeomorphisms onto their images. Since both maps coincide, when restricted to $\mathcal{O}_X(X)$, we deduce that the subspace topologies on $\mathcal{O}_X(X)$ induced by the coverings are the same. \square

Corollary 4.3. *If X is an affinoid then τ coincides with the Banach space topology on $\mathcal{O}_X(X)$.*

Proof. This follows from the fact that $X \xrightarrow{\text{id}} X$ is a covering of X by affinoids. \square

Remark 4.4. If a covering of X by admissible open affinoids for the analytic topology is also a covering in \mathcal{T} then Lemma 4.2 implies that both topologies on $\mathcal{O}_X(X)$ coincide. In particular, there is no difference for the topology on $\mathcal{O}_X(X)$ in the analytic and étale topologies.

Remark 4.5. We may identify \widehat{G} with a Zariski closed subset of \mathbb{A}^n for some n . Then $\widehat{G}(\mathcal{O}_{U_i}(U_i))$ is a closed subset of $\mathcal{O}_{U_i}(U_i)^{\oplus n}$. If P is a \widehat{G} -torsor over X for \mathcal{T} and $\mathcal{U} = \{U_i\}_{i \in I}$ is a covering of X by affinoids trivialising P , then we may identify $P^{\text{ad}}(U_i)$ with $\widehat{G}(\mathcal{O}_{U_i}(U_i))$. This identification is canonical up to conjugation by elements of $\widehat{G}(\mathcal{O}_{U_i}(U_i))$, thus the induced topology on $P^{\text{ad}}(U_i)$ does not depend on choices. The argument in Lemma 4.2 shows that the subspace topology on $P^{\text{ad}}(X)$ induced by the inclusion $P^{\text{ad}}(X) \subset \prod_i P^{\text{ad}}(U_i)$ does not depend on the choice of the cover \mathcal{U} .

4.2. The Sen operator in the affinoid case. Let $U = \text{Sp}(A)$ be an affinoid and let

$$\rho_U : \text{Gal}_F \rightarrow {}^L G_f(A) = \widehat{G}(A) \rtimes \Gamma$$

be an admissible representation. If $r : {}^L G_f \rightarrow \text{GL}(V)$ is an algebraic representation of ${}^L G_f$ on a finite dimensional L -vector space V , we let

$$\Theta_{\text{Sen}, r, U} \in \text{End}_{\mathbb{C}_p \widehat{\otimes}_A}((\mathbb{C}_p \widehat{\otimes}_A) \otimes_L V)$$

be the Sen operator $\Theta_{\text{Sen}, V \otimes_L A}$ defined in Definition 3.3 for the representation $r \circ \rho_U : \text{Gal}_F \rightarrow \text{GL}(V)$.

Let $\widehat{\mathfrak{g}}$ be the Lie algebra of \widehat{G} , which we identify with the Lie algebra of ${}^L G_f$.

Lemma 4.6. *There is a unique $\Theta_{\text{Sen}, U} \in (\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}}$ such that*

$$\text{Lie}(r)(\Theta_{\text{Sen}, U}) = \Theta_{\text{Sen}, r, U}$$

for every algebraic representation (r, V) of ${}^L G_f$ over L . Moreover, if $U' = \text{Sp}(A') \rightarrow U$ is a map of affinoid varieties then the map

$$(\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}} \rightarrow (\mathbb{C}_p \widehat{\otimes} A') \otimes_L \widehat{\mathfrak{g}},$$

induced by extension of scalars, sends $\Theta_{\text{Sen}, U}$ to $\Theta_{\text{Sen}, U'}$.

Proof. As explained in [68, Lem. 2.2.5], given the transformation properties of the Sen operator with respect to the direct sum and tensor product proved in Lemma 3.5, the assertion follows from the Tannaka duality for Lie algebras. Patrikis refers for this to [45]. We note that alternatively one could follow Milne [59, Prop. 6.11] and deduce the statement from the Tannaka duality for algebraic groups. Informally, it says that if R is an L -algebra then to give an element of ${}^L G_f(R)$ is equivalent to giving a compatible family $\{g_V \in \text{Aut}_R(R \otimes_L V)\}_V$ indexed by algebraic representation V of ${}^L G_f$ defined over L . The statement is made precise in the references cited below. We may identify

$$(18) \quad R \otimes_L \widehat{\mathfrak{g}} = \text{Ker}({}^L G_f(R[\varepsilon]) \rightarrow {}^L G_f(R)),$$

where $R[\varepsilon] = R[X]/(X^2)$ is the algebra of dual numbers over R , and consider

$$(19) \quad 1 + \varepsilon \Theta_{\text{Sen}, r, U} \in \text{Aut}_{(\mathbb{C}_p \widehat{\otimes} A)[\varepsilon]}((\mathbb{C}_p \widehat{\otimes} A)[\varepsilon] \otimes_L V).$$

The existence and uniqueness of $\Theta_{\text{Sen}, U}$ follows from [25, Prop. 2.8], see also [59, §2.10]. The last assertion follows from Lemma 3.6. \square

The natural A -linear action of Gal_F on $\mathbb{C}_p \widehat{\otimes} A$ induces an action of Gal_F on $(\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}}$, defined by

$$(20) \quad \gamma * (z \otimes X) = \gamma(z) \otimes X$$

for $z \in \mathbb{C}_p \widehat{\otimes} A$, $X \in \widehat{\mathfrak{g}}$ and $\gamma \in \text{Gal}_F$.

On the other hand, the group ${}^L G_f(\mathbb{C}_p \widehat{\otimes} A)$ acts on $(\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}}$ via the adjoint action of ${}^L G_f$ on its Lie algebra $\widehat{\mathfrak{g}}$. Since ${}^L G_f$ is defined over \mathbb{Q}_p , the action of Gal_F on $\mathbb{C}_p \widehat{\otimes} A$ induces an action of Gal_F on ${}^L G_f(\mathbb{C}_p \widehat{\otimes} A)$, denoted $(\gamma, g) \rightarrow \gamma(g)$. These three actions are related by

$$(21) \quad \gamma * (\text{Ad}(g)(X)) = \text{Ad}(\gamma(g))(\gamma * X)$$

for $\gamma \in \text{Gal}_F$, $g \in {}^L G_f(\mathbb{C}_p \widehat{\otimes} A)$, $X \in (\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}}$. Indeed, it suffices to check this after an L -embedding of ${}^L G_f$ in some GL_n , where it becomes obvious. The above relation (21) shows in particular that the adjoint action of ${}^L G_f(A) \subset {}^L G_f(\mathbb{C}_p \widehat{\otimes} A)^{\text{Gal}_F}$ commutes with the action of Gal_F .

Finally, we define another action of Gal_F on $(\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}}$, via

$$(22) \quad \gamma \cdot X = \text{Ad}(\mu_G(\bar{\gamma}))(X),$$

where $\bar{\gamma}$ is the image of γ in Γ . Since $\mu_G(\bar{\gamma}) \in {}^L G_f(A)$, the discussion in the previous paragraph shows that this action commutes with the star action $(\gamma, X) \rightarrow \gamma * X$ of Gal_F on $(\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}}$.

For all $\gamma \in \text{Gal}_F$ we write $\rho_U(\gamma) = (c_\gamma, \bar{\gamma})$ with $c_\gamma \in \widehat{G}(A)$ as in subsection 2.2.

Lemma 4.7. *For all $\gamma \in \text{Gal}_F$ we have*

$$\text{Ad}(c_\gamma)(\gamma * (\gamma \cdot \Theta_{\text{Sen},U})) = \Theta_{\text{Sen},U}.$$

Proof. For simplicity let $\Theta = \Theta_{\text{Sen},U}$ and let $\rho = \rho_U$. We identify Γ and \widehat{G} with their images in ${}^L G_f$, so that $\rho(\gamma) = c_\gamma \bar{\gamma}$. By the discussion preceding the lemma

$$\text{Ad}(c_\gamma)(\gamma * (\gamma \cdot \Theta)) = \text{Ad}(c_\gamma)(\gamma \cdot (\gamma * \Theta)) = \text{Ad}(c_\gamma \mu_G(\bar{\gamma}))(\gamma * \Theta) = \text{Ad}(\rho(\gamma))(\gamma * \Theta).$$

Thus we need to check that $\text{Ad}(\rho(\gamma))(\gamma * \Theta) = \Theta$. It suffices to check the equality after applying $\text{Lie}(r)$ to both sides, where $r : {}^L G_f \rightarrow \text{GL}(V)$ is a faithful representation of ${}^L G_f$. Note that for all $g \in {}^L G_f$, $\gamma \in \text{Gal}_F$ and $u \in (\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}}$ we have

$$\text{Lie}(r)(\text{Ad}(g)(u)) = \text{Ad}(r(g))(\text{Lie}(r)(u)), \quad \text{Lie}(r)(\gamma * u) = \gamma * \text{Lie}(r)(u).$$

Letting $\Theta_r = \Theta_{\text{Sen},r,U} = \text{Lie}(r)(\Theta)$, we need to check that $\text{Ad}(r \circ \rho(\gamma))(\gamma * \Theta_r) = \Theta_r$ for $r \circ \rho : \text{Gal}_F \rightarrow \text{GL}(V \otimes A)$. This assertion is proved in Lemma 3.7. \square

Let $S(\widehat{\mathfrak{g}}^*)$ be the symmetric algebra in the L -linear dual of $\widehat{\mathfrak{g}}$. We think of $S(\widehat{\mathfrak{g}}^*)$ as polynomial functions on $\widehat{\mathfrak{g}}$. Let $S(\widehat{\mathfrak{g}}^*)^{\widehat{G}}$ be the subring \widehat{G} -invariant polynomial functions of $\widehat{\mathfrak{g}}$.

If V is a finite dimensional L -vector space and R is an L -algebra then

$$\text{Hom}_{L\text{-alg}}(S(V^*), R) \cong \text{Hom}_L(V^*, R) \cong V \otimes_L R.$$

If $\lambda \in V \otimes_L R$ then we will denote the corresponding L -algebra homomorphism by ev_λ . To give a homomorphism of L -algebras from $S(\widehat{\mathfrak{g}}^*)$ to an L -algebra R is the same as to give an L -linear map from $\widehat{\mathfrak{g}}^*$ to R , which is the same as to give an element of $\widehat{\mathfrak{g}} \otimes_L R$. Thus the elements $\Theta_{\text{Sen},U}$ give us a compatible system of homomorphism of L -algebras $\theta_U : S(\widehat{\mathfrak{g}}^*) \rightarrow \mathbb{C}_p \widehat{\otimes} A$, which we may restrict to the subrings considered above.

Lemma 4.8. *The map $\theta_U : S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \rightarrow \mathbb{C}_p \widehat{\otimes} A$ takes values in $E \otimes A$. Moreover,*

$$(23) \quad \gamma(\theta_U(f)) = \theta_U(\gamma \cdot f), \quad \forall \gamma \in \text{Gal}_F, \quad \forall f \in S(\widehat{\mathfrak{g}}^*)^{\widehat{G}},$$

where $\gamma \cdot$ denotes the L -linear action of Gal_F on $S(\widehat{\mathfrak{g}}^*)$ via μ_G .

Proof. Let $\Theta = \Theta_{\text{Sen},U}$. Thinking of elements of $S(\widehat{\mathfrak{g}}^*)^{\widehat{G}}$ as functions on $\widehat{\mathfrak{g}}$, we need to prove that $\gamma(f(\Theta)) = (\gamma \cdot f)(\Theta)$ for all $f \in S(\widehat{\mathfrak{g}}^*)^{\widehat{G}}$. Since $(\gamma \cdot f)(u) = f(\gamma^{-1} \cdot u)$, for all $u \in \mathbb{C}_p \widehat{\otimes} A \otimes_L \widehat{\mathfrak{g}}$, it is enough to prove that $\gamma(f(\Theta)) = f(\gamma^{-1} \cdot \Theta)$. By observing that $f(\gamma * u) = \gamma(f(u))$ for all $u \in \mathbb{C}_p \widehat{\otimes} A \otimes_L \widehat{\mathfrak{g}}$, we obtain

$$f(\Theta) = f(c_\gamma \cdot (\gamma * (\gamma \cdot \Theta))) = f(\gamma * (\gamma \cdot \Theta)) = \gamma(f(\gamma \cdot \Theta)),$$

where the first equality follows from Lemma 4.7 and the second from the \widehat{G} -invariance of f . Replacing γ by γ^{-1} yields the result. \square

4.3. The Sen operator and twisted families of L -parameters. Let X be a rigid analytic variety and let P be a \widehat{G} -torsor on X with respect to a Grothendieck topology satisfying the conditions in Section 4.1. Let $\rho : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(X)$ be an admissible representation, see section 2.3.

Let $\mathcal{U} = \{U_i \rightarrow X\}_{i \in I}$ be a covering of X by affinoids together with local sections $s_i : U_i \rightarrow P$. Lemma 2.11 gives a family of admissible representations $\rho_{U_i} : \text{Gal}_F \rightarrow {}^L G(\mathcal{O}_X(U_i))$ satisfying

$$\rho_{U_i}(\gamma) = g_{ij} \rho_{U_j}(\gamma) g_{ij}^{-1}, \quad \forall \gamma \in \text{Gal}_F, \quad \forall i, j \in I,$$

where $g_{ij} \in \widehat{G}(\mathcal{O}_X(U_{ij}))$ come from the glueing data for the torsor. For each U_i , Lemma 4.6 gives us $\Theta_{\text{Sen}, U_i} \in (\mathbb{C}_p \widehat{\otimes} \mathcal{O}_X(U_i)) \otimes_L \widehat{\mathfrak{g}}$, which behave well under maps between affinoid varieties. The uniqueness of the Sen operator implies that

$$\Theta_{\text{Sen}, U_i} = g_{ij} \Theta_{\text{Sen}, U_j} g_{ij}^{-1}, \quad \forall i, j \in I$$

in $(\mathbb{C}_p \widehat{\otimes} \mathcal{O}_X(U_{ij})) \otimes_L \widehat{\mathfrak{g}}$. Hence, they glue to $\Theta_{\text{Sen}, \rho} \in \mathbb{C}_p \widehat{\otimes} (\text{Lie } P^{\text{ad}})(X)$, use (19).

Lemma 4.9. *If $f : Y \rightarrow X$ is a map of rigid varieties over L , and $\rho_Y : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(Y)$ is the extension of scalars of ρ to Y , see Definition 2.10, then $\Theta_{\text{Sen}, \rho_Y}$ is the image $\Theta_{\text{Sen}, \rho}$ in $\mathbb{C}_p \widehat{\otimes} (\text{Lie } P^{\text{ad}})(Y)$.*

Proof. One can check this on an affinoid cover trivialising P , and there the assertion follows from the last part of Lemma 4.6. \square

Lemma 4.10. *The maps $\theta_{U_i} : S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \rightarrow E \otimes_{\mathbb{Q}_p} \mathcal{O}_X(U_i)$ given by Lemma 4.8 glue to a homomorphism of L -algebras*

$$(24) \quad \theta : S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \rightarrow E \otimes_{\mathbb{Q}_p} \mathcal{O}_X(X),$$

which satisfies the transformation property in (23). Moreover, θ depends only on the equivalence class of ρ .

Proof. Since the restrictions of Θ_{Sen, U_i} and Θ_{Sen, U_j} to U_{ij} are conjugate by an element of $\widehat{G}(\mathcal{O}_X(U_{ij}))$, the evaluation maps at Θ_{Sen, U_i} and Θ_{Sen, U_j} coincide when restricted to $S(\widehat{\mathfrak{g}}^*)^{\widehat{G}}$ and hence glue. The transformation property in (23) holds, since it holds on a cover.

If we choose different sections $s'_i = g_i s_i$ then Θ_{Sen, U_i} get replaced by $g_i \Theta_{\text{Sen}, U_i} g_i^{-1}$, so we obtain the same θ_{U_i} and hence the same θ . Clearly, θ does not change if we refine the cover. Thus θ does not depend on the choices made at the beginning of the subsection.

If ρ and ρ' are conjugate by an element $u \in P^{\text{ad}}(X)$ then the representations ρ'_{U_i} and ρ_{U_i} are also conjugate. More precisely, $\rho'_{U_i}(\gamma) = q_i \rho_{U_i}(\gamma) q_i^{-1}$, for all $\gamma \in \text{Gal}_F$ and $i \in I$, where $q_i \in \widehat{G}(\mathcal{O}_X(U_i))$ are uniquely determined by $u(s_i) = q_i^{-1} s_i$, see [10, §1, (1.1.4)]. Hence, we obtain the same θ . \square

Lemma 4.11. *If $f : Y \rightarrow X$ is a map of rigid varieties over L and $\theta_Y : S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \rightarrow E \otimes_{\mathbb{Q}_p} \mathcal{O}_Y(Y)$ is the map (24) for ρ_Y , see Definition 2.10, then θ_Y is equal to θ composed with the map $\text{id} \otimes f^\# : E \otimes_{\mathbb{Q}_p} \mathcal{O}_X(X) \rightarrow E \otimes_{\mathbb{Q}_p} \mathcal{O}_Y(Y)$.*

Proof. It is enough to check it on the cover, where it follows from Lemma 3.6. \square

4.4. Chevalley's restriction theorem. Let $\widehat{\mathfrak{t}}$ be the Lie algebra of \widehat{T} and let W be the Weyl group associated to the root system. We note that W is also the Weyl group of the dual root system, [51, §2.9].

Proposition 4.12 (Chevalley's restriction theorem). *The restriction to $\widehat{\mathfrak{t}}$ induces an isomorphism of rings*

$$(25) \quad S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \xrightarrow{\cong} S(\widehat{\mathfrak{t}}^*)^W.$$

Proof. If $\widehat{\mathfrak{g}}$ is semi-simple both assertions are proved in §23.1 and the Appendix to §23 in [50]. The same proof carries over when $\widehat{\mathfrak{g}}$ is reductive. \square

Recall that the group $\Gamma = \text{Gal}(E/F)$ acts on the based root datum. Recall that by choosing a pinning the group Γ acts through $\text{Aut}(\widehat{G}, \widehat{T})$. If $\theta \in \text{Aut}(\widehat{G}, \widehat{T})$ and $g \in N_{\widehat{G}}(\widehat{T})$ then $\theta(g)t\theta(g)^{-1} = \theta(g\theta^{-1}(t)g^{-1}) \in \widehat{T}$ for all $t \in \widehat{T}$ and thus $\text{Aut}(\widehat{G}, \widehat{T})$ normalizes $N_{\widehat{G}}(\widehat{T})$ and thus acts on $S(\widehat{\mathfrak{t}}^*)^W$. The isomorphism (25) is Γ -equivariant, since it is obtained by restriction of functions on $\widehat{\mathfrak{g}}$ to $\widehat{\mathfrak{t}}$.

4.5. Harish-Chandra homomorphism. Since G_E is split we may assume that the triple (G_E, B, T) is defined over E . Let \mathfrak{t} be the Lie algebra of T . We will now recall the construction of Harish-Chandra homomorphism

$$(26) \quad \psi : Z(\mathfrak{g}_E) \xrightarrow{\cong} S(\mathfrak{t})^W.$$

Let $(h_i)_{1 \leq i \leq n}$ be an E -basis of \mathfrak{t} . We extend it to a basis of \mathfrak{g}_E , $\{h_i, x_\beta, y_\beta : 1 \leq i \leq n, \beta \in \Phi^+\}$, such that $tx_\beta t^{-1} = \beta(t)x_\beta$ and $ty_\beta t^{-1} = \beta^{-1}(t)y_\beta$ for all $\beta \in \Phi^+$ and all $t \in T$. By PBW theorem $U(\mathfrak{g}_E)$ has an E -basis consisting of monomials $\prod_{\beta \in \Phi^+} y_\beta^{j_\beta} \prod_{i=1}^n h_i^{k_i} \prod_{\beta \in \Phi^+} x_\beta^{j_\beta}$ with $i_\alpha, k_i, j_\alpha \geq 0$. Let

$$\xi : U(\mathfrak{g}_E) \rightarrow S(\mathfrak{t})$$

be a linear map, which sends all the monomials with $i_\alpha > 0$ or $j_\alpha > 0$ to zero and is identity on the monomials with $i_\alpha = j_\alpha = 0$.

Let $\lambda \in X^*(T)_+$ and let $V(\lambda)$ be an irreducible representation of G_E of highest weight λ . The highest weight theory implies that $V(\lambda)$ is absolutely irreducible as representation of G_E and \mathfrak{g}_E . In particular, $\text{End}_{\mathfrak{g}_E}(V(\lambda)) = E$ and hence the action of $U(\mathfrak{g}_E)$ on $V(\lambda)$ induces a ring homomorphism

$$(27) \quad \chi_\lambda : Z(\mathfrak{g}_E) \rightarrow E.$$

As explained in [54, §VIII.5] we have

$$(28) \quad \text{Lie}(\lambda)(\xi(z)) = \chi_\lambda(z), \quad \forall z \in Z(\mathfrak{g}_E), \quad \forall \lambda \in X^*(T)_+.$$

Let $\mathfrak{t} \rightarrow S(\mathfrak{t})$ be the E -linear map sending $h \mapsto h - \delta(h)1$ for $h \in \mathfrak{t}$ with $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. This induces an isomorphism of E -algebras $\eta : S(\mathfrak{t}) \xrightarrow{\cong} S(\mathfrak{t})$. We define $\psi := \eta \circ \xi : Z(\mathfrak{g}_E) \rightarrow S(\mathfrak{t})$. It follows from [54, Th. 8.18] that ψ induces an isomorphism of E -algebras $Z(\mathfrak{g}_E) \cong S(\mathfrak{t})^W$ which is the isomorphism (26).

Each $\lambda \in X^*(T) \otimes_{\mathbb{Z}} E = \mathfrak{t}^*$ defines a linear map $\mathfrak{t} \rightarrow E$, $t \mapsto \text{Lie}(\lambda)(t)$ and hence a ring homomorphism $S(\mathfrak{t}) \rightarrow E$, which we denote by ev_λ . We note it follows from the construction of ψ that

$$(29) \quad \text{ev}_{\lambda+\delta}(\psi(z)) = \chi_\lambda(z), \quad \forall \lambda \in X^*(T)_+, \quad \forall z \in Z(\mathfrak{g}_E),$$

where δ is a half sum of positive roots. The ring $S(\mathfrak{t})$ is reduced and the set of closed points $\text{ev}_\lambda : S(\mathfrak{t}) \rightarrow E$, for $\lambda \in X^*(T)_+$, is Zariski dense in $\text{Spec } S(\mathfrak{t})$. Thus ψ is the unique isomorphism (26) such that (29) holds.

Lemma 4.13. *The isomorphism ψ defined in (26) is Γ -equivariant.*

Proof. The group $\text{Aut}(G_E)$ acts on \mathfrak{g}_E and hence on $U(\mathfrak{g}_E)$ and $Z(\mathfrak{g}_E)$. If V is an algebraic representation of G_E , then for $\gamma \in \Gamma$ we define V^γ to be the action of G_E on V given by $(g, v) \mapsto \gamma^{-1}(g) \cdot v$. For $\lambda \in X^*(T)_+$, we have $V(\lambda)^\gamma \simeq V(\gamma \cdot \lambda)$. We deduce from relation (29) that

$$\text{ev}_{\lambda+\delta}(\gamma(\psi(z))) = \text{ev}_{\gamma^{-1} \cdot \lambda + \delta}(\psi(z)) = \chi_{\gamma^{-1} \cdot \lambda}(z), \quad \forall z \in Z(\mathfrak{g}_E).$$

It follows from (28) that $Z(\mathfrak{g}_E)$ acts on $V(\gamma^{-1} \cdot \lambda) \simeq V(\lambda)^{\gamma^{-1}}$ via $\chi_\lambda(\gamma(z))$. Finally we have

$$\text{ev}_{\lambda+\delta}(\gamma(\psi(z))) = \chi_\lambda(\gamma(z))$$

for all $z \in Z(\mathfrak{g}_E)$, this implies $\gamma \circ \psi = \psi \circ \gamma$ and the result. \square

Let \mathfrak{g} be the Lie algebra of G as an algebraic group over F , let $U(\mathfrak{g})$ be its universal enveloping algebra and let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Then \mathfrak{g}_E , where the subscript E -denotes the extension of scalars from F to E , is the Lie algebra of G_E . It follows from its universal property [27, Lem. 2.1.3] that $U(\mathfrak{g}_E) = U(\mathfrak{g})_E$.

Lemma 4.14. *Let A, B be central algebras over a field F . The natural map*

$$Z(A) \otimes_F Z(B) \rightarrow Z(A \otimes_F B)$$

is an isomorphism. In particular, the centre $Z(\mathfrak{g}_E)$ of $U(\mathfrak{g}_E)$ is equal to $Z(\mathfrak{g})_E$.

Proof. Let $x \in Z(A \otimes_F B)$ and let $(a_i)_{i \in I}$ be a basis of A over F . Writing $x = \sum_{i \in I} a_i \otimes b_i$ and imposing $x(1 \otimes b) = (1 \otimes b)x$ yields $\sum_{i \in I} a_i \otimes (bb_i - b_i b) = 0$, thus $b_i \in Z(B)$ for all $i \in I$ and $x \in A \otimes_F Z(B)$. Now pick a basis of $Z(B)$ over F and repeat the argument to get $x \in Z(A) \otimes_F Z(B)$. \square

4.6. Definition of infinitesimal character. Let us fix an embedding $\sigma : F \hookrightarrow L$ and choose an embedding $\tau : E \hookrightarrow L$, such that $\tau|_F = \sigma$.

We have canonical identifications $\mathfrak{t} = X_*(T) \otimes_{\mathbb{Z}} E$ and $\widehat{\mathfrak{t}} = X_*(\widehat{T}) \otimes_{\mathbb{Z}} L = X^*(T) \otimes_{\mathbb{Z}} L$. Thus an isomorphism of L -vector spaces

$$\mathfrak{t} \otimes_{E, \tau} L \cong \widehat{\mathfrak{t}}^*,$$

which is equivariant for Γ and W actions. By base changing (26) along τ and using Lemma 4.14 we obtain an isomorphism of L -algebras

$$(30) \quad \kappa_\tau : Z(\mathfrak{g}) \otimes_{F, \sigma} L \xrightarrow{\cong} S(\widehat{\mathfrak{t}}^*)^W.$$

Since T is split over E we have canonically $X^*(T) = X^*(T \times_{E, \tau} L)$. If V is an irreducible representation of $G \times_{F, \sigma} L$ and λ is the highest weight of V with respect to $(B, T) \times_{E, \tau} L$ then the above identifications allows to view λ_τ as an element of $\widehat{\mathfrak{t}}$, and thus induces a homomorphism $\text{ev}_{\lambda_\tau} : S(\widehat{\mathfrak{t}}^*) \rightarrow L$. Since $G \times_{F, \sigma} L$ is split V is absolutely irreducible and thus by the same argument as in (27) we obtain a homomorphism $\chi_V : Z(\mathfrak{g}) \otimes_{F, \sigma} L \rightarrow L$.

Lemma 4.15. *The isomorphism κ_τ is uniquely characterised by the property*

$$\text{ev}_{\lambda_\tau + \delta} \circ \kappa_\tau = \chi_V$$

for all irreducible representations V of $G \times_{F, \sigma} L$.

Proof. The assertion follows from (29) and Zariski density as explained after (29). \square

Lemma 4.16. *For every $\gamma \in \Gamma$ the composition*

$$Z(\mathfrak{g}) \otimes_{F,\sigma} L \xrightarrow{\kappa_\tau} S(\widehat{\mathfrak{t}}^*)^W \xrightarrow{\gamma \cdot} S(\widehat{\mathfrak{t}}^*)^W$$

is equal to $\kappa_{\tau \circ \gamma^{-1}}$, where \cdot denotes the action of Γ on $S(\widehat{\mathfrak{t}}^)^W$.*

Proof. Let V be an irreducible representation of $G \times_{F,\sigma} L$. If λ is the highest weight of V with respect to $(B, T) \times_{E,\tau} L$ then $\gamma(\lambda)$ is the highest weight of V with respect to $(B, T) \times_{E,\tau \circ \gamma^{-1}} L$. This assertion follows from the definition of the action of Γ on $X^*(T)$ after identifying E and L via τ . The claim then follows from Lemma 4.15. \square

Corollary 4.17. *If $\lambda \in \widehat{\mathfrak{t}}^\Gamma$ then the composition*

$$Z(\mathfrak{g}) \otimes_{F,\sigma} L \xrightarrow{\kappa_\tau} S(\widehat{\mathfrak{t}}^*)^W \xrightarrow{\text{ev}_\lambda} L$$

is independent of the choice of embedding τ .

Lemma 4.18. *The composition*

$$Z(\mathfrak{g}) \otimes_{F,\sigma} L \xrightarrow{\kappa_\tau} S(\widehat{\mathfrak{t}}^*)^W \xrightarrow[(25)]{\cong} S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \xrightarrow[(24)]{\theta} E \otimes \mathcal{O}_X(X) \xrightarrow{m_\tau} \mathcal{O}_X(X),$$

where the last arrow is the map $x \otimes a \mapsto \tau(x)a$, is independent of the choice of τ above σ .

Proof. This follows from Lemma 4.16 and Γ -equivariance of (25) and (24). \square

Definition 4.19. *Let P be a \widehat{G}_X -torsor over X and let $\rho : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(X)$ be an admissible representation. If $\sigma : F \hookrightarrow L$ is an embedding of fields then we define a homomorphism of L -algebras*

$$\zeta_{\rho,\sigma} : Z(\mathfrak{g}) \otimes_{F,\sigma} L \rightarrow \mathcal{O}_X(X)$$

as the composition of the maps in Lemma 4.18.

Remark 4.20. We remind the reader if $P = \widehat{G}_X$ then ${}^L P^{\text{ad}}(X) = {}^L G_f(\mathcal{O}_X(X))$ and ρ is an admissible representation in the sense of section 2.2.

Let $\text{Res}_{F/\mathbb{Q}_p} \mathfrak{g}$ be the Lie algebra of $\text{Res}_{F/\mathbb{Q}_p} G$. We may identify $\text{Res}_{F/\mathbb{Q}_p} \mathfrak{g}$ with \mathfrak{g} as a \mathbb{Q}_p -Lie algebra. Since

$$(\text{Res}_{F/\mathbb{Q}_p} G) \times_{\mathbb{Q}_p} L \cong \prod_{\sigma:F \hookrightarrow L} G \times_{F,\sigma} L,$$

using Lemma 4.14, we obtain isomorphisms of L -algebras

$$\begin{aligned} U(\text{Res}_{F/\mathbb{Q}_p} \mathfrak{g}) \otimes_{\mathbb{Q}_p} L &\cong \bigotimes_{\sigma:F \hookrightarrow L} U(\mathfrak{g}) \otimes_{F,\sigma} L \\ Z(\text{Res}_{F/\mathbb{Q}_p} \mathfrak{g}) \otimes_{\mathbb{Q}_p} L &\cong \bigotimes_{\sigma:F \hookrightarrow L} Z(\mathfrak{g}) \otimes_{F,\sigma} L. \end{aligned}$$

Definition 4.21. *Let P be a \widehat{G}_X -torsor over X and let $\rho : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(X)$ be an admissible representation. Then we define an L -algebra homomorphism*

$$\zeta_\rho : Z(\text{Res}_{F/\mathbb{Q}_p} \mathfrak{g}) \otimes_{\mathbb{Q}_p} L \rightarrow \mathcal{O}_X(X)$$

as $\zeta_\rho := \otimes_\sigma \zeta_{\rho,\sigma}$ using the isomorphism above and Definition 4.19.

Lemma 4.22. *If $Y \rightarrow X$ is a map of rigid varieties over L then ζ_{ρ_Y} is equal to ζ_ρ composed with the map $\mathcal{O}_X(X) \rightarrow \mathcal{O}_Y(Y)$.*

Proof. This follows from Lemma 4.11. \square

4.7. Modification for C -groups. There are two issues at hand. Firstly, one expects that the Galois representations attached to C -algebraic automorphic forms take values in the C -group and not the L -group, see [12, Conj. 5.3.4]. We would like to consider a local p -adic analog of this situation, but our Definition 4.21 does not cover it. Secondly, if we take $F = L = \mathbb{Q}_p$, X a point, $G = \mathrm{GL}_2$, so that ${}^L G_f = \mathrm{GL}_2$, and

$$\rho : \mathrm{Gal}_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(\mathbb{Q}_p), \quad g \mapsto \begin{pmatrix} \chi_{\mathrm{cyc}}(g)^a & 0 \\ 0 & \chi_{\mathrm{cyc}}(g)^b \end{pmatrix},$$

where $a > b$ are integers, then the Sen operator is the matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, but the character $\zeta_\rho : Z(\mathfrak{g}) \rightarrow \mathbb{Q}_p$ is not an infinitesimal character of an algebraic representation of GL_2 . The problem is caused by the shift by δ in (29). We will resolve both of these issues by modifying Definition 4.21.

We use the notation introduced in section 2.1. Let $\widehat{\mathfrak{g}}^T$ be the Lie algebra of \widehat{G}^T . We may identify $\widehat{\mathfrak{g}}^T = \widehat{\mathfrak{g}} \oplus \mathrm{Lie} \mathbb{G}_m$ as F -vector spaces. Since taking square roots out of 2δ on the Lie algebra level is just dividing by 2, the map

$$\widehat{\mathfrak{g}}^T \rightarrow \widehat{\mathfrak{g}} \oplus \mathrm{Lie} \mathbb{G}_m, \quad (g, t) \mapsto (g + t\delta, t)$$

is a Γ -equivariant isomorphism of Lie algebras. Let $\alpha : \widehat{\mathfrak{g}}^T \rightarrow \widehat{\mathfrak{g}}$ be the composition of the above map with the projection to $\widehat{\mathfrak{g}}$. The map induces a \widehat{G}^T -equivariant homomorphism of L -algebras $\alpha : S(\widehat{\mathfrak{g}}^*) \rightarrow S((\widehat{\mathfrak{g}}^T)^*)$. Since \mathbb{G}_m acts trivially on \widehat{T} , it follows from the Chevalley's restriction theorem that $S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} = S(\widehat{\mathfrak{g}}^*)^{\widehat{G}^T}$. We thus obtain a Γ -equivariant homomorphism of L -algebras

$$\alpha : S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \rightarrow S((\widehat{\mathfrak{g}}^T)^*)^{\widehat{G}^T}.$$

Let P be a \widehat{G}_X -torsor on X and let $\rho : \mathrm{Gal}_F \rightarrow {}^C P^{\mathrm{ad}}(X)$ be an admissible representation. We define

$$(31) \quad \theta' : S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \xrightarrow{\alpha} S((\widehat{\mathfrak{g}}^T)^*)^{\widehat{G}^T} \xrightarrow{\theta^T} E \otimes \mathcal{O}_X(X)$$

where θ^T is the map defined in (24) with G^T instead of G and regarding ρ as a representation of $\rho : \mathrm{Gal}_F \rightarrow {}^L P^{\mathrm{ad}}(X)$, see Remark 2.12. We note that θ' is Γ -equivariant since both α and θ^T are.

Definition 4.23. *If $\rho : \mathrm{Gal}_F \rightarrow {}^C P^{\mathrm{ad}}(X)$ is an admissible representation and $\sigma : F \hookrightarrow L$ is an embedding of fields then we define a homomorphism of L -algebras*

$$\zeta_{\rho, \sigma}^C : Z(\mathfrak{g}) \otimes_{F, \sigma} L \rightarrow \mathcal{O}_X(X)$$

as the composition of the maps in Lemma 4.18, but using θ' instead of θ . We define a homomorphism of L -algebras

$$\zeta_\rho^C : Z(\mathrm{Res}_{F/\mathbb{Q}_p} \mathfrak{g}) \otimes_{\mathbb{Q}_p} L \rightarrow \mathcal{O}_X(X),$$

as $\zeta_\rho^C := \otimes_\sigma \zeta_{\rho, \sigma}^C$.

Remark 4.24. We remind the reader if $P = \widehat{G}_X$ then ${}^C P^{\mathrm{ad}}(X) = {}^C G_f(\mathcal{O}_X(X))$ and ρ is an admissible representation in the sense of section 2.2.

Remark 4.25. Although the definition makes sense for all admissible ρ , we will apply it to those ρ , where the composition $d \circ \rho : \mathrm{Gal}_F \rightarrow \mathcal{O}_X(X)^*$, where d is defined in (14), is equal to χ_{cyc} , since the Galois representations associated to C -algebraic automorphic forms should satisfy this condition according to [12, Conj. 5.3.4]. In this case, if $U = \mathrm{Sp} A \rightarrow X$ is an affinoid such that $P|_U$ is trivial, the Sen operator

$\Theta_{\text{Sen},U}^T \in (\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}}^T$ is of the form $(M, 1 \otimes 1)$, with $M \in (\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}}$, $1 \otimes 1 \in (\mathbb{C}_p \widehat{\otimes} A) \otimes \text{Lie } \mathbb{G}_m$ and $\text{id} \otimes \alpha$ maps $\Theta_{\text{Sen},U}^T$ to $M + 1 \otimes \delta \in (\mathbb{C}_p \widehat{\otimes} A) \otimes_L \widehat{\mathfrak{g}}$.

Lemma 4.26. *If $Y \rightarrow X$ is a map of rigid varieties over L then $\zeta_{\rho_Y}^C$ is equal to ζ_ρ^C composed with the map $\mathcal{O}_X(X) \rightarrow \mathcal{O}_Y(Y)$.*

Proof. This follows from Lemma 4.22. \square

If there is $\tilde{\delta} \in X_*(\widehat{T})$ such that its image in $X_*(\widehat{T}/Z_{\widehat{G}})$ is equal to δ_{ad} then we have an isomorphism

$$(32) \quad \text{tw}_{\tilde{\delta}} : \widehat{G}^T \cong \widehat{G} \times \mathbb{G}_m, \quad (g, t) \mapsto (g\tilde{\delta}(t), t)$$

If $\tilde{\delta}$ is Γ -invariant then (32) yields ${}^C G_f \cong {}^L G_f \times \mathbb{G}_m$, $P_f^T \cong P_f \times \mathbb{G}_{m,X}$ and

$$(33) \quad \text{tw}_{\tilde{\delta}} : {}^C P^{\text{ad}}(X) \cong {}^L P^{\text{ad}}(X) \times \mathcal{O}_X(X)^*$$

If $\rho : \text{Gal}_F \rightarrow {}^L P^{\text{ad}}(X)$ is an admissible representation then we may define $\rho^C : \text{Gal}_F \rightarrow {}^C P^{\text{ad}}(X)$ via $\rho^C := \text{tw}_{\tilde{\delta}}^{-1} \circ (\rho \boxtimes \chi_{\text{cyc}})$. We note that $d \circ \rho^C = \chi_{\text{cyc}}$ and, if $U = \text{Sp } A \rightarrow X$ is an affinoid such that $P|_U$ is trivial, using Remark 4.25 we obtain that

$$(34) \quad (\text{id} \otimes \alpha)(\Theta_{\text{Sen},U}^T) = \Theta_{\text{Sen},U} - 1 \otimes \tilde{\delta} + 1 \otimes \delta,$$

where $\Theta_{\text{Sen},U}$ is the Sen operator attached to ρ in Lemma 4.6, and we consider $\delta, \tilde{\delta} \in \widehat{\mathfrak{t}}$ via the identification $\widehat{\mathfrak{t}} = X_*(\widehat{T}) \otimes_{\mathbb{Z}} L$.

Going back to the GL_2 example at the beginning of the subsection, we see that if we choose $\tilde{\delta}(t) := \begin{pmatrix} t^{n+1} & 0 \\ 0 & t^n \end{pmatrix}$ for some $n \in \mathbb{Z}$ then $\zeta_{\rho^C}^C$ is equal to the infinitesimal character of $\text{Sym}^{a-b-1} \otimes \det^{b-n}$.

4.8. The archimedean case. We check that our definition is compatible with the archimedean case.

Assume now that F is an archimedean local field and let \overline{F} be some algebraic closure of F , so that $[\overline{F} : F]$ is 1 or 2. We recall that the Weil group of F is the semidirect product $W_F = \overline{F}^\times \rtimes \text{Gal}_F$. If $F = \mathbb{R}$, we denote by c the non trivial element of Gal_F .

Let $\rho : W_F \rightarrow {}^L G(\mathbb{C})$ be some admissible representation. The restriction of ρ to \overline{F}^\times takes values in $\widehat{G}(\mathbb{C})$ and, up to conjugate by an element of $\widehat{G}(\mathbb{C})$, we can assume that $\rho(\overline{F}^\times) \subset \widehat{T}(\mathbb{C})$. Let σ_1 and σ_2 be the two isomorphisms of \overline{F} with \mathbb{C} . Identifying $X_*(\widehat{T}) \otimes \mathbb{C}$ with the Lie algebra of $\widehat{T}(\mathbb{C})$, we see that there exists two elements λ_{σ_1} and λ_{σ_2} in $X_*(\widehat{T}) \otimes \mathbb{C}$ such that

$$\rho(z) = \exp(\log(\sigma_1(z))\lambda_{\sigma_1} + \log(\sigma_2(z))\lambda_{\sigma_2})$$

for $z \in \overline{F}^\times$. We can assume that $\lambda_{\sigma_1} - \lambda_{\sigma_2} \in X_*(\widehat{T})$ so that this quantity doesn't depend on the choice of the branch of log.

If $F = \mathbb{R}$, then we have $\lambda_\tau = \text{Ad}(\rho(c))\lambda_\sigma$ so that λ_τ and λ_σ defines the same element of $(X_*(\widehat{T}) \otimes \mathbb{C}/W)$. Evaluation on this element gives us a linear map $\theta : S(\widehat{\mathfrak{t}}^*)^W \rightarrow \mathbb{C}$ which, by composition with Harish-Chandra isomorphism $Z(\mathfrak{g})_{\mathbb{C}} \simeq S(\widehat{\mathfrak{t}}^*)^W$ gives rise to a character $\zeta_\rho : Z(\mathfrak{g})_{\mathbb{C}} \rightarrow \mathbb{C}$.

If $F = \mathbb{C}$, then we use the elements λ_σ and λ_τ in $X_*(\widehat{T}) \otimes \mathbb{C}$ to define a character $\zeta_{\rho,\sigma} : Z(\mathfrak{g}) \otimes_{\overline{F},\sigma} \mathbb{C} \rightarrow \mathbb{C}$ for each embedding σ of \overline{F} in \mathbb{C} . We finally define

$$\zeta_\rho : Z(\mathrm{Res}_{F/\mathbb{R}} \mathfrak{g})_{\mathbb{C}} \xrightarrow{\zeta_{\rho,\sigma_1} \otimes \zeta_{\rho,\sigma_2}} \mathbb{C}.$$

Remark 4.27. If ρ is associated to an irreducible representation of $G(F)$, then the center $Z(\mathrm{Res}_{F/\mathbb{R}} \mathfrak{g})$ acts on π by ζ_ρ (see Propositions 7.4 and 7.10 in [87] which follow from [56]).

5. HODGE–TATE REPRESENTATIONS

We will study Hodge–Tate representations following [79], [3] and [12, §2.4]⁸. Let L be a finite extension of \mathbb{Q}_p and let V be a finite dimensional L -vector space with a continuous L -linear action of Gal_F . Consider the semi-linear diagonal action of Gal_F on $W = V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ and let $W(i)$ be the i th Tate twist of W . The natural map

$$\bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} W(-i)^{\mathrm{Gal}_F} \rightarrow W$$

is injective and we say that V is Hodge–Tate if this map is an isomorphism. Suppose that this is the case and let W_i be the image of $\mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} (W(-i))^{\mathrm{Gal}_F}$ in W , i.e. the \mathbb{C}_p -span of the set of $v \in W$ such that

$$gv = \chi^i(g)v, \quad \forall g \in \mathrm{Gal}_F,$$

where χ is the p -adic cyclotomic character. Then $W = \bigoplus_{i \in \mathbb{Z}} W_i$ and the Hodge–Tate cocharacter μ_V of V is the algebraic morphism $\mathbb{G}_{m,\mathbb{C}_p} \rightarrow \mathrm{GL}(W)$ defined by

$$\mu_V(z) = \sum_{i \in \mathbb{Z}} z^i \mathrm{id}_{W_i}.$$

Note that the Sen operator of V is simply multiplication by i on W_i , in particular

$$(35) \quad \mathrm{Lie}(\mu_V)(1) = \Theta_{\mathrm{Sen},V}$$

Let H be a reductive group defined over L and let $\mathrm{Rep}(H)_L$ be the category of algebraic representations of H defined over L . Let $\rho : \mathrm{Gal}_F \rightarrow H(L)$ be a continuous representation. We say that ρ is Hodge–Tate if the action of Gal_F on V obtained by composing ρ with $r : H \rightarrow \mathrm{GL}(V)$ is Hodge–Tate for every $(r, V) \in \mathrm{Rep}(H)_L$.

Let $\rho : \mathrm{Gal}_F \rightarrow H(L)$ be a Hodge–Tate representation. Given a \mathbb{C}_p -algebra R and $z \in R^\times$, for each $(r, V) \in \mathrm{Rep}(H)_L$ we obtain an automorphism $\mathrm{id} \otimes \mu_V(z)$ of $R \otimes_{\mathbb{C}_p} (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V) \simeq R \otimes_{\mathbb{Q}_p} V \simeq (R \otimes_{\mathbb{Q}_p} L) \otimes_L V$, compatible with tensor products when we vary V . It follows that there is a unique $\mu_\rho(z) \in H(R \otimes_{\mathbb{Q}_p} L)$ such that $\mathrm{id} \otimes \mu_V(z) = r(\mu_\rho(z))$ for all $(V, r) \in \mathrm{Rep}(H)_L$. Varying R and observing that $H(R \otimes_{\mathbb{Q}_p} L) = (\mathrm{Res}_{L/\mathbb{Q}_p} H)_{\mathbb{C}_p}(R)$, we obtain a cocharacter

$$\mu_\rho : \mathbb{G}_{m,\mathbb{C}_p} \rightarrow (\mathrm{Res}_{L/\mathbb{Q}_p} H)_{\mathbb{C}_p}.$$

Using the decomposition

$$(\mathrm{Res}_{L/\mathbb{Q}_p} H)_{\mathbb{C}_p} \simeq \prod_{v \in \mathrm{Hom}(L, \mathbb{C}_p)} H \times_{L,v} \mathbb{C}_p,$$

we can write $\mu_\rho = (\mu_{\rho,v})_{v \in \mathrm{Hom}(L, \mathbb{C}_p)}$ with $\mu_{\rho,v} : \mathbb{G}_{m,\mathbb{C}_p} \rightarrow H \times_{L,v} \mathbb{C}_p$. Note that since \mathbb{G}_m is connected, $\mu_{\rho,v}$ factors through the connected component of $H \times_{L,v} \mathbb{C}_p$.

⁸We note that the published version of §2.4 in [12] has been updated in <https://arxiv.org/pdf/1009.0785.pdf>

Arguing as in the proof of lemma 4.6 one associates to *any* continuous representation $\rho : \text{Gal}_F \rightarrow H(L)$ a Sen operator $\Theta_{\text{Sen},\rho} \in (\mathbb{C}_p \otimes_{\mathbb{Q}_p} L) \otimes_L \mathfrak{h} \simeq \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathfrak{h}$, where \mathfrak{h} is the Lie algebra of H . When ρ is Hodge–Tate relation (1) combined with the Tannakian description of both $\Theta_{\text{Sen},\rho}$ and μ_ρ yields⁹

$$(36) \quad \text{Lie}(\mu_\rho)(1) = \Theta_{\text{Sen},\rho}$$

Lemma 5.1. *A continuous representation $\rho : \text{Gal}_F \rightarrow H(L)$ is Hodge–Tate if and only if there is $\mu : \mathbb{G}_{m,\mathbb{C}_p} \rightarrow (\text{Res}_{L/\mathbb{Q}_p} H)_{\mathbb{C}_p}$ such that $\Theta_{\text{Sen},\rho} = \text{Lie}(\mu)(1)$, in which case we necessarily have $\mu = \mu_\rho$.*

Proof. The only fact which hasn’t already been explained is that the existence of μ forces ρ being Hodge–Tate. Let $(r, V) \in \text{Rep}_L(H)$. We want to prove that V endowed with the action of Gal_F via composition with ρ is Hodge–Tate. The cocharacter μ and the representation r give rise to an algebraic morphism $r \circ \mu : \mathbb{C}_p^\times \rightarrow (\text{Res}_{L/\mathbb{Q}_p} H)_{\mathbb{C}_p}(\mathbb{C}_p) = H(\mathbb{C}_p \otimes_{\mathbb{Q}_p} L) \rightarrow \text{GL}(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)$ and $\Theta_{\text{Sen},V} = \text{Lie}(r)(\Theta_{\text{Sen},\rho})(1) = \text{Lie}(r \circ \mu)(1)$. It follows that $\Theta_{\text{Sen},V}$ is semi-simple with integer eigenvalues and V is Hodge–Tate by the corollary to theorem 6 in [76]. \square

5.1. Galois representations valued in L -groups. Now we assume that G is a connected reductive group over F and let $\rho : \text{Gal}_F \rightarrow {}^L G(L)$ be an admissible representation, so that we are in the situation of Section 4 with X equal to a point, so that $\Gamma(X, \mathcal{O}_X) = L$. We assume that ρ is Hodge–Tate and apply the above discussion to $H = {}^L G_f$. Let $T \subset G_{\overline{F}}$ a maximal split torus and let \widehat{T} be a dual torus in \widehat{G} . For each v , the cocharacter $\mu_{\rho,v}$ is conjugate inside $\widehat{G}(\mathbb{C}_p)$ to a cocharacter $\mathbb{G}_{m,\mathbb{C}_p} \rightarrow \widehat{T}_{\mathbb{C}_p}$, which is uniquely determined up to the action of the Weyl group W . Consequently we obtain a well defined element

$$\nu_{\rho,v} \in X^*(T)/W = X_*(\widehat{T})/W.$$

We may think of $\nu_{\rho,v}$ as a W -orbit of an element in $\widehat{\mathfrak{t}} \otimes_{L,v} \mathbb{C}_p = X_*(\widehat{T}) \otimes_{\mathbb{Z}} \mathbb{C}_p$.

Proposition 5.2. *The composition*

$$(37) \quad S(\widehat{\mathfrak{t}}^*)^W \xrightarrow[(25)]{\cong} S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \xrightarrow[(24)]{\theta} \mathbb{C}_p \otimes L \xrightarrow{m_v} \mathbb{C}_p$$

where the last arrow is given by $x \otimes y \mapsto xv(y)$, is equal to the evaluation map at $\nu_{\rho,v} \in (\widehat{\mathfrak{t}} \otimes_{L,v} \mathbb{C}_p)/W$.

Proof. Let $\Theta_{\text{Sen}} \in (\mathbb{C}_p \otimes L) \otimes_L \widehat{\mathfrak{g}} = \mathbb{C}_p \otimes_{\mathbb{Q}_p} \widehat{\mathfrak{g}}$ be the Sen operator defined in Lemma 4.6. Since X is a point we will suppress the affinoid U from the notation. Since $\Theta_{\text{Sen}} = \text{Lie}(\mu_\rho)(1)$, the image of Θ_{Sen} in $\widehat{\mathfrak{g}} \otimes_{L,v} \mathbb{C}_p$ is equal to $\text{Lie}(\mu_{\rho,v})(1)$. Since $\mu_{\rho,v}$ and $\nu_{\rho,v}$ are conjugate by an element of $\widehat{G}(\mathbb{C}_p)$, the proposition is proved. \square

Recall that the group Gal_F acts on the pinned root datum, which induces an action on $X_*(\widehat{T})/W$, which we denote by \cdot . We can now reprove [12, Lem. 2.4.1].

Proposition 5.3. *For $g \in \text{Gal}_F$, we have $\nu_{\rho,gv} = g \cdot \nu_{\rho,v}$ in $X_*(\widehat{T})/W$.*

Proof. Let $W = (\mathbb{C}_p \otimes L) \otimes_L \widehat{\mathfrak{g}}$. We refer to the discussion preceding lemma 4.7 for the definitions of the various actions of Gal_F and $\widehat{G}(\mathbb{C}_p \otimes L)$ on W used in this proof. For each embedding $v : L \rightarrow \mathbb{C}_p$ let $W_v = \mathbb{C}_p \otimes_{L,v} \widehat{\mathfrak{g}}$. Let $X_v \in W_v$ be

⁹Once we identify the \mathbb{Q}_p -Lie algebras $\text{Lie}(\text{Res}_{L/\mathbb{Q}_p}(H))$ and \mathfrak{h} .

the image of $X \in W$ via the natural isomorphism $W \simeq \prod_{v:L \rightarrow \mathbb{C}_p} W_v$. One easily checks that the map $\tilde{\gamma}_v : W_v \rightarrow W_{\gamma v}$ defined by $\tilde{\gamma}_v(x \otimes y) = \gamma(x) \otimes y$ satisfies

$$(38) \quad (\gamma * X)_{\gamma v} = \tilde{\gamma}_v(X_v)$$

In particular, if $X = (1 \otimes u_v)_v \in \prod_{v:L \rightarrow \mathbb{C}_p} W_v \simeq W$, with $u_v \in \hat{\mathfrak{g}}$, then for all $\gamma \in \text{Gal}_F$ we have $[\gamma * (\gamma \cdot X)]_{\gamma v} = 1 \otimes \gamma \cdot u_v$, where $\gamma \cdot u_v$ is the action of γ on u_v induced by μ_G .

Now let $\Theta = \Theta_{\text{Sen}, U}$ and consider

$$X = (1 \otimes \text{Lie}(\nu_{\rho, v})(1))_v \in \prod_{v:L \rightarrow \mathbb{C}_p} W_v \simeq W.$$

By definition of $\nu_{\rho, v}$ and relation (36), there is $g \in \hat{G}(\mathbb{C}_p \otimes L) \simeq \prod_{v:L \rightarrow \mathbb{C}_p} \hat{G}(\mathbb{C}_p)$ such that $\Theta = \text{Ad}(g)(X)$. Lemma 4.7 combined with relation (21) show that there exists $h \in \hat{G}(\mathbb{C}_p \otimes L)$ (more precisely $h = g^{-1} c_\gamma s g s^{-1}$, with $s = \mu_G(\tilde{\gamma})$) such that

$$(39) \quad \text{Ad}(h)(\gamma * (\gamma \cdot X)) = X.$$

Writing $h = (h_v)_v \in \prod_{v:L \rightarrow \mathbb{C}_p} \hat{G}(\mathbb{C}_p)$, projecting relation (39) in $W_{\gamma v}$ and taking into account the discussion above yields

$$\text{Ad}(h_{\gamma v})(1 \otimes \gamma \cdot \text{Lie}(\nu_{\rho, v})(1)) = 1 \otimes \text{Lie}(\nu_{\rho, \gamma v})(1).$$

It follows that $\gamma \cdot \nu_{\rho, v}$ and $\nu_{\rho, \gamma v}$ are the same in $X_*(\hat{T})/W$. \square

Remark 5.4. Let us note that Gal_F -orbits of embeddings $v : L \hookrightarrow \mathbb{C}_p$ are canonically in bijection with the set of embeddings $\tau : F \hookrightarrow L$. Concretely, $F = v(L)^{\text{Gal}_F}$ and $\tau = v^{-1}|_F$. Similarly, Gal_E -orbits of embeddings $v : L \hookrightarrow \mathbb{C}_p$ are in bijection with the set of embeddings $\tau : E \hookrightarrow L$. It follows from Proposition 5.3 that if $v, v' : L \hookrightarrow \mathbb{C}_p$ lie in the same Gal_E -orbit then $\nu_{\rho, v} = \nu_{\rho, v'}$.

5.2. Galois representations valued in C -groups. We will consider the set up of §2.1. Since \mathbb{G}_m acts trivially on \hat{T} , $\hat{T} \times \mathbb{G}_m$ is a maximal torus of \hat{G}^T , and $\hat{B} \rtimes \mathbb{G}_m$ is a Borel subgroup of \hat{G}^T . We also recall that ${}^C G$ is the L -group of G^T , which is a central extension of G by \mathbb{G}_m over F , so that the previous discussion in this section applies.

Proposition 5.5. *Let $\rho : \text{Gal}_F \rightarrow {}^C G_f(L)$ be an admissible representation such that $d \circ \rho = \chi_{\text{cyc}}$, let $\sigma : F \hookrightarrow L$, $\tau : E \hookrightarrow L$, $v : L \hookrightarrow \mathbb{C}_p$ be field embeddings such that $v \circ \tau = \text{id}_E$ and $\tau|_F = \sigma$.*

If ρ is Hodge–Tate and the image of $\nu_{\rho, v}$ under

$$(40) \quad X_*(\hat{T} \times \mathbb{G}_m)/W \rightarrow X_*(\hat{T})/W = X^*(T)/W = X^*(T \times_{E, \tau} L)/W$$

contains a character λ_τ , which is dominant with respect to $B \times_{E, \tau} L$, then $\zeta_{\rho, \sigma}^C$, defined in Definition 4.23, is an infinitesimal character of irreducible representation of highest weight λ_τ with respect to $B \times_{E, \tau} L$.

If $\zeta_{\rho, \sigma}^C$ is an infinitesimal character of irreducible algebraic representation V_σ of $G \times_{F, \sigma} L$ for every $\sigma : F \hookrightarrow L$, and the highest weight λ_τ of V_σ with respect to $B \times_{E, \tau} L$ is regular for all τ (equivalently, for at least one τ above every σ) then ρ is Hodge–Tate and $\nu_{\rho, v}$ maps to the orbit of λ_τ under (40), where $v \circ \tau|_E = \text{id}_E$.

Proof. We first recall the definition of $\zeta_{\rho,\sigma}^C$. Let $\Theta_{\text{Sen}}^T \in (\mathbb{C}_p \otimes L) \otimes_L \widehat{\mathfrak{g}}^T$. As explained in Remark 4.25, Θ_{Sen}^T is of the form $(M, 1 \otimes 1)$ with $M \in (\mathbb{C}_p \otimes L) \otimes_L \widehat{\mathfrak{g}}$. Let $\Theta_{\text{Sen},v}^T$ be the image of Θ_{Sen}^T in $\widehat{\mathfrak{g}}^T \otimes_{L,v} \mathbb{C}_p$ and let M_v be the image of M in $\widehat{\mathfrak{g}}^T \otimes_{L,v} \mathbb{C}_p$ under the base change along $m_v : \mathbb{C}_p \otimes L \rightarrow \mathbb{C}_p$, so that $\Theta_{\text{Sen},v}^T = (M_v, 1 \otimes 1)$.

Then $\zeta_{\rho,\sigma}^C$ is equal to the composition

$$Z(\mathfrak{g}) \otimes_{F,\sigma} L \xrightarrow{\kappa_\tau} S(\widehat{\mathfrak{t}}^*)^W \xrightarrow{\cong} S(\widehat{\mathfrak{g}}^*)^{\widehat{G}} \xrightarrow{\text{ev}_{M_v+\delta}} v(L) \xrightarrow{v^{-1}} L.$$

Let us point out that (37) in our situation is the map

$$S((\widehat{\mathfrak{t}} \oplus \text{Lie } \mathbb{G}_m)^*)^W \xrightarrow{\cong} S((\widehat{\mathfrak{g}} \oplus \text{Lie } \mathbb{G}_m)^*)^{\widehat{G}^T} \xrightarrow{\text{ev}_{(M_v,1)}} v(L).$$

Let us assume ρ is Hodge–Tate and that $\lambda^T \in \nu_{\rho,v}$ maps to a dominant weight $\lambda \in X^*(T \times_{E,\tau} L)$ with respect to $B \times_{E,\tau} L$. We will denote the images of λ in $X_*(\widehat{T})$ and $X^*(T)$ by the same letter. It follows from Proposition 5.2 that $\zeta_{\rho,\sigma}^C = \text{ev}_{\lambda+\delta} \circ \kappa_\tau$ and Lemma 4.15 implies that $\zeta_{\rho,\sigma}^C$ is the infinitesimal character of $V(\lambda)$.

Let us assume that $\zeta_{\rho,\sigma}^C$ is equal to the infinitesimal character of irreducible representation V_σ of $G \times_{E,\sigma} L$, which is of highest weight λ_τ with respect to $B \times_{E,\tau} L$. Let us additionally assume that λ_τ is regular. Lemma 4.15 implies that the maps ev_{λ_τ} and ev_{M_v} coincide, when restricted to $S(\widehat{\mathfrak{g}}^*)^{\widehat{G}}$. This implies that if we write $M_v = x_s + x_u \in \widehat{\mathfrak{g}} \otimes_{L,v} \mathbb{C}_p$, where x_s is semisimple, x_u is nilpotent and $[x_s, x_u] = 0$, then x_s is conjugate to λ_τ by an element of $\widehat{G}(\mathbb{C}_p)$, see for example [48, Prop. 10.3.1]. Since λ_τ is regular, $x_u = 0$ and thus M_v is conjugate to λ_τ by an element of $\widehat{G}(\mathbb{C}_p)$. This implies that there is $\mu'_v : \mathbb{G}_{m,\mathbb{C}_p} \rightarrow \widehat{G}^T \times_{L,v} \mathbb{C}_p$, such that $d \circ \mu'_v = \text{id}_{\mathbb{G}_m}$ and $\text{Lie}(\mu'_v) = \Theta_{\text{Sen},v}^T$. Thus $\mu := (\mu'_v)_v : \mathbb{G}_{m,\mathbb{C}_p} \rightarrow (\text{Res}_{L/\mathbb{Q}_p} \widehat{G}^T)_{\mathbb{C}_p}$ satisfies $\text{Lie}(\mu) = \Theta_{\text{Sen}}^T$. It follows from lemma 5.1 that ρ is Hodge–Tate and $\mu' = \mu_\rho$. On then checks as in the proof of Proposition 5.2 that $\nu_{\rho,v}$ maps to the W -orbit of λ_τ under (40). \square

Lemma 5.6. *Let $\rho, \rho' : \text{Gal}_F \rightarrow {}^C G_f(L)$ be admissible representations, such that $d \circ \rho = d \circ \rho' = \chi_{\text{cyc}}$. We write $\rho(\gamma) = (c_\gamma, \bar{\gamma})$, $\rho'(\gamma) = (c'_\gamma, \bar{\gamma})$ for all $\gamma \in \text{Gal}_F$, where $c_\gamma, c'_\gamma \in \widehat{G}^T(L)$. Assume that*

$$\text{tr}_V(r(c_\gamma)) = \text{tr}_V(r(c'_\gamma)), \quad \forall \gamma \in \text{Gal}_E,$$

for all algebraic representations (r, V) of \widehat{G}^T . Then $\zeta_\rho^C = \zeta_{\rho'}^C$.

Proof. Let $\Theta = \Theta_{\text{Sen},\rho}^T, \Theta' = \Theta_{\text{Sen},\rho'}^T \in (\mathbb{C}_p \otimes L) \otimes_L \widehat{\mathfrak{g}}^T$ be the respective Sen operators, where we consider ${}^C G_f$ as the L -group of G^T . We claim that $f(\Theta) = f(\Theta')$ for all $f \in S((\widehat{\mathfrak{g}}^T)^*)^{\widehat{G}^T}$. The claim implies that the maps θ' in (31) associated to representations ρ and ρ' coincide, which implies the assertion.

To prove the claim we note that the polynomial function $x \mapsto \text{tr}_V(\text{Lie}(r)(x)^k)$ is an element of $S((\widehat{\mathfrak{g}}^T)^*)^{\widehat{G}^T}$ and it follows from Chevalley’s restriction theorem that these functions, for all algebraic representations (r, V) of \widehat{G}^T and all $k \geq 1$, span $S((\widehat{\mathfrak{g}}^T)^*)^{\widehat{G}^T}$ as an L -vector space. Thus it suffices to show that

$$\text{tr}(\text{Lie}(r)(\Theta)^k) = \text{tr}(\text{Lie}(r)(\Theta')^k), \quad \forall k \geq 1,$$

which in turn is equivalent to $\text{tr}(\Theta_{\text{Sen},r \circ \rho}^k) = \text{tr}(\Theta_{\text{Sen},r \circ \rho'}^k)$ for all $k \geq 1$. By assumption the Galois representations $r \circ \rho$ and $r \circ \rho'$ restricted to Gal_E have the

same semi-simplification. Note that the restriction to Gal_E does not change the Sen operator. It follows from (15) that the functor $W \mapsto D_{\text{Sen}}(W)$ is exact. This implies that for every embedding $v : L \hookrightarrow \mathbb{C}_p$ the semi-simple parts of $\Theta_{\text{Sen}, r \circ \rho, v}$ and $\Theta_{\text{Sen}, r \circ \rho', v}$ are conjugate in $\text{GL}(V \otimes_{L, v} \mathbb{C}_p)$ and the assertion follows. \square

5.3. Algebraic representations and regular Hodge–Tate weights. We would like to spell out a part of the proof of Proposition 5.5, since this is important for the p -adic Langlands correspondence.

We assume the setup at the beginning of section 4 and let $\rho : \text{Gal}_F \rightarrow {}^C G_f(L)$ be a continuous admissible representation, satisfying $d \circ \rho = \chi_{\text{cyc}}$. We assume that ρ is Hodge–Tate. For each embedding $v : L \hookrightarrow \mathbb{C}_p$ we get a W -orbit $\nu_{\rho, v}^T \in X_*(\widehat{T} \times \mathbb{G}_m)$. Each $\lambda^T \in \nu_{\rho, v}^T$ is of the form $(\lambda, \text{id}_{\mathbb{G}_m})$ with $\lambda \in X_*(\widehat{T})$. We let $\nu_{\rho, v}^C$ be the image of $\nu_{\rho, v}^T$ in $X_*(\widehat{T})$. It follows from Remark 5.4 that $\nu_{\rho, v}^C$ depends only on Gal_E -orbit of v and these are canonically in bijection with the set of embeddings $\tau : E \hookrightarrow L$. Let us reindex the W -orbits by the embeddings $\tau : E \hookrightarrow L$. We may identify $X_*(\widehat{T}) = X^*(T) = X^*(T \times_{E, \tau} L)$, so that $\nu_{\rho, \tau}^C \subset X^*(T \times_{E, \tau} L)$.

Definition 5.7. *We say that ρ has regular Hodge–Tate weights if for each $\tau, \nu_{\rho, \tau}^C$ contains $\lambda_\tau \in X^*(T \times_{E, \tau} L)$, which is dominant with respect to $B \times_{E, \tau} L$.*

Remark 5.8. Since the action of Gal_F on the root system permutes the positive roots, Corollary 5.3 implies that it is enough to check this condition for at least one τ above every $\sigma : F \hookrightarrow L$.

Definition 5.9. *If ρ has regular Hodge–Tate weights then we define an irreducible algebraic representation $\pi_{\text{alg}}(\rho)$ of $(\text{Res}_{F/\mathbb{Q}_p} G)_L \cong \prod_{\sigma} G \times_{F, \sigma} L$ by*

$$\pi_{\text{alg}}(\rho) := \bigotimes_{\sigma: F \hookrightarrow L} V_{\sigma},$$

where V_{σ} is an irreducible algebraic representation of $G \times_{F, \sigma} L$ of highest weight λ_{τ} with respect to $B \times_{E, \tau} L$, for any $\tau : E \hookrightarrow L$ above σ .

Remark 5.10. The definition of V_{σ} is independent of the choice of τ : any two lie in the same Gal_F -orbit and Corollary 5.3 together with the proof of Lemma 4.16 imply the assertion.

Remark 5.11. It follows from the proof of Proposition 5.5 that under the above assumptions ζ_{ρ}^C , defined in Definition 4.23, is the infinitesimal character of $\pi_{\text{alg}}(\rho)$.

We will discuss the notion of regular Hodge–Tate weights in the presence of the twisting element, see the end of subsection 4.7. Recall that $\delta_{\text{ad}} \in X_*(\widehat{T}/Z_{\widehat{G}})$ is the unique element such that $2\delta_{\text{ad}}$ is equal to the image of 2δ . We assume that there is $\tilde{\delta} \in X_*(\widehat{T})$ such that its image in $X_*(\widehat{T}/Z_{\widehat{G}})$ is equal to δ_{ad} . Let $\rho : \text{Gal}_F \rightarrow {}^L G(L)$ be an admissible representation, which we assume to be Hodge–Tate. Let $\nu_{\rho, \tau}$ be the W -orbit corresponding to Gal_E -orbit of $v : L \hookrightarrow \mathbb{C}_p$.

Lemma 5.12. *Let $\rho^C : \text{Gal}_F \rightarrow {}^C G_f(L)$ be the representation*

$$\rho^C := \text{tw}_{\tilde{\delta}}^{-1} \circ (\rho \boxtimes \chi_{\text{cyc}}),$$

where $\text{tw}_{\tilde{\delta}}$ is defined in (32). Then ρ^C is Hodge–Tate and its Hodge–Tate weights are regular if and only if for every $\tau : E \hookrightarrow L$ there is $\lambda_{\tau} \in \nu_{\rho, \tau} \subset X^*(\widehat{T} \times_{E, \tau} L)$ such that $\lambda_{\tau} - \tilde{\delta}$ is dominant with respect to $B \times_{E, \tau} L$.

Proof. The assertion follows from (34). \square

Example 5.13. Let G be an inner form of $\mathrm{GL}_{n,F}$, B the upper-triangular matrices over E and T the diagonal matrices over E . Then $\widehat{G} = \mathrm{GL}_{n,L}$, \widehat{B} the upper-triangular matrices over L and \widehat{T} the diagonal matrices over L . Let $\rho : \mathrm{Gal}_F \rightarrow \mathrm{GL}_n(L)$ be a Hodge–Tate representation. Since G is an inner form, the action of Gal_F on the root system is trivial, so all $\nu_{\rho,\tau}$ above depend only on $\tau|_F$. We may identify $X_*(\widehat{T})$ with the set of n -tuples of integers, where (k_1, \dots, k_n) corresponds to $t \mapsto \mathrm{diag}(t^{k_1}, \dots, t^{k_n})$. The action of $W = S_n$ permutes the entries. With the above identification let

$$\tilde{\delta} := (0, -1, \dots, 1 - n).$$

If $\lambda \in \nu_{\rho,\sigma}$ corresponds to an n -tuple $(k_{1,\sigma}, \dots, k_{n,\sigma})$ then $\lambda - \tilde{\delta}$ is dominant with respect to $B \times_{E,\tau} L$ if and only if

$$k_{1,\sigma} \geq k_{2,\sigma} + 1 \geq \dots \geq k_{n,\sigma} + n - 1,$$

which is equivalent to $k_{1,\sigma} > k_{2,\sigma} > \dots > k_{n,\sigma}$. This is the usual notion of regular Hodge–Tate weights in the p -adic Hodge theory.

Remark 5.14. If $G = \mathrm{GL}_{n,F}$ and $\rho : \mathrm{Gal}_F \rightarrow \mathrm{GL}_n(L)$ is a Hodge–Tate representation with regular Hodge–Tate weights in the traditional sense then the algebraic representation $\pi_{\mathrm{alg}}(\rho)$ defined in [15, §1.8] coincides with $\pi_{\mathrm{alg}}(\rho^C)$ defined in Definition 5.9 with $\tilde{\delta}$ as above. We caution the reader that in order to verify this one has to bear in mind that in [15] the Hodge–Tate weight of the cyclotomic character is -1 and in this paper it is 1 .

5.4. Automorphic representations. In this section we fix an embedding $\iota : L \rightarrow \mathbb{C}$. Now let F be a number field, G a connected reductive group over F and \mathfrak{g} its Lie algebra. Let π be some automorphic representation of $G(\mathbb{A}_F)$ which is C -algebraic in the sense of [12, Def. 3.1.2]. Let π_∞ be the representation of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ which is the archimedean part of π . Then the center $Z(\mathrm{Res}_{F/\mathbb{Q}} \mathfrak{g})_{\mathbb{C}}$ of the enveloping algebra $U(\mathrm{Res}_{F/\mathbb{Q}} \mathfrak{g})_{\mathbb{C}}$ acts on the subset of smooth vectors of π_∞ by a character χ . The decomposition

$$Z(\mathrm{Res}_{F/\mathbb{Q}} \mathfrak{g})_{\mathbb{C}} \simeq \bigotimes_{\kappa: F \hookrightarrow \mathbb{C}} Z(\mathfrak{g}) \otimes_{F,\kappa} \mathbb{C}$$

gives a decomposition $\chi = \bigotimes_{\kappa} \chi_{\kappa}$ of the character χ .

Proposition 5.15. *Let $\rho_{\pi,\iota} : \mathrm{Gal}_F \rightarrow {}^C G(L)$ be a continuous admissible representation satisfying the assumptions of Conjecture 5.3.4 in [12]. Let v be a place of F dividing p and let $\rho_v := \rho_{\pi,\iota}|_{\mathrm{Gal}_{F_v}}$. Then we have*

$$\zeta_{\rho_v}^C \otimes_{L,\iota} \mathbb{C} = \bigotimes_{\kappa: F_v \hookrightarrow \overline{\mathbb{Q}}_p} \chi_{\iota \circ \kappa}.$$

Proof. Let $\xi : \mathbb{G}_m \rightarrow {}^C G$ be the cocharacter defined by $t \mapsto (2\delta(t^{-1}), t^2)$.

If v is an archimedean place of F , the local Langlands correspondence gives us an admissible morphism $\rho_v : W_{F_v} \rightarrow {}^L G(\mathbb{C})$. We recalled in section 4.8 how to associate to an embedding κ of F in \mathbb{C} above v , an element of $\lambda_{\kappa} \in X_*(\widehat{T}) \otimes \mathbb{C}$, well defined up to the action of the Weyl group. This gives us $\lambda_{\kappa} \in X_*(\widehat{T}) \otimes \mathbb{C}$ for each embedding κ of F in \mathbb{C} . Saying that π is C -algebraic is equivalent to the condition $\lambda_{\sigma} + \frac{1}{2}\xi \in X_*(\widehat{T} \times \mathbb{G}_m)$ for each $\kappa : F \hookrightarrow \mathbb{C}$. Note that we have $\lambda_{\kappa} \in X_*(\widehat{T}) \otimes \mathbb{Q}$.

Now let v be a place of F dividing p . According to Conjecture 5.3.4 and Remark 5.3.5 in [12], the representation ρ is Hodge–Tate at v and the Hodge–Tate cocharacter of ρ associated to an embedding $\kappa : F_v \hookrightarrow L$ is $\lambda_{\iota \circ \kappa} + \frac{1}{2}\xi$.

Recall from Remark 4.27 that for an embedding $\kappa : F \hookrightarrow \mathbb{C}$ the character χ_κ is the composite

$$Z(\mathfrak{g})_{\mathbb{C}} \xrightarrow{\sim} S(\widehat{\mathfrak{t}}_{\mathbb{C}}^*)^W \xrightarrow{\lambda_\kappa} \mathbb{C}$$

where the first map is the Harish-Chandra isomorphism over \mathbb{C} which can be obtained from the Harish-Chandra isomorphism over L after base change along $\iota : L \hookrightarrow \mathbb{C}$.

If $\kappa : F_v \hookrightarrow L$, the character $\zeta_{\rho, \kappa}^C$ is the composite

$$Z(\mathfrak{g}) \otimes_{F, \kappa} L \xrightarrow{\sim} S(\widehat{\mathfrak{t}}^*)^W \xrightarrow{\lambda_\kappa} L$$

since λ_κ is exactly the image of $(\lambda_\kappa + \frac{1}{2}\xi, 1)$ in $\widehat{\mathfrak{g}}_L$ by the map α of section 4.7 (note that since $\lambda_\kappa \in X_*(\widehat{T}) \otimes \mathbb{Q}$, we land in $L \subset E \otimes L$). It follows that $\chi_{\iota \circ \kappa} = \zeta_{\rho, \kappa}^C \otimes_{L, \iota} \mathbb{C}$ and hence the result. \square

6. FIBRES OF COHEN–MACAULAY MODULES

Let (R, \mathfrak{m}) be a complete local noetherian \mathcal{O} -algebra with residue field k , which we assume to be \mathcal{O} -torsion free. Let R^{rig} be the ring of global functions on the rigid space $\mathfrak{X}^{\text{rig}}$, associated to the formal scheme $\text{Spf } R$, see [53, §7.1].

Proposition 6.1. *Let M be a faithful, finitely generated R -module and let $M^{\text{rig}} := M \otimes_R R^{\text{rig}}$. If R is reduced and M is Cohen–Macaulay then the natural map*

$$M^{\text{rig}} \rightarrow \prod_{x \in \mathfrak{X}^{\text{rig}}} M^{\text{rig}} \otimes_{R^{\text{rig}}} \kappa(x)$$

is injective.

Proof. Since R is \mathcal{O} -torsion free we may choose a finite injective map of \mathcal{O} -algebras $S := \mathcal{O}[[x_1, \dots, x_d]] \rightarrow R$, where necessarily $d + 1 = \dim R$, see [58, Thm. 29.4 (iii)] and the Remark following it. The assumptions on M imply that M is a faithful, Cohen–Macaulay module over S . Since S is regular it follows from the Auslander–Buchsbaum theorem, [58, Thm. 19.1], that M is free over S . The finite injective map $S[1/p] \rightarrow R[1/p]$ induces a finite surjective morphism

$$\varphi : X := \text{Spec } R[1/p] \rightarrow Y := \text{Spec } S[1/p].$$

The fibre of the generic point η of Y is geometrically reduced. Indeed, it suffices to check that it is reduced (since it lives in characteristic zero), which is clear since it is a localization of the reduced ring $R[1/p]$. Thus Lemma 37.24.4 in [83, Tag 0574] yields a nonzero $f \in R[1/p]$ such that all fibres over the non-vanishing locus of f are reduced.

Let Σ be the set of closed points y of Y such that $f(y) \neq 0$. For any $y \in \Sigma$ and any $x \in \varphi^{-1}(y)$ we have $x \in \mathfrak{m}\text{-Spec } R[1/p]$, thus x corresponds to a point of $\mathfrak{X}^{\text{rig}}$ with the same residue field, see [53, Lem. 7.1.9]. Moreover, for such y we have an isomorphism $\kappa(y) \otimes_S R \cong \prod_{\varphi(x)=y} \kappa(x)$, since the fibre over y is finite and reduced. Using the isomorphism $R^{\text{rig}} \cong R \otimes_S S^{\text{rig}}$ provided by [53, Lem. 7.2.2], we obtain an isomorphism

$$\prod_{\varphi(x)=y} M^{\text{rig}} \otimes_{R^{\text{rig}}} \kappa(x) \cong M^{\text{rig}} \otimes_{S^{\text{rig}}} \kappa(y).$$

It suffices therefore to prove that the natural map $M^{\text{rig}} \rightarrow \prod_{y \in \Sigma} M^{\text{rig}} \otimes_{S^{\text{rig}}} \kappa(y)$ is injective. Since M is finite free over S , this reduces to the case $M = S$, i.e. we need to prove that a nonzero rigid analytic function g on the open unit disc $\mathfrak{Y}^{\text{rig}}$ cannot vanish at all $y \in \Sigma$. In that case, fg would vanish at all points of $\mathfrak{Y}^{\text{rig}}$. It follows from [8, Prop. 5.1.3/3] that the restriction of fg to every closed disc of radius $1/p^{1/n}$ is zero. Since the union of such discs for $n \geq 1$ give an admissible covering of the open unit disc, we deduce that $fg = 0$. Since $f \neq 0$, $g \neq 0$ and S^{rig} is a domain, we get a contradiction. \square

7. DENSITY OF ALGEBRAIC VECTORS

Let X be a smooth affine scheme over $\text{Spec } \mathbb{Q}_p$ of finite type of dimension d such that $X(\mathbb{Q}_p)$ is Zariski dense in X . Let $A = \Gamma(X, \mathcal{O}_X)$ be the ring of global sections on X . We may choose a presentation $A = \mathbb{Q}_p[T_1, \dots, T_n]/I$. This allows us to consider $X(\mathbb{Q}_p)$ as a closed subset of \mathbb{Q}_p^n with the induced topology. If U is an open subset of $X(\mathbb{Q}_p)$ we let $\mathcal{C}(U, L)$ be the space of continuous functions from U to L . Since $X(\mathbb{Q}_p) = \text{Hom}_{\mathbb{Q}_p\text{-alg}}(A, \mathbb{Q}_p)$, evaluation induces a bilinear map $X(\mathbb{Q}_p) \times A \rightarrow \mathbb{Q}_p$. This induces a map $A \otimes_{\mathbb{Q}_p} L \rightarrow \mathcal{C}(X(\mathbb{Q}_p), L)$ and we denote the image by $\mathcal{C}^{\text{alg}}(X(\mathbb{Q}_p), L)$.

Definition 7.1. *Let U be an open and closed subset of $X(\mathbb{Q}_p)$. We let $\mathcal{C}^{\text{alg}}(U, L)$ be the image of $\mathcal{C}^{\text{alg}}(X(\mathbb{Q}_p), L)$ in $\mathcal{C}(U, L)$ under the restriction map $f \mapsto f|_U$.*

We will equip $X(\mathbb{Q}_p)$ with the structure of a locally analytic manifold. Let f_1, \dots, f_d be a sequence of generators of I . They induce a map $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{Q}_p^d$ which is polynomial and in particular map of locally analytic manifold. The smoothness of X and the Jacobian criterion imply that φ is a subimmersion in the sense of [78, Part. II, chap. III.4]) at all points of $X(\mathbb{Q}_p)$. It follows from the Theorem in III.11C) of *loc. cit.* that $X(\mathbb{Q}_p) = \varphi^{-1}(0)$ is a submanifold of \mathbb{Q}_p^n and in particular carries a canonical structure of locally analytic manifold. Moreover it follows from III.11.A) in *loc. cit.* that, for each point $x \in X(\mathbb{Q}_p)$, there exists an open neighborhood U_x of x in \mathbb{Q}_p^n and a locally analytic isomorphism $\alpha_x = (\alpha_{x,1}, \dots, \alpha_{x,n})$ from U_x onto an open subset V_x of \mathbb{Q}_p^r such that

$$(41) \quad U_x \cap X(\mathbb{Q}_p) = \{y \in U_x \mid \alpha_{x,r+1}(y) = \dots = \alpha_{x,n}(y) = 0\}.$$

It follows that the inverse β_x of the map $(\alpha_{x,1}, \dots, \alpha_{x,r})$ induces a locally analytic map from V_x to U_x whose image is $U_x \cap X(\mathbb{Q}_p)$.

The main result of the section, Theorem 7.4, says that $\mathcal{C}^{\text{alg}}(U, L)$ is a dense subspace of $\mathcal{C}^{\text{la}}(U, L)$ locally \mathbb{Q}_p -analytic functions on U , when U is an open and closed subspace of $X(\mathbb{Q}_p)$. This holds if for example U is open and compact. We then look closer at the example, when X is a connected reductive group scheme and U is a compact open subgroup of $X(\mathbb{Q}_p)$.

Lemma 7.2. *Let M be a paracompact manifold, $U \subset M$ an open and closed subspace, and let S be a dense subset of $\mathcal{C}^{\text{la}}(M, L)$. Then the image of S is dense in $\mathcal{C}^{\text{la}}(U, L)$.*

Proof. Since U is open, it naturally carries a structure of a paracompact manifold. Since $M \setminus U$ is also open, we have a homeomorphism:

$$\mathcal{C}^{\text{la}}(M, L) \cong \mathcal{C}^{\text{la}}(U, L) \times \mathcal{C}^{\text{la}}(M \setminus U, L)$$

for the product topology on the right-hand side, [72, Prop. 12.5]. Hence, the restriction map $r : \mathcal{C}^{\text{la}}(M, L) \rightarrow \mathcal{C}^{\text{la}}(U, L)$, $f \mapsto f|_U$ is continuous and surjective. Hence, if $V \subset \mathcal{C}^{\text{la}}(U, L)$ is non-empty and open, then $r^{-1}(V) \subset \mathcal{C}^{\text{la}}(M, L)$ is non-empty and open. The intersection $S \cap r^{-1}(V)$ is non-empty, since S is dense in $\mathcal{C}^{\text{la}}(M, L)$. Hence, $r(S) \cap V$ is also non-empty and so $r(S)$ is dense in $\mathcal{C}^{\text{la}}(U, L)$. \square

Lemma 7.3. $\mathcal{C}^{\text{alg}}(\mathbb{A}^n(\mathbb{Q}_p), L)$ is dense in $\mathcal{C}^{\text{la}}(\mathbb{A}^n(\mathbb{Q}_p), L)$.

Proof. Let $\mathbb{A}^n(\mathbb{Q}_p) = \cup_{i \in I} U_i$ be a disjoint covering by open subsets, where each U_i is a closed ball of radius ϵ around some point in $\mathbb{A}^n(\mathbb{Q}_p)$. By [72, Prop. 12.5] we have a homeomorphism of topological vector spaces:

$$\mathcal{C}^{\text{la}}(\mathbb{A}^n(\mathbb{Q}_p), L) \cong \prod_{i \in I} \mathcal{C}^{\text{la}}(U_i, L)$$

with the product topology on the right-hand side. Thus it is enough to show that the image of $\mathcal{C}^{\text{alg}}(\mathbb{A}^n(\mathbb{Q}_p), L)$ in $\prod_{i \in J} \mathcal{C}^{\text{la}}(U_i, L)$ is dense for every finite subset $J \subset I$. Given such J there will exist an $m \geq 0$ such that $\cup_{i \in J} U_i$ is contained in $p^{-m}\mathbb{Z}_p^n$.

It follows from the theorem of Amice, see [57, (III.1.3.8)] that the image of $\mathcal{C}^{\text{alg}}(\mathbb{A}^n(\mathbb{Q}_p), L)$ is dense in the Banach space of locally analytic functions on $p^{-m}\mathbb{Z}_p^n$ of radius of convergence p^{-h} , for any $h \geq 0$. These Banach spaces are denoted by $\mathcal{F}_{\mathcal{I}}(E)$ in [72], with $E = L$ in our situation. The topology on $\mathcal{C}^{\text{la}}(p^{-m}\mathbb{Z}_p^n, L)$ is the locally convex inductive limit topology defined by the family of Banach spaces $\{\mathcal{F}_{\mathcal{I}}(L)\}_{\mathcal{I}}$, see [72, §II.12]. If $\ell : \mathcal{C}^{\text{la}}(p^{-m}\mathbb{Z}_p^n, L) \rightarrow L$ is a continuous linear form, then its restriction to $\mathcal{F}_{\mathcal{I}}(L)$ is continuous and hence if ℓ vanishes on $\mathcal{C}^{\text{alg}}(\mathbb{A}^n(\mathbb{Q}_p), L)$ it vanishes on $\mathcal{F}_{\mathcal{I}}(L)$ for all \mathcal{I} , and thus $\ell = 0$. Hahn–Banach implies that the image of $\mathcal{C}^{\text{alg}}(\mathbb{A}^n(\mathbb{Q}_p), L)$ is dense in $\mathcal{C}^{\text{la}}(p^{-m}\mathbb{Z}_p^n, L)$.

Since $\cup_{i \in J} U_i$ is both open and compact, it will be closed in $p^{-m}\mathbb{Z}_p^n$. Lemma 7.2 implies that the image of $\mathcal{C}^{\text{alg}}(\mathbb{A}^n(\mathbb{Q}_p), L)$ in $\mathcal{C}^{\text{la}}(\cup_{i \in J} U_i, L) = \prod_{i \in J} \mathcal{C}^{\text{la}}(U_i, L)$ is dense. \square

Theorem 7.4. $\mathcal{C}^{\text{alg}}(X(\mathbb{Q}_p), L)$ is dense in $\mathcal{C}^{\text{la}}(X(\mathbb{Q}_p), L)$.

Proof. We consider a closed embedding $X \hookrightarrow \mathbb{A}^n$ as before. Since the inclusion $X(\mathbb{Q}_p) \hookrightarrow \mathbb{A}^n(\mathbb{Q}_p)$ is a map of locally analytic manifolds, it induces a continuous map

$$(42) \quad \mathcal{C}^{\text{la}}(\mathbb{A}^n(\mathbb{Q}_p), L) \rightarrow \mathcal{C}^{\text{la}}(X(\mathbb{Q}_p), L),$$

by [72, Prop. 12.4 (ii)]. Since $\mathcal{C}^{\text{alg}}(X(\mathbb{Q}_p), L)$ is equal to the image $\mathcal{C}^{\text{alg}}(\mathbb{A}^n(\mathbb{Q}_p), L)$ via (42) and $\mathcal{C}^{\text{alg}}(\mathbb{A}^n(\mathbb{Q}_p), L)$ is dense in $\mathcal{C}^{\text{la}}(\mathbb{A}^n(\mathbb{Q}_p), L)$ by Lemma 7.3, it is enough to show that (42) is surjective, see the proof of Lemma 7.2.

To show the surjectivity of (42) let $\mathbb{A}^n(\mathbb{Q}_p) = \bigcup_{i \in I} U_i$ be a covering by pairwise disjoint open subsets such that each U_i is of the form U_x as in (41). It is enough to prove that the induced map

$$(43) \quad \mathcal{C}^{\text{la}}(U_x, L) \rightarrow \mathcal{C}^{\text{la}}(X(\mathbb{Q}_p) \cap U_x, L)$$

is surjective. After composition with $\alpha_x : U_x \xrightarrow{\sim} V_x$ it is induced by a map of the form $(x_1, \dots, x_r) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$ which has a locally analytic section. Hence (43) is surjective. \square

Corollary 7.5. *If U is an open and closed subset of $X(\mathbb{Q}_p)$ then $\mathcal{C}^{\text{alg}}(U, L)$ is dense in $\mathcal{C}^{\text{la}}(U, L)$.*

Proof. This follows from Theorem 7.4 and Lemma 7.2. \square

Let G be a connected reductive group scheme defined over \mathbb{Q}_p . It follows from [5, Cor. 18.3] that $G(\mathbb{Q}_p)$ is dense in G . Let $\text{Irr}_G(L)$ be the set of isomorphism classes of irreducible algebraic representations of G_L . We assume that G splits over L , which implies that all representations in $\text{Irr}_G(L)$ are absolutely irreducible. Let K be a compact open subgroup of $G(\mathbb{Q}_p)$. By evaluating V at L we get an action on V by $G(\mathbb{Q}_p)$, and hence by K , via the embedding $G(\mathbb{Q}_p) \hookrightarrow G(L)$.

Corollary 7.6. *The evaluation map*

$$(44) \quad \bigoplus_{[V] \in \text{Irr}_G(L)} \text{Hom}_K(V, \mathcal{C}^{\text{la}}(K, L)) \otimes V \rightarrow \mathcal{C}^{\text{la}}(K, L),$$

is injective and the image is equal to $\mathcal{C}^{\text{alg}}(K, L)$. In particular, the image of (44) is a dense subspace of $\mathcal{C}^{\text{la}}(K, L)$.

Proof. The first part follows from [64, Prop. A.3], where we have shown an analogous statement with $\mathcal{C}^{\text{la}}(K, L)$ replaced with $\mathcal{C}(K, L)$. The density result follows from Corollary 7.5. \square

Let τ be a continuous representation of K on a finite dimensional L -vector space. If V is an irreducible representation of G_L we will write $V(\tau) := V \otimes \tau$ with the diagonal K -action.

Corollary 7.7. *Let Π be an admissible unitary L -Banach space representation of K , which is a direct summand of $\mathcal{C}(K, L)^{\oplus m}$ for some $m \geq 1$. Let Π^{la} be the subspace of locally analytic vectors in Π . Then the evaluation map*

$$(45) \quad \bigoplus_{[V] \in \text{Irr}_G(L)} \text{Hom}_K(V(\tau), \Pi^{\text{la}}) \otimes V(\tau) \rightarrow \Pi^{\text{la}},$$

is injective and its image is a dense subspace of Π^{la} .

Proof. We note that the subspace of locally analytic vectors in $\mathcal{C}(K, L)$ is equal to $\mathcal{C}^{\text{la}}(K, L)$ by [31, Prop. 3.3.4]. Thus Π^{la} is a direct summand of $\mathcal{C}^{\text{la}}(K, L)^{\oplus m}$. If τ is the trivial representation then the assertion can be easily deduced from Corollary 7.6. We will reduce the assertion to this case.

Let W be the closure of the image of (45) in Π^{la} . Then W can be characterised as the smallest closed subspace of Π^{la} such that

$$\text{Hom}_K(V(\tau), W) = \text{Hom}_K(V(\tau), \Pi^{\text{la}}), \quad \forall V \in \text{Irr}_G(L).$$

We deduce that

$$\text{Hom}_K(V, W \otimes \tau^*) = \text{Hom}_K(V, \Pi^{\text{la}} \otimes \tau^*), \quad \forall V \in \text{Irr}_G(L).$$

Thus $W \otimes \tau^*$ contains the closure of the image of the evaluation map

$$(46) \quad \bigoplus_{[V] \in \text{Irr}_G(L)} \text{Hom}_K(V, \Pi^{\text{la}} \otimes \tau^*) \otimes V \rightarrow \Pi^{\text{la}} \otimes \tau^*.$$

The projection formula gives us an isomorphism $\mathcal{C}(K, L) \otimes \tau^* \cong \mathcal{C}(K, L)^{\oplus \dim \tau}$. Thus $\Pi \otimes \tau^*$ is a direct summand of a direct sum of finitely many copies of $\mathcal{C}(K, L)$. Since τ is finite dimensional we have $(\Pi \otimes \tau^*)^{\text{la}} = \Pi^{\text{la}} \otimes \tau^*$. From the special case considered above we deduce that $W \otimes \tau^* = \Pi^{\text{la}} \otimes \tau^*$ and hence $W = \Pi^{\text{la}}$. \square

We want to have a variant of Corollary 7.7 with fixed central character. Let Z be the centre of G and let $\psi : K \cap Z(\mathbb{Q}_p) \rightarrow \mathcal{O}^\times$ be a continuous group homomorphism. We let $\mathcal{C}_\psi(K, L)$ be the closed subspace of $\mathcal{C}(K, L)$ consisting of functions on which $K \cap Z(\mathbb{Q}_p)$ acts by ψ . We assume that τ has central character ψ .

Corollary 7.8. *Let Π be an admissible unitary L -Banach space representation of K , which is a direct summand of $\mathcal{C}_\psi(K, L)^{\oplus m}$ for some $m \geq 1$. Let Π^{la} be the subspace of locally analytic vectors in Π . Then the image of the evaluation map*

$$(47) \quad \bigoplus_{[V] \in \text{Irr}_{G/Z}(L)} \text{Hom}_K(V(\tau), \Pi^{\text{la}}) \otimes V(\tau) \rightarrow \Pi^{\text{la}},$$

is a dense subspace of Π^{la} .

Proof. Since the central character of τ^* is equal to ψ^{-1} , projection formula gives us an isomorphism $\mathcal{C}_\psi(K, L) \otimes \tau^* \cong \mathcal{C}(K/K \cap Z(\mathbb{Q}_p), L)^{\oplus \dim \tau}$. Thus arguing as in the proof of previous Corollary we may reduce the assertion to the case when τ and ψ are both trivial. Since the image of K in $(G/Z)(\mathbb{Q}_p)$ is a compact open subgroup, the assertion follows from Corollary 7.6 applied to G/Z . \square

8. FAMILIES OF BANACH SPACE REPRESENTATIONS

Let G be a connected reductive group over \mathbb{Q}_p , split over L and let K be a compact open subgroup of $G(\mathbb{Q}_p)$. An example relating this set up to section 4, would be $G = \text{Res}_{F/\mathbb{Q}_p} G'$, where G' is a connected reductive group over F , and K compact open subgroup of $G'(F)$.

Let (R, \mathfrak{m}) be a complete local noetherian \mathcal{O} -algebra with residue field k . Let M be a finitely generated $R[[K]]$ -module. If V is an irreducible algebraic representation of G_L then by evaluating at L we get an action on V by $G(\mathbb{Q}_p)$, and hence by K , via the embedding $G(\mathbb{Q}_p) \hookrightarrow G(L)$. Let τ be a representation of the form $\sigma \otimes \eta$, where σ is a smooth absolutely irreducible representation of K and $\eta : K \rightarrow \mathcal{O}^\times$ is a continuous character. Then $V(\tau) := V \otimes \tau$ is an absolutely irreducible representation of K . Since K is compact there is a K -invariant \mathcal{O} -lattice Θ in $V(\tau)$. We let

$$M(\Theta) := \text{Hom}_{\mathcal{O}[[K]]}^{\text{cont}}(M, \Theta^d) \cong \Theta \otimes_{\mathcal{O}[[K]]} M,$$

where $(\cdot)^d := \text{Hom}_{\mathcal{O}}^{\text{cont}}(\cdot, \mathcal{O})$ and in the tensor product Θ is considered as a left $\mathcal{O}[[K]]$ -module via $g \mapsto g^{-1}$. Since M is a finitely generated $R[[K]]$ -module, $M(\Theta)$ is a finitely generated A -module, see [63, Prop. 2.15]. Moreover, $M(\Theta)[1/p]$ is independent of the choice of lattice Θ . Let $R_{V(\tau)}$ be the quotient of R , which acts faithfully on $M(\Theta)$ and let

$$\Sigma_{V(\tau)} := \text{m-Spec } R_{V(\tau)}[1/p].$$

We note that $R_{V(\tau)}$ and $\Sigma_{V(\tau)}$ also depend on M . Since M is finitely generated over $R[[K]]$, it is compact. Thus

$$\Pi := \text{Hom}_{\mathcal{O}}^{\text{cont}}(M, L)$$

equipped with the supremum norm is a unitary L -Banach space representation of K . If M is not finitely generated over $\mathcal{O}[[K]]$ then Π is not admissible. However, one can get around this by a trick introduced in [63, §2.1]. We choose a presentation $\mathcal{O}[[x_1, \dots, x_s]] \twoheadrightarrow R$. This induces a surjection $\mathcal{O}[[\mathbb{Z}_p^s \times K]] \twoheadrightarrow R[[K]]$, and thus Π is an admissible unitary $\mathbb{Z}_p^s \times K$ -Banach space representation. Following [11, Def. 3.2]

we define $\Pi^{R\text{-la}}$ as the subspace of locally analytic vectors for $\mathbb{Z}_p^s \times K$ -action on Π . It will become apparent in the proof of the Theorem below that it is actually better to think of $\mathcal{O}[[x_1, \dots, x_s]]$ as the completed group algebra of $((1 + 2p\mathbb{Z}_p)^s, \times)$ instead of $(\mathbb{Z}_p^s, +)$.

If $x \in \mathfrak{m}\text{-Spec } R[1/p]$ then we denote by \mathfrak{m}_x the corresponding maximal ideal and by $\kappa(x)$ its residue field. Let $\Pi[\mathfrak{m}_x]$ be the subspace of Π consisting of vectors killed by \mathfrak{m}_x . Since R acts on Π by continuous endomorphisms, $\Pi[\mathfrak{m}_x]$ is a closed subspace of Π . It follows from [63, Lem. 2.20, 2.21] applied with $\mathfrak{m} = \kappa(x)$ that

$$\Pi[\mathfrak{m}_x] \cong \text{Hom}_{\mathcal{O}}^{\text{cont}}(M \otimes_R \mathcal{O}_{\kappa(x)}, L)$$

and $\Pi[\mathfrak{m}_x]$ is an admissible unitary Banach space representation of K .

We will now explain how to put a topology on $\text{Hom}_K(V(\tau), \Pi^{R\text{-la}}) \otimes V(\tau)$. The module $M(\Theta) \otimes \Theta^d$ is finitely generated over $R[[K]]$. Hence, $\text{Hom}_{\mathcal{O}}^{\text{cont}}(M(\Theta) \otimes \Theta^d, L)$ is an admissible unitary Banach space representation of $\mathbb{Z}_p^s \times K$. It follows from [63, Eq. (10), (11)] that we have an isomorphisms of Banach space representations:

$$(48) \quad \begin{aligned} \text{Hom}_{\mathcal{O}}^{\text{cont}}(M(\Theta) \otimes \Theta^d, L) &\cong \text{Hom}_{\mathcal{O}[[K]]}^{\text{cont}}(M, V(\tau)^*) \otimes V(\tau) \\ &\cong \text{Hom}_K(V(\tau), \Pi) \otimes V(\tau). \end{aligned}$$

Since both $\text{Hom}_K(V(\tau), \Pi) \otimes V(\tau)$ and Π are admissible unitary Banach space representations of $\mathbb{Z}_p^s \times K$, the evaluation map

$$(49) \quad \text{Hom}_K(V(\tau), \Pi) \otimes V(\tau) \rightarrow \Pi$$

is continuous and the image is closed in Π .

Lemma 8.1. *The evaluation map (49) is injective.*

Proof. Since as K -representation $\text{Hom}_K(V(\tau), \Pi) \otimes V(\tau)$ is isomorphic to a direct sum of copies of $V(\tau)$, which is irreducible, the kernel \mathcal{K} is also isomorphic to a direct sum of copies of $V(\tau)$. Since $V(\tau)$ is absolutely irreducible and finite dimensional we have $\text{Hom}_K(V(\tau), V(\tau)) = L$. By applying the functor $\text{Hom}_K(V(\tau), \cdot)$ to (49) we obtain an isomorphism. Hence, $\text{Hom}_K(V(\tau), \mathcal{K}) = 0$ and so $\mathcal{K} = 0$. \square

By applying the functor of R -locally analytic vectors to (49) we obtain injection

$$(50) \quad \text{Hom}_K(V(\tau), \Pi)^{R\text{-la}} \otimes V(\tau) = (\text{Hom}_K(V(\tau), \Pi) \otimes V(\tau))^{R\text{-la}} \hookrightarrow \Pi^{R\text{-la}}$$

where $\text{Hom}_K(V(\tau), \Pi)^{R\text{-la}}$ is the subspace of locally analytic vectors for \mathbb{Z}_p^s -action on $\text{Hom}_K(V(\tau), \Pi)$, thus is naturally endowed with a topology, see [73].

Lemma 8.2. $\text{Hom}_K(V(\tau), \Pi)^{R\text{-la}} \otimes V(\tau) = \text{Hom}_K(V(\tau), \Pi^{R\text{-la}}) \otimes V(\tau)$,

Proof. We identify $\text{Hom}_K(V(\tau), \Pi) \otimes V(\tau)$ with its image in Π . Then its R -locally analytic vectors are equal to $(\text{Hom}_K(V(\tau), \Pi) \otimes V(\tau)) \cap \Pi^{R\text{-la}}$. The inclusions

$$\text{Hom}_K(V(\tau), \Pi^{R\text{-la}}) \otimes V(\tau) \subset (\text{Hom}_K(V(\tau), \Pi) \otimes V(\tau)) \cap \Pi^{R\text{-la}} \subset \Pi^{R\text{-la}}$$

become isomorphisms after applying $\text{Hom}_K(V(\tau), \cdot)$. Since as representations of K both $\text{Hom}_K(V(\tau), \Pi^{R\text{-la}}) \otimes V(\tau)$ and $(\text{Hom}_K(V(\tau), \Pi) \otimes V(\tau)) \cap \Pi^{R\text{-la}}$ are isomorphic to a direct sum of copies of $V(\tau)$, we conclude that they are equal. \square

The Lemma above and (50) allows us to put a topology on $\text{Hom}_K(V(\tau), \Pi^{R\text{-la}}) \otimes V(\tau)$. It follows from the theory of admissible locally analytic representations, see [73, Prop. 6.4], that the image of $\text{Hom}_K(V(\tau), \Pi^{R\text{-la}}) \otimes V(\tau)$ under (49) is closed in $\Pi^{R\text{-la}}$ and (49) induces a homeomorphism between $\text{Hom}_K(V(\tau), \Pi^{R\text{-la}}) \otimes V(\tau)$ and its image in $\Pi^{R\text{-la}}$.

Proposition 8.3. *If $R_{V(\tau)}$ is reduced and $M(\Theta)$ is Cohen-Macaulay then*

$$\bigoplus_{x \in \Sigma_{V(\tau)}} \text{Hom}_K(V(\tau), \Pi[\mathfrak{m}_x]^{\text{la}}) \otimes V(\tau)$$

is a dense subspace of $\text{Hom}_K(V(\tau), \Pi^{R\text{-la}}) \otimes V(\tau)$.

Proof. Let us fix $x \in \Sigma_{V(\tau)}$. Since $\Pi[\mathfrak{m}_x]$ is admissible as K -representation [11, Prop. 3.8] implies that $\Pi[\mathfrak{m}_x]^{\text{la}} = \Pi[\mathfrak{m}_x]^{R\text{-la}}$. We thus have an inclusion $\Pi[\mathfrak{m}_x]^{\text{la}} \subset \Pi^{R\text{-la}}$. If $x, x_1, \dots, x_n \in \Sigma_{V(\tau)}$ are pairwise distinct then $(\bigoplus_{i=1}^n \Pi[\mathfrak{m}_{x_i}]) \cap \Pi[\mathfrak{m}_x] = 0$, as $1 \in \mathfrak{m}_x + \bigcap_{i=1}^n \mathfrak{m}_{x_i}$. We thus have an inclusion $\bigoplus_{x \in \Sigma_{V(\tau)}} \Pi[\mathfrak{m}_x]^{\text{la}} \subset \Pi^{R\text{-la}}$. By applying $\text{Hom}_K(V(\tau), \cdot)$ we deduce that

$$(51) \quad \bigoplus_{x \in \Sigma_{V(\tau)}} \text{Hom}_K(V(\tau), \Pi[\mathfrak{m}_x]^{\text{la}}) \subset \text{Hom}_K(V(\tau), \Pi^{R\text{-la}}).$$

By Hahn–Banach theorem, see [71, Cor. 9.3], it is enough to show that any continuous linear form on $\text{Hom}_K(V(\tau), \Pi^{R\text{-la}})$, which vanishes on the left hand-side of (51), is zero. Below we will denote by W' the continuous linear dual of a locally convex L -vector space W .

Going back to (48) we think of $\text{Hom}_K(V(\tau), \Pi^{R\text{-la}})$ as locally analytic vectors for the \mathbb{Z}_p^s -action on $\text{Hom}_{\mathcal{O}}^{\text{cont}}(M(\Theta), L)$. Thus

$$\text{Hom}_K(V(\tau), \Pi^{R\text{-la}})' \cong M(\Theta) \otimes_{\mathcal{O}[\mathbb{Z}_p^s]} D(\mathbb{Z}_p^s) \cong M(\Theta) \otimes_{R_{V(\tau)}} R_{V(\tau)}^{\text{rig}} =: M(\Theta)^{\text{rig}}.$$

We may think of $\text{Hom}_K(V(\tau), \Pi[\mathfrak{m}_x]^{\text{la}})$ as $\text{Hom}_K(V(\tau), \Pi^{R\text{-la}})[\mathfrak{m}_x]$. Thus

$$\text{Hom}_K(V(\tau), \Pi[\mathfrak{m}_x]^{\text{la}})' \cong M(\Theta)^{\text{rig}} \otimes_{R_{V(\tau)}^{\text{rig}}} \kappa(x).$$

The assertion follows from Proposition 6.1. \square

Theorem 8.4. *Assume that there is an M -regular sequence y_1, \dots, y_h contained in \mathfrak{m} , such that $M/(y_1, \dots, y_h)M$ is a finitely generated projective $\mathcal{O}[[K]]$ -module. Then the evaluation map*

$$(52) \quad \bigoplus_{[V] \in \text{Irr}_G(L)} \text{Hom}_K(V(\tau), \Pi^{R\text{-la}}) \otimes V(\tau) \rightarrow \Pi^{R\text{-la}},$$

is injective and its image is a dense subspace of $\Pi^{R\text{-la}}$. Moreover, the map

$$(53) \quad \bigoplus_{[V] \in \text{Irr}_G(L)} \bigoplus_{x \in \Sigma_{V(\tau)}} \text{Hom}_K(V(\tau), \Pi[\mathfrak{m}_x]^{\text{la}}) \otimes V(\tau) \rightarrow \Pi^{R\text{-la}},$$

is injective and if additionally $R_{V(\tau)}$ is reduced for all $V \in \text{Irr}_G(L)$ then the image of (53) is a dense subspace of $\Pi^{R\text{-la}}$.

Proof. Since $\text{Hom}_K(V(\tau), \Pi^{R\text{-la}}) \otimes V(\tau)$ is $V(\tau)$ -isotypic to prove the injectivity of (52) it is enough to do so for a single summand. That follows from Lemma 8.2 and injectivity of (50). Since the left hand-side of (53) is a subspace of the left hand-side of (52), see Proposition 8.3, we deduce that (53) is injective. We will now show that image of (52) is dense by exhibiting a dense subspace.

Let $S = \mathcal{O}[[x_1, \dots, x_h]]$. By mapping x_i to y_i we obtain a ring homomorphism $S \rightarrow R$, which makes M into a finitely generated $S[[K]]$ -module. We claim that M is a projective as $S[[K]]$ -module. To prove the claim it is enough to show that M is a free $S[[P]]$ -module, where P is a pro- p Sylow of K . By topological Nakayama's lemma we may choose a surjection $S[[P]]^{\oplus m} \twoheadrightarrow M$, which becomes an isomorphism

after applying the functor $k \otimes_S \llbracket P \rrbracket$, and let \mathcal{K} denote the kernel. We argue by induction on h that such map has to be an isomorphism. If $h = 0$ then $S = \mathcal{O}$ and M is projective as $S \llbracket K \rrbracket$ -module, and so the surjection has a section. Hence, $k \otimes_S \llbracket P \rrbracket \mathcal{K} = 0$ and topological Nakayma's lemma implies that $\mathcal{K} = 0$. Let h be arbitrary. Since x_h is M -regular, we have an exact sequence of $S/(x_h) \llbracket P \rrbracket$ -modules

$$0 \rightarrow \mathcal{K}/x_h \mathcal{K} \rightarrow S/(x_h) \llbracket P \rrbracket^{\oplus m} \rightarrow M/x_h M \rightarrow 0.$$

Since $S/(x_h) \cong \mathcal{O} \llbracket x_1, \dots, x_{h-1} \rrbracket$ and the sequence x_1, \dots, x_{h-1} in $M/x_h M$ -regular, we deduce that $M/x_h M$ is a free $S/(x_h) \llbracket P \rrbracket$ -module and so the surjection has a splitting. By the same argument as above we deduce that $\mathcal{K}/x_h \mathcal{K} = 0$. Topological Nakayama's lemma implies that $\mathcal{K} = 0$. This finishes the proof of the claim.

The claim implies that M is a direct summand of $S \llbracket K \rrbracket^{\oplus m}$ for some $m \geq 1$. By dualizing and identifying S with the completed group algebra of $(1 + 2p\mathbb{Z}_p)^h$, we may consider Π as an admissible unitary Banach space representation of $K' := (1 + 2p\mathbb{Z}_p)^h \times K$, which is a direct summand $\mathcal{C}(K', L)^{\oplus m}$. The locally analytic vectors for the action of K' on Π are equal to $\Pi^{S\text{-la}}$ and since M is finitely generated over $S \llbracket K \rrbracket$, [11, Prop. 3.8] implies that $\Pi^{R\text{-la}} = \Pi^{S\text{-la}}$. Since K' is a compact open subgroup of $G' := \mathbb{G}_m^h \times G$, Corollary 7.7 implies that the evaluation map

$$(54) \quad \bigoplus_{[V'] \in \text{Irr}_{G'}(L)} \text{Hom}_{K'}(V'(\tau), \Pi^{R\text{-la}}) \otimes V'(\tau) \rightarrow \Pi^{R\text{-la}},$$

is injective and its image is a dense subspace of $\Pi^{R\text{-la}}$. Every V' is of the form $\chi \boxtimes V$, where $V \in \text{Irr}_G(L)$ and $\chi : \mathbb{G}_m^h \rightarrow \mathbb{G}_m$ is an algebraic representation. Such χ induces a continuous group homomorphism $\chi : (1 + 2p\mathbb{Z}_p)^s \rightarrow L^\times$ and hence a maximal ideal of $S[1/p]$, which we denote by \mathfrak{m}'_χ . We thus may rewrite (54) as a K -equivariant map

$$(55) \quad \bigoplus_{[V] \in \text{Irr}_G(L)} \bigoplus_{\chi \in \text{Irr}_{\mathbb{G}_m^h}(L)} \text{Hom}_K(V(\tau), \Pi[\mathfrak{m}'_\chi]^{\text{la}}) \otimes V(\tau) \rightarrow \Pi^{R\text{-la}}.$$

Since $\Pi[\mathfrak{m}'_\chi]^{\text{la}} = \Pi[\mathfrak{m}'_\chi]^{R\text{-la}} \subset \Pi^{R\text{-la}}$, the image of (55) will be contained in the image of (52). Since the image of (55) is dense, we conclude that the image of (52) is dense.

The proof of [15, Lem. 4.18 1)] shows that $M(\Theta)$ is a free S -module of finite rank. Thus y_1, \dots, y_h is a regular sequence of parameters for $M(\Theta)$ and hence $M(\Theta)$ is Cohen–Macaulay as $R_{V(\tau)}$ -module. If $R_{V(\tau)}$ is reduced then Proposition 8.3 implies that the closure of the image of

$$\bigoplus_{x \in \Sigma_{V(\tau)}} \text{Hom}_K(V(\tau), \Pi[\mathfrak{m}_x]^{\text{la}}) \otimes V(\tau) \rightarrow \Pi^{R\text{-la}}$$

is equal to $\text{Hom}_K(V(\tau), \Pi^{R\text{-la}}) \otimes V(\tau)$. Hence the closure of the image of (53) will contain the image of (52) and hence is equal to $\Pi^{R\text{-la}}$. \square

Let $Z(\mathfrak{g})_L$ be the centre of the enveloping algebra of the Lie algebra of G_L and let $\zeta : Z(\mathfrak{g})_L \rightarrow R^{\text{rig}}$ be a homomorphism of L -algebras, where R^{rig} is the ring of global functions on the rigid space $\mathfrak{X}^{\text{rig}}$, associated to the formal scheme $\text{Spf } R$, see [53, §7.1]. The map $R[1/p] \rightarrow R^{\text{rig}}$ induces a bijection between $\text{m-Spec } R[1/p]$ and points of $\mathfrak{X}^{\text{rig}}$ and isomorphism on the completions of local rings by [53, Lem. 7.1.9].

Thus we may specialise ζ at any $x \in \mathfrak{m}\text{-Spec } R[1/p]$ to obtain a ring homomorphism:

$$\zeta_x : Z(\mathfrak{g})_L \xrightarrow{\zeta} R^{\text{rig}} \rightarrow \kappa(x).$$

Theorem 8.5. *Assume that the following hold:*

- (i) *there is an M -regular sequence y_1, \dots, y_h in \mathfrak{m} , such that $M/(y_1, \dots, y_h)M$ is a finitely generated projective $\mathcal{O}[[K]]$ -module;*
- (ii) *the rings $R_{V(\tau)}$ are reduced, $\forall V \in \text{Irr}_G(L)$;*
- (iii) *the action of $Z(\mathfrak{g})_L$ on $V(\tau)$ is given by ζ_x , $\forall V \in \text{Irr}_G(L)$, $\forall x \in \Sigma_{V(\tau)}$.*

Then $Z(\mathfrak{g})_L$ acts on $\Pi^{R\text{-la}}$ via ζ . In particular, for all $y \in \mathfrak{m}\text{-Spec } R[1/p]$, $Z(\mathfrak{g})_L$ acts on $\Pi[\mathfrak{m}_y]^{\text{la}}$ via the infinitesimal character ζ_y .

Proof. It follows from [11, Lem. 3.3] that we have a continuous $R^{\text{rig}} \widehat{\otimes} D(K, L)$ action on $\Pi^{R\text{-la}}$ by continuous endomorphisms. This induces an action of $R^{\text{rig}} \otimes Z(\mathfrak{g})_L$, see [74, Prop. 3.7]. Let \mathfrak{a} be the ideal of $R^{\text{rig}} \otimes Z(\mathfrak{g})_L$ generated by the elements of the form $\zeta(z) \otimes 1 - 1 \otimes z$ for $z \in Z(\mathfrak{g})_L$. Assumption (iii) implies that such elements will kill

$$\bigoplus_{[V] \in \text{Irr}_G(L)} \bigoplus_{x \in \Sigma_{V(\tau)}} \text{Hom}_K(V(\tau), \Pi[\mathfrak{m}_x]^{\text{la}}) \otimes V(\tau).$$

Since the assumptions (i) and (ii) imply via Theorem 8.4 that the subspace is dense we conclude that the action of $R^{\text{rig}} \otimes Z(\mathfrak{g})_L$ on $\Pi^{R\text{-la}}$ factors through the action of $(R^{\text{rig}} \otimes Z(\mathfrak{g})_L)/\mathfrak{a} \cong R^{\text{rig}}$. Thus the action of $Z(\mathfrak{g})$ on $\Pi^{R\text{-la}}$ factors through ζ . If $y \in \mathfrak{m}\text{-Spec } R[1/p]$ then the action of $(R^{\text{rig}} \otimes Z(\mathfrak{g})_L)/\mathfrak{a}$ on $\Pi[\mathfrak{m}_y]^{\text{la}} = \Pi[\mathfrak{m}_y]^{R\text{-la}} = \Pi^{R\text{-la}}[\mathfrak{m}_y]$ factors through the specialisation of ζ at y . \square

We want to prove a variant of Theorem 8.5 with a fixed central character. Let Z be the centre of G and let $\psi : K \cap Z(\mathbb{Q}_p) \rightarrow \mathcal{O}^\times$ be a continuous group homomorphism. Let $\text{Mod}_{K, \psi}^{\text{pro}}(\mathcal{O})$ be the category of linearly compact $\mathcal{O}[[K]]$ -modules on which $K \cap Z(\mathbb{Q}_p)$ acts by ψ^{-1} . We assume that the central character of τ is equal to ψ .

Theorem 8.6. *Assume that the following hold:*

- (o) *M is in $\text{Mod}_{K, \psi}^{\text{pro}}(\mathcal{O})$;*
- (i) *there is an M -regular sequence y_1, \dots, y_h in \mathfrak{m} , such that $M/(y_1, \dots, y_h)M$ is a finitely generated $\mathcal{O}[[K]]$ -module, which is projective in $\text{Mod}_{K, \psi}^{\text{pro}}(\mathcal{O})$;*
- (ii) *the rings $R_{V(\tau)}$ are reduced, $\forall V \in \text{Irr}_{G/Z}(L)$;*
- (iii) *the action of $Z(\mathfrak{g})_L$ on $V(\tau)$ is given by ζ_x , $\forall V \in \text{Irr}_{G/Z}(L)$, $\forall x \in \Sigma_{V(\tau)}$.*

Then $Z(\mathfrak{g})_L$ acts on $\Pi^{R\text{-la}}$ via ζ . In particular, for all $y \in \mathfrak{m}\text{-Spec } R[1/p]$, $Z(\mathfrak{g})_L$ acts on $\Pi[\mathfrak{m}_y]^{\text{la}}$ via the infinitesimal character ζ_y .

Proof. One proves the analog of Theorem 8.4 using Corollary 7.8 instead of Corollary 7.7. Then the proof is the same as the proof of Theorem 8.5. \square

9. GLOBAL APPLICATIONS

The abstract Theorem 8.5 is motivated by the study of Hecke eigenspaces in the completed cohomology. We will show that if we are in the situation where the Hecke eigenvalues contribute only in one cohomological degree and we can attach Galois representations to the Hecke eigenspaces in a way to be made precise below, then the locally analytic vectors in the Hecke eigenspaces in the completed cohomology afford an infinitesimal character.

9.1. Locally symmetric spaces. We are motivated by Emerton's ICM talk [34] and his paper [32]. Let G be a connected reductive group over \mathbb{Q} , let Z be its centre, let A denote the maximal split torus in Z . Let $G_p = G(\mathbb{Q}_p)$, let $G_\infty = G(\mathbb{R})$, let $A_\infty = A(\mathbb{R})$, let $Z_\infty = Z(\mathbb{R})$ be the centre of G_∞ and let K_∞ denote a choice of maximal compact subgroup of G_∞ . For any Lie group H , we let H° denote the subgroup consisting of the connected component at the identity.

The quotient $G_\infty/Z_\infty^\circ K_\infty^\circ$ is a symmetric space on which G_∞ acts. We denote its dimension by d .

Let \mathbb{A} denote the ring of adèles over \mathbb{Q} , \mathbb{A}_f the ring of finite adèles, and \mathbb{A}_f^p the ring of prime-to- p adèles.

Definition 9.1. *A compact open subgroup $K_f \subset G(\mathbb{A}_f)$ is sufficiently small if $G(\mathbb{Q})$ acts on $G(\mathbb{A})/A_\infty^\circ K_\infty^\circ K_f$ with trivial stabilisers.*

Lemma 9.2. *If $K_f = \prod_\ell K_{f,\ell}$ with $K_{f,\ell} \subset G(\mathbb{Q}_\ell)$ compact open and $K_{f,\ell}$ is torsion free for some ℓ then K_f is sufficiently small.*

Proof. Let us first assume that Z is \mathbb{Q} -anisotropic, so that A_∞° is trivial. The stabiliser of $gK_\infty^\circ K_f$ is equal to $G(\mathbb{Q}) \cap gK_\infty^\circ K_f g^{-1}$ and is both compact and discrete, hence finite. By considering the projection onto $gK_{f,\ell} g^{-1}$, we deduce it is trivial. Let us return to the general case. Since the centre of G/A is anisotropic, we deduce that the $G(\mathbb{Q})$ -stabilizer of $gK_\infty^\circ K_f A_\infty^\circ$ is contained in $A(\mathbb{A})$ and hence in $A(\mathbb{Q})$. Since $A(\mathbb{Q}) \cap gK_\infty^\circ A_\infty^\circ K_f g^{-1}$ is finite we conclude by the same argument. \square

Remark 9.3. An easy way to ensure that the condition in Lemma 9.2 holds both for K_f and its image in $G(\mathbb{A}_f)/Z(\mathbb{A}_f)$ is to embed $G \subset \mathrm{GL}_n$ by choosing a faithful representation over \mathbb{Q} . If $\ell > n+1$ then the pro- ℓ Sylow in $\mathrm{GL}_n(\mathbb{Q}_\ell)$ is ℓ -saturable by [57, III (3.2.7)] and hence torsion free. If we assume that $K_{f,\ell}$ is pro- ℓ , for example a pro- ℓ Iwahori subgroup if G is split over \mathbb{Q}_ℓ , then it will be torsion free. To make sure that the assertion also holds for the image of K_f in $G(\mathbb{A}_f)/Z(\mathbb{A}_f)$, pick an embedding $G/Z \hookrightarrow \mathrm{GL}_{n'}$ and repeat the argument. Thus if $\ell > \max(n, n') + 1$ and $K_{f,\ell}$ is pro- ℓ then both conditions are satisfied.

Let us assume that K_f satisfies the condition of Lemma 9.2 and its image in $(G/Z)(\mathbb{A}_f)$ is sufficiently small with respect to G/Z and let

$$\Lambda = \Lambda(K_f) := Z(\mathbb{Q}) \cap K_f K_\infty^\circ Z_\infty^\circ.$$

Lemma 9.4. *Λ is a discrete cocompact subgroup of Z_∞/A_∞ . In particular, $\Lambda \cong \mathbb{Z}^r$, where r is the split \mathbb{R} -rank of Z_∞/A_∞ .*

Proof. By assumption on K_f , Λ is torsion free and $Z(\mathbb{Q}) \cap K_f K_\infty^\circ A_\infty^\circ$ is trivial. Thus the map induced by the projection to the ∞ -component induces an injection $\Lambda \hookrightarrow Z_\infty/A_\infty$. Since the torus Z/A is \mathbb{Q} -anisotropic, it follows from [69, Thm. 4.11] that the quotient is compact. \square

Remark 9.5. Emerton in [32, §2.4] assumes that $r = 0$. We don't do this because we would like to cover the case of Shimura curves in the applications.

Lemma 9.6. *The action of $G(\mathbb{Q})$ on $G(\mathbb{A})/Z_\infty^\circ K_\infty^\circ K_f$ factors through the action of $G(\mathbb{Q})/\Lambda$, which acts with trivial stabilisers.*

Proof. The first part is clear. Since the image of K_f in $G(\mathbb{A}_f)/Z(\mathbb{A}_f)$ is sufficiently small, $G(\mathbb{Q})/Z(\mathbb{Q})$ acts with trivial stabilisers on $G(\mathbb{A})/Z(\mathbb{A})K_\infty^\circ K_f$. The rest of the proof is an exercise for the reader. \square

If K_f is a compact open subgroup of $G(\mathbb{A})$, we write

$$Y(K_f) := G(\mathbb{Q}) \backslash G(\mathbb{A}) / A_\infty^\circ K_\infty^\circ K_f, \quad \tilde{Y}(K_f) := G(\mathbb{Q}) \backslash G(\mathbb{A}) / Z_\infty^\circ K_\infty^\circ K_f.$$

It follows from Lemma 9.6 that the map $q : Y(K_f) \rightarrow \tilde{Y}(K_f)$ makes $Y(K_f)$ into a torus bundle for the compact torus

$$\mathcal{T} := (Z_\infty^\circ / A_\infty^\circ (Z_\infty^\circ \cap K_\infty^\circ)) / \Lambda \cong \mathbb{R}_{>0}^r / \Lambda \cong \mathbb{R}^r / \mathbb{Z}^r.$$

Let G_∞^1 be the intersection of $\text{Ker } \chi^2$ for all $\chi : G_{\mathbb{R}} \rightarrow \mathbb{G}_{m,\mathbb{R}}$. We then have

$$(56) \quad G_\infty = G_\infty^1 Z_\infty^\circ,$$

see [6, 4.3.1]. Let $G(\mathbb{A})^1 := G(\mathbb{A}_f) G_\infty^1 A_\infty^\circ$ and let $G(\mathbb{Q})^1 := G(\mathbb{Q}) \cap G(\mathbb{A})^1$ and let

$$Y^1(K_f) := G(\mathbb{Q})^1 \backslash G(\mathbb{A})^1 / A_\infty^\circ K_\infty^\circ K_f.$$

Lemma 9.7. *The natural map $Y^1(K_f) \rightarrow Y(K_f)$ identifies the source with the closed subset of the target. The map $\mathcal{T} \times Y^1(K_f) \rightarrow Y(K_f)$, $(t, y) \mapsto ty$ is a homeomorphism. In particular, we obtain a homeomorphism*

$$Y^1(K_f) \xrightarrow{\cong} Y(K_f) / \mathcal{T} = \tilde{Y}(K_f).$$

Proof. The inclusion $G(\mathbb{A})^1 \subset G(\mathbb{A})$ induces an injection

$$G(\mathbb{Q})^1 \backslash G(\mathbb{A})^1 \subset G(\mathbb{Q}) \backslash G(\mathbb{A}),$$

and hence an injection $Y^1(K_f) \hookrightarrow Y(K_f)$. Since the map $G(\mathbb{A}) \rightarrow Y(K_f)$ is open and $G(\mathbb{A})^1$ is a closed subset of $G(\mathbb{A})$ we deduce that $Y^1(K_f)$ is a closed subset of $Y(K_f)$. Since $Y(K_f)$ is a \mathcal{T} -bundle over $\tilde{Y}(K_f)$, it is enough to show that the map $Y^1(K_f) \rightarrow \tilde{Y}(K_f)$ is bijective. From (56) we obtain an equality:

$$Y^1(K_f) = G(\mathbb{Q})^1 \backslash G(\mathbb{A}) / Z_\infty^\circ K_\infty^\circ K_f = G(\mathbb{Q})^1 Z_\infty^\circ \backslash G(\mathbb{A}) / Z_\infty^\circ K_\infty^\circ K_f.$$

Thus it is enough to show that $G(\mathbb{Q}) \subset G(\mathbb{Q})^1 Z_\infty^\circ$, which follows by considering the map $G(\mathbb{A}) \rightarrow G_\infty \rightarrow G_\infty / G_\infty^1 A_\infty^\circ$ and using (56) again. \square

If M is a K_f -module then following [32, Def. 2.2.3] we define a local system

$$\mathcal{M} := (M \times (G(\mathbb{Q}) \backslash G(\mathbb{A}) / A_\infty^\circ K_\infty^\circ)) / K_f, \quad \tilde{\mathcal{M}} := (M \times (G(\mathbb{Q}) \backslash G(\mathbb{A}) / Z_\infty^\circ K_\infty^\circ)) / K_f,$$

on $Y(K_f)$ and $\tilde{Y}(K_f)$ respectively. We have $q_* \mathcal{M} = \tilde{\mathcal{M}}$. If N is the maximal quotient of M on which Λ acts trivially then $\tilde{\mathcal{M}} = \tilde{\mathcal{N}}$.

Lemma 9.8. *If Λ acts trivially on M then for all $n \geq 0$ we have a natural isomorphism*

$$H^n(Y(K_f), \mathcal{M}) \cong \bigoplus_{i+j=n} H^i(\tilde{Y}(K_f), \tilde{\mathcal{M}}) \otimes H^j(\mathcal{T}, \mathbb{Z}).$$

Proof. Since $H^j(\mathcal{T}, \mathbb{Z})$ is a free \mathbb{Z} -module for all $j \geq 0$, the assertion follows from Künneth formula, see for example [26, Thm. 15.10]. \square

9.2. Completed cohomology. It follows from Lemma 9.6 that if K'_f is an open normal subgroup of K_f then $\tilde{Y}(K'_f) \rightarrow \tilde{Y}(K_f)$ is Galois covering with Galois group $K_f/\Lambda(K_f)K'_f$. We want to vary K_f by shrinking the subgroup at p . We fix a compact open subgroup $K_f^p \subset G(\mathbb{A}_f^p)$ and assume that for some sufficiently large ℓ , as in Remark 9.3, $K_{f,\ell}^p$ is pro- ℓ . As a consequence we have that for all compact open subgroups $K_p \subset G(\mathbb{Q}_p)$ both $K_f^p K_p$ and its image in $G(\mathbb{A}_f)/Z(\mathbb{A}_f)$ are sufficiently small and $K_f^p K_p \cap G(\mathbb{Q})$ is torsion free. We set up things this way, because we don't want to put any restrictions on the subgroup K_p , but all three properties hold if K_f^p is arbitrary and K_p is small enough.

We then define the completed (co)homology by

$$\begin{aligned} \tilde{H}^i(\mathcal{O}/\varpi^s) &:= \varinjlim_{K_p} H^i(\tilde{Y}(K_f^p K_p), \mathcal{O}/\varpi^s), & \tilde{H}^i &:= \varprojlim_s \tilde{H}^i(\mathcal{O}/\varpi^s) \\ \tilde{H}_i(\mathcal{O}/\varpi^s) &:= \varprojlim_{K_p} H_i(\tilde{Y}(K_f^p K_p), \mathcal{O}/\varpi^s), & \tilde{H}_i &:= \varinjlim_s \tilde{H}_i(\mathcal{O}/\varpi^s). \end{aligned}$$

where in the limits K_p ranges over all compact open subgroups of $G(\mathbb{Q}_p)$. The topologies are as follows: discrete on $\tilde{H}^i(\mathcal{O}/\varpi^s)$, p -adic on \tilde{H}^i , projective limit topology on $\tilde{H}_i(\mathcal{O}/\varpi^s)$ and \tilde{H}_i . Note that $\tilde{H}^i(\mathcal{O}/\varpi^s)$ and $\tilde{H}_i(\mathcal{O}/\varpi^s)$ are related by the Pontryagin duality. If K_p is a compact open subgroup of $G(\mathbb{Q}_p)$ then \tilde{H}_i is a finitely generated $\mathcal{O}[[K_p]]$ -module, [34, Thm. 2.2].

We will consider K_p -modules M as $K_f^p K_p$ -modules by making K_f^p act trivially. We can then obtain local systems \mathcal{M} and $\tilde{\mathcal{M}}$ as in the previous subsection. Let Λ_p be the closure of $\Lambda(K_f^p K_p)$ in K_p . If $M = \text{Ind}_{K_p \Lambda_p}^{K_f^p} M'$ then $\tilde{\mathcal{M}}$ is equal to $\tilde{\mathcal{M}}'$ defined with respect to K_p . Since cohomology commutes with direct limits, see for example [44, Lem. 2.5], by writing $\mathcal{C}(K_p/\Lambda_p, \mathcal{O}/\varpi^s) \cong \varinjlim_{K'_p} \text{Ind}_{K'_p \Lambda_p}^{K_f^p} \mathcal{O}/\varpi^s$ we obtain

$$(57) \quad \tilde{H}^i(\mathcal{O}/\varpi^s) \cong H^i(\tilde{Y}(K_f^p K_p), \mathcal{C}(K_p/\Lambda_p, \mathcal{O}/\varpi^s)).$$

Lemma 9.9. *Let M be a finite \mathcal{O}/ϖ^s -module with continuous K_p/Λ_p -action. Then there is a spectral sequence*

$$E_2^{ij} = \text{Ext}_{K_p/\Lambda_p}^i(M, \tilde{H}^j(\mathcal{O}/\varpi^s)) \implies H^{i+j}(\tilde{Y}(K_f^p K_p), \tilde{\mathcal{M}}^\vee),$$

where $\tilde{\mathcal{M}}^\vee$ is the local system associated to the Pontryagin dual of M .

Proof. We use an argument due to Hill [47] in an easier setting, alternatively see [32, (2.1.10)]. We pick a triangulation of $\tilde{Y}(K_f^p K_p)$ and write down the Čech complex computing the cohomology of the local system associated to $\mathcal{C}(K_p/\Lambda_p, \mathcal{O}/\varpi^s)$. We then apply $\text{Hom}_{K_p/\Lambda_p}(M, *)$ to obtain a complex computing the cohomology of $\tilde{\mathcal{M}}^\vee$. This yields the desired spectral sequence. \square

9.3. Hecke algebra. We fix a finite set of places S containing p and ∞ , such that G is unramified over \mathbb{Q}_ℓ for all $\ell \notin S$. We assume that we may factor $K_f^p = K_f^{p,\ell} K_\ell$ with K_ℓ hyperspecial for all $\ell \notin S$. Let \mathcal{H}_ℓ be the double coset algebra $\mathcal{O}[K_\ell \backslash G(\mathbb{Q}_\ell) / K_\ell]$ and let

$$\mathbb{T}^{\text{univ}} := \bigotimes_{\ell \notin S} \mathcal{H}_\ell$$

where the tensor product (defined by its universal property) is taken over \mathcal{O} . One may think of \mathbb{T}^{univ} as a polynomial ring over \mathcal{O} in infinitely many variables. If M is an $\mathcal{O}[K_p]$ -module then \mathbb{T}^{univ} acts on the cohomology of the associated local system. Following [34] we define \mathbb{T} to be the closure of the image of \mathbb{T}^{univ} in

$$(58) \quad \prod_{K_p} \prod_M \prod_i \text{End}_{\mathcal{O}}(H^i(\tilde{Y}(K_f^p K_p), \tilde{\mathcal{M}})),$$

where the product is taken over all compact open subgroups $K_p \subset G(\mathbb{Q}_p)$, all finite \mathcal{O} -torsion modules M with continuous K_p/Λ_p -action, and all degrees i , where the target is equipped with the profinite topology.

Lemma 9.10. *Let K_p be an open pro- p subgroup of $G(\mathbb{Q}_p)$, let \mathfrak{m} be an open maximal ideal of \mathbb{T} and let i be a non-negative integer. Then the following are equivalent:*

- (i) $H^i(\tilde{Y}(K_f^p K_p), \mathcal{O}/\varpi)_{\mathfrak{m}} = 0$;
- (ii) $H^i(\tilde{Y}(K_f^p K_p), \mathcal{M})_{\mathfrak{m}} = 0$, for all representations of K_p on finitely generated \mathcal{O} -torsion modules M ;
- (iii) $H^i(\tilde{Y}(K_f^p K'_p), \mathcal{O}/\varpi)_{\mathfrak{m}} = 0$, for all open normal subgroups $K'_p \subset K_p$.

Proof. Since K_p is pro- p any such M has a composition series with graded pieces isomorphic to \mathcal{O}/ϖ with the trivial K_p -action. Since localisation is exact part (i) implies (ii). Part (ii) implies (iii) by considering the induced module. Part (iii) trivially implies (i). \square

Corollary 9.11. *\mathbb{T} has only finitely many open maximal ideals.*

Proof. If \mathfrak{m} is open then it has to lie in the support of some $H^i(\tilde{Y}(K_f^p K'_p), \mathcal{M})$. By embedding M into an induction from the trivial representation of a smaller subgroup, we may assume that K'_p is an open normal subgroup of K_p . Lemma 9.10 implies that \mathfrak{m} lies in the support of $H^i(\tilde{Y}(K_f^p K_p), \mathcal{O}/\varpi)$. Thus if we let $\bar{\mathbb{T}}$ be the image of \mathbb{T} in the finite dimensional k -algebra $\text{End}_k(\oplus_{i=0}^d H^i(\tilde{Y}(K_f^p K_p), \mathcal{O}/\varpi))$ then the set of maximal ideals of $\bar{\mathbb{T}}$ coincide with the set of open maximal ideals of \mathbb{T} . \square

Remark 9.12. By applying the Chinese remainder theorem at each finite level of (58), we obtain an isomorphism $\mathbb{T} \cong \prod_{\mathfrak{m}} \hat{\mathbb{T}}_{\mathfrak{m}}$, where the product is taken over the open maximal ideals and $\hat{\mathbb{T}}_{\mathfrak{m}} := \varprojlim_n \mathbb{T}/\mathfrak{m}^n$ with the topology induced by the limit. Since the product is finite the completion coincides with localisation, so that $\hat{\mathbb{T}}_{\mathfrak{m}} = \mathbb{T}_{\mathfrak{m}}$. This explains why localisation at \mathfrak{m} behaves well for various topological modules of \mathbb{T} . Below subscript \mathfrak{m} always means localisation at \mathfrak{m} . If $\mathbb{T}_{\mathfrak{m}}$ is noetherian then the topology will coincide with the \mathfrak{m} -adic topology.

9.4. Weakly non-Eisenstein ideals.

Definition 9.13. *We say that an open maximal ideal \mathfrak{m} of \mathbb{T} is weakly non-Eisenstein if there is an integer q_0 such that the equivalent conditions of Lemma 9.10 hold for all $i \neq q_0$.*

Remark 9.14. If G_{∞}/Z_{∞} is compact then every open maximal ideal is weakly non-Eisenstein.

Remark 9.15. We note that one does not expect weakly non-Eisenstein ideals to exist unless the rank of G_∞ is equal to the rank of $Z_\infty K_\infty$, which corresponds to the assumption $l_0 = 0$ for the derived subgroup of G_∞ . In that case, d is even and $q_0 = d/2$.

Lemma 9.16. *If \mathfrak{m} is weakly non-Eisenstein then the following hold:*

- (i) $\tilde{H}_{q_0, \mathfrak{m}}$ is a projective finitely generated $\mathcal{O}[[K_p/\Lambda_p]]$ -module;
- (ii) $\mathbb{T}_{\mathfrak{m}}$ acts faithfully on $\tilde{H}_{q_0, \mathfrak{m}}$;
- (iii) There are natural $\mathbb{T}_{\mathfrak{m}}[G(\mathbb{Q}_p)]$ -equivariant homeomorphisms:

$$\tilde{H}_{\mathfrak{m}}^{q_0} \cong \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\tilde{H}_{q_0, \mathfrak{m}}, \mathcal{O}), \quad \tilde{H}_{q_0, \mathfrak{m}} \cong \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\tilde{H}_{\mathfrak{m}}^{q_0}, \mathcal{O}).$$

Proof. Since the functor $M \mapsto H^i(\tilde{Y}(K_f^p K_p), \tilde{\mathcal{M}})$ commutes with direct limits, we obtain from (57) that $\tilde{H}^j(\mathcal{O}/\varpi^s)_{\mathfrak{m}} = 0$ for all $j \neq q_0$ and all $s \geq 0$. This implies that $\tilde{H}_{\mathfrak{m}}^j = 0$ for $j \neq q_0$ and the spectral sequence in Lemma 9.9 degenerates to give:

$$(59) \quad \mathrm{Hom}_{K_p}(M, \tilde{H}^{q_0}(\mathcal{O}/\varpi^s)_{\mathfrak{m}}) \cong H^{q_0}(\tilde{Y}(K_f^p K_p), \tilde{\mathcal{M}})_{\mathfrak{m}}.$$

The assumption on \mathfrak{m} implies that $M \mapsto H^i(\tilde{Y}(K_f^p K_p), \tilde{\mathcal{M}})_{\mathfrak{m}}$ is exact, thus we deduce that $\tilde{H}^{q_0}(\mathcal{O}/\varpi^s)_{\mathfrak{m}}$ is injective in the category smooth representations of K_p/Λ_p on \mathcal{O}/ϖ^s -modules. By Pontryagin duality we get that $\tilde{H}_{q_0}(\mathcal{O}/\varpi^s)_{\mathfrak{m}}$ is projective in the category of compact $\mathcal{O}/\varpi^s[[K_p/\Lambda_p]]$ -modules. By passing to the limit we obtain part (i). Any M as in Lemma 9.9 may be embedded into $\mathcal{C}(K_p/\Lambda_p, \mathcal{O}/\varpi^s)^{\oplus m}$ for some m . Using (57) we get an embedding $H^{q_0}(\tilde{Y}(K_f^p K_p), \tilde{\mathcal{M}})_{\mathfrak{m}} \hookrightarrow \tilde{H}^{q_0}(\mathcal{O}/\varpi^s)_{\mathfrak{m}}^{\oplus m}$. This yields part (ii). It follows from part (i) that $\tilde{H}_{q_0, \mathfrak{m}}$ is \mathcal{O} -torsion free. Part (iii) follows from Schikhof duality, see the discussion in [67, §2], together with the identity $\tilde{H}^{q_0}(\mathcal{O}/\varpi^s)_{\mathfrak{m}}^{\vee} \cong \tilde{H}_{q_0}(\mathcal{O}/\varpi^s)_{\mathfrak{m}} \cong \tilde{H}_{q_0, \mathfrak{m}}/\varpi^s$; alternatively one could use [13, Thm. 1.1 3)]. \square

9.5. Automorphic forms. Let V be an irreducible algebraic representation of G over L . As in the previous section we evaluate V at L , and make $G(\mathbb{Q}_p)$ and K_p act on it via $G(\mathbb{Q}_p) \hookrightarrow G(L)$. We assume that Λ_p acts trivially on V . We fix a K_p -invariant lattice M in V . Let \mathcal{M}^d the local system associated to $M^d := \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$, let \mathcal{V}^* the local system associated to V^* on $Y(K_p^f K_p)$ and let i be a non-negative integer. We let

$$H^i(\mathcal{V}^*) := \varinjlim_{K_p'} H^i(Y(K_f^p K_p'), \mathcal{V}^*), \quad H^i(\mathcal{M}^d) := \varinjlim_{K_p'} H^i(Y(K_f^p K_p'), \mathcal{M}^d).$$

Then $H^i(\mathcal{V}^*) = H^i(\mathcal{M}^d) \otimes_{\mathcal{O}} L$.

Let τ be a smooth absolutely irreducible representation of K_p on an L -vector space with the property that if π_p is a smooth irreducible \bar{L} -representation of $G(\mathbb{Q}_p)$ then $\mathrm{Hom}_{K_p}(\tau, \pi_p) \neq 0$ implies that π_p is supercuspidal. We will call such representations *supercuspidal types*. If p is bigger than the Coxeter number of G then examples of supercuspidal types for every open K_p can be obtained by inducing representations considered in [38, Thm. 2.2.15] and letting τ be an irreducible subquotient; if $G = \mathrm{GL}_n$ then there is no need to put restrictions on p , see [36, Prop. 3.19]. We assume that Λ_p acts trivially on τ .

We fix an embedding $L \hookrightarrow \mathbb{C}$. Following [34, §2.1.6], we let

$$\mathcal{A}(K_f^p) := \varinjlim_{K'_p} \mathcal{A}(K_f^p K'_p),$$

where $\mathcal{A}(K_f^p K'_p)$ is the space of automorphic forms on $G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f^p K'_p$. Let χ be the character through which A_∞° acts on $V_\mathbb{C}$. Let $\mathcal{A}(K_f^p)_\chi$ be the subspace of $\mathcal{A}(K_f^p)$ on which A_∞° acts through χ . Franke's theorem [39] implies, see [40, Thm. 2.3], that

$$(60) \quad H^i(\mathcal{V}^*) \otimes_L \mathbb{C} \cong H^i(\tilde{\mathfrak{g}}, \mathfrak{k}; \mathcal{A}(K_f^p)_\chi \otimes V_\mathbb{C}^*),$$

where \tilde{G}_∞ is the group of real points of the intersection of the kernels of all the rational characters of G , $\tilde{\mathfrak{g}}$ is its Lie algebra and \mathfrak{k} is the Lie algebra of K_∞ . The space $\mathcal{A}(K_f^p)_\chi$ decomposes into the cuspidal part $\mathcal{A}_{\text{cusp}}(K_f^p)_\chi$ and its orthogonal complement $\mathcal{A}_{\text{Eis}}(K_f^p)_\chi$. Since τ is a supercuspidal type, it follows from the description of $\mathcal{A}_{\text{Eis}}(K_f^p)_\chi$ in the course of the proof of [40, Prop. 3.3, Eqn. (3), (4)] as a quotient of a direct sum of parabolically induced representations, that $\text{Hom}_{K_p}(\tau_\mathbb{C}, \mathcal{A}_{\text{Eis}}(K_f^p)_\chi) = 0$. Thus we obtain a $\mathbb{T}_\mathbb{C}^{\text{univ}}$ -equivariant isomorphism:

$$(61) \quad \text{Hom}_{K_p}(\tau, H^i(\mathcal{V}^*)) \otimes_L \mathbb{C} \cong \text{Hom}_{K_p}(\tau_\mathbb{C}, H^i(\tilde{\mathfrak{g}}, \mathfrak{k}; \mathcal{A}_{\text{cusp}}(K_f^p)_\chi \otimes V_\mathbb{C}^*)).$$

If we let $\mathcal{A}_{\text{cusp}} = \varinjlim_{K_f^p} \mathcal{A}_{\text{cusp}}(K_f^p)$ then $\mathcal{A}_{\text{cusp}}$ decomposes into a direct sum of irreducible representations $\pi = \otimes'_v \pi_v$ of $G(\mathbb{A})$. Moreover, $\mathcal{A}_{\text{cusp}}^{K_f^p} = \mathcal{A}_{\text{cusp}}(K_f^p)$. We thus obtain a $\mathbb{T}_\mathbb{C}^{\text{univ}}$ -equivariant isomorphism

$$(62) \quad \begin{aligned} & \text{Hom}_{K_p}(\tau_\mathbb{C}, H^i(\tilde{\mathfrak{g}}, \mathfrak{k}; \mathcal{A}_{\text{cusp}}(K_f^p)_\chi \otimes V_\mathbb{C}^*)) \\ & \cong \bigoplus_{\pi} H^i(\tilde{\mathfrak{g}}, \mathfrak{k}; \pi_\infty \otimes V_\mathbb{C}^*) \otimes \text{Hom}_{K_p}(\tau_\mathbb{C}, \pi_p) \otimes (\pi^{p, \infty})^{K_f^p}, \end{aligned}$$

where the sum is taken over all irreducible subrepresentations π of $\mathcal{A}_{\text{cusp}}$ counted with multiplicities, such that A_∞° acts by χ on π_∞ , and, where $\pi^{\infty, p} = \otimes'_{v|p\infty} \pi_v$.

Remark 9.17. The supercuspidal type τ is only used to ensure that $\mathcal{A}_{\text{Eis}}(K_f^p)_\chi$ does not contribute to Hecke eigenvalues. If the group G is anisotropic then $\mathcal{A}_{\text{Eis}}(K_f^p)_\chi$ is zero and the use of supercuspidal type is redundant. See also the discussion in [34, 3.1.2].

Let $\tilde{\mathcal{M}}^d$ the local system associated to $M^d := \text{Hom}_{\mathcal{O}}(M, \mathcal{O})$ and $\tilde{\mathcal{V}}^*$ the local system associated to V^* on $\tilde{Y}(K_f^p K_p)$. If K'_p is an open normal subgroup of K_p then

$$(63) \quad \begin{aligned} \text{Hom}_{K'_p}(M, \tilde{H}_\mathfrak{m}^{q_0}) & \cong \varprojlim_s \text{Hom}_{K'_p}(M, \tilde{H}^{q_0}(\mathcal{O}/\varpi^s)_\mathfrak{m}) \\ & \stackrel{(59)}{\cong} \varprojlim_s H^i(\tilde{Y}(K_f^p K_p), (\tilde{\mathcal{M}}/\varpi^s)^\vee)_\mathfrak{m} \\ & \cong H^i(\tilde{Y}(K_f^p K_p), \tilde{\mathcal{M}}^d)_\mathfrak{m}, \end{aligned}$$

where the last isomorphism follows from Mittag-Leffler. Thus we have an isomorphism of \mathbb{T} -modules

$$(64) \quad H^{q_0}(\tilde{Y}(K_f^p K_p), \tilde{\mathcal{M}}^d) \cong \text{Hom}_{K'_p}(M, \tilde{H}_\mathfrak{m}^{q_0}) \oplus \bigoplus_{\mathfrak{m}' \neq \mathfrak{m}} H^{q_0}(\tilde{Y}(K_f^p K_p), \tilde{\mathcal{M}}^d)_{\mathfrak{m}'},$$

where the direct sum is over open maximal ideals of \mathbb{T} different from \mathfrak{m} .

It follows from (64) that $\mathrm{Hom}_{K_p}(V(\tau), \tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L)$ is a direct summand as a \mathbb{T} -module of $\mathrm{Hom}_{K_p}(\tau, H^{q_0}(\tilde{\mathcal{V}}^*))$, where $V(\tau) := V \otimes_L \tau$. It follows from Lemma 9.8 that we have a $G(\mathbb{A}_f)$ -equivariant isomorphism

$$\varinjlim_{K_f} H^{q_0}(Y(K_f), \mathcal{V}^*) \cong \bigoplus_{i+j=q_0} (\varinjlim_{K_f} H^i(\tilde{Y}(K_f), \tilde{\mathcal{V}}^*)) \otimes_L H^j(\mathbb{R}^r/\mathbb{Z}^r, L).$$

After taking K_f^p -invariants we deduce that $\mathrm{Hom}_{K_p}(\tau, H^{q_0}(\tilde{\mathcal{V}}^*))$ is a direct summand of $\mathrm{Hom}_{K_p}(\tau, H^{q_0}(\mathcal{V}^*))$ as a \mathbb{T} -module.

Let $\mathbb{T}_{\mathfrak{m}, V(\tau)}$ be the quotient of $\mathbb{T}_{\mathfrak{m}}$ acting faithfully on $\mathrm{Hom}_{K_p}(V(\tau), \tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L)$.

Lemma 9.18. *The algebra $\mathbb{T}_{\mathfrak{m}, V(\tau)}$ is reduced. Moreover, if we fix an isomorphism $\overline{\mathbb{Q}}_p \xrightarrow{\cong} \mathbb{C}$ then for every L -algebra homomorphism $x : \mathbb{T}_{\mathfrak{m}, V(\tau)}[1/p] \rightarrow \overline{\mathbb{Q}}_p$ there is a cuspidal automorphic representation $\pi_x = \otimes'_v \pi_{x,v}$ of $G(\mathbb{A})$, such that the following hold:*

- (i) $\mathbb{T}_{\mathfrak{m}}$ acts on $\pi_x^{K_f^p}$ via x ;
- (ii) $\mathrm{Hom}_{K_p}(\tau, \pi_{x,p}) \neq 0$;
- (iii) A_{∞}° acts on $\pi_{x,\infty}$ via χ ;
- (iv) $H^{q_0}(\mathfrak{g}, \mathfrak{k}; \pi_{x,\infty} \otimes V_{\mathbb{C}}^*) \neq 0$.

Proof. Let $\mathbb{T}'_{V(\tau)}$ be the L -subalgebra of $\mathrm{End}_L(\mathrm{Hom}_{K_p}(\tau, H^{q_0}(\mathcal{V}^*)))$ generated by the image of \mathbb{T} . As explained above $\mathrm{Hom}_{K_p}(V(\tau), \tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L)$ is a direct summand of $\mathrm{Hom}_{K_p}(\tau, H^{q_0}(\mathcal{V}^*))$ as a \mathbb{T} -module and hence it is enough to prove the assertion for $\mathbb{T}'_{V(\tau)}$. Since $\mathrm{Hom}_{K_p}(\tau, H^{q_0}(\mathcal{V}^*))$ is finite dimensional, $\mathbb{T}'_{V(\tau)}$ is a quotient of $\mathbb{T}^{\mathrm{univ}}[1/p]$. It follows from (61) and (62) that the assertion of the Lemma holds for $\mathbb{T}'_{V(\tau)}$. \square

Lemma 9.19. *For all π_x in Lemma 9.18, $\pi_{x,\infty}$ and V have the same infinitesimal character.*

Proof. It follows from Lemma 9.18 (iv) and [7, Cor. I.4.2] that $Z(\mathfrak{g})$ acts on $(\mathfrak{g}, \mathfrak{k})$ -module of $\pi_{x,\infty}$ and V by the same infinitesimal character. Since A_{∞}° acts on both representations by χ and $\tilde{G}_{\infty} A_{\infty}^{\circ}$ is of finite index in G_{∞} , see for example [6, 4.3.1], the assertion follows. \square

9.6. Main result. Part (iv) of Lemma 9.18 implies that π_x is cohomological, and such automorphic forms are C -algebraic by [12, Lem. 7.2.2]. Thus according to the Conjecture 5.3.4 of [12], that there should be an admissible Galois representation $\rho_x : \mathrm{Gal}_{\mathbb{Q}} \rightarrow {}^C G(\overline{\mathbb{Q}}_p)$ attached to π_x (or rather to the local information detailed in Lemma 9.18).

Theorem 9.20. *We assume that the following hold:*

- (o) Z_{∞}/A_{∞} is compact;
- (i) $\mathbb{T}_{\mathfrak{m}}$ is noetherian;
- (ii) there is an admissible representation $\rho : \mathrm{Gal}_{\mathbb{Q}} \rightarrow {}^C G_f(\mathbb{T}_{\mathfrak{m}}^{\mathrm{rig}})$, such that for all $V \in \mathrm{Irr}_G(L)$ and all $x : \mathbb{T}_{\mathfrak{m}, V(\tau)}[1/p] \rightarrow \overline{\mathbb{Q}}_p$, the specialisation of ρ at x matches π_x according [12, Conj. 5.3.4];
- (iii) the composition $d \circ \rho$ is equal to the p -adic cyclotomic character.

Then for all $y \in \mathrm{m}\text{-Spec } \mathbb{T}_{\mathfrak{m}}[1/p]$ the centre $Z(\mathfrak{g})$ acts on $(\tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{m}_y]^{\mathrm{la}}$ by the infinitesimal character $\zeta_{\rho,y}^C$.

Proof. We will show that the conditions of Theorem 8.5 holds for $M = \tilde{H}_{q_0, \mathfrak{m}}$, $R = \mathbb{T}_{\mathfrak{m}}$ and $\zeta = \zeta_{\rho}^C$. The assumption that Z_{∞}/A_{∞} is compact implies that Λ_p is trivial via Lemma 9.4. Part (i) of Theorem 8.5 holds by Lemma 9.16 with $h = 0$. Part (ii) is given by Lemma 9.18. To show that part (iii) holds we denote by ρ_x the specialisation of ρ at x . It follows from Lemma 4.26 that the specialisation of ζ_{ρ}^C at x is equal to $\zeta_{\rho_x}^C$. Proposition 5.15 implies that $\zeta_{\rho_x}^C$ is equal to the infinitesimal character of $\pi_{x, \infty}$ which is equal to the infinitesimal character of V by Lemma 9.19. Since τ is a smooth representation, part (iii) of Theorem 8.5 is satisfied. \square

Remark 9.21. Let us note that we do not need the full force of the [12, Conj. 5.3.4], just the part relating the Hodge–Tate cocharacter of the Galois representation at places above p to the infinitesimal character of the automorphic representation at the archimedean places, as discussed in [12, Rem. 5.3.5] in the amended version of the paper.

The most difficult condition to check is part (ii). In the known cases one obtains such representations by expressing $\mathbb{T}_{\mathfrak{m}}$ as a quotient of a Galois deformation ring. This then automatically implies part (i). Part (iii) is forced upon us by the conjecture of Buzzard–Gee, since according to it part (iii) should hold for representations associated to x in part (ii), and it follows from Theorem 8.4 that such points are Zariski dense in $\mathbb{T}_{\mathfrak{m}}[1/p]$.

We will now prove a version of the theorem above with a fixed central character. It will enable us to remove the assumption that Z_{∞}/A_{∞} is compact.

Lemma 9.22. *The closure of the image of $Z(\mathbb{A}^S)$ inside $Z(\mathbb{A})/Z(\mathbb{Q})$ is a subgroup of finite index. Moreover, if $Z = \text{Res}_{F/\mathbb{Q}} Z'$, where Z' is a split torus defined over a number field F , then the image is dense.*

Proof. Since the map $Z(\mathbb{A}) \rightarrow Z(\mathbb{A})/Z(\mathbb{Q})$ is continuous the inverse image of the closure is closed in $Z(\mathbb{A})$ and contains $Z(\mathbb{A}^S)Z(\mathbb{Q})$. Thus it is enough to show that the closure of $Z(\mathbb{A}^S)Z(\mathbb{Q})$ is of finite index in $Z(\mathbb{A})$. Since $Z(\mathbb{A}^S)$ is closed in $Z(\mathbb{A})$, this closure is equal to $Z(\mathbb{A}^S)C$, where C is the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_S)$. Since C is of finite index in $Z(\mathbb{A}_S)$ by [70, Cor. 3.5], we are done. If $Z = \text{Res}_{F/\mathbb{Q}} Z'$, where Z' is split over F , then it follows from [69, Prop. 7.8] applied to Z' that $C = Z(\mathbb{A}_S)$. Thus $Z(\mathbb{A}^S)Z(\mathbb{Q})$ is dense in $Z(\mathbb{A})$. \square

For each $\ell \notin S$ let \mathcal{Z}_{ℓ} be the \mathcal{O} -subalgebra of \mathcal{H}_{ℓ} generated by functions with support in $Z(\mathbb{Q}_{\ell})K_{\ell}$, let $\mathcal{Z}^{\text{univ}} \subset \mathbb{T}^{\text{univ}}$ be the tensor product of these algebras over \mathcal{O} . If $\psi : Z(\mathbb{A}) \rightarrow \mathcal{O}^{\times}$ is a character then we will denote by $(\tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L)_{\psi}$ the ψ -eigenspace for the action of $Z(\mathbb{A})$ on $\tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L$.

Lemma 9.23. *Let \mathfrak{a} be the kernel of an \mathcal{O} -algebra homomorphism $\varphi : \mathcal{Z}^{\text{univ}} \rightarrow \mathcal{O}$, such that $(\tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{a}] \neq 0$. There exist finitely many characters $\psi_i : Z(\mathbb{A}) \rightarrow \mathcal{O}^{\times}$, $1 \leq i \leq n$, such that*

$$(\tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{a}] = \bigoplus_{i=1}^n (\tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L)_{\psi_i}.$$

Moreover, if $Z = \text{Res}_{F/\mathbb{Q}} Z'$, where Z' is a split torus defined over a number field F , then $n = 1$.

Proof. The action of $Z(\mathbb{A})$ on $\tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L$ factors through $Z(\mathbb{A})/Z(\mathbb{Q})$. Let J be the closure of the image of $Z(\mathbb{A}^S)$ inside $Z(\mathbb{A})/Z(\mathbb{Q})$. Lemma 9.22 says that J is

of finite index in $Z(\mathbb{A})/Z(\mathbb{Q})$. Thus it is enough to show that there is a character $\psi : J \rightarrow \mathcal{O}^\times$, such that $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{a}]$ is equal to the ψ -eigenspace for the action of J on $\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L$.

Let $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{a}]^0$ be the unit ball and let $v \in (\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{a}]^0/\varpi^n$. Since the topology on $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{a}]^0/\varpi^n$ is discrete and the action of J is continuous, the J -stabiliser $\text{Stab}(v)$ of v is open in J . Since $Z(\mathbb{A}^S)$ is dense in J , the map $Z(\mathbb{A}^S) \rightarrow J/\text{Stab}(v)$ is surjective. On the other hand if $\ell \notin S$ then $Z(\mathbb{Q}_\ell)$ acts on $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{a}]$ by the character $g \mapsto \varphi(gK_\ell)$, thus J acts on v by the character ψ_n , uniquely determined by the formula $\psi_n(g) \equiv \varphi(gK_\ell) \pmod{\varpi^n}$ for all $g \in Z(\mathbb{Q}_\ell)$ and for all $\ell \notin S$. The uniqueness implies that J acts on $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{a}]^0/\varpi^n$ via ψ_n . By passing to the limit we obtain a character $\psi : J \rightarrow \mathcal{O}^\times$, by which J acts on $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{a}]$, and which satisfies $\psi(g) = \varphi(gK_\ell)$ for all $g \in Z(\mathbb{Q}_\ell)$ for all $\ell \notin S$. In particular, ψ -eigenspace is contained in $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{a}]$ and thus the two coincide.

If $Z = \text{Res}_{F/\mathbb{Q}} Z'$, where Z' is a split torus defined over F , then $J = Z(\mathbb{A})/Z(\mathbb{Q})$ by Lemma 9.22 and the assertion follows. \square

We fix V_0 such that $\Lambda(K_f^p K_p)$ acts trivially on it and choose an \mathcal{O} -algebra homomorphism $x_0 : \mathbb{T}_{m, V_0(\tau)} \rightarrow \mathcal{O}$. Let \mathfrak{a} be the kernel of the composition

$$\mathcal{Z}^{\text{univ}} \rightarrow \mathbb{T}_{m, V_0(\tau)} \xrightarrow{x_0} \mathcal{O}.$$

Then $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{a}]$ is non-zero, since it contains $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{m}_{x_0}]$, which is non-zero as x_0 lies in the support $\text{Hom}_{K_p}(V_0(\tau), \tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)$. Let $\psi : Z(\mathbb{A}) \rightarrow \mathcal{O}^\times$ be one of the characters in Lemma 9.23, such that $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)_\psi$ is non-zero. Note that $\psi|_{Z(\mathbb{Q}_p) \cap K_p}$ is the central character of $V_0(\tau)$. Let \mathbb{T}_m^ψ be the quotient of \mathbb{T}_m acting faithfully on $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)_\psi$, and let $\mathbb{T}_{m, V(\tau)}^\psi$ be the quotient of \mathbb{T}_m^ψ acting faithfully on

$$\text{Hom}_{K_p}(V(\tau), (\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)_\psi).$$

If the action of $Z(\mathbb{Q}_p) \cap K_p$ on $V(\tau)$ is not given by ψ then $\mathbb{T}_{m, V(\tau)}^\psi$ will be zero.

Theorem 9.24. *We assume that the following hold:*

- (i) \mathbb{T}_m^ψ is noetherian;
- (ii) there is an admissible representation $\rho : \text{Gal}_{\mathbb{Q}} \rightarrow {}^C G_f(\mathbb{T}_m^{\psi, \text{rig}})$, such that for all $V \in \text{Irr}_G(L)$ and all $x : \mathbb{T}_{m, V(\tau)}^\psi[1/p] \rightarrow \overline{\mathbb{Q}_p}$, the specialisation of ρ at x matches π_x according [12, Conj. 5.3.4];
- (iii) the composition $d \circ \rho$ is equal to the p -adic cyclotomic character.

Then for all $y \in \mathfrak{m}\text{-Spec } \mathbb{T}_m^\psi[1/p]$ the centre $Z(\mathfrak{g})$ acts on $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{m}_y]^{1a}$ by the infinitesimal character $\zeta_{\rho, y}^C$.

Proof. We first note that since \mathfrak{m}_x is an ideal of $\mathbb{T}_m^\psi[1/p]$, the subspaces annihilated by \mathfrak{m}_x in $\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L$ and in $(\tilde{H}_m^{q_0} \otimes_{\mathcal{O}} L)_\psi$ coincide. It follows from Lemma 9.16 that $(\tilde{H}_m^{q_0})_\psi$ is a direct summand of $\mathcal{C}_\psi(K_p, \mathcal{O})^{\oplus m}$. Thus its Schikhof dual M is projective in $\text{Mod}_{K, \psi}^{\text{pro}}(\mathcal{O})$ and is equal to the largest quotient of $\tilde{H}_{q_0, m}$ on which $Z(\mathbb{Q}_p) \cap K_p$ acts by ψ^{-1} . The algebras $\mathbb{T}_{m, V(\tau)}$ are finite dimensional over L and reduced by Lemma 9.18, hence products of finite field extensions of L . Thus the quotients $\mathbb{T}_{m, V(\tau)}^\psi$ are also reduced. The rest of the proof is the same as the proof of Theorem 9.20 using Theorem 8.6. \square

In sections 9.7, 9.8, 9.9 and 9.10 we will discuss some examples, where the conditions of Theorems 9.20 and 9.24 are satisfied.

9.7. Modular curves. Let $G = \mathrm{GL}_2$. Then $Z_\infty = A_\infty$, $d = 2$, $q_0 = 1$, $K_\ell = \mathrm{GL}_2(\mathbb{Z}_\ell)$ and $\mathbb{T}^{\mathrm{univ}} = \mathcal{O}[T_\ell, S_\ell^{\pm 1} : \ell \notin S]$. Let \mathfrak{m} be an open maximal ideal of \mathbb{T} . After extending scalars we may assume that the residue field of \mathfrak{m} is k . Since H^2 is dual to H_c^0 , which is contained in H^0 , \mathfrak{m} is weakly non-Eisenstein if and only if $H^0(Y(K_f^p K_p), \mathcal{O}/\varpi)_{\mathfrak{m}} = 0$.

Lemma 9.25. $H^0(Y(K_f^p K_p), \mathcal{O}/\varpi)_{\mathfrak{m}} \neq 0$ if and only if there is a character $\psi : \widehat{\mathbb{Z}}^* / \det(K_f^p K_p) \rightarrow k^*$ such that

$$T_\ell \equiv (\ell + 1)\psi(\ell) \pmod{\mathfrak{m}}, \quad S_\ell \equiv \psi(\ell^2) \pmod{\mathfrak{m}}, \quad \forall \ell \notin S.$$

Proof. We may identify $H^0(Y(K_f^p K_p), \mathcal{O}/\varpi)$ with the set of maps from the set of connected components of $Y(K_f^p K_p)$ to k . The set of connected components can be identified with $\widehat{\mathbb{Z}}^* / \det(K_f^p K_p)$. The action of $\mathrm{GL}_2(\mathbb{A}_f^p)$ on $\varinjlim_{K_f^p} H^0(Y(K_f^p K_p), \mathcal{O}/\varpi)$ factors through the determinant. Since $H^0(Y(K_f^p K_p), \mathcal{O}/\varpi)$ is the K_f^p -invariants of this representation we obtain the assertion. \square

We assume that \mathfrak{m} is weakly non-Eisenstein. It follows from Deligne–Serre lemma that after extending scalars there is a homomorphism of \mathcal{O} -algebras $x : \mathbb{T}_{\mathfrak{m}, V(\tau)} \rightarrow \mathcal{O}$, lifting $\mathbb{T}_{\mathfrak{m}}^{\mathrm{univ}} \rightarrow k$. By composing it with our fixed embedding $L \hookrightarrow \mathbb{C}$ we obtain a cuspidal eigenform f . To it Deligne associates a Galois representation $\rho_f : \mathrm{Gal}_{\mathbb{Q}, S} \rightarrow \mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ such that

$$\mathrm{tr} \rho_f(\mathrm{Frob}_\ell) \equiv T_\ell \pmod{\mathfrak{m}_x}, \quad \det \rho_f(\mathrm{Frob}_\ell) \equiv \ell S_\ell \pmod{\mathfrak{m}_x}, \quad \forall \ell \notin S.$$

Let $\bar{\rho} : \mathrm{Gal}_{\mathbb{Q}, S} \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be the reduction of ρ_f modulo p . Then

$$\mathrm{tr} \bar{\rho}(\mathrm{Frob}_\ell) \equiv T_\ell \pmod{\mathfrak{m}}, \quad \det \bar{\rho}(\mathrm{Frob}_\ell) \equiv \ell S_\ell \pmod{\mathfrak{m}}, \quad \forall \ell \notin S.$$

After extending scalars we may assume that $\bar{\rho}$ takes values in $\mathrm{GL}_2(k)$.

We assume that $\bar{\rho}$ is absolutely irreducible. It is explained in [33, §5.2] there is a surjection $R_{\bar{\rho}} \twoheadrightarrow \mathbb{T}_{\mathfrak{m}}$, where $R_{\bar{\rho}}$ is the universal Galois deformation ring of $\bar{\rho} : \mathrm{Gal}_{\mathbb{Q}, S} \rightarrow \mathrm{GL}_2(k)$. This implies that $\mathbb{T}_{\mathfrak{m}}$ is noetherian. Let $\rho : \mathrm{Gal}_{\mathbb{Q}, S} \rightarrow \mathrm{GL}_2(\mathbb{T}_{\mathfrak{m}})$ be the corresponding Galois representation. This is a representation into an L -group. By using the twisting element $\tilde{\delta}(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix}$ we obtain a representation $\rho^C : \mathrm{Gal}_{\mathbb{Q}, S} \rightarrow {}^C G_f(\mathbb{T}_{\mathfrak{m}})$ as in section 4.7. This representation satisfies the condition (ii) of Theorem 9.20, but it is not so easy to extract this statement from [12]. Instead, we observe that it follows from [37, Thm. 7] that the conditions in Remark 9.21 are satisfied. Thus Theorem 9.20 holds when $G = \mathrm{GL}_2$ and \mathfrak{m} is associated to an absolutely irreducible Galois representation $\bar{\rho} : \mathrm{Gal}_{\mathbb{Q}, S} \rightarrow \mathrm{GL}_2(k)$.

9.8. Shimura curves. Let D be a quaternion algebra over a totally real field F split at only one infinite place. Let G' be the group over F defined by $G'(A) = (D \otimes_F A)^*$ for F -algebras A and let $G = \mathrm{Res}_{F/\mathbb{Q}_p} G'$. Then $d = 2$, $q_0 = 1$ and the split \mathbb{R} -rank of Z_∞/A_∞ is equal to $[F : \mathbb{Q}] - 1$. We assume that S contains the ramification primes of F/\mathbb{Q} and the primes below the ramification places of D . Let S' be all the places of F above the places in S . Then $\mathbb{T}^{\mathrm{univ}} = \mathcal{O}[T_v, S_v^{\pm 1} : v \notin S']$. Let \mathfrak{m} be a weakly non-Eisenstein ideal of \mathbb{T} . Arguing as in the previous section and

using the results of Carayol [17] we may assume that there is a Galois representation $\bar{\rho} : \text{Gal}_{F,S'} \rightarrow \text{GL}_2(k)$, such that

$$(65) \quad \text{tr } \bar{\rho}(\text{Frob}_v) \equiv T_v \pmod{\mathfrak{m}}, \quad \det \bar{\rho}(\text{Frob}_v) \equiv q_v S_v \pmod{\mathfrak{m}}, \quad \forall v \notin S',$$

where q_v denotes the number of elements in the residue field of F_v .

Let $x_0 : \mathbb{T}_{\mathfrak{m}, V_0(\tau)} \rightarrow \mathcal{O}$ be an \mathcal{O} -algebra homomorphism and let $\psi : \mathbb{A}_{f,F}^\times / F^\times \rightarrow \mathcal{O}^\times$ be as in section 9.6. If ρ_0 is the Galois representation attached to the automorphic form corresponding to x_0 , then $\det \rho_0 = \psi \chi_{\text{cyc}}$.

Let $R_{\text{tr } \bar{\rho}}$ be the universal pseudodeformation ring deforming the characteristic polynomial of $\bar{\rho}$ as in [19], and let $R_{\text{tr } \bar{\rho}}^\psi$ be the quotient of $R_{\text{tr } \bar{\rho}}$ which corresponds to the determinant $\chi_{\text{cyc}} \psi$. Arguing as in [66] we obtain a surjection $R_{\text{tr } \bar{\rho}}^\psi \rightarrow \mathbb{T}_{\mathfrak{m}}^\psi$, which implies that $\mathbb{T}_{\mathfrak{m}}^\psi$ is noetherian.

We will assume that $\bar{\rho}$ is absolutely irreducible. Then $R_{\text{tr } \bar{\rho}}^\psi$ is the universal deformation ring of $\bar{\rho}$ and thus by specialising the universal deformation along $R_{\text{tr } \bar{\rho}}^\psi \rightarrow \mathbb{T}_{\mathfrak{m}}^\psi$ we obtain a Galois representation $\rho : \text{Gal}_{F,S'} \rightarrow \text{GL}_2(\mathbb{T}_{\mathfrak{m}}^\psi)$.

As explained in [4] there is a canonical bijection between the equivalence classes of admissible representations of $\text{Gal}_{\mathbb{Q},S}$ into the L -group of G and the admissible representations of $\text{Gal}_{F,S'}$ into the L -group of G' . So we will work with $\text{Gal}_{F,S'}$.

By using the twisting element $\tilde{\delta}(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix}$ we obtain a representation $\rho^C : \text{Gal}_{F,S'} \rightarrow {}^C G_f(\mathbb{T}_{\mathfrak{m}}^\psi)$ as in section 4.7. Specialisations of this representation at $x \in \mathfrak{m}\text{-Spec } \mathbb{T}_{\mathfrak{m}, V(\tau)}^\psi[1/p]$ satisfies [12, Conj. 5.3.4], but again it is not so easy to extract this statement from [12]. Instead, we observe that the compatibility at p and ∞ discussed in Remark 9.21 follows from [82].

We conclude that the conditions of Theorem 9.24 are satisfied, when \mathfrak{m} is associated to an absolutely irreducible Galois representation $\bar{\rho} : \text{Gal}_{F,S'} \rightarrow \text{GL}_2(k)$, and $Z(\mathfrak{g})$ acts on $(\tilde{H}_{\mathfrak{m}}^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]^{1a}$ by the infinitesimal character $\zeta_{\rho^C, x}^C$, for all $x \in \mathfrak{m}\text{-Spec } \mathbb{T}_{\mathfrak{m}}^\psi[1/p]$.

Remark 9.26. Let \mathfrak{m} be an open maximal ideal of \mathbb{T} and let $\bar{\rho} : \text{Gal}_{F,S'} \rightarrow \text{GL}_2(k)$ be a Galois representation satisfying (65). If $\bar{\rho}$ is absolutely irreducible then \mathfrak{m} is weakly non-Eisenstein, see [60].

9.9. Definite unitary groups. Let F be a totally real field and let E be a quadratic totally imaginary extension of F . We assume that $[F : \mathbb{Q}]$ is even, every place of F above p is split in E and every finite place of F is unramified in E . Since $[F : \mathbb{Q}]$ is even there exists a unitary group G' over F which is an outer form of GL_n with respect to the quadratic extension E/F such that G' is quasi-split at all finite places and anisotropic at all infinite places. Let $G = \text{Res}_{F/\mathbb{Q}} G'$. Then $d = 0$ and Z_∞ is compact. We assume that S contains the ramification places of F/\mathbb{Q} . Let S' be the set of places of E above the places in S . Let $\mathbb{T}^{\text{Spl}, \text{univ}}$ be the subalgebra of \mathbb{T}^{univ} generated by subalgebras $\mathcal{H}(G'(F_v)/K_v)$ for those places v of F , which split completely in E and are not above places in S . Let \mathbb{T}^{Spl} be the closure of the image of $\mathbb{T}^{\text{Spl}, \text{univ}}$ in \mathbb{T} .

Since $d = 0$ every open maximal ideal of \mathbb{T}^{Spl} is weakly non-Eisenstein. As explained in Remark 9.12 the set of such ideals is finite and we have an isomorphism $\mathbb{T}^{\text{Spl}} \cong \prod_{\mathfrak{m}} \mathbb{T}_{\mathfrak{m}}^{\text{Spl}}$. As explained in [66, Lem. C.7] every open maximal ideal of \mathfrak{m} corresponds to a characteristic polynomial of a Galois representation $\bar{\rho} : \text{Gal}_{E,S'} \rightarrow \text{GL}_n(\mathbb{F}_p)$. Let $R_{\text{tr } \bar{\rho}}^{\text{ps}}$ be the universal deformation ring parameterising

pseudorepresentations (or determinants) lifting the characteristic polynomial of $\bar{\rho}$ as in [19]. It is shown in [66, Thm. C.3] that $\mathbb{T}_m^{\text{Sp1}}$ is a quotient of $R_{\text{tr } \bar{\rho}}^{\text{ps}}$ and hence is noetherian.

Let us assume that \mathfrak{m} corresponds to an absolutely irreducible representation $\bar{\rho}$. Then $R_{\text{tr } \bar{\rho}}^{\text{ps}}$ is also the universal deformation ring of $\bar{\rho}$. Let $\rho : \text{Gal}_{E, S'} \rightarrow \text{GL}_n(\mathbb{T}_m^{\text{Sp1}})$ be the representation obtained from the universal deformation representation of $\bar{\rho}$ by extending scalars along the surjection $R_{\text{tr } \bar{\rho}}^{\text{ps}} \twoheadrightarrow \mathbb{T}_m^{\text{Sp1}}$. If $x : \mathbb{T}_{\mathfrak{m}, V(\tau)}^{\text{Sp1}} \rightarrow \overline{\mathbb{Q}}_p$ is a homomorphism of \mathcal{O} -algebras then ρ_x is the Galois representation associated to π_x by Clozel, see [38, Thm. 3.3.1]. The local-global compatibility between π_x and ρ_x is summarised in [35, Thm. 7.2.1], see the references in its proof for proper attributions. In particular, ρ_x is potentially semi-stable with Hodge–Tate cocharacter $\lambda + \tilde{\delta}$ (note that the convention concerning Hodge–Tate numbers is different in *loc. cit.*), where λ is the highest weight of V and $\tilde{\delta} = (0, -1, \dots, 1 - n)$. Let $\rho^C : \text{Gal}_{E, S'} \rightarrow {}^C G'_f(\mathbb{T}_m^{\text{Sp1}})$ be the representation obtained from ρ using the twisting element $\tilde{\delta}$. It follows from Proposition 5.5 that $Z(\mathfrak{g})$ acts on V through the character $\zeta_{\rho^C}^C$.

It follows from Theorem 9.20 that if \mathfrak{m} is associated to an absolutely irreducible $\bar{\rho}$ then for all $x \in \text{m-Spec } \mathbb{T}_m^{\text{Sp1}}[1/p]$, $Z(\mathfrak{g})$ acts on $(H_m^0 \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]^{\text{la}}$ via $\zeta_{\rho_x^C}^C$.

9.10. Compact unitary Shimura varieties. Let F be totally real field and E a CM quadratic extension of F . We assume that E contains a quadratic imaginary number field. Let G be some anisotropic similitude unitary group over \mathbb{Q} with similitude factor $c : G \rightarrow \mathbb{G}_{m, \mathbb{Q}}$ and let $H = \ker(c)$. We have $H = \text{Res}_{F/\mathbb{Q}} H'$ with H' an unitary group over F such that H'_E is an inner form of GL_n . Note that we have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ where \mathfrak{s} is the Lie algebra of the maximal \mathbb{Q} -split normal torus S contained in G .

As in section 9.9, we define the Hecke algebra \mathbb{T}^{Sp1} using places v of F which are split in E and such that H' is quasi-split at v . Let \mathfrak{m} be some open maximal ideal in \mathbb{T}^{Sp1} and let $\rho_{\mathfrak{m}} : \text{Gal}_E \rightarrow \text{GL}_n(\overline{\mathbb{F}}_p)$ be Galois representation associated to \mathfrak{m} in [16, Thm. 1.1]. Up to enlarging L , we can assume that $\rho_{\mathfrak{m}}$ takes values in $\text{GL}_n(k)$. We assume from now that $\rho_{\mathfrak{m}}$ is decomposed generic (see Definition 1.9 in *loc. cit.*) so that, by Thm. 1.1 in *loc. cit.*, the ideal \mathfrak{m} is non Eisenstein.

Let V be some algebraic irreducible representation of G such that $\mathbb{T}_{\mathfrak{m}, V}^{\text{Sp1}} \neq 0$. If $x : \mathbb{T}_{\mathfrak{m}, V}^{\text{Sp1}} \rightarrow \mathbb{C}$ is a character of $\mathbb{T}_{\mathfrak{m}, V}^{\text{Sp1}}$, we fix some automorphic representation π_x of $G(\mathbb{A}_{\mathbb{Q}})$ whose existence is assured by Lemma 9.18. It follows from [81, Thm. A.1] that we can define the base change Π of π which is an isobaric sum of conjugate self dual C -algebraic cuspidal automorphic representations of $\text{GL}_{n, E}$. The main Theorem of [20] gives us some admissible representation $\rho_x : \text{Gal}_E \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$ associated to Π . Assume now that $\rho_{\mathfrak{m}}$ is absolutely irreducible and that $p > 2$. Reasoning as in the proof of [21, Prop. 3.4.4], we can construct a continuous representation $\rho : \text{Gal}_{E, S} \rightarrow \text{GL}_n(\mathbb{T}_m^{\text{Sp1}})$ such that, for all $x : \mathbb{T}_{\mathfrak{m}, V}^{\text{Sp1}} \rightarrow \overline{\mathbb{Q}}_p$, ρ_x is associated to π_x by the previous construction. As in subsection 9.9, for all $x \in \text{m-Spec } \mathbb{T}_m^{\text{Sp1}}[1/p]$, the center $Z(\mathfrak{h})$ acts on $(\tilde{H}^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]^{\text{la}}$ by $\zeta_{\rho_x^C}^C$, the representation ρ_x^C being obtained from ρ_x using $\tilde{\delta} = (0, -1, \dots, 1 - n)$. As $S(\mathbb{Q}_p)$ acts on $(\tilde{H}^{q_0} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]^{\text{la}}$ by a character by Lemma 9.23, it follows that $Z(\mathfrak{g})$ acts by a character on this space.

9.11. Patched module. In this subsection let F be a finite extension of \mathbb{Q}_p . Let $n \geq 1$ be such that $p \nmid 2n$, let $\bar{\rho} : \text{Gal}_F \rightarrow \text{GL}_n(k)$ be a Galois representation and let $R_{\bar{\rho}}^{\square}$ be the framed deformation ring of $\bar{\rho}$.

Let $K = \text{GL}_n(\mathcal{O}_F)$ and let \mathfrak{g} be the \mathbb{Q}_p -linear Lie algebra of $G = \text{GL}_n(F)$. Let M_{∞} be the patched module constructed in [15] by patching automorphic forms on definite unitary groups. This is a compact $R_{\infty}[[K]]$ -module carrying an R_{∞} -linear action of G extending the action of K , where R_{∞} is a complete local noetherian $R_{\bar{\rho}}^{\square}$ -algebra with residue field k , which is flat over $R_{\bar{\rho}}^{\square}$.

We let $\Pi_{\infty} := \text{Hom}_{\mathcal{O}}^{\text{cont}}(M_{\infty}, L)$. If $y \in \text{m-Spec } R_{\infty}[1/p]$ then $\Pi_{\infty}[\mathfrak{m}_y]$ is an admissible unitary $\kappa(y)$ -Banach space representation of G . By letting x be the image of y in $\text{m-Spec } R_{\bar{\rho}}^{\square}[1/p]$, we obtain a Galois representation $\rho_x : \text{Gal}_F \rightarrow \text{GL}_n(\kappa(x))$. The expectation is that $\Pi_{\infty}[\mathfrak{m}_y]$ and ρ_x should be related by the hypothetical p -adic Langlands correspondence, see [15, §6].

Let $\rho : \text{Gal}_F \rightarrow \text{GL}_n(R_{\infty})$ be the Galois representation obtained from the universal framed deformation of $\bar{\rho}$ by extending scalars to R_{∞} . Let

$$\rho^C := \text{tw}_{\tilde{\delta}}^{-1} \circ (\rho \boxtimes \chi_{\text{cyc}}),$$

where $\tilde{\delta}$ is the twisting element $(0, -1, \dots, 2-n, 1-n)$ of GL_n , see Remark 5.14.

Theorem 9.27. *The algebra $Z(\mathfrak{g})$ acts on $\Pi_{\infty}[\mathfrak{m}_y]^{\text{la}}$ through the character $\zeta_{\rho_x^C}$.*

Proof. We will show that Theorem 8.5 applies with $M = M_{\infty}$, $R = R_{\infty}$ and $\zeta = \zeta_{\rho^C}$. By [15, Prop. 2.10], there is a morphism of local rings

$$S_{\infty} = \mathcal{O}[[y_1, \dots, y_h]] \rightarrow R_{\infty}$$

such that M_{∞} is a finitely generated projective $S_{\infty}[[K]]$ -module. Thus the sequence (y_1, \dots, y_h) is M_{∞} -regular and $M_{\infty}/(y_1, \dots, y_h)$ is a finitely generated projective $\mathcal{O}[[K]]$ -module. This gives hypothesis (i).

Let V be an irreducible algebraic representation of $\text{Res}_{F/\mathbb{Q}_p} \text{GL}_n$ over L . Let $R_{\infty}(V)$ be the quotient of R_{∞} acting faithfully on $\text{Hom}_K(V, \Pi_{\infty})$. Then $R_{\infty}(V)$ is reduced by [15, Lem. 4.17] and we have (ii).

If $y \in \Sigma_V$, the \mathcal{O} -algebra homomorphism $x : R_{\infty} \rightarrow \mathcal{O}_{\kappa(y)}$ factors through $R_{\infty}(V)$ and, by [15, Prop. 4.33], the representation ρ_x is crystalline with Hodge–Tate cocharacter $\lambda + \tilde{\delta}$ where λ is the highest weight of V and x is the image of y in $\text{m-Spec } R_{\bar{\rho}}^{\square}[1/p]$. By Proposition 5.5, see also Remark 5.14, the algebra $Z(\mathfrak{g})$ acts on V via the character $\zeta_{\rho_x^C}$. Thus part (iii) of Theorem 8.5 is satisfied.

The specialisation of ζ_{ρ^C} at $y \in \text{m-Spec } R_{\infty}[1/p]$ is equal to $\zeta_{\rho_x^C}$ by Lemma 4.26, where x is the image of y in $\text{m-Spec } R_{\bar{\rho}}^{\square}[1/p]$, thus we obtain the result. \square

Remark 9.28. It is not known in general whether the Banach space representation $\Pi_{\infty}[\mathfrak{m}_y]$ depends only on the Galois representation ρ_x . However, this is expected to be true, as the p -adic Langlands correspondence should not depend on the choices made in the patching process. The theorem above shows that the infinitesimal character of $\Pi_{\infty}[\mathfrak{m}_y]^{\text{la}}$ depends only on ρ_x , thus adding evidence that the expectation should be true.

9.12. The p -adic Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$. It is shown in [62] for $p \geq 5$ and in [23] in general that Colmez’s Montreal functor $\Pi \mapsto \check{\mathbf{V}}(\Pi)$ induces a bijection between the equivalence classes of absolutely irreducible admissible L -Banach space representations Π of $G = \text{GL}_2(\mathbb{Q}_p)$, which do not arise as subquotients

of parabolic inductions of unitary characters, and the equivalence classes of absolutely irreducible Galois representations $\rho : \text{Gal}_{\mathbb{Q}_p} \rightarrow \text{GL}_2(L)$. The correspondence is normalised so that local class field theory matches the central character of $\Pi(\rho)$ with $\chi_{\text{cyc}}^{-1} \det \rho$. Let $\rho^C : \text{Gal}_{\mathbb{Q}_p} \rightarrow {}^C G_f(L)$ be the Galois representation attached to ρ using the twisting element $\tilde{\delta} = (1, 0)$.

The following result proved by one of us (G.D.) was an important motivation for this paper and we will give a new proof of it.

Theorem 9.29 ([28], Thm.1.2). *Let Π be as above and let $\rho = \check{\mathbf{V}}(\Pi)$ then the action of $Z(\mathfrak{g})$ on Π^{la} is given by $\zeta_{\rho^C}^C$.*

Proof. We will use the recent results of Shen-Ning Tung [85], [86] proved in his thesis. We may assume that ρ is the specialisation at $x \in \text{m-Spec } R_{\bar{\rho}}^{\square}[1/p]$ of the universal framed deformation of some $\bar{\rho} : \text{Gal}_{\mathbb{Q}_p} \rightarrow \text{GL}_2(k)$. If $p > 2$ then let M_{∞} and R_{∞} be as in the previous section with $n = 2$ and $F = \mathbb{Q}_p$. Since R_{∞} is flat over $R_{\bar{\rho}}^{\square}$ there is an \mathcal{O} -algebra homomorphism $y : R_{\infty} \rightarrow \mathcal{O}$ extending x . It follows from [85, Thm.4.1] that there is an irreducible subquotient Π' of $\Pi_{\infty}[\mathfrak{m}_y]$, such that $\check{\mathbf{V}}(\Pi') = \rho$. Theorem 9.27 implies that the action of $Z(\mathfrak{g})$ on $\Pi_{\infty}[\mathfrak{m}_y]^{\text{la}}$, and hence also on $(\Pi')^{\text{la}}$, is given by $\zeta_{\rho^C}^C$. Using the bijectivity of the correspondence we obtain that $\Pi = \Pi'$. If $p = 2$ then in [86] Tung carries out the patching construction himself to obtain the analog of M_{∞} , see [86, Prop.6.1.2], which says that the patched module satisfies parts (o) and (i) of Theorem 8.6. The same argument as in the case $p > 2$ can be carried over using [86, Thm.6.3.7]. \square

Remark 9.30. The proof of bijectivity in [23] uses the results of [28] in an essential way. However, if one is willing to assume that $p \geq 5$ or if $p = 2$ or $p = 3$ then $\bar{\rho}^{\text{ss}} \neq \chi \oplus \chi\omega$, for any character $\chi : \text{Gal}_{\mathbb{Q}_p} \rightarrow k^{\times}$, where ω is the reduction of χ_{cyc} modulo p , then the bijectivity follows from [62, Thm.1.3], [65, Cor.1.4]. The papers [62] and [65] use only Colmez's functor $\check{\mathbf{V}}$, which goes from $\text{GL}_2(\mathbb{Q}_p)$ -representations to Galois representations, see [22, §IV], and not the construction $V \mapsto \Pi(V)$, which uses the p -adic Hodge theory in a deeper way and is used in [28].

9.13. A conjectural picture. In this subsection we formulate a conjecture, which describes the infinitesimal characters of the subspace of locally analytic vectors of Hecke eigenspaces in completed cohomology in the general setting, i.e. in the setting when one does not expect weakly non-Eisenstein ideals to exist.

Let \mathbb{T} be the Hecke algebra defined in subsection 9.3. Let $x : \mathbb{T}[1/p] \rightarrow \overline{\mathbb{Q}_p}$ be a continuous homomorphism of \mathcal{O} -algebras with kernel \mathfrak{m}_x , such that the image of x is a finite extension of \mathbb{Q}_p . Note that this condition is satisfied if \mathbb{T} is noetherian. If ℓ a prime number which is not in S , then we will denote by $x_{\ell} : \mathcal{H}_{\ell} \rightarrow \overline{\mathbb{Q}_p}$ the composition of x with the natural map $\mathcal{H}_{\ell} \rightarrow \mathbb{T}$.

There is a version of the Satake isomorphism using the C -group which is defined in [88] (see Proposition 5 and Remark 6 in *loc. cit.*). As ℓ is invertible in \mathcal{O} , it takes the form of an isomorphism of \mathcal{O} -algebras

$$\mathcal{H}_{\ell} \simeq \mathcal{O}[\widehat{G}^T|_{d=\ell} \rtimes \{\text{Frob}_{\ell}\}]^{\widehat{G}}$$

where Frob_{ℓ} is a geometric Frobenius at ℓ and $\widehat{G}^T|_{d=\ell}$ is the subscheme of \widehat{G}^T which is the inverse image of ℓ under $d : \widehat{G}^T \rightarrow \mathbb{G}_m$.

Using this isomorphism, we can associate to x_{ℓ} a semisimple $\widehat{G}(\overline{\mathbb{Q}_p})$ -conjugation class $\text{CC}(x_{\ell})$ in $\widehat{G}(\overline{\mathbb{Q}_p}) \rtimes (\{\ell\} \times \{\text{Frob}_{\ell}\})$.

Inspired by [12, Conj. 5.3.4] we have the (very) optimistic conjecture :

Conjecture 9.31. *There exists an admissible representation*

$$\rho : \text{Gal}_{\mathbb{Q}} \longrightarrow {}^C G_f(\overline{\mathbb{Q}}_p)$$

such that

- (i) $d \circ \rho$ is the cyclotomic character;
- (ii) ρ is unramified outside of S ;
- (iii) for $\ell \notin S$, the semisimplification of $\rho(\text{Frob}_{\ell})$ is in

$$\text{CC}(x_{\ell})\xi(\chi_{\text{cyc}}(\text{Frob}_{\ell})) \subset \widehat{G}(\overline{\mathbb{Q}}_p) \rtimes (\{\ell^{-1}\} \times \{\text{Frob}_{\ell}\})$$

where $\xi : \mathbb{G}_m \rightarrow \widehat{G}^T$ is the cocharacter is $t \mapsto ((2\delta)(t^{-1}), t^2)$, where 2δ is the sum of positive roots.

Remark 9.32. The cocharacter ξ is central in \widehat{G}^T , thus is independent of the choice of \widehat{B} and \widehat{T} and $\text{CC}(x_{\ell})\xi(\chi_{\text{cyc}}(\text{Frob}_{\ell}))$ is a $\widehat{G}^T(\overline{\mathbb{Q}}_p)$ -conjugacy class.

Remark 9.33. If $\iota \circ x : \mathbb{T} \rightarrow \mathbb{C}$, where $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ is a fixed isomorphism, is associated to a C -algebraic automorphic form then the existence of ρ satisfying the conditions of Conjecture 9.31 is conjectured in [12, Conj. 5.3.4].

Assume that ρ and ρ' are two admissible representations associated to x as in Conjecture 9.31, then $\rho(\text{Frob}_{\ell})^{\text{ss}}$ and $\rho'(\text{Frob}_{\ell})^{\text{ss}}$ are conjugate by an element of $\widehat{G}(\overline{\mathbb{Q}}_p)$. Let E be a finite Galois extension of \mathbb{Q} unramified outside S such that Gal_E acts trivially on the root datum of \widehat{G} . If $\gamma \in \text{Gal}_E$, then $\rho(\gamma) = (c_{\gamma}, 1)$ and $\rho'(\gamma) = (c'_{\gamma}, 1)$ with $c_{\gamma}, c'_{\gamma} \in \widehat{G}^T(\overline{\mathbb{Q}}_p)$ and thus

$$g\rho(\gamma)g^{-1} = (gc_{\gamma}g^{-1}, 1), \quad \forall g \in \widehat{G}(\overline{\mathbb{Q}}_p).$$

Thus if ℓ splits completely in E then

$$\text{tr}_V(r(c_{\text{Frob}_v})) = \text{tr}_V(r(c'_{\text{Frob}_v})),$$

for all places v of E above ℓ and all algebraic representations (r, V) of \widehat{G}^T . By Čebotarev density we have $\text{tr}_V(r(c_{\gamma})) = \text{tr}_V(r(c'_{\gamma}))$ for all $\gamma \in \text{Gal}_E$ and thus $\zeta_{\rho}^C = \zeta_{\rho'}^C$, by Lemma 5.6. This proves that if Conjecture 9.31 is true, there is a well defined character $Z(\mathfrak{g}) \rightarrow \overline{\mathbb{Q}}_p$ associated to x .

Conjecture 9.34. *Let ρ be an admissible representation associated to $x : \mathbb{T}[1/p] \rightarrow \overline{\mathbb{Q}}_p$ as in Conjecture 9.31, and let n be a non-negative integer. Then $Z(\mathfrak{g})$ acts on $(\widetilde{H}^n \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]^{\text{la}} \otimes_{\mathbb{T}, x} \overline{\mathbb{Q}}_p$ via ζ_{ρ}^C .*

Remark 9.35. Note that $(\widetilde{H}^n \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]$ can be zero, in which case the statement of the Conjecture 9.34 holds trivially.

Remark 9.36. In the general case, even after localising \widetilde{H}_n at a maximal ideal of the Hecke algebra, we do not expect to get a projective $\mathcal{O}[[K_p]]$ -module. So Theorems 8.5 and 8.6 cannot be applied directly. However, one might hope to be able to apply our results to the patched homology groups obtained via the patching method of Calegari–Geraghty [14]. The most accessible case, when weakly non-Eisenstein maximal ideal are not expected to exist, is when $G = \text{PGL}_2$ over a quadratic imaginary field F , such that p splits completely in F , studied by Gee–Newton in [43]. It follows from [43, Prop. 5.3.1] and its proof that under the assumptions made

there the patched homology, denoted by $H_{q_0}(\tilde{\mathcal{C}}(\infty))$ in [43], satisfies the conditions of Theorem 8.5. We do not pursue this further, just remark that in that setting instead of applying Theorem 8.5 to $H_{q_0}(\tilde{\mathcal{C}}(\infty))$ it might be easier to use local–global compatibility at p and appeal to the results on the infinitesimal character in the p -adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, see Theorem 9.29.

REFERENCES

- [1] K. ARDAKOV AND S. WADSLEY, On irreducible representations of compact p -adic analytic groups, *Annals of Mathematics*, 178 (2013) 453–557.
- [2] R. BELLOVIN, p -adic Hodge theory in rigid analytic families, *Algebra Number Theory*, Volume 9, Number 2 (2015), 371–433.
- [3] L. BERGER AND P. COLMEZ, Familles de représentations de de Rham et monodromie p -adique, *Représentations p -adiques de groupes p -adiques. I. Représentations galoisiennes et (φ, Γ) -modules*. Astérisque No. 319 (2008), 303–337.
- [4] A. BOREL, Automorphic L -functions, *Automorphic Forms, Representations and L -functions*, (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61.
- [5] A. BOREL, *Linear Algebraic Groups* (second edition), Springer 1991.
- [6] A. BOREL, Automorphic forms on reductive groups. *Automorphic forms and applications*, 7–39, IAS/Park City Math. Ser., 12, Amer. Math. Soc., Providence, RI, 2007.
- [7] A. BOREL AND N. WALLACH, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000.
- [8] S. BOSCH, U. GÜNTZER AND R. REMMERT, *Non-Archimedean analysis*, Grundlehren der Mathematischen Wissenschaften, 261. Springer-Verlag, Berlin, 1984. xii+436 pp.
- [9] N. BOURBAKI, *Commutative Algebra*, Springer, 1989.
- [10] L. BREEN AND W. MESSING, Differential geometry of gerbes, *Adv. Math.* 198 (2005), no. 2, 732–846.
- [11] C. BREUIL, E. HELLMANN AND B. SCHRAEN, Une interprétation modulaire de la variété triangulaire, *Math. Annalen* 367, 2017, 1587–1645.
- [12] K. BUZZARD AND T. GEE, The conjectural connections between automorphic representations and Galois representations. *Automorphic forms and Galois representations*. Vol. 1, 135–187, London Math. Soc. Lecture Note Ser., 414, Cambridge Univ. Press, Cambridge, 2014.
- [13] F. CALEGARI AND M. EMERTON, Completed cohomology - a survey, *Non-abelian fundamental groups and Iwasawa theory*, CUP 2011, pp 239–257.
- [14] F. CALEGARI AND D. GERAGHTY, Modularity lifting beyond the Taylor-Wiles method, *Inventiones Mathematicae*, 211 (2018) No. 1, 297–433.
- [15] A. CARAIANI, M. EMERTON, T. GEE, D. GERAGHTY, V. PAŠKŪNAS AND S.W. SHIN, Patching and the p -adic Langlands program, *Cambridge J. Math.* 4 (2016), No. 2, 197–287.
- [16] A. CARAIANI AND P. SCHOLZE, On the generic part of the cohomology of compact Shimura varieties, *Annals of Mathematics*, 186 (2017), 649–766.
- [17] H. CARAYOL, Sur les représentations ℓ -adiques associées aux formes modulaires de Hilbert, *Ann. Sci. École Norm. Sup.* (4) 19 (1986), no. 3, 409–468.
- [18] G. CHENEVIER, Familles p -adiques de formes automorphes pour $\mathrm{GL}(n)$, *Journal für die reine und angewandte Mathematik* 570, 143–217 (2004).
- [19] G. CHENEVIER, The p -adic analytic space of pseudocharacters of a profinite group, and pseudorepresentations over arbitrary rings, London Math. Soc. Lecture Notes Series 414, Proceedings of the LMS Durham Symposium 2011, *Automorphic forms and Galois representations* vol. 1, 221–285 (2014).
- [20] G. CHENEVIER AND M. HARRIS, Construction of automorphic Galois representations. II, *Camb. J. Math.* 1 (2013), No. 1, 53–73.
- [21] L. CLOZEL, M. HARRIS AND R. TAYLOR, Automorphy for some l -adic lifts of automorphic mod l Galois representations, *Pub. Math. I.H.É.S.* 108 (2008), 1–181.
- [22] P. COLMEZ, Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ, Γ) -modules, *Astérisque* 330 (2010) 281–509.
- [23] P. COLMEZ, G. DOSPINESCU AND V. PAŠKŪNAS, The p -adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, *Camb. J. Math.* 2 (2014), no. 1, 1–47.

- [24] B. CONRAD, Reductive group schemes. *Autour des schémas en groupes*. Vol. I, 93–444, Panor. Synthèses, 42/43, Soc. Math. France, Paris, 2014.
- [25] P. DELIGNE AND J. S. MILNE, Tannakian Categories, in *Hodge Cycles, Motives, and Shimura Varieties*, LNM 900, Springer, 1982, 101–228.
- [26] J.-P. DEMAILLY, *Complex Analytic and Differential Geometry*, available at <https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf>.
- [27] J. DIXMIER, *Enveloping algebras*. Revised reprint of the 1977 translation. Graduate Studies in Mathematics, 11. American Mathematical Society, Providence, RI, 1996. xx+379 pp.
- [28] G. DOSPINESCU, Actions infinitésimales dans la correspondance de Langlands locale p -adique *Mathematische Annalen* 354 (2012), 627–657.
- [29] G. DOSPINESCU, V. PAŠKŪNAS AND B. SCHRAEN, Infinitesimal characters and Banach representations of p -adic reductive groups, in preparation.
- [30] G. DOSPINESCU, B. SCHRAEN Endomorphism algebras of admissible p -adic representations of p -adic Lie groups, *Representation Theory* 17 (2013), 237–246.
- [31] M. EMERTON, Locally analytic vectors in representations of locally p -adic analytic groups, to appear in *Memoirs of the AMS*.
- [32] M. EMERTON, On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. *Invent. Math.* 164 (2006), no. 1, 1–84.
- [33] M. EMERTON, Local-global compatibility in the p -adic Langlands programme for GL_2/\mathbb{Q} , preprint (2011).
- [34] M. EMERTON, Completed cohomology and the p -adic Langlands program. *Proceedings of the International Congress of Mathematicians–Seoul 2014*. Vol. II, 319–342, Kyung Moon Sa, Seoul, 2014.
- [35] M. EMERTON, T. GEE AND F. HERZIG, Weight cycling and Serre-type conjectures for unitary groups, *Duke Math. J.* **162**, no. 9, 1649–1722 (2013).
- [36] M. EMERTON AND V. PAŠKŪNAS, On the density of supercuspidal points of fixed regular weight in local deformation rings and global Hecke algebras, *Journal de l'École polytechnique – Mathématiques*, Tome 7 (2020), pp. 337–371.
- [37] G. FALTINGS, Hodge–Tate structures and modular forms, *Math. Ann.*, 278, 133–149 (1987).
- [38] J. FINTZEN AND S.W. SHIN, Congruences of algebraic automorphic forms and supercuspidal representations, Preprint 2020, arXiv:2009.08476.
- [39] J. FRANKE, Harmonic analysis in weighted L_2 -spaces. *Ann. scient. Ec. Norm. Sup.*, 4e série 31, 181–279 (1998).
- [40] J. FRANKE AND J. SCHWERMER, A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups, *Math. Ann.* 311, 765–790 (1998).
- [41] J. FRESNEL AND M. VAN DER PUT, *Rigid Analytic Geometry and Its Applications*, Progress in Mathematics 218, Birkhäuser, 2004.
- [42] T. GEE, F. HERZIG, T. LIU AND D. SAVITT, Potentially crystalline lifts of certain prescribed types, *Documenta Mathematica* 22 (2017), 397–422.
- [43] T. GEE AND J. NEWTON, Patching and the completed homology of locally symmetric spaces, *Journal de l'Institut de Mathématiques de Jussieu*, to appear.
- [44] D. HANSEN AND C. JOHANSSON, Perfectoid Shimura varieties and the Calegari–Emerton conjectures, Preprint 2020.
- [45] HARISH-CHANDRA, Lie algebras and the Tannaka duality theorem, *Ann. of Math.* (2) 51 (1950), 299–330.
- [46] E. HELMANN AND B. SCHRAEN, Density of potentially crystalline representations of fixed weight, *Compos. Math.*, Vol. 152 (2016), pp. 1609–1647.
- [47] R. HILL, On Emerton's p -adic Banach Spaces, *International Mathematics Research Notices*, Vol. 2010, No. 18, pp. 3588–3632.
- [48] R. HOTTA, K. TAKEUCHI AND T. TANISAKI, *D-Modules, Perverse Sheaves and Representation Theory*, Progress in Mathematics 236, Birkhäuser Boston, 2008.
- [49] S. HOWE, The p -adic Jacquet–Langlands correspondence and a question of Serre, <https://arxiv.org/abs/1806.06807>.
- [50] J. E. HUMPHREYS, *Introduction to Lie algebras and representation theory*. Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York–Berlin, 1972. xii+169 pp.
- [51] J. E. HUMPHREYS, *Reflection groups and Coxeter groups*. Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990. xii+204 pp.

- [52] J. C. JANTZEN, *Representations of algebraic groups*, Mathematical Surveys and Monographs, Volume: 107; 2003; 576 pp.
- [53] A. J. DE JONG, Crystalline Dieudonné module theory via formal and rigid geometry, *Publications Mathématiques de l’IHÉS*, Volume 82 (1995), p. 5–96.
- [54] A. W. KNAPP, *Representation theory of semisimple groups. An overview based on examples*. Reprint of the 1986 original. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001. xx+773 pp..
- [55] E. KNIGHT, A p -adic Jacquet–Langlands correspondence, <http://www.math.toronto.edu/~eknight/Thesis.pdf>.
- [56] R. P. LANGLANDS, *On the classification of representations of real algebraic groups*, in *Representation Theory and harmonic analysis on semisimple Lie Groups*, Mathematical Surveys and Monographs **31**, pp. 101–170.
- [57] M. LAZARD, Groupes analytiques p -adiques, *Inst. Hautes Études Sci. Publ. Math.* No. 26 1965 389–603.
- [58] H. MATSUMURA, Commutative ring theory, Second edition, *Cambridge Studies in Advanced Mathematics*, 8. CUP, Cambridge, 1989.
- [59] J. S. MILNE, Semisimple Lie Algebras, Algebraic Groups and Tensor Categories, Notes dated May 9, 2007 available at www.jmilne.org/math/articles/2007a.pdf.
- [60] J. NEWTON, Completed cohomology of Shimura curves and a p -adic Jacquet–Langlands correspondence, *Math. Ann.* (2013) 355:729–763.
- [61] L. PAN, On locally analytic vectors of the completed cohomology of modular curves, <https://arxiv.org/abs/2008.07099>.
- [62] V. PAŠKŪNAS, The image of Colmez’s Montreal functor, *Publ. Math. Inst. Hautes Études Sci.* 118 (2013), 1–191.
- [63] V. PAŠKŪNAS, On the Breuil–Mézard Conjecture, *Duke Math. J.* **164**, no. 2, 297–359 (2015).
- [64] V. PAŠKŪNAS, Blocks for mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, *Automorphic forms and Galois representations*, Vol. 2, 231–247, London Math. Soc. Lecture Note Ser., 415, Cambridge Univ. Press, Cambridge, 2014.
- [65] V. PAŠKŪNAS, On 2-dimensional 2-adic Galois representations of local and global fields, *Algebra Number Theory* 10 (2016), no. 6, 1301–1358.
- [66] V. PAŠKŪNAS, A note on Galois representations associated to automorphic forms, Appendix C to [38].
- [67] V. PAŠKŪNAS, On some consequences of a theorem of J. Ludwig, <https://arxiv.org/abs/1804.07567>.
- [68] S. PATRIKIS, Variation on a theorem of Tate, *Memoirs of AMS* to appear.
- [69] V. PLATONOV AND A. RAPINCHUK, *Algebraic Groups and Number Theory*, Academic Press Inc., 1994.
- [70] J.-J. SANSUC, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, *Journal für die reine und angewandte Mathematik* 327 (1981): 12–80.
- [71] P. SCHNEIDER, *Nonarchimedean Functional Analysis*, Springer 2002.
- [72] P. SCHNEIDER, *p -adic Lie groups*, Grundlehren der mathematischen Wissenschaften 344, Springer 2011.
- [73] P. SCHNEIDER AND J. TEITELBAUM, Algebras of p -adic distributions and admissible representations, *Invent. math.* 153, 145–196 (2003)
- [74] P. SCHNEIDER AND J. TEITELBAUM, Locally analytic distributions and p -adic representation theory, with applications to GL_2 ,
- [75] P. SCHOLZE, On the p -adic cohomology of the Lubin-Tate tower, *Ann. Sci. Éc. Norm. Supér.* (4) 51 (2018), no. 4, 811–863.
- [76] S. SEN, Continuous Cohomology and p -Adic Galois Representations, *Inventiones mathematicae* (1980/81) Volume 62, page 89–116.
- [77] J.-P. SERRE, Groupes de Grothendieck des schémas en groupes réductifs déployés, *Inst. Hautes Études Sci. Publ. Math.* No. 34 1968 37–52.
- [78] J.-P. SERRE, *Lie algebras and Lie groups*, Springer 1992.
- [79] J.-P. SERRE, Groupes algébriques associés aux modules de Hodge–Tate, *Astérisque*, no. 65 (1979), p. 155–188.
- [80] J.-P. SERRE, *Abelian l -adic representations and elliptic curves*, New York, W. A. Benjamin, 1968.

- [81] S. W. SHIN, On the cohomological base change for unitary similitude groups, Appendix to W. GOLDRING, Galois representations associated to holomorphic limits of discrete series, *Compositio Math.* 150 (2014), 191–228.
- [82] C. SKINNER, A note on the p -adic Galois representations attached to Hilbert modular forms, *Doc. Math.* 14 (2009), 241–258.
- [83] THE STACKS PROJECT AUTHORS, *The Stacks project*, <https://stacks.math.columbia.edu>, 2020.
- [84] J. TITS, Représentations linéaires irréductibles d’un groupe réductif sur un corps quelconque, *Journal für die reine und angewandte Mathematik (Crelles Journal)*, Volume 1971, Issue 247, Pages 196–220.
- [85] S.-N. TUNG, On the automorphy of 2-dimensional potentially semi-stable deformation rings of $G_{\mathbb{Q}_p}$, arXiv:1803.07451.
- [86] S.-N. TUNG, On the modularity of 2-adic potentially semi-stable deformation rings, *Math. Z.* to appear, <https://doi.org/10.1007/s00209-020-02588-4>.
- [87] D. A. VOGAN JR, The local Langlands correspondende, *Contemp. Math.*, **145** (1993), pp. 305–379.
- [88] XINWEN ZHU, A note on integral Satake isomorphisms, Preprint 2020, arXiv:2005.13056.

ECOLE NORMALE SUPÉRIEURE DE LYON, 46 ALLÉE D’ITALIE, 69 364 LYON CEDEX 07, FRANCE
Email address: gabriel.dospinescu@ens-lyon.fr

UNIVERSITÄT DUISBURG-ESSEN, FAKULTÄT FÜR MATHEMATIK, THEA-LEYMANN-STR. 9, 45127
ESSEN, GERMANY
Email address: paskunas@uni-due.de

UNIVERSITÉ PARIS-SACLAY, CNRS, LABORATOIRE DE MATHÉMATIQUES D’ORSAY, 91405, OR-
SAY, FRANCE
Email address: benjamin.schraen@universite-paris-saclay.fr