Patching and multiplicities of p-adic eigenforms

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Abstract

We prove the existence of non-classical p-adic automorphic eigenforms associated to a classical system of eigenvalues on definite unitary groups in 3 variables. These eigenforms are associated to Galois representations which are crystalline but very critical at p. We use patching techniques related to the trianguline variety of local Galois representations and its local model. The new input is a comparison of the coherent sheaves appearing in the patching process with coherent sheaves on the Grothendieck–Springer version of the Steinberg variety given by a functor constructed by Bezrukavnikov.

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1 Introduction

The aim of this paper is to unravel (and explain) a new phenomenon in the theory of p-adic automorphic forms. Given a reductive group $\underline{G}$ over a number field (overconvergent) p-adic automorphic forms are p-adic avatars of automorphic forms on $\underline{G}$. We usually refer to the latter as classical automorphic forms in order to distinguish them from their p-adic limits. Additional structures on spaces of automorphic forms, such as the Hecke-action, naturally extend to the L-vector spaces of overconvergent p-adic automorphic forms $S^\dagger(K^p), S^\dagger_\kappa(K^p)$, where the field of coefficients L is a finite extension of $\mathbb{Q}_p$ and $K^p \subset G(\mathbb{A}^p)$ is a compact open subgroup (referred to as the *tame level*) and κ is a *weight*. A central question about p-adic automorphic forms is to clarify whether a given overconvergent p-adic automorphic form (of algebraic weight) that is an eigenform for the Hecke action is a classical automorphic form. Often this question can be answered in terms of the Hecke eigenvalues. Coleman's *small slope implies classical* result [Col97] and generalizations thereof (see e.g. [Kas06], [Che11], [BPS16]) asserts that this question can be purely decided using the Hecke action at p if the p-adic valuation of the Hecke eigenvalues at p is small compared to the weight. Beyond the *numerically non critical slope* it is known that this fails. However, one can ask the same question taking into account the full Hecke action (as opposed to the Hecke action at p).

Assume that we are in a situation where we can construct the Galois representation $\rho_f = \rho_\chi$ attached to a p-adic eigenform f, respectively to the Hecke character χ giving the system of Hecke eigenvalues of f. Then the Hecke action away from p encodes all the information about the p-adic Galois representation ρ_f, including the p-adic Hodge theoretic information at places dividing p (though this is encoded in a rather indirect and mysterious way). The naive generalization of the classicality question about overconvergent p-adic automorphic forms can hence be phrased as follows (though we phrase the question in a rather informal way):

Question A: Let f be an overconvergent p-adic eigenform of dominant algebraic weight such that the corresponding Galois representation ρ_f is de Rham at places dividing p. Is it true that f is a classical automorphic form?

We note that a softer version of this question is the following expectation that is implied by the Fontaine–Mazur conjecture. Again we state the expectation in a rather informal way – it might fail without more precise assumptions on the group the level, etc. (see e.g. [BHS19, Conj. 5.1.1] for a precise formulation).

Rough Expectation B: Let $S^\dagger_\kappa(K^p)[\chi] \subset S^\dagger_\kappa(K^p)$ be an eigensystem (for the action of the full Hecke algebra $\mathbb{T}$ generated by Hecke operators at p and away from p) in the space $S^\dagger_\kappa(K^p)$ of overconvergent p-adic automorphic forms of weight κ on $\underline{G}$. We assume that the eigensystem χ at p is of *finite slope*. Assume that κ is dominant algebraic and that the Galois representation ρ_χ associated to the Hecke character $\chi : \mathbb{T} \to L$ is de Rham at places dividing p. Then $S^\dagger_\kappa(K^p)[\chi]$ contains a classical automorphic form, i.e. its subspace $S^{\mathrm{cl}}_\kappa(K^p)[\chi]$ of classical forms is non-zero.

Question A then can be rephrased as the question whether $S^{\mathrm{cl}}_\kappa(K^p)[\chi] = S^\dagger_\kappa(K^p)[\chi]$ in Expectation B. It is known that Question A does not have an affirmative answer in general. Ludwig [Lud18] and Johansson–Ludwig [JL23] have shown that there are counterexamples for SL_2. The reason for these counterexamples however, is of global (endoscopic) nature and it remains a reasonable question to ask Question A for groups where these phenomena do not apply, e.g. for definite unitary groups.

Expectation B has been verified for GL_2 (this is basically [Kis03]), and generalizations of Kisins' result were proven by Bellaïche and his coauthors ([BC06],[Bel12] and [BD16]). For definite unitary groups, and under Taylor–Wiles assumptions, these results were vastly generalized in [BHS17a], [BHS19]. We point out that in the cases treated in [BHS17a] the results imply that $S^{\mathrm{cl}}_\kappa(K^p)[\chi] = S^\dagger_\kappa(K^p)[\chi]$, while the more general case in [BHS19] only allows to construct some classical form in the eigensystem (though no counterexample to Question A is constructed in loc. cit.). The reason for this difference is due to a phenomenon in the geometry of eigenvarieties (i.e. rigid analytic spaces parametrizing the systems of Hecke eigenvalues in the space of overconvergent p-adic automorphic forms of finite slope), respectively in the geometry of their local Galois-theoretic counterparts (the so-called trianguline variety of [BHS17b]). In the case treated in [BHS17a] the trianguline variety is smooth at the Galois representations in question (and hence the eigenvariety is local complete intersection). In general the trianguline variety is not smooth, and as a consequence one can construct non-smooth points on the corresponding eigenvarieties, see [BHS19, Thm. 5.4.2]. It is this failure of smoothness that prevents [BHS19] from identifying $S^{\mathrm{cl}}_\kappa(K^p)[\chi]$ and $S^\dagger_\kappa(K^p)[\chi]$.

In this paper we prove that the answer to Question A is *no* for definite unitary groups in three variables (see Theorem 1.2 below for a more precise formulation).

Theorem 1.1. *There exists a unitary group in three variables U, a tame level K^p, a dominant algebraic weight κ and a Hecke character $\chi : \mathbb{T} \to L$ that occurs in the space $S^\dagger_\kappa(K^p)_{\mathrm{fs}}$ of overconvergent automorphic forms of finite slope and weight κ such that the eigenspace $S^\dagger_\kappa(K^p)[\chi]$ contains classical as well as non-classical eigenforms.*

The construction of this example also clarifies the role of the singularities of the trianguline variety X_{tri}. The precise results we prove suggest that the answer to Question A is no, whenever the dualizing sheaf $\omega_{X_{\mathrm{tri}}}$ is not locally free at the point defined by ρ (and the refinement associated to χ), i.e. whenever X_{tri} is non-Gorenstein at this point (we refer to Theorem 1.3 below for the precise link with $\omega_{X_{\mathrm{tri}}}$). In the three dimensional case, this results in a precise comparison of the dimensions of the eigenspaces $S^{\mathrm{cl}}_\kappa(K^p)[\chi] \subset S^\dagger_\kappa(K^p)[\chi]$.

We point out that, in contrast to [Lud18] and [JL23] this is a purely local *p-adic* phenomenon. Moreover, the theorem implies that the usual invariants (i.e. the Hecke action, respectively the p-adic Hodge theoretic information of the associated Galois representation) can not distinguished between classical and non-classical forms. We like to refer to the non-classical forms in such eigensystems as *undercover* automorphic forms.

The main result, and in particular the occurrence of the dualizing sheaf $\omega_{X_{\mathrm{tri}}}$ therein, is inspired by the categorical point of view in the p-adic Langlands program, see [EGH23]. The space of overconvergent p-adic automorphic forms of finite slope $S^\dagger(K^p)_{\mathrm{fs}}$ can be viewed as the topological dual of the global sections of a coherent sheaf (that we simply refer to as the *sheaf of p-adic automorphic forms*) on the rigid analytic generic fiber of the universal deformation space of Galois representations (more precisely, on the product of this space with the space of continuous characters of a maximal torus $\underline{T}(\mathbb{Q}_p) \subset \underline{G}(\mathbb{Q}_p)$ at p). The support of this sheaf is, by definition, the corresponding eigenvariety. The local-global-compatibility conjectures [EGH23, Conj. 9.6.8 and Conj. 9.6.16] give a precise description of this sheaf in terms of the geometry of moduli stacks of (φ, Γ)-modules (that are closely related to the trianguline variety). More precisely, the categorical approach to the p-adic Langlands program asks for a functor from certain (locally analytic) representations of $\underline{G}(\mathbb{Q}_p)$ to sheaves on stacks of (φ, Γ)-modules, and the sheaf of p-adic automorphic forms is the globalization of the evaluation of this functor on a specific representation. One of the punchlines of [EGH23] (see section 1.6 therein for a more detailed discussion) is that avatars of the envisioned functor have been around in number theory during the past decades in the context of the Taylor–Wiles patching method, in particular *patching functors* as used for example in [EGS15] (or also in [BHS19, 5.]) A crucial point in the proof of the main theorem is the identification of such a patching functor with an explicit local functor, see Theorem 1.4 below. This partially confirms expectations in the categorical picture, see [EGH23, Expectation 6.2.27].

Note that the multiplicity result in Theorem 1.2 has some striking consequence for the p-adic Langlands Program for $\mathrm{GL}_3(\mathbb{Q}_p)$. It implies that the locally analytic representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ on the Hecke eigenspace of overconvergent p-adic modular forms over $\underline{G}$ corresponding to a Galois representation ρ as in Theorem 1.2 contains locally algebraic vectors which are *not* in the socle of the representation (see Remark 7.31). After finishing this works, the authors learned that Ding also proved examples of this penomena for generic Galois representations (see [Din]).

We now describe our results in more detail. Let F be a totally real number field and let E/F be a CM (imaginary) quadratic extension in which every place $v|p$ in F splits in E. Let U be a unitary group (over $\mathbb{Q}$) in n variables for the quadratic extension E/F which is compact at infinity. By the hypothesis on p the group $U_{\mathbb{Q}_p}$ is a product of general linear groups over finite extensions of $\mathbb{Q}_p$ and we denote $\underline{T}$ a maximal torus of $U_{\mathbb{Q}_p}$. We also fix a finite extension $L/\mathbb{Q}_p$ which is big enough to split E. Let $\mathcal{O}_L \subset L$ be its ring of integers, π_L a uniformizer and k_L its residue field.

For any continuous character $\delta : \underline{T}(\mathbb{Q}_p) \longrightarrow L^\times$, we can define a weight κ (which is given by the derivative of δ at 1) and a character of the Atkin–Lehner ring $\mathcal{A}(p)$ (the ring of Hecke-operators at p, see Definition 5.4) that we still denote by δ. We will assume that $\delta_{|T^0}$ is algebraic where $T^0 \subset \underline{T}(\mathbb{Q}_p)$ is the maximal compact subgroup. Let $K^p \subset U(\mathbb{A}^p)$ be a tame level and let S be a finite set, containing places above p, away from which K^p is hyperspecial. We write $\mathbb{T}^S$ for the unramified Hecke algebra at places not in S and $\mathbb{T} = \mathbb{T}^S \otimes_{\mathbb{Z}} \mathcal{A}(p)$. Associated to these data we consider the spaces $S^\dagger_\kappa(K^p)$ and $S^{\mathrm{cl}}_\kappa(K^p)$, see Definition 5.7 for the precise definition, which come equipped with an action of $\mathbb{T}^S$

and $\mathcal{A}(p)$.

Given a character $\chi^S : \mathbb{T}^S \to L$ let $\chi = \chi^S \otimes \delta$ and consider the eigenspaces $S_\kappa^\dagger(K^p)[\chi]$ and $S_\kappa^{\mathrm{cl}}(K^p)[\chi]$. We note that the classical subspace $S_\kappa^{\mathrm{cl}}(K^p)[\chi]$ is zero unless κ is dominant algebraic. To an eigenvector $f \in S_\kappa^\dagger(K^p)[\chi]$ we can associate a Galois representation $\rho = \rho_f = \rho_\chi : \mathrm{Gal}_E \coloneqq \mathrm{Gal}(\overline{E}/E) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$. For the precise form of the main result we introduce the following (strong) Taylor–Wiles hypothesis. Let $\overline{\rho} : \mathrm{Gal}_E \longrightarrow \mathrm{GL}_n(k_L)$ be the semisimplification of the reduction modulo the maximal ideal of $\mathcal{O}_L$ of ρ. We assume that (see Hypothesis 5.10 in the text)

$$\begin{cases} \bullet \quad p > 2, \\ \bullet \quad E/F \text{ is unramified and } \zeta_p \notin E, \\ \bullet \quad U \text{ is quasi-split at all finite places of } F, \\ \bullet \quad \text{if a place } v \text{ of } F \text{ is inert in } E, \text{ then } K_v \text{ is hyperspecial}, \\ \bullet \quad \overline{\rho} \text{ is absolutely irreducible and } \overline{\rho}(\mathrm{Gal}_{E(\zeta_p)}) \text{ is adequate.} \end{cases} \tag{1}$$

For simplicity of the exposition we assume now that p is totally split in F (in the core of the paper we work in the general case). If the representation ρ is crystalline at $v|p$, ρ_v can be described by its associated *filtered isocrystal* which is a finite dimensional L-vector space $D_{\mathrm{cris}}(\rho_v)$ endowed with a linear automorphism $\varphi \in \mathrm{GL}(D_{\mathrm{cris}}(\rho_v))$ and a complete flag $D^\bullet$, called the Hodge–Tate filtration (in our case, this is a complete flag as we will assume that ρ_v has regular Hodge–Tate weights). We say that ρ_v is *φ-generic* if the ratio of two of its eigenvalues is not in $\{1, p\}$. In this case the character δ determines an order of the eigenvalues of φ (that is called a *refinement* of ρ_v) which in turn (using the fact that the φ-eigenvalues are pairwise distinct) defines another complete flag $\mathcal{F}_\bullet$ on $D_{\mathrm{cris}}(\rho_v)$ which is φ-stable. We denote $w_{\rho,\delta,v} \in \mathfrak{S}_n$ the relative position of the flags $\mathcal{F}_\bullet$ and $D^\bullet$ in the flag variety of $D_{\mathrm{cris}}(\rho_v)$. When $w_{\rho,\delta,v} = w_0$ is the longest element of $\mathfrak{S}_n$, i.e. when the two flags $D^\bullet$ and $\mathcal{F}_\bullet$ are in generic position, we say that f is *non-critical* at v. The "most critical case" is the case where $w_{\rho,\delta,v} = 1$, i.e. when the two flags coincide. In this case we say that f is *very critical* at v.

Theorem 1.2. *Assume $n = 3$. Let $\delta : \underline{T}(\mathbb{Q}_p) \to L^\times$ be a continuous character of weight κ dominant algebraic. Let $\chi^S : \mathbb{T}^S \to L$ be a character and let $\chi = \chi^S \otimes \delta$. We assume that the eigenspace $S_\kappa^\dagger(K^p)[\chi]$ is non-zero and that for any $v|p$ the local Galois representation $\rho_v = \rho_\chi|_{\mathrm{Gal}_{E_v}} : \mathrm{Gal}_{E_v} \longrightarrow \mathrm{GL}_3(\overline{\mathbb{Q}_p})$ is crystalline with distinct Hodge-Tate weights and is φ-generic. Assume moreover that the Taylor–Wiles hypothesis (1) is satisfied. Let r be the number of places $v|p$ in F such that $w_{\rho_\chi,\delta,v} = 1$. Then*

$$\dim S_\kappa^\dagger(K^p)[\chi] = 2^r \dim S_\kappa^{\mathrm{cl}}(K^p)[\chi].$$

We refer to Corollary 7.28 for a more general statement where p is not necessarily totally split in F.

Theorem 1.2 would be vacuous without proving the existence of characters χ and δ (and a group U and a tame level K^p) such that the corresponding eigenspace $S_\kappa^{\mathrm{cl}}(K^p)[\chi]$

is non-zero and consists of very critical forms. As there exist only countably many classical automorphic forms, but uncountably many flags it doesn't seem very easy to construct an f with $w_{\rho_f,\delta} = 1$. This is Corollary 8.13, the main result of section 8, which uses global automorphic methods that are rather disjoint from the methods of the other parts of the paper. The Galois representation corresponding to the constructed Hecke character is induced from a degree 3 extension of E.

We finally discuss the relation of these results with patching functors and the categorical approach to a p-adic Langlands correspondence. Assume that $\delta = \delta_\lambda \delta_{\mathcal{R}}^{\mathrm{sm}}$ is the product of a dominant algebraic character δ_λ and a smooth unramified character $\delta_{\mathcal{R}}^{\mathrm{sm}}$ (which is in fact implied by the assumption that ρ_v is crystalline). As the notation suggests, the character $\delta_{\mathcal{R}}^{\mathrm{sm}}$ corresponds to the choice of a refinement $\mathcal{R}$ of $\rho_p := (\rho_v)_{v|p}$. Let $\mathcal{X}_{\rho_p} = \mathrm{Spec}(R_{\rho_p})$ be the scheme associated to the universal deformation ring of ρ_p. Using results of [BHS19], we can construct a subscheme

$$\mathcal{X}^{\mathrm{qtri}}_{\rho_p,\mathcal{R}} = \mathrm{Spec}(R^{\mathrm{qtri}}_{\rho_p,\mathcal{R}}) \subset \mathcal{X}_{\rho_p}$$

of "quasi-trianguline" deformations of ρ_p associated to the refinement $\mathcal{R}$. By loc. cit. this scheme has a local model modeled on the Steinberg variety (or rather its "Grothendieck–Springer" variant) and its irreducible components $\mathcal{X}^{\mathrm{qtri},w}_{\rho,\mathcal{R}}$ are labeled by the Weyl group W of $\prod_{v|p} \mathrm{GL}_3$. It is known that these irreducible components are normal and Cohen–Macaulay.

Let's denote $\lambda = \delta|_{T^0} (= \delta_{\lambda|T^0})$, this is a dominant algebraic character. Using hypothesis (1) the Taylor–Wiles method, as extended to the setting of completed cohomology in [CEG+16], can be used ([BHS19, 5.]) to construct coherent sheaves $\mathcal{M}_\infty(L(\lambda))$ and $\mathcal{M}_\infty(M(w\cdot\lambda))$ for $w \in W$ over $\mathcal{X}^{\mathrm{qtri}}_{\infty,\rho,\mathcal{R}} = \mathrm{Spec}(R^{\mathrm{qtri}}_{\rho_p,\mathcal{R}}[[x_1,\dots,x_g]])$ for some $g \geqslant 0$, that "patch" the duals of the spaces of classical, respectively p-adic, automorphic forms. More precisely

$$\begin{aligned}\mathcal{M}_\infty(L(\lambda)) \otimes k(\rho_p) &= \mathrm{Hom}_L(S^{\mathrm{cl}}_\lambda(K^p)[\chi], L),\\ \mathcal{M}_\infty(M(w\cdot\lambda)) \otimes k(\rho_p) &= \mathrm{Hom}_L(S^{\dagger}_{w\cdot\lambda}(K^p)[\chi], L).\end{aligned}$$

These coherent sheaves are in a certain precise sense associated to the $U(\mathfrak{g})$-modules $L(\lambda)$ (the algebraic representation of highest weight λ) respectively the Verma modules $M(w\cdot\lambda)$, where $\mathfrak{g}$ is the Lie algebra of $U_L \cong \prod_{v|p} \mathrm{GL}_3$. The results of [BHS19] show that the coherent sheaves $\mathcal{M}_\infty(M(w\cdot\lambda))$ have generic rank (when nonzero) equal to $\dim_L S^{\mathrm{cl}}_\lambda(K^p)[\chi]$. Denote $\mathcal{X}^{\mathrm{qtri},w}_{\infty,\rho,\mathcal{R}} := \mathcal{X}^{\mathrm{qtri}}_{\infty,\rho,\mathcal{R}} \times_{\mathcal{X}^{\mathrm{qtri}}_{\rho,\mathcal{R}}} \mathcal{X}^{\mathrm{qtri},w}_{\rho,\mathcal{R}}$. The key to the proof of Theorem 1.2 is the following result:

Theorem 1.3. *Under the assumptions of Theorem 1.2, let $m = \dim_L S^{\mathrm{cl}}_\lambda[\chi]$. For any $w \in W$, there is an isomorphism*

$$\mathcal{M}_\infty(M(w\cdot\lambda)) \cong \omega^{\oplus m}_{\overline{\mathcal{X}}^{\mathrm{qtri},ww_0}_{\infty,\rho,\mathcal{R}}}.$$

Here $\omega_{\overline{\mathcal{X}}^{\mathrm{qtri},ww_0}_{\infty,\rho\mathcal{R}}}$ is the dualizing sheaf of a complete intersection $\overline{\mathcal{X}}^{\mathrm{qtri},ww_0}_{\infty,\rho,\mathcal{R}} \subset \mathcal{X}^{\mathrm{qtri},ww_0}_{\infty,\rho,\mathcal{R}}$.

In order to prove Theorem 1.3, we extend $\mathcal{M}_\infty$ to a functor on the whole category $\mathcal{O}_\lambda$, the block of the BGG category $\mathcal{O}$ containing $L(\lambda)$. This is the *patching functor* alluded to above. More precisely, assuming that ρ_p is crystalline with regular Hodge-Tate weights, and δ is φ-generic, we construct an exact functor

$$\mathcal{M}_\infty : \mathcal{O}_\lambda \longrightarrow \mathrm{Coh}(\mathcal{X}^{\mathrm{qtri}}_{\infty,\rho,\mathcal{R}}),$$

such that, for every $M \in \mathcal{O}_\lambda$ the sheaf $\mathcal{M}_\infty(M)$ is Cohen–Macaulay of the expected dimension.

In spirit of the categorical approach to the p-adic Langlands correspondence the functor $\mathcal{M}_\infty$ should be a "local" functor, that is (up to multiplicities coming from contributions at the places away from p) the functor $\mathcal{M}_\infty$ should be the pullback, denoted $\mathcal{B}_\infty$, of a functor

$$\mathcal{B}_p : \mathcal{O}_\lambda \longrightarrow \mathrm{Coh}(\mathcal{X}^{\mathrm{qtri}}_{\rho_p,\mathcal{R}}).$$

This functor $\mathcal{B}_p$ can be written down explicitly using the local model for $\mathcal{X}^{\mathrm{qtri}}_{\rho_p,\mathcal{R}}$ and a functor constructed by Bezrukavnikov [Bez16], see 7.2 for details. Our main local result compares $\mathcal{M}_\infty$ and $\mathcal{B}_p$ (see Corollary 7.17 for the general version):

Theorem 1.4. *Under the assumptions of Theorem 1.2, let $m = \dim_L S^{\mathrm{cl}}_\lambda[\chi]$. Then there is an isomorphism of functors $\mathcal{M}_\infty \simeq \mathcal{B}_\infty^{\oplus m}$. As a consequence, we have*

1) for all $w \in W$, $\mathcal{M}_\infty(M(w\cdot\lambda)^\vee) \simeq \mathcal{O}^{\oplus m}_{\mathcal{X}^{\mathrm{qtri},ww_0}_{\infty,\rho,\mathcal{R}}}$;

2) for all $w \in W$, $\mathcal{M}_\infty(M(w\cdot\lambda)) \simeq \omega^{\oplus m}_{\mathcal{X}^{\mathrm{qtri},ww_0}_{\infty,\rho,\mathcal{R}}}$;

3) for all $M \in \mathcal{O}$, we have $\mathcal{M}_\infty(M^\vee) \simeq \mathcal{M}_\infty(M)^\vee$ where $(\cdot)^\vee$ denote both the dual in $\mathcal{O}_\lambda$ and the Serre dual in the category of coherent sheaves.

Remark 1.5. We can only prove Theorem 1.4 in the three dimensional case. However, we expect an isomorphism $\mathcal{M}_\infty \cong \mathcal{B}_\infty^{\oplus m}$ for higher dimensional definite unitary groups as well.

In fact $\mathcal{B}_p$ should factor through the category of locally analytic representations, and is expected to extend to a functor with values in coherent sheaves on the stack of all (φ,Γ)-modules (compare [EGH23, Conjecture 6.2.4 and Expectation 6.2.27]). Theorem 1.4 should be viewed as some partial evidence for these expectations. In fact, in view of the conjectures in [EGH23] we can formulate an expectation how Theorem 1.2 (and its technical key input Theorem 1.3) should generalize beyond the case of GL_3, respectively a unitary group in three variables.

Conjecture 1.6. *Assume the situation of Theorem 1.3 but drop the assumption that $n = 3$. Then, for any $w \in W$, there is an isomorphism*

$$\mathcal{M}_\infty(M(w\cdot\lambda)) \cong \omega^{\oplus m}_{\mathcal{X}^{\mathrm{qtri},ww_0}_{\infty,\rho,\mathcal{R}}},$$

where $m = \dim_L S_\lambda^{\mathrm{cl}}[\chi]$*. In particular, in the situation of Theorem 1.2 (but dropping the assumption* $n = 3$*) the dimension of the eigenspace* χ*-eigenspace in the space of overconvergent automorphic forms* $S_\kappa^\dagger(K^p)$ *of weight* κ *and level* K^p *can be computed as*

$$\dim S_\kappa^\dagger(K^p)[\chi] = \big(\dim \omega_{\mathcal{X}_{\rho,\mathcal{R}}^{\mathrm{qtri},w_0}} \otimes k(\rho_p, \mathcal{R})\big)\big(\dim S_\kappa^{\mathrm{cl}}(K^p)[\chi]\big).$$

Let us return to the three dimensional case and indicate how to prove our main results. The key to proving Theorem 1.4 is to extend the functor $\mathcal{M}_\infty$ to a larger category $\mathcal{O}_{\mathrm{alg}}^\infty$ and to a deformation $\widetilde{\mathcal{O}}_{\mathrm{alg}}$ as introduced in [Soe92], which we think of as a *deformed version* of $\mathcal{O}_{\mathrm{alg}}$. We would like to emphasize that we first prove 1) and we deduce the isomorphism $\mathcal{M}_\infty \simeq \mathcal{B}_\infty^{\oplus m}$ from this in a second time . The proof of 1) is based on a dévissage whose roots can be found in the paper [EGS15]. We first prove the result in the case where $\mathcal{X}_{\infty,\mathcal{R}}^{\mathrm{qtri},w}$ is smooth and then proceed inductively. Note that the *existence* of Bezrukavnikov's functor $\mathcal{B}_\infty$ plays a key role in this induction. The second main input into this induction is the computation of $\mathcal{M}_\infty(M_I(w\cdot\lambda))$ where $M_I(w\cdot\lambda)$ is a generalized Verma module (corresponding to some parabolic P_I). These sheaves, that are related to sheaves of p-adic automorphic forms on the partial eigenvarieties constructed by Wu [Wu], are supported on "partially de Rham quasi-trianguline" deformation spaces $\mathcal{X}_{\rho_p,\mathcal{R}}^{I-\mathrm{qtri}}$ which have been studied by Breuil and Ding in [BD].

We finally note that the component $\mathcal{X}_{\infty,\rho,\mathcal{R}}^{\mathrm{qtri},w_0}$ is not Gorenstein and its dualizing sheaf has a 2^r-dimensional fiber at ρ_p, which is the reason for the factor 2^r in Theorem 1.2.

Note that, while many constructions in the body of the paper work for arbitrary dimension n, there are severe difficulties in generalizing our proof to a full proof of Conjecture 1.6: one of the main inputs in the proof of Theorem 1.4 is that we know that $\mathcal{X}_{\infty,\mathcal{R}}^{\mathrm{qtri},w}$ is smooth unless $w = (w_v)_{v|p} \in W = \prod_{v|p} S_3$ has the property that $w_v \in S_3$ is the longest element for at least one $v|p$.

We now describe the content of the article. In section 2 we introduce the category $\mathcal{O}_{\mathrm{alg}}$ and its deformed versions. Section 3 studies Emerton's Jacquet functor and gives the abstract framework to construct patching functors. In section 4, we recall the quasi-trianguline deformation spaces of [BHS19], their local models, and their parabolic version ([BD, Wu]). Section 5 recalls the definitions of the global objects like completed cohomology, overconvergent automorphic forms and their patched versions. Section 6 is devoted to the further study of the functor $\mathcal{M}_\infty$ and its factorization through $\mathcal{X}_{\infty,\rho,\mathcal{R}}^{\mathrm{qtri}}$, the (global) quasi-trianguline deformation space. In section 7, we study the supports of the sheaves $\mathcal{M}_\infty(M)$ for specific objects of $\mathcal{O}_{\mathrm{alg}}$ (and their deformed version), and we recall results on Bezrukavnikov's functor before deducing Theorem 1.4 (in the three dimensional case). Finally, in section 8 we explain how to explicitly construct very critical forms satisfying the assumptions in Theorem 1.2 for $n = 3$.

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Notations

We fix p a prime number. When K is a field, we fix K^{sep} a separable closure of K we write $\mathrm{Gal}_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. We fix L a finite extension of $\mathbb{Q}_p$ which will be chosen sufficiently large in the text.

If $\mathfrak{h}$ is a Lie algebra we note $\mathfrak{h}^{\mathrm{ss}}$ its derived Lie algebra.

2 Variants of the BGG-category $\mathcal{O}$

In this section, we fix L to be a field of characteristic 0. Let $\underline{G}$ be a split reductive group over L. Let $\underline{B}$ be a Borel subgroup, $\underline{T}$ a maximal split torus of $\underline{G}$ contained in $\underline{B}$ and $\underline{N}$ the radical of $\underline{B}$. We use the notation $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{n}$... for the Lie algebras of $\underline{G}$, $\underline{B}$, $\underline{T}$, $\underline{N}$... We denote by $X^*(\underline{T})$ the finite free abelian group $\mathrm{Hom}(\underline{T}, \mathbb{G}_{m,L})$ of characters of $\underline{T}$. This abelian group can be identified with a $\mathbb{Z}$-lattice in $\mathfrak{t}^* := \mathrm{Hom}_L(\mathfrak{t}, L)$. For $\lambda \in X^*(\underline{T})$, we also write λ for the character of $\mathfrak{t}$ induced by λ. Let Φ be the set of roots of the pair $(\underline{G}, \underline{T})$, $\Phi^+ \subset \Phi$ the subset of positive roots with respect to $\underline{B}$ and $\Delta \subset \Phi^+$ the subset of simple roots. If $\alpha \in \Phi$, we denote $\mathfrak{g}_\alpha$ the α-eigenspace in $\mathfrak{g}$. We write $\delta_G \in X^*(\underline{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for the half sum of positive roots. Let W be the Weyl group of $(\underline{G}, \underline{T})$. For $w \in W$, we write $\lambda \mapsto w \cdot \lambda$ for the *dot action* of W on $X^*(\underline{T})$ (with respect to $\underline{B}$, that is $w \cdot \lambda := w(\lambda + \delta_G) - \delta_G$). We equip W with the Bruhat order corresponding to the choice of $\underline{B}$ and we denote $w_0 \in W$ the longest element for this order.

If $I \subset \Delta$ is a subset of simple roots, we denote by $\Phi_I \subset \Phi$ the subset of roots which are sums of elements of I, $\underline{P}_I \supset \underline{B}$ the standard parabolic subgroup of $\underline{G}$ such that $\mathfrak{p}_I = \mathfrak{b} + \sum_{\alpha \in \Phi_I} \mathfrak{g}_\alpha$ and $\underline{N}_I$ its unipotent radical. Let $\underline{L}_I$ be the standard Levi subgroup of $\underline{P}_I$ and $\underline{Z}_I$ be the center of $\underline{L}_I$. We say that a character $\lambda \in X^*(\underline{T})$ is *dominant with respect to* $\underline{P}_I$ if $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for $\alpha \in I$ and we denote $X^*(\underline{T})_I^+$ the set of such characters.

When $I = \Delta$, we have $\underline{P}_\Delta = \underline{G}$ and we write $X^*(\underline{T})^+ = X^*(\underline{T})^+_\Delta$. We define an order relation on $X^*(\underline{T})$, by saying that $\lambda \geqslant \mu$ if and only if $\lambda - \mu \in \sum_{\alpha \in \Phi^+} \mathbb{N}\alpha$.

We write W_I for the Weyl group of $(\underline{L}_I, \underline{T})$; it is the subgroup of W generated by the simple reflections s_α for $\alpha \in I$. Given $w \in W$, we denote $w^{\min}$ (resp. $w^{\max}$) the unique minimal (resp. maximal) element for the Bruhat order having the same class as w in $W_I \backslash W$. This definition depends on I (and on the fact that the quotient is on the left) but we hope our notation will cause no confusion. We have $(ww_0)^{min} = w^{max}w_0$ and $(ww_0)^{max} = w^{min}w_0$ for any $w \in W$. Finally, we write ${}^IW$ for the set of minimal length representatives of $W_I \backslash W$ in W, i.e. ${}^IW = \{w^{\min} \mid w \in W\}$.

2.1 Recollections

For $I \subset \Delta$, we consider the full subcategory $\mathcal{O}^{I,\infty}$ of the category $U(\mathfrak{g})$-mod of $U(\mathfrak{g})$-modules whose objects are all finitely generated $U(\mathfrak{g})$-modules M such that

- for any $m \in M$, the L-vector space $U(\mathfrak{p}_I)m$ is finite dimensional;
- for any $h \in \mathfrak{t}$ and any h-stable finite dimensional L-vector subspace $V \subset M$, the characteristic polynomial of $h_{|V}$ is split in $L[X]$.

This is the category $\mathcal{O}^{\mathfrak{p}_I,\infty}$ in [AS22, §3.1].

For M in $\mathcal{O}^{I,\infty}$ and $\mu \in \mathrm{Hom}_L(\mathfrak{t}, L)$, we write $M^\mu \subset M$ for the L-subspace of those $v \in M$ such that, for any $h \in \mathfrak{t}$, $(h - \mu(h))^n \cdot v = 0$ for some $n \geqslant 1$. We have

$$M = \bigoplus_{\mu \in \mathrm{Hom}_L(\mathfrak{t}, L)} M^\mu.$$

We write $\mathcal{O}^{I,\infty}_{\mathrm{alg}}$ for the full subcategory of $\mathcal{O}^{I,\infty}$ whose objects M satisfy $M^\mu = 0$ for $\mu \notin X^*(\underline{T})$.

Moreover, we write $\mathcal{O}^I_{\mathrm{alg}} \subset \mathcal{O}^{I,\infty}_{\mathrm{alg}}$ for the full subcategory whose objects are direct sums of finitely generated semisimple $U(\mathfrak{l}_I)$-modules (when seen as $U(\mathfrak{l}_I)$-modules). This coincides with the usual parabolic (algebraic) category $\mathcal{O}$, which is denoted $\mathcal{O}^{\mathfrak{p}_I}_{\mathrm{alg}}$ in [OS15]). When $I = \emptyset$ we simply use the notations $\mathcal{O}^\infty_{\mathrm{alg}}$ and $\mathcal{O}_{\mathrm{alg}}$ for $\mathcal{O}^{\emptyset,\infty}_{\mathrm{alg}}$ and $\mathcal{O}^\emptyset_{\mathrm{alg}}$. Note that $\mathcal{O}^{I,\infty}_{\mathrm{alg}} \subset \mathcal{O}^\infty_{\mathrm{alg}}$ for any $I \subset \Delta$. As these categories depend on the choices of $\mathfrak{g}$ and $\mathfrak{b}$ we write $\mathcal{O}^{\mathfrak{g},\mathfrak{b}}$ (with additional decorations) instead of $\mathcal{O}$, when the context is unclear.

These categories are stable by subobject and quotients in the category of $U(\mathfrak{g})$-modules. Moreover the category $\mathcal{O}^{I,\infty}_{\mathrm{alg}}$ is stable under extensions.

For any character $\lambda \in X^*(\underline{T})^+_I$, we write $L_I(\lambda)$ for the simple $U(\mathfrak{l}_I)$-module of highest weight λ. This is a finite dimensional L-vector space and we define the *generalized Verma module* of highest weight λ as

$$M_I(\lambda) \coloneqq U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} L_I(\lambda).$$

The generalized Verma module is an object of $\mathcal{O}^I_{\mathrm{alg}}$ and has a unique simple quotient which is isomorphic to $L(\lambda) = L_\Delta(\lambda)$. When $I = \emptyset$, we simply write $M(\lambda) = M_\emptyset(\lambda)$ and say that $M(\lambda)$ is a *Verma module*. We also denote by $P(\lambda)$ the projective cover of the simple module $L(\lambda)$. If λ is dominant with respect to $\underline{B}$, we call $P(w_0 \cdot \lambda)$ the *antidominant projective* (with respect to λ).

2.2 Nilpotent action of $U(\mathfrak{t})$

Given $I \subset \Delta$ we denote by $\mathfrak{m}_I$ the augmentation ideal of $U(\mathfrak{z}_I)$ and set

$$A_I \coloneqq U(\mathfrak{z}_I)_{\mathfrak{m}_I}$$
$$A \coloneqq A_\emptyset \coloneqq U(\mathfrak{t})_{\mathfrak{m}}.$$

The canonical Lie algebra decomposition $\mathfrak{l}_I = \mathfrak{z}_I \oplus \mathfrak{l}_I^{\mathrm{ss}}$ defines a canonical morphism of Lie algebras $p_I : \mathfrak{l}_I \twoheadrightarrow \mathfrak{z}_I$ which extends to a morphism $U(\mathfrak{l}_I) \twoheadrightarrow U(\mathfrak{z}_I)$ of L-algebras also denoted by p_I. This morphism induces a surjective morphism $A \twoheadrightarrow A_I$ of A_I-algebras.

We show that the category $\mathcal{O}^{I,\infty}$ naturally embeds into the category $U(\mathfrak{g})_{A_I}$-mod, where $U(\mathfrak{g})_{A_I} \coloneqq U(\mathfrak{g}) \otimes_L A_I$.

Let M be an object of the category $\mathcal{O}^{I,\infty}$. Let $h \in \mathfrak{t}$. For $v \in M$ the element h defines an L-linear endomorphism of the finite dimensional L-vector space $U(\mathfrak{t})v$ and we write $h = D_{h,v} + N_{h,v}$ for its Jordan decomposition with semisimple part $D_{h,v}$ and nilpotent part $N_{h,v}$. As M is locally $U(\mathfrak{t})$-finite, uniqueness of the Jordan decomposition implies that these endomorphisms "glue" into an endomorphism D_h and a locally nilpotent endomorphism N_h of M such that $D_{h,v}$ (resp. $N_{h,v}$) is the restriction of D_h (resp. N_h) to $U(\mathfrak{t})v$ for any $v \in M$.

Lemma 2.1. *The endomorphism N_h is $U(\mathfrak{g})$-equivariant.*

Proof. By construction N_h and D_h commute with the action of $\mathfrak{t}$ and stabilize each M^μ. Let $\alpha \in \Phi$ and $x \in \mathfrak{g}_\alpha$. For $v \in M^\mu$, we have $x \cdot v \in M^{\mu+\alpha}$ and $[h, x] = \alpha(h)x$ so that

$$D_h x \cdot v + N_h x \cdot v = x D_h \cdot v + x N_h \cdot v + \alpha(h) x v.$$

By definition of M^μ, we have $D_h \cdot v = \mu(h)v$ for any $v \in M^\mu$. This implies $D_h x \cdot v = (\mu(h) + \alpha(h))x \cdot v$ and $x D_h \cdot v = \mu(h) x \cdot v$. Therefore $N_h x \cdot v = x N_h \cdot v$. We conclude that N_h commutes with the endomorphism of M induced by x. Therefore N_h is $U(\mathfrak{g})$-equivariant. □

Given $M \in \mathcal{O}^{I,\infty}$ Lemma 2.1 implies that we can define an $U(\mathfrak{t})$-module structure on M by letting $h \in \mathfrak{t} \subset U(\mathfrak{t})$ act via N_h. As the action of each h on M is locally nilpotent, this action extends to an A-module structure.

Lemma 2.2. *Let M be an object of $\mathcal{O}^{I,\infty}$, then the A-action on M factors through A_I. Moreover, this A_I-module structure makes $\mathcal{O}^{I,\infty}$ into a full subcategory of $U(\mathfrak{g})_{A_I}$-mod.*

Proof. In order to prove that the A-action factors through $A \to A_I$ it is enough to prove that for $h \in \mathfrak{t} \cap \mathfrak{l}_I^{\rm ss}$ the endomorphism N_h is zero. This is a direct consequence of the fact that $\mathfrak{l}_I^{\rm ss}$ is a semi-simple Lie algebra and that the L-vector space $U(\mathfrak{l}_I^{\rm ss})v$ is finite dimensional for any $v \in M$ (by definition of $\mathcal{O}^{I,\infty}$). As the $U(\mathfrak{g})$-action commutes with the A-action by Lemma 2.1 the module M is an $U(\mathfrak{g})_{A_I}$-module. Finally we note that, given $h \in \mathfrak{t}$, the construction of N_h is functorial in M. $\square$

Remark 2.3. Let $M \in \mathcal{O}^{I,\infty}$ and $\mu \in \mathrm{Hom}_L(\mathfrak{t}, L)$ then the above construction implies that

$$M^\mu = \{v \in M \mid hv = (\mu(h)v) + p_I(h))v \ \forall h \in \mathfrak{t}\}.$$

Let $M \in \mathcal{O}_{\rm alg}^\infty$. Lemma 2.1 also implies that we can define another structure of $U(\mathfrak{g})$-module on M where an element $h \in \mathfrak{t}$ acts through the semisimple part D_h and the action of an element $x \in \mathfrak{g}_\alpha$ for $\alpha \in \Phi$ is not modified. We denote this $U(\mathfrak{g})$-module structure by $M^{\rm ss}$. Then $M^{\rm ss}$ is an object of $\mathcal{O}_{\rm alg}$ and [OS15, Lemm. 3.2] implies that there is a unique structure of algebraic $\underline{B}$-module on M lifting the structure of $U(\mathfrak{b})$-module on $M^{\rm ss}$. The compatibility of this $\underline{B}$-action with the original $U(\mathfrak{g})$-module structure on M is made explicit in the following Lemma.

Lemma 2.4. *Let M be an object of $\mathcal{O}_{\rm alg}^\infty$ endowed with the $\underline{B}$-module structure defined above. Then*

$$b \cdot (X \cdot (b^{-1} \cdot v)) = (\mathrm{Ad}(b)X) \cdot v$$

for any $b \in \underline{B}(L)$, $X \in \mathfrak{g}$ and $v \in M$.

Proof. It is sufficient to prove the formula for $b \in \underline{N}(L)$ and for $b \in \underline{T}(L)$. If $b \in \underline{N}(L)$, then $b = \exp(n)$ for some $n \in \mathfrak{n}$. It follows that $\mathrm{Ad}(b)X$ is equal to the finite sum $\sum_{k \geqslant 0} \frac{1}{k!} \mathrm{ad}(n)^k X$ and that the action of b on M is given by the series $\sum_{k\geqslant 0} \frac{1}{k!} n^k$ (which is locally finite). Therefore we have,

$$\begin{aligned} b \cdot (X \cdot (b^{-1} \cdot v)) &= \sum_{k \geqslant 0, \ell \geqslant 0} (-1)^\ell \frac{1}{k!\ell!} n^k X n^\ell \cdot v \\ &= \sum_{m \geqslant 0} \frac{1}{m!} \sum_{k+\ell=m} (-1)^{k-m} \binom{m}{k} n^k X n^\ell \cdot v \\ &= \sum_{m \geqslant 0} \frac{1}{m!} (\mathrm{ad}(n)^m X) \cdot v = \mathrm{Ad}(b) X \cdot v. \end{aligned}$$

If $b \in \underline{T}(L)$, then if $\alpha \in \Phi \cup \{0\}$ and $X \in \mathfrak{g}_\alpha$, and if $v \in M^\mu$, we have

$$\begin{aligned} b \cdot (X \cdot (b^{-1} \cdot v)) = b \cdot (X \cdot (\mu(b^{-1})v)) &= (\mu + \alpha)(b) \mu(b^{-1}) X \cdot v \\ &= \alpha(b) X \cdot v = \mathrm{Ad}(b) X \cdot v \end{aligned}$$

as $\mathrm{Ad}(b)X = \alpha(b)X$. $\square$

For later use, we note that we can resolve objects in $\mathcal{O}_{\rm alg}^\infty$ as follows:

Lemma 2.5. *Let M be an object of $\mathcal{O}^{\infty}_{\mathrm{alg}}$. Then there exist finite dimensional $U(\mathfrak{b})$-modules V_0 and V_1 such that for any $h \in \mathfrak{t}$, the characteristic polynomials of $h_{|V_0}$ and $h_{|V_1}$ are split in $L[X]$, and an exact sequence of $U(\mathfrak{g})$-modules*

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_1 \longrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_0 \longrightarrow M \longrightarrow 0. \tag{2}$$

Moreover $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_i$ is in $\mathcal{O}^{\infty}$ for $i \in \{0,1\}$ and this exact sequence is $\underline{B}$-equivariant for the $\underline{B}$-actions (on each of the three terms) defined just before Lemma 2.4.

Proof. The existence of a finite dimensional $U(\mathfrak{b})$-module V_0 and a surjective map $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_0 \twoheadrightarrow M$ is a consequence of the fact that M is a finitely generated $U(\mathfrak{g})$-module and locally finite as a $U(\mathfrak{b})$-module (it also comes from Proposition 2.14 below). The existence of V_1 and of the map $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_1 \to U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_0$ follows the fact that $U(\mathfrak{g})$ is noetherian and $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_0$ is locally $U(\mathfrak{b})$-finite. The $\underline{B}$-equivariance is a direct consequence of the definition of the algebraic action of $\underline{B}$-action on each term of the sequence (2). $\square$

2.3 Deformations of the category $\mathcal{O}$

Fix $I \subset \Delta$ and let M be some $U(\mathfrak{g})_{A_I}$-module. For $\mu \in X^*(\underline{T})$, we define the A_I-submodule

$$M^{\mu} := \{v \in M \mid \forall h \in \mathfrak{t},\, h \cdot v = (p_I(h) + \mu(h))v\}.$$

We note that for $M \in \mathcal{O}^{I,\infty}$ this coincides with the generalized eigenspace for μ by Remark 2.3. Inspired by the construction of [Soe92, §3.1], we define $\widetilde{\mathcal{O}}^{I}_{\mathrm{alg}}$ as the category of $U(\mathfrak{g})_{A_I}$-modules M such that

- M is finitely generated over $U(\mathfrak{g})_{A_I}$;
- $M = \bigoplus_{\mu \in X^*(\underline{T})} M^{\mu}$ and each M^{μ} is a finite free A_I-module ;
- for any $m \in M$ the A_I-submodule $(U(\mathfrak{p}_I) \otimes_L A_I)m$ is finitely generated.

Lemma 2.6. *Let M be an object of $\widetilde{\mathcal{O}}^{I}_{\mathrm{alg}}$. Then for any $n \geqslant 0$, the $U(\mathfrak{g})$-module $M/\mathfrak{m}_I^n M$ is an object of $\mathcal{O}^{I,\infty}_{\mathrm{alg}}$ and $M/\mathfrak{m}_I M$ is in $\mathcal{O}^{I}_{\mathrm{alg}}$.*

Proof. This is a direct consequence of the definitions. $\square$

For $\lambda \in X^*(\underline{T})^+_I$ we define the *deformed generalized Verma module* of weight λ as

$$\widetilde{M}_I(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} (L_I(\lambda) \otimes_L A_I)$$

where $U(\mathfrak{p}_I)$ acts on A_I via the composition $U(\mathfrak{p}_I) \to U(\mathfrak{l}_I) \xrightarrow{p_I} U(\mathfrak{z}_I) \to A_I$. The module $\widetilde{M}_I(\lambda)$ is an object of $\widetilde{\mathcal{O}}^{I}_{\mathrm{alg}}$ and we have an isomorphism of $U(\mathfrak{g})_{A_I}$-modules

$$\widetilde{M}_I(\lambda) \otimes_{A_I} (A_I/\mathfrak{m}_I) = M_I(\lambda).$$

2.3.1 Duality

Recall that there exists an internal duality functor $M \mapsto M^\vee$ on the category $\mathcal{O}_{\mathrm{alg}}$ (see [Hum08, §3.2]). We will define an analogue on $\widetilde{\mathcal{O}}_{\mathrm{alg}}$. Let M be an object of the category $\widetilde{\mathcal{O}}^I_{\mathrm{alg}}$. We define an action of $U(\mathfrak{g})$ on $M^* := \mathrm{Hom}_{A_I}(M, A_I)$ by $x \cdot f(m) = f(\tau(x)m)$ where τ is the anti-involution of $U(\mathfrak{g})$ defined in [Hum08, §0.5]. We then define $M^\vee$ to be the sub-$U(\mathfrak{g})$-module of M^* given by

$$M^\vee := \bigoplus_{\mu \in X^*(\underline{T})} (M^*)^\mu.$$

Lemma 2.7. *If M is an object of the category $\widetilde{\mathcal{O}}^I_{\mathrm{alg}}$, then so is $M^\vee$ and there is a canonical isomorphism of $U(\mathfrak{g})$-modules $M^\vee/\mathfrak{m}_I M^\vee \simeq (M/\mathfrak{m}_I M)^\vee$. Consequently is in the category $\mathcal{O}^I_{\mathrm{alg}}$. Moreover there is a canonical isomorphism $M \xrightarrow{\sim} (M^\vee)^\vee$ of $U(\mathfrak{g})_{A_I}$-modules.*

Proof. We have a canonical isomorphism of A_I-modules

$$M^* \simeq \prod_{\mu \in X^*(\underline{T})} \mathrm{Hom}_{A_I}(M^\mu, A_I)$$

and we easily check that $(M^*)^\mu = \mathrm{Hom}_{A_I}(M^\mu, A_I)$ for $\mu \in X^*(\underline{T})$. As any M^μ is a finite free A_I-module, so is $(M^*)^\mu = (M^\vee)^\mu$. By definition, we have $M^\vee = \bigoplus_{\mu \in X^*(\underline{T})} M^{\vee,\mu}$ and $M^\vee/\mathfrak{m}_I M^\vee = \bigoplus_{\mu \in X^*(\underline{T})} M^{\vee,\mu}/\mathfrak{m}_I M^{\vee,\mu}$. Therefore we have $M^{\vee,\mu}/\mathfrak{m}_I M^{\vee,\mu} \subset (M^\vee/\mathfrak{m}_I M^\vee)^\mu$. As the eigenspaces $(M^\vee/\mathfrak{m}_I M^\vee)^\mu$ are in direct sum, we must have, for any $\mu \in X^*(\underline{T})$,

$$M^{\vee,\mu}/\mathfrak{m}_I M^{\vee,\mu} = (M^\vee/\mathfrak{m}_I M^\vee)^\mu.$$

The reduction modulo $\mathfrak{m}_I$ induces a sequence of morphisms of $U(\mathfrak{g})$-modules

$$\begin{aligned} M^\vee/\mathfrak{m}_I M^\vee \to M^*/\mathfrak{m}_I M^* = \mathrm{Hom}_{A_I}(M, A_I)/\mathfrak{m}_I \to \mathrm{Hom}_{A_I}(M, A_I/\mathfrak{m}_I) \\ = \mathrm{Hom}(M/\mathfrak{m}_I M, L). \end{aligned}$$

We claim that the composite of these maps induces an isomorphism from $M^\vee/\mathfrak{m}_I M^\vee$ onto $(M/\mathfrak{m}_I M)^\vee$. In order to see this it is sufficient to check that, for any $\mu \in X^*(\underline{T})$, it maps $M^{\vee,\mu}/\mathfrak{m}_I M^{\vee,\mu}$ onto $\mathrm{Hom}_L(M^\mu/\mathfrak{m}_I M^\mu, L) \simeq (M/\mathfrak{m}_I M)^{\vee,\mu}$. This comes from the fact that, as the A_I-module M^μ is finite free, all maps in the following sequence are isomorphisms

$$M^{\vee,\mu}/\mathfrak{m}_I M^{\vee,\mu} \xrightarrow{\sim} \mathrm{Hom}_{A_I}(M^\mu, A_I/\mathfrak{m}_I) \xrightarrow{\sim} \mathrm{Hom}_L(M^\mu/\mathfrak{m}_I M^\mu, L).$$

Let $\mathfrak{n}_I^- := \bigoplus_{\alpha \in -\Phi^+ \backslash \Phi_I} \mathfrak{g}_\alpha$ denote the nilpotent radical of the parabolic Lie subalgebra opposite to $\mathfrak{p}_I$. Note that a $U(\mathfrak{g})_{A_I}$-module M such that $M = \bigoplus_\mu M^\mu$ with M^μ finite free over A_I is in $\widetilde{\mathcal{O}}^I_{\mathrm{alg}}$ if and only if we can write $M = U(\mathfrak{n}_I^-) \cdot \left(\bigoplus_{\mu \in S} M^\mu\right)$ for some finite set $S \subset X^*(\underline{T})$. By Lemma 2.6, the object $M/\mathfrak{m}_I M$ lies in the category $\mathcal{O}^I_{\mathrm{alg}}$ and it follows from [Hum08, §9.3] that $(M/\mathfrak{m}_I M)^\vee$ lies in $\mathcal{O}^I_{\mathrm{alg}}$. This implies that there exists a finite set $S \subset X^*(\underline{T})$ such that

$$(M/\mathfrak{m}_I M)^\vee = U(\mathfrak{n}_I^-) \cdot \Big(\bigoplus_{\mu \in S} (M/\mathfrak{m}_I M)^{\vee,\mu}\Big).$$

It follows that for any μ such that $(M/\mathfrak{m}_I M)^{\vee,\mu} \neq 0$, the map

$$\bigoplus_{\substack{\nu \in \sum_{\alpha \in -\Phi \setminus \Phi_I} \mathbb{N}\alpha \\ \mu' \in S \\ \mu' + \nu = \mu}} (M/\mathfrak{m}_I M)^{\vee,\mu'} \longrightarrow (M/\mathfrak{m}_I M)^{\vee,\mu}$$

given by the action of the corresponding element of $U(\mathfrak{n}_I^-)$ on each summand, is surjective. As M^μ is a finite free A_I-module and A_I is a local ring, we deduce from Nakayama's Lemma that the map

$$\bigoplus_{\substack{\nu \in \sum_{\alpha \in -\Phi \setminus \Phi_I} \mathbb{N}\alpha \\ \mu' \in S \\ \mu' + \nu = \mu}} M^{\vee,\mu'} \longrightarrow M^{\vee,\mu}$$

is surjective and thus that $M^\vee = U(\mathfrak{n}_I^-) \cdot \left(\bigoplus_{\mu' \in S} M^{\vee,\mu'}\right)$. This implies that $M^\vee$ is a finitely generated $U(\mathfrak{g})_{A_I}$-module and we also deduce from this equality that $M^\vee$ is locally $U(\mathfrak{p}_I)_{A_I}$-finite.

In order to prove that $M \xrightarrow{\sim} (M^\vee)^\vee$ we note that the natural map $M \longrightarrow (M^\vee)^*$ of $U(\mathfrak{g})_{A_I}$-modules factors through $(M^\vee)^\vee$ and respects the weight decomposition. Moreover as M^μ is free over A_I for all μ, the induced bi-duality $M^\mu \xrightarrow{\sim} (M^{\mu,*})^*$ morphism is an isomorphism. □

2.3.2 Blocks

Let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$ and let $\chi : Z(\mathfrak{g}) \to L$ be a character of $Z(\mathfrak{g})$. Let $\mathcal{O}_\chi$ be the subcategory of objects M of $\mathcal{O}_{\mathrm{alg}}$ such that $z - \chi(z)$ acts nilpotently on M for any $z \in Z(\mathfrak{g})$. For $I \subset \Delta$, we denote by $\mathcal{O}^I_\chi$ the full subcategory of objects of $\mathcal{O}^I_{\mathrm{alg}}$ which are also in $\mathcal{O}_\chi$. We deduce from [Hum08, Prop. 1.12] that there is a decomposition into blocks

$$\mathcal{O}^I_{\mathrm{alg}} = \bigoplus_\chi \mathcal{O}^I_\chi.$$

We write $\widetilde{\mathcal{O}}^I_\chi$ for the subcategory of objects M of $\widetilde{\mathcal{O}}^I_{\mathrm{alg}}$ such that $M/\mathfrak{m}_I M$ lies in $\mathcal{O}^I_\chi$, and similarly $\mathcal{O}^{I,\infty}_\chi$.

Remark 2.8. For $\lambda \in X^*(\underline{T})$, let χ be the character χ_λ defined in [Hum08, §1.7]. Then *loc. cit.* implies that $\widetilde{M}_I(\lambda)$ is in $\widetilde{\mathcal{O}}^I_{\chi_\lambda}$. Note that $\chi_\lambda = \chi_\mu$ if, and only if, there is $w \in W$ such that $w \cdot \lambda = \mu$.

Lemma 2.9. *We have decompositions* $\widetilde{\mathcal{O}}^I_{\mathrm{alg}} = \bigoplus_\chi \widetilde{\mathcal{O}}^I_\chi$ *and* $\mathcal{O}^{I,\infty}_{\mathrm{alg}} = \bigoplus_\chi \mathcal{O}^{I,\infty}_\chi$.

Proof. Let M be an object of $\widetilde{\mathcal{O}}^I_{\mathrm{alg}}$. For a character $\chi : Z(\mathfrak{g}) \to L$ and $\mu \in X^*(\underline{T})$, let $M^{\mu,\chi}$ denote the subset of elements $x \in M^\mu$ such that $(z - \chi(z))^n x \to 0$ for the $\mathfrak{m}_I$-adic topology on the finite free A_I-module M^μ. We easily check that $M^\chi := \bigoplus_{\mu \in X^*(\underline{T})} M^{\mu,\chi}$ is an $U(\mathfrak{g})_{A_I}$-submodule of M which lies in $\widetilde{\mathcal{O}}^I_\chi$ and that $M = \bigoplus_\chi M^\chi$. The case of $\mathcal{O}^{I,\infty}_{\mathrm{alg}}$ is similar. $\square$

Lemma 2.10. *Let $\lambda_1, \lambda_2 \in X^*(\underline{T})$. Assume that $\widetilde{M}_I(\lambda_1)$ and $\widetilde{M}_I(\lambda_2)$ are in the same block $\widetilde{\mathcal{O}}^I_\chi$ for a character $\chi : Z(\mathfrak{g}) \to L$. Then there exists $w \in W$ such that $w \cdot \lambda_1 = \lambda_2$.*

Proof. By Remark 2.8, the claim follows from the same claim in the category $\mathcal{O}^I_\chi$. As $M_I(\lambda_1)$ and $M_I(\lambda_2)$ are quotients of $M(\lambda_1)$ and $M(\lambda_2)$, this is a consequence of [Hum08, Thm. 1.10]. $\square$

When λ is a character of $\mathfrak{t}$, we often write by abuse of notation $\mathcal{O}_\lambda$ (resp. $\mathcal{O}^{I,\infty}_\lambda, \widetilde{\mathcal{O}}^I_\lambda$) for the block $\mathcal{O}_{\chi_\lambda}$ (resp. $\mathcal{O}^{I,\infty}_{\chi_\lambda}, \widetilde{\mathcal{O}}^I_{\chi_\lambda}$) where χ_λ is the character defined in Remark 2.8.

Corollary 2.11. *Let $\lambda \in X^*(\underline{T})$ be a dominant weight and let χ_λ be the associated character of $Z(\mathfrak{g})$. If M is an object of $\widetilde{\mathcal{O}}^I_{\chi_\lambda}$ (resp. $\mathcal{O}^{I,\infty}_{\chi_\lambda}$), then $M^\lambda = (M^\lambda)^{\mathfrak{n}}$.*

Proof. Assume that this is false. Then there exists $\alpha \in \Phi^+$ and $x \in \mathfrak{g}_\alpha$ such that $xM^\lambda \neq 0$. Thus there exists $\mu > \lambda$ such that $M^\mu \neq 0$. As M lies in the category $\widetilde{\mathcal{O}}^I_{\mathrm{alg}}$ (resp. $\mathcal{O}^{I,\infty}_\chi$), we can choose μ to be maximal which then implies $\mathfrak{n}M^\mu = 0$. As $M^\mu \neq 0$ Nakayama's lemma implies that there exists $v \in M^\mu$ which is non zero in $M^\mu/\mathfrak{m}M^\mu$. Then v defines a map $\widetilde{M}(\mu) \to M$ with $\mu > \lambda$, which is non-zero after reduction by $\mathfrak{m}$. Thus it induces a non-zero map $M(\mu) \longrightarrow M/\mathfrak{m}M \in \mathcal{O}_{\chi_\lambda}$. It follows that $\mu = w \cdot \lambda$ which is a contradiction. $\square$

2.3.3 Deformed Verma modules

Let $\lambda \in X^*(\underline{T})$ and let V be a finite dimensional $U(\mathfrak{g})$-module. Then we have an isomorphism of $U(\mathfrak{g})_A$-modules

$$\widetilde{M}(\lambda) \otimes_L V \simeq U(\mathfrak{g})_A \otimes_{U(\mathfrak{b})_A} (V_{|\mathfrak{b}} \otimes_L A(\lambda)).$$

Indeed there is a canonical map from the left to the right, which then is easily checked to be an isomorphism. As $V_{|\mathfrak{b}}$ is a successive extension of one dimensional $U(\mathfrak{b})$-modules, and as $U(\mathfrak{g})_A \otimes_{U(\mathfrak{b})_A} (-)$ is an exact functor (as follows from the PBW Theorem), we have a filtration (Fil_i) of $\widetilde{M}(\lambda) \otimes_L V$ such that each subquotient $\mathrm{Fil}_i / \mathrm{Fil}_{i-1}$ is isomorphic to $\widetilde{M}(\lambda + \nu_i)$ for ν_i a weight of V. Moreover the family (ν_i) is the family of weights of V (counted with multiplicity).

Proposition 2.12. *Let K denote the fraction field of A. Then the filtration* $(\mathrm{Fil}_i \otimes_A K)$ *of* $(\widetilde{M}(\lambda) \otimes_L V) \otimes_A K$ *splits in the category of* $U(\mathfrak{g})_K$*-modules, i.e. there exists an isomorphism of* $U(\mathfrak{g})_K$*-modules*

$$(\widetilde{M}(\lambda) \otimes_L V) \otimes_A K \simeq \bigoplus_i (\widetilde{M}(\lambda + \nu_i) \otimes_A K)$$

compatible with the filtration $(\mathrm{Fil}_i \otimes_A K)$.

Proof. This is a consequence of the paragraph preceding [Soe92, Thm. 8]. □

Lemma 2.13. *Let $\lambda \in X^*(\underline{T})_I^+$ be a dominant weight (with respect to $\underline{P}_I$) and let V be a finite dimensional $U(\mathfrak{g})$-module. Let M be an object of $\mathcal{O}_{\mathrm{alg}}^{I,\infty}$. Then the map*

$$\mathrm{Hom}_{U(\mathfrak{g})_{A_I}}(\widetilde{M}_I(\lambda) \otimes_L V, M) \to \mathrm{Hom}_{U(\mathfrak{g})}(M_I(\lambda) \otimes_L V, M/\mathfrak{m}_I M)$$

given by reduction modulo $\mathfrak{m}_I$ is surjective.

Proof. The L-vector space $\mathrm{Hom}_L(V, L)$ has the structure of an $U(\mathfrak{g})$-module induced by $\mathfrak{g}$-action defined by $x \cdot \phi = -\phi(x \cdot)$ for $x \in \mathfrak{g}$ and $\phi \in \mathrm{Hom}_L(V, L)$. For any $U(\mathfrak{g})$-modules M_1 and M_2, the adjunction isomorphism $\mathrm{Hom}_L(M_1 \otimes_L V, M_2) \simeq \mathrm{Hom}_L(M_1, M_2 \otimes_L \mathrm{Hom}_L(V, L))$ is $\mathfrak{g}$-equivariant and hence induces an isomorphism,

$$\mathrm{Hom}_{U(\mathfrak{g})}(M_1 \otimes_L V, M_2) \simeq \mathrm{Hom}_{U(\mathfrak{g})}(M_1, M_2 \otimes_L \mathrm{Hom}_L(V, L)).$$

Thus, as $M \otimes_L \mathrm{Hom}_L(V, L)$ lies in $\mathcal{O}_{\mathrm{alg}}^{I,\infty}$, we can assume that $V = L$. Using Lemma 2.9, we can assume that M is in $\mathcal{O}_{\chi}^{I,\infty}$ for some character χ and by Remark 2.8, it is sufficient to consider the case where $\chi = \chi_\lambda$. By construction of the deformed generalized Verma modules we have $\mathrm{Hom}_{U(\mathfrak{g})_{A_I}}(\widetilde{M}_I(\lambda), M) = (M^\lambda)^{\mathfrak{n}_I}$ and $\mathrm{Hom}_{U(\mathfrak{g})}(M_I(\lambda), M/\mathfrak{m}_I M) = ((M/\mathfrak{m}_I M)^\lambda)^{\mathfrak{n}_I}$. However it follows from Corollary 2.11 that $(M^\lambda)^{\mathfrak{n}_I} = M^\lambda$ and $((M/\mathfrak{m}_I M)^\lambda)^{\mathfrak{n}_I} = (M/\mathfrak{m}_I)^\lambda$. It is thus sufficient to prove that the map $M^\lambda \to (M/\mathfrak{m}_I M)^\lambda$ is surjective, which is obvious. □

Proposition 2.14. *Let M be an object of the category $\mathcal{O}_{\mathrm{alg}}^{I,\infty}$. Then there exist weights $\lambda_1, \dots, \lambda_r \in X^*(\underline{T})_I^+$ and finite dimensional $U(\mathfrak{g})$-modules $W_1, \dots, W_r$ and a surjective map of $U(\mathfrak{g})_{A_I}$-modules*

$$(\widetilde{M}_I(\lambda_1) \otimes_L W_1) \oplus \cdots \oplus (\widetilde{M}_I(\lambda_r) \otimes_L W_r) \twoheadrightarrow M. \tag{3}$$

In particular M is a quotient of an object of the category $\widetilde{\mathcal{O}}_{\mathrm{alg}}^I$. Moreover there exists an integer $N \geqslant 0$ such that the map (3) factors through

$$\Big((\widetilde{M}_I(\lambda_1) \otimes_L W_1) \oplus \cdots \oplus (\widetilde{M}_I(\lambda_r) \otimes_L W_r)\Big) \otimes_{A_I} (A_I/\mathfrak{m}_I^N).$$

Proof. By [Hum08, Thm. 9.8] (and its proof), there exist dominant weights $\lambda_1, \dots, \lambda_r$, finite dimensional $U(\mathfrak{g})$-modules $W_1, \dots, W_r$ and a surjective map

$$(M_I(\lambda_1) \otimes_L W_1) \oplus \cdots \oplus (M_I(\lambda_r) \otimes_L W_r) \twoheadrightarrow M/\mathfrak{m}_I M.$$

By Lemma 2.13, this map can be lifted into a $U(\mathfrak{g})_{A_I}$-equivariant map

$$\widetilde{M}_I(\lambda_1) \otimes_L W_1 \oplus \cdots \widetilde{M}_I(\lambda_r) \otimes_L W_r \twoheadrightarrow M$$

which is surjective by Nakayama's Lemma. The last assertion is a consequence of the fact that M is finitely generated as a $U(\mathfrak{g})$-module and all its elements are killed by some power of $\mathfrak{m}_I$ so that M is killed by $\mathfrak{m}_I^N$ for some $N \geqslant 0$. $\square$

2.4 Bimodule structure

Let $\xi : Z(\mathfrak{g}) \to U(\mathfrak{t})$ be the Harish-Chandra map. Recall that it is defined as follows: for $x \in Z(\mathfrak{g})$ there exists a unique element $\xi(x) \in U(\mathfrak{t})$ such that $x \in \xi(x) + U(\mathfrak{g})\mathfrak{n}$ (see [Kna01, Lem. 8.17]). For any $\nu \in X^*(\underline{T})$ we denote by t_ν the unique endomorphism of $U(\mathfrak{t})$ such that $t_\nu(x) = x + \nu(x)$ for $x \in \mathfrak{t}$. Note that $t_{-\delta_G} \circ \xi$ induces an isomorphism from $Z(\mathfrak{g})$ on to $U(\mathfrak{t})^W$ (see [Kna01, Thm. 6.18]). For a dominant weight $\lambda \in X^*(\underline{T})$ we define a map

$$h_\lambda : A \otimes_L Z(\mathfrak{g}) \xrightarrow{\mathrm{Id}\otimes\xi} A \otimes_L U(\mathfrak{t}) \xrightarrow{\mathrm{Id}\otimes t_\lambda} A \otimes_{A^W} A$$

following [Soe92, §3.2], It follows from [Soe92, Thm. 9] that h_λ is surjective (note that $\mathcal{W}_\lambda$ in *loc. cit.* is trivial in our situation). If $I \subset \Delta$ is a finite subset, tensorization on the left with $p_I : A \twoheadrightarrow A_I$ yields a map $h_\lambda : A_I \otimes_L Z(\mathfrak{g}) \to A_I \otimes_{A^W} A$.

For $w \in W$, let $I_w \subset A_I \otimes_L Z(\mathfrak{g})$ denote the kernel of the map

$$h_{\lambda,w} : A_I \otimes_L Z(\mathfrak{g}) \xrightarrow{\mathrm{Id}\otimes h_\lambda} A_I \otimes_{A^W} A \xrightarrow{x\otimes y \mapsto (xp_I(\mathrm{Ad}(w)y))} A_I.$$

It is not hard to see that this kernel only depends on the choice of $\overline{w} \in W_I \backslash W$.

Proposition 2.15. *For $w \in {}^I W$, the $A_I \otimes_L Z(\mathfrak{g})$-modules $\widetilde{M}_I(w \cdot \lambda)$ and $\widetilde{M}_I(w \cdot \lambda)^\vee$ are annihilated by I_w.*

Proof. The result for $\widetilde{M}_I(w \cdot \lambda)^\vee$ follows from the result for $\widetilde{M}_I(w \cdot \lambda)$ and the inclusion

$$\widetilde{M}_I(w \cdot \lambda)^\vee \subset \mathrm{Hom}_A(\widetilde{M}_I(w \cdot \lambda), A).$$

Hence it is enough to check that the action of $A_I \otimes_L Z(\mathfrak{g})$ on $\widetilde{M}_I(w \cdot \lambda)$ factors through $h_{\lambda,w}$. As this action is central and $\widetilde{M}_I(w \cdot \lambda)$ is generated by $\widetilde{M}_I(w \cdot \lambda)^{w \cdot \lambda}$ as an $U(\mathfrak{g})_{A_I}$-module, it is sufficient to check that the action $A_I \otimes_L Z(\mathfrak{g})$ on $\widetilde{M}_I(w \cdot \lambda)^{w \cdot \lambda}$ factors through $h_{\lambda,w}$. Using the fact that $\mathfrak{n}$ acts trivially on $\widetilde{M}_I(w \cdot \lambda)^{w \cdot \lambda}$, an element $x \in Z(\mathfrak{g})$ acts on this space via ξ. For the clarity of the computation let us write $\varepsilon_\nu : U(\mathfrak{t}) \to A_I$

for the L-algebra homomorphism associated to an L-linear map $\nu : \mathfrak{t} \to A_I$ and let $\iota : \mathfrak{t} \hookrightarrow A \twoheadrightarrow A_I$. Then for $x \in Z(\mathfrak{g})$ and $v \in \widetilde{M}_I(w \cdot \lambda)$, we have

$$\begin{aligned}\varepsilon_{w\cdot\lambda+\iota}(\xi(x)) = \varepsilon_{w(\lambda+\delta_G+w^{-1}(\iota))}(t_{-\delta_G}(\xi(x))) &= \varepsilon_{\lambda+\delta_G+w^{-1}(\iota)}(t_{-\delta_G}(\xi(x))) \\ &= \varepsilon_{w^{-1}(\iota)}(h_\lambda(x)) = p_I(\mathrm{Ad}(w)(h_\lambda(x)))\end{aligned}$$

(where we use that the image of $t_{-\delta_G} \circ \xi$ lies in $U(\mathfrak{t})^W$). As an element $y \in U(\mathfrak{t})$ acts by multiplication by $\varepsilon_{w\cdot\lambda+\iota}(y)$ on $\widetilde{M}_I(w \cdot \lambda)^{w\cdot\lambda}$, we conclude that an element $x \otimes z \in A_I \otimes_L Z(\mathfrak{g})$ acts by multiplication by $xp_I(\mathrm{Ad}(w)(h_\lambda(x)))$ on $\widetilde{M}_I(w \cdot \lambda)^{w\cdot\lambda}$, which is the desired formula. □

Remark 2.16. The ring $U(\mathfrak{t})$ (resp. $U(\mathfrak{z}_I)$) is the affine coordinate ring of the (affine) L-scheme associated to the dual $\mathfrak{t}^*$ of $\mathfrak{t}$ (resp. to the dual $\mathfrak{z}_I^*$ of $\mathfrak{z}_I$) so that A (resp. A_I) is the stalk of the structure sheaf of $\mathfrak{t}^*$ (resp. of $\mathfrak{z}_I^*$) at the origin. The ideal I_w is the ideal defining the irreducible component $T_{I,w}$ of $(\mathfrak{z}_I^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^*)_{(0,0)}$ consisting of pairs $(\lambda, \mu) \in \mathfrak{z}_I^* \times \mathfrak{t}^*$ of characters such that $\mu = w(\lambda)$.

Later in the paper we will view the L-scheme $\mathfrak{t}^*$ as the Lie algebra $\mathfrak{t}^\vee$ of the dual torus $\underline{T}_L^\vee$ of the Langlands dual group $\underline{G}_L^\vee$, that we consider as an algebraic group over L. As we will later specialize to the case where $\underline{G}$ is isomorphic to a product of r copies of GL_n the reductive group $\underline{G}$ is self dual and we will identify $\mathfrak{t}^* = \mathfrak{t}^\vee$ with $\mathfrak{t}$ in order to avoid the additional $(-)^\vee$ in the notation. In particular we will consider $U(\mathfrak{t})$ as the affine coordinate ring of $\mathfrak{t}$. The inclusion $\mathfrak{z}_I^* \hookrightarrow \mathfrak{l}_I^*$ induced by the projection $p_I : \mathfrak{l}_I \to \mathfrak{z}_I$ is then identified with the inclusion $\mathfrak{z}_I^\vee \hookrightarrow \mathfrak{l}_I^\vee$ of the center of the Lie algebra of the Langlands dual group of $\underline{L}$ and again we use self duality (in the case of products of copies of GL_n) to identify this map with $\mathfrak{z}_I \hookrightarrow \mathfrak{l}_I$. Hence we obtain a canonical map $\mathfrak{z}_I \hookrightarrow \mathfrak{t}$ of L-schemes corresponding to the morphism $U(\mathfrak{t}) \to U(\mathfrak{z}_I)$. With this identification the ideal I_w defines the irreducible component $T_{I,w}$ of $(\mathfrak{z}_I \times_{\mathfrak{t}/W} \mathfrak{t})_{(0,0)}$ whose points are the pairs $(x, y) \in \mathfrak{t}^2$ such that $y = w^{-1}(x)$.

We finally recall the following result of Soergel (Endomorphismensatz 7 [Soe90]).

Proposition 2.17. *The action of $Z(\mathfrak{g})$ on $P(w_0 \cdot \lambda)$ factors through the map $t_\lambda \circ \xi : Z(\mathfrak{g}) \twoheadrightarrow L \otimes_{A^W} A$ and induces an isomorphism $L \otimes_{A^W} A \simeq \mathrm{End}_{\mathcal{O}}(P(w_0 \cdot \lambda))$.*

3 The Emerton–Jacquet functor

Let $\underline{G}$ be a quasi-split reductive group defined over $\mathbb{Q}_p$. Let $\underline{B}$ be a Borel subgroup and $\underline{T}$ be a maximal torus of $\underline{G}$ contained in $\underline{B}$. We set $G \coloneqq \underline{G}(\mathbb{Q}_p)$, $B \coloneqq \underline{B}(\mathbb{Q}_p)$, $T \coloneqq \underline{T}(\mathbb{Q}_p)$. We also fix L a finite extension of $\mathbb{Q}_p$ which will be the coefficient field of our representations. We assume that L is big enough so that the torus $\underline{T} \times_{\mathbb{Q}_p} L$ is split (and then $\underline{G} \times_{\mathbb{Q}_p} L$ is split). We denote $\mathfrak{g}$, $\mathfrak{b}$ etc. the Lie algebras of $\underline{G} \times_{\mathbb{Q}_p} L$, $\underline{B} \times_{\mathbb{Q}_p} L$ etc. For a $\mathbb{Q}_p$-analytic Lie group H, we consider the category $\mathrm{Rep}_L^{\mathrm{la}} H$ of locally analytic representations of H on locally convex L-vector spaces of compact type. In [Eme06a, Def. 3.4.5] Emerton constructs a functor

$$J_B : \operatorname{Rep}_L^{\mathrm{la}} G \to \operatorname{Rep}_L^{\mathrm{la}} T$$

that we refer to as the Emerton–Jacquet functor. We briefly recall its definition. Let N_0 be a compact open subgroup of N and let $T^+ := \{t \in T \mid tN_0t^{-1} \subset N_0\}$. If V is a L-linear representation of B, we endow the L-vector space V^{N_0} with the action of the monoid T^+ defined by

$$[t]v := \frac{1}{[N_0 : tN_0t^{-1}]} \sum_{u \in N_0/tN_0t^{-1}} ut(v).$$

Then $J_B(V)$ is the finite slope space (see [Eme06a, Def. 3.2.1]) $(V^{N_0})_{\mathrm{fs}}$ of V^{N_0} with respect to the action of T^+ on which the T^+-action extends to a locally analytic representation of T. Of course this construction does not depend on the choice of N_0.

3.1 Families of locally analytic representations of the Borel subgroup

Let $s \in \mathbb{Z}_{\geqslant 0}$ be an integer and let Π be an object of $\operatorname{Rep}_K^{\mathrm{la}}(\mathbb{Z}_p^s \times B)$. We consider the following hypothesis on Π.

Hypothesis 3.1. There exists a locally analytic representation of N_0 on a locally convex L-vector space of compact type V such that

$$\Pi_{|\mathbb{Z}_p^s \times N_0} \simeq \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L) \hat{\otimes}_L V.$$

Given s, we set $S := \mathcal{O}_L[[\mathbb{Z}_p^s]]$ and write $\operatorname{Spf}(S)^{\mathrm{rig}}$ for the rigid analytic generic fiber of $\operatorname{Spf}(S)$. This space is a rigid analytic open polydisc and we write

$$S^{\mathrm{rig}} = \Gamma(\operatorname{Spf}(S)^{\mathrm{rig}}, \mathcal{O}_{\operatorname{Spf} S^{\mathrm{rig}}})$$

for its ring of rigid analytic functions, which is a Fréchet L-algebra (when endowed with its natural topology). We note that a finitely generated projective S^{rig}-module C defines a vector bundle on $\operatorname{Spf}(S)^{\mathrm{rig}}$. As every vector bundle on a rigid analytic polydisc over L is free (see [Gru68, §V]), it follows that C is free as well, i.e. every finitely generated projective S^{rig}-module is finite free. Moreover the following lemma implies that finite dimensional quotients of S^{rig} admit resolutions by a perfect complexes.

Lemma 3.2. *Let $\mathfrak{a} \subset S^{\mathrm{rig}}$ be a closed strict ideal such that $\dim_L S^{\mathrm{rig}}/\mathfrak{a} < \infty$. Then there exists a perfect complex $C_\bullet$ of S^{rig}-modules which is a resolution of $S^{\mathrm{rig}}/\mathfrak{a}$ and such that $C_0 = S^{\mathrm{rig}}$.*

Proof. As $S[1/p]$ is dense in S^{rig}, its image in $S^{\mathrm{rig}}/\mathfrak{a}$ is a dense L-vector subspace and, as $S^{\mathrm{rig}}/\mathfrak{a}$ is finite dimensional, is in fact equal to $S^{\mathrm{rig}}/\mathfrak{a}$. Setting $\mathfrak{a}_0 := \mathfrak{a} \cap S[1/p]$, we have $S[1/p]/\mathfrak{a}_0 \simeq S^{\mathrm{rig}}/\mathfrak{a}$. As S^{rig} is a flat $S[1/p]$-module, it is sufficient to prove that $S[1/p]/\mathfrak{a}_0$ has a finite resolution by finite projective $S[1/p]$-modules, which is a consequence of the fact that $S[1/p]$ is a regular noetherian ring. $\square$

Let $C_\bullet$ be a complex of finite free S^{rig}-modules. If V is an object of $\mathrm{Rep}_L^{\mathrm{la}}(\mathbb{Z}_p^s)$, a choice of S^{rig}-basis of C_n induces an isomorphism of $\mathrm{Hom}_{S^{\mathrm{rig}}}(C_n, V)$ with a direct sum of copies of V. We endow $\mathrm{Hom}_{S^{\mathrm{rig}}}(C_n, V)$ with the product topology which does no depend on the choice of the basis of C_n. With this topology the differentials in the complex $\mathrm{Hom}_{S^{\mathrm{rig}}}(C_n, V)$ are continuous. The complex $\Pi^\bullet := \mathrm{Hom}_{S^{\mathrm{rig}}}(C_\bullet, \Pi)$ is then a complex of locally analytic L-representations of $\mathbb{Z}_p^s \times B$. We also set $\Pi^{N_0,\bullet} := \mathrm{Hom}_{S^{\mathrm{rig}}}(C_\bullet, \Pi^{N_0})$ and $J_B(\Pi)^\bullet := \mathrm{Hom}_{S^{\mathrm{rig}}}(C_\bullet, J_B(\Pi))$.

Lemma 3.3. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence of topological L-vector spaces of compact type (resp. nuclear Fréchet spaces) and let X be a topological L-vector space of compact type (resp. nuclear Fréchet space). Then the following sequence is exact*

$$0 \to U\hat{\otimes}_L X \to V\hat{\otimes}_L X \to W\hat{\otimes}_L X \to 0.$$

Proof. The claim follows from [Sch11, Lemm. 4.13], [ST02, Cor. 1.4] and from [Eme17, Prop. 1.1.32]. □

Lemma 3.4. *Let Π be a locally analytic representation of $\mathbb{Z}_p^s \times B$ satisfying Hypothesis 3.1. Then the two complexes $\Pi^\bullet$ and $\Pi^{N_0,\bullet}$ are complexes of L-vector spaces of compact type with strict continuous transition maps. Moreover for any integer $n \geqslant 0$, we have an isomorphism of topological T^+- modules*

$$H^n(\Pi^{N_0,\bullet}) \simeq H^n(\Pi^\bullet)^{N_0}.$$

Proof. Fix an isomorphism $\Pi_{|\mathbb{Z}_p^s \times N_0} \simeq \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L)\hat{\otimes}_L V$ whose existence comes from hypothesis 3.1. As any C_m is a finite free S^{rig}-module and as the completed tensor product $-\hat{\otimes}_L-$ commutes with finite direct sums ([Koh07, Lem. 1.2.13]), we have an isomorphism of complexes of topological representations of $\mathbb{Z}_p^s \times N_0$:

$$\Pi^\bullet \simeq \mathrm{Hom}_{S^{\mathrm{rig}}}(C_\bullet, \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L))\hat{\otimes}_L V.$$

The terms of the complex $\mathrm{Hom}_{S^{\mathrm{rig}}}(C_\bullet, \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L))$ are locally analytic representations of $\mathbb{Z}_p^s$ isomorphic to finite direct products of copies of $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L)$ (with $\mathbb{Z}_p^s$ acting by translation on the left or the right, which is equivalent as it is a commutative group) and transition maps which are continuous and $\mathbb{Z}_p^s$-equivariant. As $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L)$ is an admissible locally analytic representation of $\mathbb{Z}_p^s$, it follows from [ST03, Prop. 6.4] that the transition maps of the complex $\mathrm{Hom}_{S^{\mathrm{rig}}}(C_\bullet, \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L))$ are strict with closed images. We deduce from this fact and from Lemma 3.3 that the complex $\Pi^\bullet$ has strict transition maps and that we have topological isomorphisms $H^n(\Pi^\bullet) \simeq H^n(\mathrm{Hom}_{S^{\mathrm{rig}}}(C_\bullet, \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L)))\hat{\otimes}_L V$ for any $n \geqslant 0$. The commutation of $\hat{\otimes}_L$ with finite direct sum implies that we have a topological isomorphism of L-vector spaces for any $m \geqslant 0$:

$$\mathrm{Hom}_{S^{\mathrm{rig}}}(C_m, \Pi^{N_0}) \simeq \mathrm{Hom}_{S^{\mathrm{rig}}}(C_m, \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L))\hat{\otimes}_L V^{N_0}.$$

We deduce as before that the complex $\Pi^{\bullet,N_0}$ has strict transition maps and that we have isomorphisms

$$H^n(\Pi^{N_0,\bullet}) \simeq H^n(\Pi^\bullet)^{N_0}$$

for any $n \geqslant 0$. $\square$

Proposition 3.5. *For any integer $n \geqslant 0$, there is an isomorphism*

$$H^n(J_B(\Pi)^\bullet) \simeq J_B(H^n(\Pi^\bullet))$$

of locally analytic L-representations of $\mathbb{Z}_p^s \times T$.

Proof. It follows from [Eme06a, Prop. 3.2.4.(ii)] that there is a natural continuous T^+-equivariant map of complexes $(\Pi^{N_0,\bullet})_{\mathrm{fs}} \to \Pi^{N_0,\bullet}$ inducing a continuous T^+-equivariant morphism $H^n(\Pi_{\mathrm{fs}}^{N_0,\bullet}) \to H^n(\Pi^{N_0,\bullet})$. By *loc. cit.*, the universal property of the functor $(-)_{\mathrm{fs}}$ provides a T-equivariant map $H^n(\Pi_{\mathrm{fs}}^{N_0,\bullet}) \to H^n(\Pi^{N_0,\bullet})_{\mathrm{fs}}$. It follows from Lemma 3.4 that it is sufficient to prove that this map is a topological isomorphism.

We now deduce from [Eme06a, Prop. 3.2.27] and [Fu, Thm. 4.5] that given an exact sequence $0 \to U \to V \to W \to 0$ of spaces of compact type with continuous action of T^+, then $0 \to U_{\mathrm{fs}} \to V_{\mathrm{fs}} \to W_{\mathrm{fs}} \to 0$ is exact, the image of U_{fs} is closed in V_{fs} and the map $V_{\mathrm{fs}} \to W_{\mathrm{fs}}$ is strict. The open mapping theorem then implies that the sequence is strict exact. As the complex $\Pi^{N_0,\bullet}$ has strict transition maps by Lemma 3.4, we conclude that the map $H^n(\Pi_{\mathrm{fs}}^{N_0,\bullet}) \to H^n(\Pi^{N_0,\bullet})_{\mathrm{fs}}$ is a topological isomorphism. $\square$

Proposition 3.6. *Let Π be a locally analytic L-representation of $\mathbb{Z}_p^s \times B$ satisfying the hypothesis 3.1. Let $\mathfrak{a}$ be a closed strict ideal of S^{rig} such that $\dim_L S^{\mathrm{rig}}/\mathfrak{a} < +\infty$. Then the map*

$$\mathfrak{a} \otimes_{S^{\mathrm{rig}}} J_B(\Pi)' \longrightarrow J_B(\Pi)'$$

is injective.

Proof. By Lemma 3.2, there exists a perfect complex $C_\bullet$ of S^{rig}-modules such that, $C_0 = S^{\mathrm{rig}}$, $H_0(C_\bullet) \simeq S^{\mathrm{rig}}/\mathfrak{a}$ and $H_i(C_\bullet) = 0$ for $i > 0$. By Hypothesis 3.1, we have $\Pi|_{\mathbb{Z}_p^s \times N_0} \simeq \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L)\hat{\otimes}_L V$ for some topological L-vector space of compact type V. As $C_\bullet$ has strict transition maps, it follows from Lemma 3.3 that the complex $C_\bullet \otimes_{S^{\mathrm{rig}}} \Pi' \simeq C_\bullet \hat{\otimes}_L V'$ is a resolution of $(S^{\mathrm{rig}}/\mathfrak{a})\hat{\otimes}_L V'$. We then deduce from $\mathrm{Hom}_{S^{\mathrm{rig}}}(C_i, \Pi)' \simeq C_i \otimes_{S^{\mathrm{rig}}} \Pi'$ for any $i \geqslant 0$, that $H^i(\mathrm{Hom}_{S^{\mathrm{rig}}}(C_\bullet, \Pi)) = 0$ for $i > 0$. Therefore Proposition 3.5 implies that $H^i(\mathrm{Hom}_{S^{\mathrm{rig}}}(C_\bullet, J_B(\Pi))) = 0$ for $i > 0$. We denote by $(-)'$ the duality between spaces of compact type and Fréchet spaces. This duality implies that $H_i(C_\bullet \otimes_{S^{\mathrm{rig}}} J_B(\Pi)') = 0$ for $i > 0$. As $\mathfrak{a} = \mathrm{Coker}(C_2 \to C_1)$, we deduce that

$$\begin{aligned}\mathfrak{a} \otimes_{S^{\mathrm{rig}}} J_B(\Pi)' &= \mathrm{Coker}(C_2 \otimes_{S^{\mathrm{rig}}} J_B(\Pi)' \to C_1 \otimes_{S^{\mathrm{rig}}} J_B(\Pi)') \\ &\subset C_0 \otimes_{S^{\mathrm{rig}}} J_B(\Pi)' = J_B(\Pi)'.\end{aligned}$$ $\square$

3.2 Families of locally analytic representations of G

Let Π be an admissible locally analytic L-representation of $\mathbb{Z}_p^s \times G$. The aim of this section is to use Π in order to construct a functor

$$M \mapsto \mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi)$$

from the category $\mathcal{O}_{\mathrm{alg}}^{\infty}$ to the category of locally analytic $\mathbb{Z}_p^s \times B$-representations, and then, by composing with J_B, to locally analytic $\mathbb{Z}_p^s \times T$-representations. We will usually assume that Π satisfies the following hypothesis.

Hypothesis 3.7. There exists a uniform open pro-p-subgroup H of G, an integer $m \geqslant 0$ and a topological $\mathbb{Z}_p^s \times H$-equivariant isomorphism

$$\Pi_{|\mathbb{Z}_p^s \times H} \simeq \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s \times H, L)^m.$$

Recall from section 2.2 that if M is an object of $\mathcal{O}_{\mathrm{alg}}^{\infty}$, there is a unique algebraic action of $\underline{B}(L)$ on M which lifts the structure of $U(\mathfrak{b})$-module on M^{ss}. We endow M with the action of $B = \underline{B}(\mathbb{Q}_p)$ obtained by restriction to B.

Let M be an object of $\mathcal{O}_{\mathrm{alg}}^{\infty}$ with its semi-simplified B-action. We define an action of B on $\mathrm{Hom}_L(M, \Pi)$ by

$$b \cdot f = bf(b^{-1}-)$$

for $f \in \mathrm{Hom}_L(M, \Pi)$ and $b \in B$. It follows from Lemma 2.4 that this action preserves the subspace $\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi)$. We moreover endow $\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi)$ with the left $\mathbb{Z}_p^s$-action inherited from the one on Π. While the definition of the B-action using the semi-simplified action on M might not seem very natural at a first glance, the following lemma says that this definition applied to deformed Verma modules allows us to compute generalized eigenspaces. Given an $U(\mathfrak{t})$-module X we write

$$X[(\mathfrak{t} - \lambda)^k] = \{x \in X \mid \forall t \in \mathfrak{t}, (t - \lambda(t))^k x = 0\}.$$

With this notation we have the following result:

Lemma 3.8. *Let $\lambda \in X^*(\underline{T})_I^+$ and $M = \widetilde{M}_I(\lambda) \otimes_{A_I} (A_I/\mathfrak{m}_I^k)$. Then there is an isomorphism*

$$\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi) \simeq (\Pi^{\mathfrak{n}_I} \otimes_L L_I(\lambda)')^{U(\mathfrak{l}_I^{\mathrm{ss}})}[\mathfrak{m}_I^k]$$

of B-representations, where $(-)'$ denote the dual (algebraic) representation. In particular, when $I = \emptyset$,

$$\mathrm{Hom}_{U(\mathfrak{g})}(\widetilde{M}(\lambda) \otimes_A (A/\mathfrak{m}^k), \Pi) \simeq (\Pi^{\mathfrak{n}}(\lambda^{-1}))[\mathfrak{m}^k] \simeq (\Pi^{\mathfrak{n}}[(\mathfrak{t} - \lambda)^k])(\lambda^{-1})$$

where the symbol (μ) denotes the twist by the algebraic character μ of T seen as a quotient of B.

Proof. We compute using the $U(\mathfrak{g})$-structure

$$\begin{aligned}\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi) &= \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g})\otimes_{U(\mathfrak{p}_I)}(L_I(\lambda)\otimes_L A_I/\mathfrak{m}_I^k),\Pi)\\ &= \mathrm{Hom}_{U(\mathfrak{l}_I)}(L_I(\lambda)\otimes_L A_I/\mathfrak{m}_I^k,\Pi^{\mathfrak{n}_I})\\ &= \mathrm{Hom}_{U(\mathfrak{l}_I)}(A_I/\mathfrak{m}_I^k,\Pi^{\mathfrak{n}_I}\otimes L_I(\lambda)')\\ &= (\Pi^{\mathfrak{n}_I}\otimes L_I(\lambda)')^{U(\mathfrak{l}_I^{\mathrm{ss}})}[\mathfrak{m}_I^k].\end{aligned}$$

Moreover each equality is compatible with the semi-simplified B-actions. □

Lemma 3.9. *Let Π be a locally analytic representation of $\mathbb{Z}_p^s\times G$ and let M be an object of $\mathcal{O}^\infty_{\mathrm{alg}}$. Then the $\mathbb{Z}_p^s\times B$-representation $\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)$ is locally analytic.*

Proof. Let $U(\mathfrak{g})\otimes_{U(\mathfrak{b})}V_1\to U(\mathfrak{g})\otimes_{U(\mathfrak{b})}V_0\to M\to 0$ be a resolution as in Lemma 2.5. Then $\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)$ is the kernel of the map

$$\mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g})\otimes_{U(\mathfrak{b})}V_0,\Pi)\simeq(V_0'\otimes_L\Pi)^{\mathfrak{b}}\longrightarrow \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g})\otimes_{U(\mathfrak{b})}V_1,\Pi)\simeq(V_1'\otimes_L\Pi)^{\mathfrak{b}}$$

which is continuous and B-equivariant. Therefore $\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)$ is isomorphic to a closed B-stable subspace of $V_0'\otimes_L\Pi$. As V_0 is an algebraic finite dimensional representation of B, the representation $V_0'\otimes_L\Pi$ is locally analytic and hence so is $\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)$. □

As $\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)$ is a locally analytic representation of B this action may be derived and induces the structure of an $U(\mathfrak{b})$-module on $\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)$. Via restriction to $U(\mathfrak{t})\subset U(\mathfrak{b})$ we may view $\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)$ as an $U(\mathfrak{t})$-module.

Lemma 3.10. *Let Π be a locally analytic representation of $\mathbb{Z}_p^s\times G$ and let M be an object of $\mathcal{O}^\infty_{\mathrm{alg}}$. Then the $U(\mathfrak{t})$ action on $\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)$ factors through a finite dimensional quotient.*

Proof. By Proposition 2.14 there exist dominant weights $\lambda_1,\dots,\lambda_r$, finite dimensional $\mathfrak{g}$-modules $V_1,\dots,V_r$ and a surjective map

$$\widetilde{M}(\lambda_1)\otimes_L V_1\oplus\cdots\oplus\widetilde{M}(\lambda_r)\otimes_L V_r\twoheadrightarrow M.$$

Moreover by the same Lemma, there exists $k\geqslant 1$ such that this map factors through $\mathfrak{m}^k$ (recall that A is the localization of $U(\mathfrak{t})$ at its augmentation ideal $\mathfrak{m}$). Therefore we have an inclusion of $U(\mathfrak{t})$-modules

$$\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)\hookrightarrow\bigoplus_{i=1}^r\mathrm{Hom}_{U(\mathfrak{g})}(\widetilde{M}(\lambda_i)\otimes_A A/\mathfrak{m}^k\otimes_L V_i,\Pi).$$

By Lemma 3.8, $\mathrm{Hom}_{U(\mathfrak{g})}(\widetilde{M}(\lambda_i)/\mathfrak{m}^k\otimes_L V_i,\Pi)=(\Pi\otimes V_i(\lambda_i)')^{\mathfrak{n}}[\mathfrak{m}^k]$. Let $\mu_1,\dots,\mu_s$ be the finitely many characters which appears in the restriction to $U(\mathfrak{t})$ of $V_1(\lambda_1),\dots,V_r(\lambda_r)$. Then the action of $U(\mathfrak{t})$ on $\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)$ factors through the quotient of $U(\mathfrak{t})$ by the intersection of the k-th powers of the kernels of the μ_i. □

Lemma 3.11. *Assume that* Π *is an admissible locally analytic* L*-representation of* $\mathbb{Z}_p^s \times G$ *satisfying Hypothesis 3.7 and* $M \in \mathcal{O}_{\mathrm{alg}}^{\infty}$*. Then* $\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi)$ *satisfies Hypothesis 3.1*

Proof. We can assume that $N_0 \subset H$. As we assume Hypothesis 3.7, there is an isomorphism $\Pi \cong \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s \times H, L)^m \simeq \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L)\widehat{\otimes}_L \mathcal{C}(H, L)^m$ of $\mathbb{Z}_p^s \times H$-representation.

Let $[U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_1 \to U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_0]$ be a resolution of M as in Lemma 2.5. Then $\mathrm{Hom}_{U(\mathfrak{g})}(M, \mathcal{C}^{\mathrm{la}}(H, L)^m)$ is the kernel of the map

$$(V_0' \otimes_L \mathcal{C}^{\mathrm{la}}(H, L)^m)^{\mathfrak{b}} \to (V_1' \otimes_L \mathcal{C}^{\mathrm{la}}(H, L)^m)^{\mathfrak{b}}. \tag{4}$$

We claim that this is a strict map, then the lemma follows, as exactness of the functor $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L)\hat{\otimes}_L(-)$ implies that we have an isomorphism of locally analytic $\mathbb{Z}_p^s \times N_0$-representation

$$\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi) \simeq \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^s, L)\hat{\otimes}_L \mathrm{Hom}_{U(\mathfrak{g})}(M, \mathcal{C}^{\mathrm{la}}(H, L))^m.$$

In order to prove that (4) is strict, we use an additional H-action. We let H act on $\mathcal{C}^{\mathrm{la}}(H, L)$ by right translation and extend this to $V_i' \otimes_L \mathcal{C}^{\mathrm{la}}(H, L)$ by acting trivially on V_i'. This action commutes with the (diagonal) action of $U(\mathfrak{b})$, as the $U(\mathfrak{b})$ action on $\mathcal{C}^{\mathrm{la}}(H, L)$ is induced by left translations. It follows that $(V_i' \otimes_L \mathcal{C}^{\mathrm{la}}(H, L)^m)^{\mathfrak{b}}$ is a closed H-stable subspace of an admissible locally analytic H-representation, and hence an admissible locally analytic H-representation itself. Hence (4) is an H-equivariant map between admissible locally analytic H-representations and hence a strict map which proves the claim. $\square$

Proposition 3.12. *Let* Π *be an admissible locally analytic representation of* $\mathbb{Z}_p^s \times G$ *satisfying the hypothesis 3.7 and let* M *be an object of* $\mathcal{O}_{\mathrm{alg}}^{\infty}$*. Then the locally analytic representation* $J_B(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi))$ *of* $\mathbb{Z}_p^s \times T$ *is essentially admissible.*

Proof. Using twice Proposition 2.14 there exists a resolution

$$M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

of M in the category $\mathcal{O}_{\mathrm{alg}}^{\infty}$ where each M_i, $i \in \{0, 1\}$, is of the form

$$(\widetilde{M}(\lambda_1) \otimes_L V_1 \oplus \cdots \oplus \widetilde{M}(\lambda_r) \otimes_L V_r) \otimes (A/\mathfrak{m}^k)$$

for some dominant weights $\lambda_1, \ldots, \lambda_r$, finite dimensional $\mathfrak{g}$-modules $V_1, \ldots, V_r$ and $k \geqslant 1$. Then we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi) \longrightarrow \mathrm{Hom}_{U(\mathfrak{g})}(M_0, \Pi) \longrightarrow \mathrm{Hom}_{U(\mathfrak{g})}(M_1, \Pi)$$

of locally analytic representations of $\mathbb{Z}_p^s \times B$ (see Lemma 3.9). As the functor J_B is left exact ([Eme06a, Lem. 3.4.7.(iii)]), this induces a short exact sequence

$$0 \longrightarrow J_B(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi)) \longrightarrow J_B(\mathrm{Hom}_{U(\mathfrak{g})}(M_0, \Pi)) \longrightarrow J_B(\mathrm{Hom}_{U(\mathfrak{g})}(M_1, \Pi))$$

of locally analytic representations of $\mathbb{Z}_p^s \times T$. As the kernel of a morphism between essentially admissible representations is essentially admissible and the category of essentially admissible representations is stable under finite direct product([Eme06a, Thm. 3.1.3]), it is sufficient to prove that $J_B(\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi))$ is essentially admissible when M is of the form $V \otimes_L (\widetilde{M}(\lambda)) \otimes (A/\mathfrak{m}^k)$ for some finite dimensional algebraic representation V of G, some dominant weight λ and some integer $k \geqslant 1$. From now let's assume that M is of this form. By Lemma 3.8, we have an isomorphism of B representations

$$\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi) \simeq ((V' \otimes_L \Pi)^{\mathfrak{n}}[(\mathfrak{t}-\lambda)^k])(\lambda^{-1}).$$

The representation $V' \otimes_L \Pi$ satifies Hypothesis 3.7 so that, by [BHS17b, Prop. 3.4] (whose proof follows [Eme06a, Thm. 0.5]), the locally analytic representation $J_B(V' \otimes_L \Pi)$ of $\mathbb{Z}_p^s \times T$ is essentially admissible. As $(V' \otimes_L \Pi)^{\mathfrak{n}}[(\mathfrak{t}-\lambda)^k]$ is closed in $V' \otimes_L \Pi$, we conclude from [Eme06a, Lem. 3.4.7.(iii)] and [Eme17, Prop. 6.4.11] that $J_B((V' \otimes_L \Pi)^{\mathfrak{n}}[(\mathfrak{t}-\lambda)^k])$ is essentially admissible and so is

$$J_B(\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)) \simeq J_B((V' \otimes_L \Pi)^{\mathfrak{n}}[(\mathfrak{t}-\lambda)^k](\lambda)^{-1}) \simeq J_B((V' \otimes_L \Pi)^{\mathfrak{n}}[(\mathfrak{t}-\lambda)^k])(\lambda^{-1}). \qquad \square$$

Lemma 3.13. *Let Π be a locally analytic representation of $\mathbb{Z}_p^s \times G$ satisfying Hypothesis 3.7.*

(i) The functor $M \mapsto \mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)$ from $\mathcal{O}^{\infty}_{\mathrm{alg}}$ to the category of locally analytic representations of $\mathbb{Z}_p^s \times B$ is exact.

(ii) The functor $M \mapsto \mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi)^{N_0}$ from $\mathcal{O}^{\infty}_{\mathrm{alg}}$ to the category of locally convex L-vector spaces sends short exact sequences on short exact sequences with strict maps.

Proof. The assertion (i) is [BHS19, Lem. 5.2.5]. We recall the proof as we will need notation for the proof of (ii). Let M be an object of the category $\mathcal{O}^{\infty}_{\mathrm{alg}}$. Let $H \subset G$ be a uniform compact open pro-p-subgroup. Recall (see for example the proof of [ST03, Prop. 6.5]) that $\Pi_{|\mathbb{Z}_p^s \times H} = \varinjlim_{r<1} \Pi_r$ with

$$\Pi_r = \mathrm{Hom}_L^{\mathrm{cont}}(D_r(\mathbb{Z}_p^s \times H) \otimes_{D(\mathbb{Z}_p^s \times G, L)} \Pi', L).$$

As M is a finitely presented $U(\mathfrak{g})$-module, we have

$$\mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi) \simeq \varinjlim_{r<1} \mathrm{Hom}_{U(\mathfrak{g})}(M,\Pi_r) = \varinjlim_{r} \mathrm{Hom}_{U_r(\mathfrak{g})}(M_r,\Pi_r)$$

with $M_r := U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M$. Note that there exists an integer $m \geqslant 0$ such that $\Pi_r \simeq \mathrm{Hom}_L^{\mathrm{cont}}(D_r(\mathbb{Z}_p^s \times H), L)^m$. Therefore we have

$$\begin{aligned}&\mathrm{Hom}_{D_r(H)}(D_r(H) \otimes_{U(\mathfrak{g})} M, \Pi_r)\\ &\qquad \simeq \mathrm{Hom}_L^{\mathrm{cont}}(D_r(H) \otimes_{U_r(\mathfrak{g})} M_r, \mathrm{Hom}_L^{\mathrm{cont}}(D_r(\mathbb{Z}_p^s, L), L))^m,\end{aligned}$$

for $r < 1$. As the functor $M \mapsto M_r$ is exact and $D_r(H)$ is a finite free $U_r(\mathfrak{g})$-module, this proves (i).

Now we prove (ii). As N_0 is a compact group and L is of characteristic 0, it is equivalent to prove (ii) after replacing N_0 by an open subgroup. Therefore we can assume that $N_0 = H \cap N$ and that $H = (\overline{N} \cap H)(T \cap H)(N \cap H)$ where $\overline{N}$ is the group of $\mathbb{Q}_p$-points of the unipotent subgroup of $\underline{G}$ opposite to $\underline{N}$. Let $r < 1$. The space $\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi_r)^{N_0}$ is the space of maps from M to Π_r that are equivariant for the actions of N_0 and $U(\mathfrak{g})$. Therefore we have

$$\begin{aligned}\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi_r)^{N_0} &= \mathrm{Hom}_{U_r(\mathfrak{g}) \otimes_{U_r(\mathfrak{n})} D_r(N_0)}(M_r, \Pi_r) \\ &\simeq \mathrm{Hom}_L^{\mathrm{cont}}(D_r(H) \otimes_{(U_r(\mathfrak{g}) \otimes_{U_r(\mathfrak{n})} D_r(N_0))} M_r, \mathrm{Hom}_L^{\mathrm{cont}}(D_r(\mathbb{Z}_p^s, L), L))^m.\end{aligned}$$

As $D_r(H)$ is a finite free right $U_r(\mathfrak{g}) \otimes_{U_r(\mathfrak{n})} D_r(N_0)$-module (see [Koh07, Thm 1.4]), this proves the claim. □

Theorem 3.14. *The functor $M \mapsto J_B(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi))$ from the category $\mathcal{O}_{\mathrm{alg}}^{\infty}$ to the category of essentially admissible representations of T is exact.*

Proof. This is essentially a consequence of Lemma 3.13 (ii) and we conclude as at the end of the proof Proposition 3.5. □

3.3 The case of Banach representations with coefficients

Let R be a complete local noetherian $\mathcal{O}_L$-algebra. As above we will write R^{rig} for the ring of rigid analytic functions on $(\mathrm{Spf}\, R)^{\mathrm{rig}}$. Let Π be an R-admissible R-Banach representation of the group G (see [BHS17b, Def. 3.1]). We assume that our representations satisfies the following property.

Hypothesis 3.15. there exists an integer $s \geqslant 0$, a local morphism of $\mathcal{O}_L$-algebras $S := \mathcal{O}_L[[\mathbb{Z}_p^s]] \to R$ such that, for some (resp. any) open pro-p-subgroup $G_0 \subset G$, the $S[[G_0]][1/p]$-module $\Pi' := \mathrm{Hom}_L^{\mathrm{cont}}(\Pi, L)$ is finite free (as a consequence Π is also S-admissible).

Using the hypothesis, one shows that the R-analytic vectors $\Pi^{R-\mathrm{an}}$ and the S-analytic vectors $\Pi^{S-\mathrm{an}}$ of Π coincide and they also coincides with the subspace of $\mathbb{Z}_p^s \times G$-locally analytic vectors in Π (see [BHS17b, Prop. 3.8]). We will simply denote this subspace by Π^{la} in what follows. This is a locally analytic representation of $\mathbb{Z}_p^s \times G$ with an action of R^{rig} commuting with G. Moreover if we forget the R^{rig}-action, the representation Π^{la} satisfies Hypothesis 3.7.

In the following we will write $\widehat{T}$ for the rigid analytic space of continuous characters of T and $\widehat{T_0}$ for the space of continuous characters of the maximal compact subgroup $T_0 \subset T$. We recall that the ring of rigid analytic functions on $\widehat{T_0}$ is identified with the algebra $D(T_0, L)$ of L-valued distributions on T_0. Restriction to T_0 defines a canonical projection $\widehat{T} \to \widehat{T}_0$. Moreover, the derivative of a character at 1 defines a *weight map*

$$\mathrm{wt} : \widehat{T}_0 \to \mathfrak{t}^*, \tag{5}$$

where by abuse of notation we write $\mathfrak{t}^*$ for the rigid analytic space associated to the L-vector space $\mathfrak{t}^*$. The map wt is étale and locally finite. Moreover, étaleness implies that for any character $\delta_0 : T_0 \to L^\times$ we can identify the tangent space of $\widehat{T}_0$ at δ_0 with the L-vector space $\mathfrak{t}^*$.

Lemma 3.16. *For any object M in $\mathcal{O}^\infty_{\mathrm{alg}}$, the dual $J_B(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi^{\mathrm{la}}))'$ of the Emerton-Jacquet module $J_B(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi^{\mathrm{la}}))$ is coadmissible as an $R^{\mathrm{rig}} \hat{\otimes}_L \mathcal{O}(\widehat{T})$-module.*

Proof. This is essentially the same proof than for Proposition 3.12 using the fact that $J_B(\Pi^{\mathrm{la}})$ is essentially admissible as a representation of $\mathbb{Z}_p^{s'} \times T$ for any s' and surjection $\mathcal{O}_L[[\mathbb{Z}_p^{s'}]] \twoheadrightarrow R$ by [BHS17b, Prop. 3.4]. □

Let M be an object of $\mathcal{O}^\infty_{\mathrm{alg}}$. It follows from Lemma 3.16 that there exists a unique up to unique isomorphism coherent sheaf $\mathcal{M}_\Pi(M)$ on $\mathrm{Spf}(R)^{\mathrm{rig}} \times \widehat{T}$ such that

$$\Gamma(\mathrm{Spf}(R)^{\mathrm{rig}} \times \widehat{T}, \mathcal{M}_\Pi(M)) = J_B(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi^{\mathrm{la}}))'.$$

In particular we obtain a functor from $\mathcal{O}^\infty_{\mathrm{alg}}$ to the category of coherent sheaves on $\mathrm{Spf}(R)^{\mathrm{rig}} \times \widehat{T}$.

Theorem 3.17. *The coherent sheaf $\mathcal{M}_\Pi(M)$ on $\mathrm{Spf}(R)^{\mathrm{rig}} \times \widehat{T}$ is, locally on $\mathrm{Spf}(R)^{\mathrm{rig}} \times \widehat{T}$, finite free over $\mathrm{Spf}(S)^{\mathrm{rig}}$. In particular, if nonzero, it is Cohen–Macaulay of dimension s.*

Proof. Let T_0 be the maximal compact subgroup of T and let $\widehat{T_0}$ be the rigid analytic space of characters of T_0 over L. Set $N := J_B(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi^{\mathrm{la}}))'$. It follows from the proof of [BHS17b, Prop. 3.11] that there exists a family $\mathcal{I}$ of pairs (U, V) where U is a rational open subset of $\mathrm{Spf}(R)^{\mathrm{rig}} \times \widehat{T}$ and V is a rational open subset of $\mathrm{Spf}(S)^{\mathrm{rig}} \times \widehat{T_0}$ such that V is the image of U and such that $\mathrm{Supp}(\mathcal{M}_\Pi(M)) \subset \bigcup_{(U,V) \in \mathcal{I}} U$. Moreover, we may assume that $\Gamma(U, \mathcal{M}_\Pi(M))$ is a finite projective $\mathcal{O}(V)$-module that is a direct factor of $\mathcal{O}(V) \hat{\otimes}_{S^{\mathrm{rig}} \hat{\otimes}_L D(T_0, L)} N$.

After shrinking each U and V if necessary, we may even assume (by the construction of the family $\mathcal{I}$) that for each $(U, V) \in \mathcal{I}$, the rational open V is of the form $V_1 \times V_2$ with V_1 rational open in $\mathrm{Spf}(S)^{\mathrm{rig}}$ and V_2 rational open in $\widehat{T_0}$. It is sufficient to prove that, for any pair $(U, V_1 \times V_2) \in \mathcal{I}$, the $\mathcal{O}(V_1)$-module $\Gamma(U, \mathcal{M}(M))$ is finitely generated and flat.

The map $V_2 \to \mathfrak{t}^*$ has finite fibers (as the weight map is locally finite), and hence there are only finitely many points of V_2 lying over a given character of $U(\mathfrak{t})$. It thus follows from Lemma 3.10 that the action of $L[T_0]$ on $\Gamma(U,\mathcal{M}(M))$ factors through a finite dimensional quotient. It follows that $\Gamma(U,\mathcal{M}(M))$ is finitely generated over $\mathcal{O}(V_1)$.

Let $\mathfrak{m}\subset\mathcal{O}(V_1)$ be a maximal ideal. As $\mathcal{O}(V_1)$ is an affinoid L-algebra, $\mathfrak{m}$ is closed in $\mathcal{O}(V_1)$ and $\mathcal{O}(V_1)/\mathfrak{m}$ is a finite extension of L. As the image of S^{rig} in $\mathcal{O}(V_1)$ is dense, we have $S^{\mathrm{rig}}/(S^{\mathrm{rig}}\cap\mathfrak{m})\simeq\mathcal{O}(V_1)/\mathfrak{m}$. The ideal $\mathfrak{a}:=S^{\mathrm{rig}}\cap\mathfrak{m}$ of S^{rig} is finitely generated by Lemma 3.2, so that the sheaf $\mathfrak{a}\otimes_{S^{\mathrm{rig}}}\mathcal{M}_\Pi(M)$ is coherent and

$$\Gamma(\mathrm{Spf}(R)^{\mathrm{rig}}\times\widehat{T},\mathfrak{a}\otimes_{S^{\mathrm{rig}}}\mathcal{M}_\Pi(M))\simeq\mathfrak{a}\otimes_{S^{\mathrm{rig}}}\Gamma(\mathrm{Spf}(R)^{\mathrm{rig}}\times\widehat{T},\mathcal{M}_\Pi(M)).$$

As the functor $\mathcal{M}\mapsto\Gamma(U,\mathcal{M}_\Pi)$ is exact on the category of coherent sheaves, we have an isomorphism

$$\Gamma(U,\mathfrak{a}\otimes_{S^{\mathrm{rig}}}\mathcal{M}_\Pi(M))\simeq\mathfrak{a}\otimes_{S^{\mathrm{rig}}}\Gamma(U,\mathcal{M}_\Pi(M))\simeq\mathfrak{m}\otimes_{\mathcal{O}(V_1)}\Gamma(U,\mathcal{M}_\Pi(M)).$$

Therefore we deduce from Proposition 3.6 that the map

$$\mathfrak{m}\otimes_{\mathcal{O}(V_1)}\otimes\Gamma(U,\mathcal{M}_\Pi(M))\longrightarrow\Gamma(U,\mathcal{M}(M))$$

is injective. This implies that $\Gamma(U,\mathcal{M}_\Pi(M))$ is a flat $\mathcal{O}(V_1)$-module. □

Corollary 3.18. *Assume that the representation Π satisfies Hypothesis 3.15. Then the functor $M\mapsto\mathcal{M}_\Pi(M)$ is an exact functor from the category $\mathcal{O}^\infty_{\mathrm{alg}}$ to the category of Cohen–Macaulay sheaves on $\mathrm{Spf}(R)^{\mathrm{rig}}\times\widehat{T}$. Moreover if $\mathcal{M}_\Pi(M)$ is nonzero, its support is s-dimensional, where s is as in Hypothesis 3.15.*

3.4 Comparison with the parabolic Jacquet functor

Let Π be an R-admissible Banach representation of G satisfying hypothesis 3.15. We end this section by computing the evaluation of $\mathcal{M}_\Pi$ on generalized (deformed) Verma modules in terms of Emerton's parabolic Jacquet-module.

Let $I\subset\Delta$ be a subset of simple roots. Let $\lambda\in X^*(\underline{T})_I^+$ be an algebraic character dominant with respect to $\mathfrak{p}_I$. Recall that, by [Eme06a, §3.4], the L-representation $J_{P_I}(\Pi^{\mathrm{la}})$ of L_I is locally analytic. Following [Wu, §5.2], we define

$$\begin{aligned}J_{P_I}(\Pi^{\mathrm{la}})_\lambda&:=\mathrm{Hom}_{U(\mathfrak{l}_I^{\mathrm{ss}})}(L_I(\lambda),J_{P_I}(\Pi^{\mathrm{la}}))\otimes_L L_I(\lambda)\\J_{I,\lambda}(\Pi^{\mathrm{la}})&:=J_{B\cap L_I}(J_{P_I}(\Pi^{\mathrm{la}})_\lambda).\end{aligned}$$

Similarly to Lemma 3.16 we have the following finiteness result:

Proposition 3.19. *The $R^{\mathrm{rig}}\widehat{\otimes}_L\mathcal{O}(\widehat{T})$-module $J_{I,\lambda}(\Pi^{\mathrm{la}})'$ is coadmissible.*

Proof. This is a consequence of [Wu, Lemm. 5.1 & 5.2]. □

By the above proposition there is a coherent sheaf $\mathcal{M}_\Pi^{I,\lambda}$ on $\mathrm{Spf}(R)^{\mathrm{rig}} \times \widehat{T}$ such that

$$\Gamma(\mathrm{Spf}(R)^{\mathrm{rig}} \times \widehat{T}, \mathcal{M}_\Pi^{I,\lambda}) = J_{I,\lambda}(\Pi^{\mathrm{la}})'.$$

For $k \geqslant 1$, let $\widehat{T}_k^{\mathrm{sm}}$ be the k-th infinitesimal neighborhood of the closed subspace $\widehat{T}^{\mathrm{sm}}$ of smooth characters in $\widehat{T}$ and let i_k be the closed immersion of $\widehat{T}_k^{\mathrm{sm}}$ in $\widehat{T}$. Moreover, for $\lambda \in X^*(T) \subset \widehat{T}$, we write $t_\lambda : \widehat{T} \longrightarrow \widehat{T}$ for the map defined by $t_\lambda(\delta) = \delta\lambda$.

Proposition 3.20. *Let $\lambda \in X^*(\underline{T})_I^+$ be an algebraic character of $\underline{T}$ dominant with respect to $\underline{P}_I$ and let $M = \widetilde{M}_I(\lambda) \otimes_{A_I} A_I/\mathfrak{m}_I^k \in \mathcal{O}_{\mathrm{alg}}^\infty$. Then there is an isomorphism of coherent sheaves on* $\mathrm{Spf}(R)^{\mathrm{rig}} \times \widehat{T}$*:*

$$\mathcal{M}_\Pi(M) \simeq i_{k,*} i_k^* t_\lambda^* \mathcal{M}_\Pi^{I,\lambda}.$$

Proof. Using Lemma 3.8 and the left exactness of J_{P_I}, there are R^{rig}-equivariant isomorphisms of locally analytic representations of L_I:

$$\begin{aligned} J_{P_I}((\Pi^{\mathfrak{n}_I} \otimes_L L_I(\lambda)')^{U(\mathfrak{l}_I^{\mathrm{ss}})}[\mathfrak{m}_I^k]) &\simeq (J_{P_I}(\Pi^{\mathfrak{n}_I}) \otimes_L L_I(\lambda)')^{U(\mathfrak{l}_I^{\mathrm{ss}})}[\mathfrak{m}_I^k] \\ &\simeq \mathrm{Hom}_{U(\mathfrak{l}_I)}(L_I(\lambda) \otimes_L A_I/\mathfrak{m}_I^k, J_{P_I}(\Pi^{\mathrm{la}})). \end{aligned}$$

Note that as $\mathfrak{n}_I$ acts trivially on $(\Pi^{\mathfrak{n}_I} \otimes_L L_I(\lambda)')^{U(\mathfrak{l}_I^{\mathrm{ss}})}[\mathfrak{m}_I^k])$, this is a P_I-representation. Therefore

$$\begin{aligned} &\mathrm{Hom}_{U(\mathfrak{t})}(\lambda \otimes_L A_I/\mathfrak{m}_I^k, J_{B\cap L_I}(J_{P_I}(\Pi^{\mathrm{la}})_\lambda)) \\ &\simeq J_{B\cap L_I}(\mathrm{Hom}_{U(\mathfrak{t})}(A_I/\mathfrak{m}_I^k, \mathrm{Hom}_{U(\mathfrak{l}_I^{\mathrm{ss}})}(L_I(\lambda), J_{P_I}(\Pi^{\mathrm{la}})))) \\ &= J_{B\cap L_I}(\mathrm{Hom}_{U(\mathfrak{l}_I)}(L_I(\lambda) \otimes_L A_I/\mathfrak{m}_I^k, J_{P_I}(\Pi^{\mathrm{la}}))) \\ &\simeq J_{B\cap L_I}(J_{P_I}((\Pi^{\mathfrak{n}_I} \otimes_L L_I(\lambda)')^{U(\mathfrak{l}_I^{\mathrm{ss}})}[\mathfrak{m}_I^k])) \\ &\simeq J_B((\Pi^{\mathfrak{n}_I} \otimes_L L_I(\lambda)')^{U(\mathfrak{l}_I^{\mathrm{ss}})}[\mathfrak{m}_I^k]) \\ &\simeq J_B(\mathrm{Hom}_{U(\mathfrak{g})}(\widetilde{M}_I(\lambda) \otimes_{A_I} A_I/\mathfrak{m}_I^k, \Pi^{\mathrm{la}})) \end{aligned}$$

where the first isomorphism comes from [Wu, Lemm. 5.3]. The claim now follows form the fact that the source of this chain of isomorphisms is the dual (of the global sections) of $i_{k,*} i_k^* t_\lambda^* \mathcal{M}_\Pi^{I,\lambda}$ and the target is the dual of $\mathcal{M}_\Pi(M)$. □

4 Quasi-trianguline local deformation rings

Let F be a finite extension of $\mathbb{Q}$. We keep the notation of section 3 but we specialize ourselves to the case $\underline{G} = \mathrm{Res}_{(F\otimes_{\mathbb{Q}}\mathbb{Q}_p)/\mathbb{Q}_p}(\mathrm{GL}_{n,F\otimes_{\mathbb{Q}}\mathbb{Q}_p}) \simeq \prod_{v|p} \mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_{n,F_v}$. We fix $\underline{B}$ the upper triangular Borel subgroup and $\underline{T}$ the diagonal torus. It is therefore sufficient to choose L a finite extension of $\mathbb{Q}_p$ splitting all the F_v. We point out that, though the field L of coefficients is the same as in the preceding section, the group $\underline{G}$ in this section should be considered as the Langlands dual group of the group in section 3.

Let Σ_F be the set of embeddings of F in L. This set can be decomposed as $\Sigma_F = \coprod_{v|p} \Sigma_{F_v}$, where Σ_{F_v} is the set of $\mathbb{Q}_p$-linear embeddings of F_v into L and where the index set is the set of places v of F that divide p. We have a decomposition

$$\mathfrak{g} \simeq \Big(\bigoplus_{\tau\in\Sigma_F} \mathrm{Lie}(\underline{G}) \otimes_{F\otimes_{\mathbb{Q}}\mathbb{Q}_p,\tau} L\Big) \simeq \bigoplus_{\tau\in\Sigma_F} \mathrm{Lie}(\mathrm{GL}_{n,L}).$$

Let Δ be the set of simple roots of $\underline{G}_L$ with respect to $\underline{B}_L$. Then

$$\Delta = \coprod_{\tau\in\Sigma_F} \Delta_\tau, \quad \Delta_\tau = \{\alpha_{1,\tau}, \dots, \alpha_{n-1,\tau}\}$$

where $\alpha_{1,\tau}, \dots, \alpha_{n-1,\tau}$ are the simple roots of the copy of $\mathrm{Lie}(\mathrm{GL}_{n,L})$ corresponding to τ. For $I \subset \Delta$ we denote $\underline{P}_I$ the standard parabolic subgroup of $\underline{G}_L$ corresponding to I.

4.1 Local models

Let $\widetilde{\mathfrak{g}} \coloneqq \underline{G}_L \times^{\underline{B}_L} \mathfrak{b}$ be the Grothendieck–Springer resolution of $\mathfrak{g}$ (which is considered as a scheme over L not just as a vector space in this section). We have a closed embedding $\widetilde{\mathfrak{g}} \hookrightarrow \underline{G}_L/\underline{B} \times \mathfrak{g}$ given by $(g\underline{B}, X) \mapsto (b\underline{B}, \mathrm{Ad}(g)X)$ and set

$$X \coloneqq \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} \subset \underline{G}_L/\underline{B}_L \times \mathfrak{g} \times \underline{G}_L/\underline{B}_L.$$

More generally if $I \subset \Delta$, we set

$$\widetilde{\mathfrak{g}}^0_{\mathfrak{p}_I} \coloneqq \underline{G}_L \times^{\underline{P}_I} (\mathfrak{z}_I \oplus \mathfrak{n}_I)$$

where we recall that $\underline{P}_I$ is the parabolic subgroup of $\underline{G}$ associated to Δ and $\mathfrak{p}_I$ is its Lie algebra. Moreover, we write $\mathfrak{z}_I$ for the center of $\mathfrak{p}_I$ and $\mathfrak{n}_I$ for its unipotent radical. Again we consider all these L-vector spaces as L-schemes. We have also a closed embedding $\widetilde{\mathfrak{g}}^0_{\mathfrak{p}_I} \hookrightarrow \underline{G}_L/\underline{P}_I \times \mathfrak{g}$ given by $(g\underline{P}_I, X) \mapsto (g\underline{P}_I, \mathrm{Ad}(g)X)$ and we set

$$X_{\mathfrak{p}_I} \coloneqq \widetilde{\mathfrak{g}}^0_{\mathfrak{p}_I} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} \hookrightarrow \underline{G}_L/\underline{P}_I \times \mathfrak{g} \times \underline{G}_L/\underline{B}_L.$$

In particular we have $X_{\mathfrak{b}} = X$. The scheme $X_{\mathfrak{p}_I}$ decomposes into irreducible components as follows:

$$X_{\mathfrak{p}_I} = \bigcup_{w\in W_I\backslash W} X_{\mathfrak{p}_I,w} \subset \underline{G}_L/\underline{P}_I \times \mathfrak{g} \times \underline{G}_L/\underline{B}.$$

Here $X_{\mathfrak{p}_I,w}$ is the closure of on open subset $V_{\mathfrak{p}_I,w} \subset X_{\mathfrak{p}_I}$, which is by definition the preimage of the $\underline{G}$-orbit of $\underline{G}\cdot(1,\widetilde{w}) \subset \underline{G}/\underline{P}_I \times \underline{G}/\underline{B}$, where $\widetilde{w} \in W$ is a lift of $w \in W_I\backslash W$ (see [BD, Cor. 5.2.2] for details). In this paper we need to control the singularities of $X_{\mathfrak{p}_I}$. Even though, for our purpose, the result of [BHS19, Rk. 4.1.6] would be sufficient, we mention the following more general result.

Proposition 4.1. *Let $w \in W$. Then X_w is smooth if, and only if, w is a product of distinct simple reflections.*

Proof. We note that the natural action

$$t \cdot (g\underline{B}, h\underline{B}, N) = (g\underline{B}, h\underline{B}, tN)$$

of $\mathbb{G}_m$ on X by scaling on the $\mathfrak{g}$-factor extends to an action of the monoid $\mathbb{A}^1$. This action obviously preserves each X_w. As the singular locus is closed and stable under the action of $\mathbb{G}_m$, the singular locus, if non-empty, contains a point of the form $(g\underline{B}, h\underline{B}, 0)$. Namely let $x \in X_w(L')$ be a point of the singular locus with coefficient in a finite extension L' of L. The $\mathbb{G}_m$-action on X_w induces a map $\mathbb{G}_{m,L'} \to X_{w,L'}$. The composite $\mathbb{G}_{m,L'} \to X_{w,L'} \to (\underline{G}_{L'}/\underline{B}_{L'})^2$ extends to a map $\mathbb{A}^1_{L'} \to (\underline{G}_{L'}/\underline{B}_{L'})^2$ by the valuative criterion of properness and the composite $\mathbb{G}_{m,L'} \to X_{w,L'} \to \mathfrak{g}_{L'}$ extends to a map $\mathbb{A}^1_{L'} \to \mathfrak{g}_{L'}$ by the L'-linear structure on $\mathfrak{g}_{L'}$. As X_w is closed in $(\underline{G}_L/\underline{B}_L)^2 \times \mathfrak{g}$, the map $\mathbb{G}_{m,L'} \to X_{w,L'}$ extends uniquely to $\mathbb{A}^1_{L'}$ and the image of 0 gives us an L'-point of the singular locus of X_w of the desired form. We will thus prove the previous proposition using [BHS19] Proposition 2.5.3 (ii).

We first assume that w is a product of distinct simple reflections. In this case it is enough to prove that

a) $\overline{U}_w$ is smooth in $\underline{G}_L/\underline{B} \times \underline{G}_L/\underline{B}$;

b) $\mathfrak{t}^{ww'^{-1}}$ has codimension $\lg(w) - \lg(w')$ in $\mathfrak{t}$ for all $w' \leqslant w$ for Bruhat ordering (with lg the Bruhat length).

By Fan's Theorem [BL00, Theorem 7.2.14], if w is a product of distinct simples reflections, then $\overline{U}_w$ is smooth and a) is true. Thus we only need to prove b). For $w \in W$, let us introduce

$$\ell(w) \coloneqq \min\{k \geqslant 0 \mid w = r_1 \dots r_k, r_k \in W \text{ a reflection}\}$$

(we recall that reflection is an element of the form s_α where $\alpha \in \Phi$ is a root, but not necessarily a simple root). By [Car72, Lemma 2] and [BHS17a, Lemma 2.7] we have $\ell(w) = \dim_L \mathfrak{t} - \dim_L \mathfrak{t}^w = d_w$ (in the notations of [BHS17a]).

Claim 4.2. If w is a product of distinct simple reflections, we have

$$\ell(ww'^{-1}) = \ell(w) - \ell(w') = \lg(w) - \lg(w')$$

for all $w' \leqslant w$.

If Claim 4.2 is true, we have $\ell(ww'^{-1}) = \dim \mathfrak{t} - \dim \mathfrak{t}^{ww'^{-1}} = \lg(w) - \lg(w')$ thus Proposition 2.5.3 of [BHS19] applies and X_w is smooth. We now prove the claim. The second equality of the claim is a consequence of [Car72, Lemma 3] as w and w' are products of pairwise distinct simple reflections. Indeed, a product of pairwise distinct simple reflections $s_1 \dots s_k$ is always a composition of reflections s_i along vectors v_i such that $v_1, \dots, v_k$ are linearly independent. Thus [Car72, Lemma 3] implies $\ell(w) = \lg(w)$ and $\ell(w') = \lg(w')$.

We write $w' = s_{i_1} \dots s_{i_k}$ and $w = t_1 \dots t_b$ as reduced expressions of pairwise distinct simple roots such that there exists $a_1 \leqslant \dots \leqslant a_k$ satisfying $t_{a_j} = s_{i_j}$. For $a_t \leqslant j < a_{t+1}$ let r_j denote the reflection $r_j \coloneqq s_{i_1} \dots s_{i_t} t_j s_{i_t} \dots s_{i_1}$. We then have

$$\begin{aligned} ww'^{-1} =& t_1 \dots t_b s_{i_k} \dots s_{i_1} \\ =& t_1 \dots t_{a_1-1} \underbrace{[s_{i_1} t_{a_1+1} s_{i_1}]}_{r_{a_1+1}} \dots \underbrace{[s_{i_1} t_{a_2-1} s_{i_1}]}_{r_{a_2-1}} \underbrace{[s_{i_1} s_{i_2} t_{a_2+1} s_{i_2} s_{i_1}]}_{r_{a_2+1}} \\ & \dots [s_{i_1} \dots s_{i_k} t_{a_k+1} s_{i_k} \dots s_{i_1}] \dots \underbrace{[s_{i_1} \dots s_{i_k} t_b s_{i_k} \dots s_{i_1}]}_{r_b} \\ =& t_1 \dots t_{a_1-1} r_{a_1+1} \dots r_{a_2-1} r_{a_2+1} \dots \dots r_b. \end{aligned}$$

In particular, $\ell(ww'^{-1}) \leqslant \lg(w) - \lg(w') = \ell(w) - \ell(w')$. Now Claim 4.2 follows from

Claim 4.3. Let $w \in W$ and w' be a product of distinct simple reflections. Then $\ell(ww'^{-1}) \geqslant \ell(w) - \ell(w') = \ell(w) - \lg(w')$.

We now prove Claim 4.3. By induction on the number of simple reflections appearing in w', it is enough to prove $\ell(ws) \geqslant \ell(w) - 1$ when $w' = s$ is a simple reflection. Note that for any w we have $\dim_L \mathfrak{t}^{ws} \cap \mathfrak{t}^s \geqslant \dim_L \mathfrak{t}^{ws} - 1$ as $\mathfrak{t}^s$ is a hyperplane in $\mathfrak{t}$. Moreover, $\mathfrak{t}^{ws} \cap \mathfrak{t}^s = \mathfrak{t}^w \cap \mathfrak{t}^s \subset \mathfrak{t}^w$. Thus $\dim \mathfrak{t}^w \geqslant \dim \mathfrak{t}^{ws} - 1$. Using $\ell(w) = \dim \mathfrak{t} - \dim \mathfrak{t}^w$ we hence find

$$\ell(w) \leqslant \ell(ws) + 1.$$

Thus $\ell(ws) \geqslant \ell(w) - 1$, which proves Claim 4.3.

We now prove the converse, i.e. that X_w is singular, if w is *not* a product of distinct simple reflections. We hence assume that w is not a product of distinct simple reflections.

It is enough (but actually equivalent) to prove that X_w is singular at $(\underline{B}, \underline{B}, 0)$. We will use Mowlavi's results [Mow23]. The pair $(1, w)$ is a good pair ([Mow23]), and thus [Mow23, Theorem 6] applies. Hence [Mow23, Proposition 3.2.2] gives an exact formula for the tangent space at $x = (\underline{B}, \underline{B}, 0) \in (X_w \cap V_1)(L)$. This can be rewritten as

$$\begin{aligned} \dim_L T_x X_w &= \dim_L T_{\pi(x)} \overline{U_w} - d_w + \dim_L \mathfrak{t} + \lg(w_0) \\ &> \dim \underline{B} + \lg(w) - \lg(w) + \dim_L \mathfrak{t} + \lg(w_0), \end{aligned}$$

as w is not a product of distinct simples so $\lg(w) > d_w$ ([BHS17a] Lemma 2.7), and where we use the notation [1] $d_w = \dim_L \mathfrak{t} - \dim_L \mathfrak{t}^w$. Thus

$$\dim_L T_x X_w > 2 \dim \underline{B} + \dim_L \mathfrak{t} = \dim \underline{G}_L = \dim X_w,$$

i.e. X_w is not smooth at x. □

We write X_I for the inverse image of $X_{\mathfrak{p}_I}$ under the canonical projection $\underline{G}_L / \underline{B}_L \times \mathfrak{g} \times \underline{G}_L / \underline{B}_L \to \underline{G}_L / \underline{P}_I \times \mathfrak{g} \times \underline{G}_L / \underline{B}_L$. This scheme can also be defined as

$$X_I \coloneqq (\underline{G}_L \times^{\underline{B}_L} (\mathfrak{z}_I \oplus \mathfrak{n}_I)) \times_{\mathfrak{g}} \widetilde{\mathfrak{g}},$$

[1] see [BHS19] just before Proposition 4.1.5

in particular $X_\emptyset = X$. The map $X_I \to X_{\mathfrak{p}_I}$ is a $\underline{P}_I/\underline{B}_L$-torsor and thus is projective and smooth. We deduce that we have a decomposition in irreducible components

$$X_I = \bigcup_{w \in W_I \backslash W} X_{I,w},$$

where each $X_{I,w} \to X_{\mathfrak{p}_I,w}$ is projective and smooth. Moreover, we have a closed embedding $X_I \hookrightarrow X$ induced by the closed embedding $\mathfrak{z}_I \oplus \mathfrak{n}_I \hookrightarrow \mathfrak{b}$, and this induces a closed embedding $X_{I,w} \hookrightarrow X_{w^{\max}}$, as each fiber of $X_I \longrightarrow X_{\mathfrak{p}_I}$ over a point in $V_{\mathfrak{p}_I,w}$ contains a (dense) open subset consisting of points that lie in the Schubert cell $\underline{G}_L(1, w^{\max}) \subset \underline{G}/\underline{B} \times \underline{G}/\underline{B}$.

Lemma 4.4. *The schemes X_I and $X_{\mathfrak{p}_I}$ are generically reduced.*

Proof. As X_I is smooth over $X_{\mathfrak{p}_I}$, it suffices to prove the claim for $X_{\mathfrak{p}_I}$. For $w \in W$, let $U_w = \underline{G}_L(1, w) \subset \underline{G}_L/\underline{P}_I \times \underline{G}_L/\underline{B}$ and let $V_w \subset X_{\mathfrak{p}_I}$ be the inverse image of U_w. It follows from [BD, Prop. 5.2.1] that the V_w are smooth L-schemes, and they all have the same dimension. As they also cover $X_{\mathfrak{p}_I}$, their generic points are the generic points of the irreducible components of $X_{\mathfrak{p}_I}$. This shows that $X_{\mathfrak{p}_I}$ is generically reduced. □

Recall that we have two maps $\kappa_1, \kappa_2 : X \to \mathfrak{t}$ (see [BHS19, §2.3]) defined by $\kappa_i(g_1\underline{B}, N, g_2\underline{B}) = g_i^{-1} N g_i (\mathrm{mod}\ \mathfrak{n})$. By construction, the image of $\kappa_{1|X_I}$ lands in $\mathfrak{z}_I$ and the map $\kappa_{1|X_I}$ factors through $X_{\mathfrak{p}_I}$. This provides a commutative diagram

$$\begin{tikzcd} X_I \arrow[r, two heads] \arrow[dr, "\Theta_I"'] & X_{\mathfrak{p}_I} \arrow[d, "\Theta_{\mathfrak{p}_I}"] \\ & \mathfrak{z}_I \times_{\mathfrak{t}/W} \mathfrak{t} \end{tikzcd}$$

where Θ_I is the restriction of the map (κ_1, κ_2) to X_I.

The following result is the analogue of [BHS19, Lem. 2.5.1] in our context, with analogous proof.

Lemma 4.5. *The irreducible components of $\mathfrak{z}_I \times_{\mathfrak{t}/W} \mathfrak{t}$ are the $(T_{I,w})_{w \in W_I \backslash W}$ where*

$$T_{I,w} = \{(z, \mathrm{Ad}(w^{-1})(z)) \mid z \in \mathfrak{z}_I\}.$$

Moreover, the irreducible component $X_{I,w}$ (resp. $X_{\mathfrak{p}_I,w}$) is the unique component of X_I (resp. $X_{\mathfrak{p}_I}$) whose image under Θ_I (resp. $\Theta_{\mathfrak{p}_I}$) dominates $T_{I,w}$.

Remark 4.6. For future use, we make the following notational convention: When $F = \mathbb{Q}$, we have $\underline{G}_L = \mathrm{GL}_{n,L}$, we will use the notations X_n, $X_{n,I}$, $X_{n,I,w}$ etc. for the schemes X, X_I, $X_{I,w}$ etc.

4.2 Partially de Rham deformation rings

For each place $v|p$ of F, we fix $r_v : \mathrm{Gal}_{F_v} \to \mathrm{GL}_n(L)$ a framed φ-generic Hodge–Tate regular crystalline representation, that we assume that the (φ,Γ)-module $D_{\mathrm{rig}}(r_v)$ associated to r_v is crystalline φ-generic with regular Hodge–Tate type in the sense of [HMS, §3.3&§3.4]. We also fix a refinement $\mathcal{R}_v = (\varphi_1, \dots, \varphi_n) \in L^n$ of r_v (see *loc. cit.*). We will use the notation $r = (r_v)_{v|p}$ and $\mathcal{R} = (\mathcal{R}_v)_{v|p}$ and say that r is *φ-generic Hodge–Tate regular* and that $\mathcal{R}$ is a *refinement* of r.

Let $\mathcal{C}_L$ be the category of local artinian L-algebras. Fix $v|p$ a place of F. Let $\mathcal{X}_{r_v}^{\square}$ be the groupoid over $\mathcal{C}_L$ of deformations of r_v. It is represented by a formal scheme over L that we also denote by $\mathcal{X}_{r_v}^{\square}$ by abuse of notation. We recall from [BHS19, 3.6] that, given the refinement $\mathcal{R}_v$, the groupoid of trianguline deformations of $\mathcal{M}_{\bullet,v}$ is representable by a closed formal subscheme $\mathcal{X}_{r_v,\mathcal{R}_v}^{\mathrm{qtri}} \subset \mathcal{X}_{r_v}^{\square}$. Here $\mathcal{M}_{\bullet,v}$ the (φ,Γ)-module over $\mathcal{R}_{K,L}[1/t]$ obtained from $D_{\mathrm{rig}}(r_v)$ by inverting t which is equipped with the unique triangulation corresponding to the refinement $\mathcal{R}_v$. We set $W_v = W_{\mathrm{dR}}(D_{\mathrm{rig}}(r_v)[1/t])$ and $W_{\bullet,v} = W_{\mathrm{dR}}(\mathcal{M}_{\bullet,v})$ and let $X_{W_v,W_{\bullet,v}}$ denote the groupoid of deformations of $(W_v, W_{\bullet,v})$ as defined in [BHS19, §3.3].

Fix a finite subset $I_v \subset \Delta_v$. For an object A of $\mathcal{C}_L$, we define $X_{W_v,W_{\bullet,v}}^{\underline{P}_{I_v}}(A)$ to be the subset of all $(W_A, W_{A,\bullet}) \in X_{W_v,W_{\bullet_v}}(A)$ such that for any $\tau \in \Sigma_{F_v}$ and $\alpha_{i,\tau} \in \Delta_\tau \setminus I_v$, the $\mathbf{B}_{\mathrm{dR}}^+$-representation $W_{A,i} \otimes_{K,\tau} L / W_{A,j+1} \otimes_{K,\tau} L$ is de Rham, where j is the largest integer $< i$ such that $\alpha_{\tau,j} \notin I$ (and $j = 0$ if i is the smallest integer such that $\alpha_{i,\tau} \notin I$). It is obvious from the definition that $X_{W_v,W_{\bullet,v}}^{\underline{P}_I}$ is a subgroupoid of $X_{W_v,W_{\bullet,v}}$.

For an object A of $\mathcal{C}_L$ and $r_A \in X_{r_v,\mathcal{R}_v}^{\mathrm{qtri}}(A)$, we denote by $\mathcal{M}_{A,\bullet}$ the unique triangulation of $D_{\mathrm{rig}}(r_A)$ lifting $\mathcal{M}_{\bullet,v}$. We say that r_A is *$\underline{P}_I$-de Rham* if

$$(W_{\mathrm{dR}}(r_A), W_{\mathrm{dR}}(\mathcal{M}_{A,\bullet})) \in X_{W_v,W_{\bullet,v}}^{\underline{P}_{I_v}}(A)$$

(see [Wu, Def. 3.10]). It now follows from [Wu, Lemm. 3.11] that this functor is representable by a closed formal subscheme of $\mathcal{X}_{r_v,\mathcal{R}_v}^{\mathrm{qtri}}$ that we denote $\mathcal{X}_{r_v,\mathcal{R}_v}^{I_v-\mathrm{qtri}}$. More precisely, we have an isomorphism of groupoids

$$\mathcal{X}_{r_v,\mathcal{R}_v}^{I_v-\mathrm{qtri}} \simeq \mathcal{X}_{r_v,\mathcal{R}_v}^{\mathrm{qtri}} \times_{X_{W_v,W_{\bullet,v}}} X_{W_v,W_{\bullet,v}}^{\underline{P}_{I_v}}.$$

Fix an $L \otimes_{\mathbb{Q}_p} F_v$-basis α_v of $W_v^{\mathrm{Gal}_K}$ and let $X_{W_v}^{\square}$ be the groupoid of deformations of the pair (W_v, α_v). We set

$$\begin{aligned} X_{W_v,W_{\bullet,v}}^{\square} &= X_{W_v}^{\square} \times_{X_{W_v}} X_{W_v,W_{\bullet,v}} \\ \mathcal{X}_{r_v,\mathcal{R}_v}^{I_v-\mathrm{qtri},\square} &= \mathcal{X}_{r_v,\mathcal{R}_v}^{I_v-\mathrm{qtri}} \times_{X_{W_v}} X_{W_v}^{\square}. \end{aligned}$$

As the map $\mathcal{X}_{r_v,\mathcal{R}_v}^{\mathrm{qtri}} \to X_{W_v^+} \times_{X_{W_v}} X_{W_v,W_{\bullet,v}}$ is formally smooth by [BHS19, Cor. 3.5.6], we deduce that the map $\mathcal{X}_{r_v,\mathcal{R}_v}^{I_v-\mathrm{qtri},\square} \to X_{W_v^+} \times_{X_{W_v}} X_{W_v,W_{\bullet,v}}^{\underline{P}_{I_v},\square}$ is formally smooth as well.

If $I = \coprod_{v|p} I_v \subset \Delta$ and if $\alpha = (\alpha_v)_{v|p}$ is fixed, we set $\mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri}} := \prod_{v|p} \mathcal{X}_{r_v,\mathcal{R}_v}^{I_v-\mathrm{qtri}}$ and $\mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri},\square} := \prod_{v|p} \mathcal{X}_{r_v,\mathcal{R}_v}^{I_v-\mathrm{qtri},\square}$.

We consider the point

$$x_{\mathrm{pdR}} := (g\underline{B}_L, 0, h\underline{B}_L) \in X_I(L) \subset (\underline{G}_L/\underline{B}_L \times \mathfrak{g} \times \underline{G}_L/\underline{B}_L)(L), \tag{6}$$

where $g \in \underline{G}(L)$ (resp. h) is the matrix sending the standard flag (corresponding to our fixed basis α) of $\prod_{v|p} W_v^{\mathrm{Gal}_K}$ to the complete flag $\prod_{v|p} W_{\mathrm{dR}}(\mathcal{M}_{\bullet,v})^{\mathrm{Gal}_K}$ (resp. to the Hodge flag). We deduce the following result (see [BD, §6.3] in a slightly different context):

Theorem 4.7. *There exists a diagram of formal L-schemes with formally smooth maps*

$$\mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri}} \xleftarrow{\ g\ } \mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri},\square} \xrightarrow{\ f\ } \widehat{X}_{I,x_{\mathrm{pdR}}}$$

Proof. Let $I = \coprod_{v\in S_p} I_v$, with $I_v \subset \Delta_v$ for $v \in S_p$. Note that we have a decomposition $X_I \simeq \prod_{v\in S_p} X_{I_v}$ where X_{I_v} is the L-scheme defined in the same way as X_I but for the group $\mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_{n,F_v}$. We also write $x_{\mathrm{pdR}} = (x_{\mathrm{pdR},v})_{v\in S_p}$ where $x_{\mathrm{pdR},v}$ is the image of x_{pdR} in X_{I_v}. We just have to check that the groupoid

$$X_{W_v^+} \times_{X_{W_v}} X_{W_v,W_{\bullet,v}}^{\underline{P}_{I_v}} \times_{X_{W_v}} X_{W_v}^{\square}$$

is represented by the completion of X_{I_v} at $x_{\mathrm{pdR},v}$. This can be checked easily as in the proof of [Wu, Lemm. 3.11] using [BHS19, Cor. 3.1.9 &Thm. 3.2.5]. $\square$

We finally note that the map κ_1 from above induces a map of formal schemes $\kappa_1 : \widehat{X}_{I,x_{\mathrm{pdR}}} \to \widehat{\mathfrak{z}_I}$, where $\widehat{\mathfrak{z}_I}$ is the completion of $\mathfrak{z}_I$ at 0, and thus a map

$$\mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri},\square} \to \widehat{\mathfrak{z}_I}.$$

This maps factors into a map of formal schemes $\kappa_1 : \mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri}} \longrightarrow \widehat{\mathfrak{z}_I}$.

For $w \in W$ such that $x_{\mathrm{pdR}} \in X_{I,w}(L)$, we denote by $\mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri},w}$ the schematic image of

$$\mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri},\square} \times_{\widehat{X}_{I,x_{\mathrm{pdR}}}} \widehat{X}_{I,w,x_{\mathrm{pdR}}} \to \mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri}}$$

and by $\overline{\mathcal{X}}_{r,\mathcal{R}}^{\mathrm{qtri}}$ (resp. $\overline{\mathcal{X}}_{r,\mathcal{R}}^{I-\mathrm{qtri},w}$) the schematic inverse image of $\{0\}$ under κ_1 in $\mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri}}$ (resp. $\mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri},w}$).

The schemes $\mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri}}$ and $\mathcal{X}_{r,\mathcal{R}}^{I-\mathrm{qtri},w}$ are formal spectra of complete local noetherian rings that we denote by $R_{r,\mathcal{R}}^{I-\mathrm{qtri}}$ and $R_{r,\mathcal{R}}^{I-\mathrm{qtri},w}$. It follows from the constructions that moreover $R_{r,\mathcal{R}}^{I-\mathrm{qtri},w}$ is an integral local ring.

5 Global construction

Let F be a totally real number field and let E/F be a totally imaginary CM extension of number fields, in particular $[E:F]=2$. We assume that all places of F dividing p are unramified and split in E/F and denote by S_p the set of places above p in F. We fix a set Σ of places of E dividing p such that, for each place $v\in S_p$, there is exactly one place of Σ above v. Let U be a unitary group in n variables for E/F that we regard, via Weil restriction, as an algebraic group over $\mathbb{Q}$. We assume that $U(\mathbb{R})$ is compact and and that $U_{\mathbb{Q}_p}$ is quasi-split. This implies in particular that there exists an isomorphism $U_{\mathbb{Q}_p}\simeq\prod_{v\in S_p}\mathrm{Res}_{F_v/\mathbb{Q}_p}\mathrm{GL}_{n,F_v}$ that we fix from now on. From now we note $\underline{G}=U_{\mathbb{Q}_p}$ identified with $\prod_{v\in S_p}\mathrm{Res}_{F_v/\mathbb{Q}_p}\mathrm{GL}_{n,F_v}$ via this fixed isomorphism and we use notations of section 3, i.e. L is the choice of a field of coefficients that is assumed to be big enough so that all embeddings of E (equivalently of F) in $\overline{\mathbb{Q}}_p$ factor through L. Moreover, $\underline{B}\subset\underline{G}_L$ is the Borel subgroup of upper triangular matrices, $\underline{T}\subset\underline{B}$ is the maximal torus of diagonal matrices, $\underline{N}$ is the unipotent radical of $\underline{B}$ etc.

5.1 Classical and p-adic automorphic forms

We write $T=\underline{T}(\mathbb{Q}_p)\simeq\left(\prod_{v\in S_p}F_v^\times\right)^n$ and let $T_0\simeq\left(\prod_{v\in S_p}\mathcal{O}_{F_v}^\times\right)^n\subset T$ denote its maximal compact subgroup. We denote by $\widehat{T}$ (resp. $\widehat{T}_0$) the rigid analytic spaces over L parametrizing the continuous characters of T (resp. of T_0) and recall from 5 that there is a weight map

$$\mathrm{wt}:\widehat{T}_0\to\mathfrak{t}^*$$

with values in the dual Lie algebra $\mathfrak{t}^*$ of $\underline{T}$ (considered as a rigid space over L). We will often, by abuse of notation, also write wt for the composition of wt with the canonical projection $\widehat{T}\to\widehat{T}_0$. Recall that we had identified $X^*(\underline{T})$ with a $\mathbb{Z}$-lattice in $\mathfrak{t}^*$. Often we will identify $X^*(\underline{T})$ with $\mathbb{Z}^{n[F:\mathbb{Q}]}$.

Definition 5.1. Let $\delta\in\widehat{T}$ (resp. $\in\widehat{T_0}$) be a character.
(i) The *weight* of δ is the image $\mathrm{wt}(\delta)$ under the weight map.
(ii) The character δ is called *of algebraic weight* if $\mathrm{wt}(\delta)\in X^*(\underline{T})\subset\mathfrak{t}^*$.
(iii) The character δ is called *algebraic* if it is of the form

$$\delta_{\underline{k}}:(z_1\otimes 1,\dots,z_n\otimes 1)\longmapsto\prod_\tau\left(\tau(z_1)^{k_1^\tau}\cdots\tau(z_n)^{k_n^\tau}\right)$$

for some $\underline{k}=(k_1^\tau,\dots,k_n^\tau)_{\tau:F\hookrightarrow L}\in\mathbb{Z}^{n[F:\mathbb{Q}]}$. It is called *dominant algebraic* if $\underline{k}\in X^*(\underline{T})^+$, i.e. if $k_1^\tau\geqslant\ldots\geqslant k_n^\tau$ for all τ.

Note that $\underline{k}\mapsto\delta_{\underline{k}}$ defines a section of the weight map over the algebraic weights, and we use this map to identify $X^*(\underline{T})$ with a subset of $\widehat{T}$ (resp. $\widehat{T}_0$).

Let $K^p\subset U(\mathbb{A}^{\infty,p})$ be a compact open subgroup, called a *tame level* that we assume to be of the form $\prod_{\ell\neq p}K_\ell$ where K_ℓ is a compact open subgroup of $U(\mathbb{Q}_\ell)$. Let I_p be the Iwahori subgroup of $G = \underline{G}(\mathbb{Q}_p) = U(\mathbb{Q}_p)$ with respect to our choice of $\underline{B}$. For any compact open $K_p \subset U(\mathbb{Q}_p)$ we consider the Shimura set

$$Sh_{K^pK_p} \coloneqq U(\mathbb{Q})\backslash U(\mathbb{A}^\infty)/K^pK_p.$$

As $U(\mathbb{R})$ is compact, this is indeed a finite set of points.

Definition 5.2. The *completed cohomology* of the tower $(Sh_{K^pK_p})_{K_p\subset U(\mathbb{Q}_p)}$ of Shimura sets is:

$$\Pi \coloneqq \Pi^\circ \otimes_{\mathcal{O}_L} L, \quad \text{with} \quad \Pi^\circ \coloneqq \varprojlim_n \varinjlim_{K_p} H^0(Sh_{K^pK_p}, \mathcal{O}_L/\pi_L^n),$$

see [Eme06b].

The completed cohomology is an L-Banach space endowed with a continuous action of $U(\mathbb{Q}_p)$. This space is naturally identified with the space of continuous functions

$$f : U(\mathbb{Q})\backslash U(\mathbb{A}^\infty)/K^p \longrightarrow L. \tag{7}$$

We denote Π^{la} the subspace of locally analytic vectors in Π for $U(\mathbb{Q}_p)$. This is the subspace of functions in (7) which are locally analytic. As Π^{la} is a locally analytic representation, there is a natural $U(\mathfrak{g})$-action on Π^{la} obtained by deriving the $G = \underline{G}(\mathbb{Q}_p)$-action. Here, as above, we write $\mathfrak{g}$ for the Lie algebra of G, and $\mathfrak{b}, \mathfrak{t}, \mathfrak{n}$ for the Lie algebras of the Borel B, of the torus T and of the unipotent radical N of B.

Definition 5.3. The space of *overconvergent p-adic automorphic forms of tame weight* K^p is the space

$$S^\dagger(K^p) = (\Pi^{\mathrm{la}})^{\mathfrak{n}} = \varinjlim_{N_0\subset \underline{N}(\mathbb{Q}_p)} (\Pi^{\mathrm{la}})^{N_0},$$

where N_0 varies among the compact open subgroups of $\underline{N}(\mathbb{Q}_p)$. Given a weight $\kappa \in \mathfrak{t}^*$, the space of overconvergent p-adic automorphic forms of tame weight K^p and *weight* κ is the eigenspace

$$S^\dagger_\kappa(K^p) \subset S^\dagger(K^p)$$

of eigenvalue κ for the $U(\mathfrak{t})$-action.

Denote by $\mathbb{T}(K^p) \coloneqq \mathbb{Z}[K^p\backslash U(\mathbb{A}^{\infty,p})/K^p]$ the Hecke algebra of Hecke operators over $\mathbb{Z}$ of tame level K^p. Then $\mathbb{T}(K^p)$ acts by convolution on $S^\dagger(K^p)$ and $S^\dagger_\kappa(K^p)$. Let S be a finite set of prime numbers containing p and all the ℓ such that K_ℓ is not hyperspecial. The subalgebra $\mathbb{T}^S \coloneqq \bigotimes_{\ell\notin S} \mathbb{T}_\ell \subset \mathbb{T}(K^p)$ is commutative.

Definition 5.4. Let

$$\underline{T}(\mathbb{Q}_p)^+ \coloneqq \{\mathrm{diag}(a_1^v, \ldots, a_n^v)_v \in \underline{T}(\mathbb{Q}_p) \mid v(a_1^v) \geqslant \ldots \geqslant v(a_n^v),\ \forall v \in S_p\}.$$

The *Atkin–Lehner ring* $\mathcal{A}(p)$ is the sub-algebra of $\mathbb{Z}[\underline{T}(\mathbb{Q}_p)]$ generated by the elements $t \in \underline{T}(\mathbb{Q}_p)^+$.

Let $\delta : T \to L^\times$ be a continuous character. Then we can extend δ to a character $\mathcal{A}(p) \to L$ whose restriction to T^+ is given by δ. By abuse of notation we still write δ for this character of $\mathcal{A}(p)$.

Note that there is a cofinal system of compact open subgroups $N_0 \subset N = \underline{N}(\mathbb{Q}_p)$ such that $tN_0t^{-1} \subset N_0$ for all $t \in T^+$. We hence can define a Hecke action of $\mathcal{A}(p)$ on $S^\dagger(K^p) = (\Pi^{\mathrm{la}})^{\mathfrak{n}}$ by letting $t \in \underline{T}(\mathbb{Q}_p)^+$ act on $f \in (\Pi^{\mathrm{la}})^{N_0}$ via

$$[t]f := \left(x \mapsto \frac{1}{[N_0 : tN_0t^{-1}]} \sum_{n \in N_0/tN_0t^{-1}} f(xnt) \right),$$

where N_0 is a sufficiently small compact open subgroup of N such that $f \in (\Pi^{\mathrm{la}})^{N_0}$ and such that $tN_0t^{-1} \subset N_0$.

Let $\mathbb{T}$ be the commutative algebra $\mathbb{T}^S \otimes_{\mathbb{Z}} \mathcal{A}(p)$. Definition 5.4 provides a structure of $\mathbb{T}$-module on $S^\dagger(K^p)$ and $S^\dagger_\kappa(K^p)$.

Definition 5.5. An overconvergent p-adic automorphic form $f \in S^\dagger(K^p) = (\Pi^{\mathrm{la}})^{\mathfrak{n}}$ is called a *finite slope eigenvector* for the $\mathcal{A}(p)$-action if, for any $t \in \underline{T}(\mathbb{Q}_p)^+$, there exists $a_t \in L^\times$ such that

$$[t]f = a_t f.$$

More generally f is of *finite slope* for the $\mathcal{A}(p)$-action if for all $t \in \underline{T}(\mathbb{Q}_p)^+$, there exists a polynomial $P \in L[X]$ such that $P(0) \neq 0$ and $P([t])f = 0$.

Given a continuous character $\delta : T \to L^\times$, we write $S^\dagger(K^p)[\delta]$ for the eigenspace with respect to the $\mathcal{A}(p)$-action of eigensystem $\delta : \mathcal{A}(p) \to L$. Note, that by definition this eigensystem is automatically of finite slope and of weight $\kappa = \mathrm{wt}(\delta)$. Moreover, the $\mathcal{A}(p)$-action on $S^\dagger(K^p)[\delta]$ uniquely extends to an action of $\mathbb{Z}[T(\mathbb{Q}_p)]$.

Remark 5.6. An overconvergent automorphic form of tame level K^p with eigenvalue $\delta : T \to L^\times$ for the Hecke-action at p (i.e. for the action of the Atkin–Lehner ring) is thus the same as a locally analytic function

$$f : U(\mathbb{Q}) \backslash U(\mathbb{A}^\infty)/K^p \longrightarrow L,$$

such that there exists a compact open subgroup $N_0 \subset \underline{N}(\mathbb{Q}_p)$ so that, for all $g \in U(\mathbb{A}^\infty), t \in T_0, n \in N_0$,

$$f(gtn) = \delta(t)f(g),$$

and such that moreover, for all $t \in \underline{T}(\mathbb{Q}_p)^+$, $[t]f = \delta(t)f$.

Definition 5.7. The space of *classical automorphic forms of tame level* K^p is the subspace $S^{\mathrm{cl}}(K^p) = (\Pi^{\mathrm{cl}})^{\mathfrak{n}}$ of $S^\dagger(K^p) = (\Pi^{\mathrm{la}})^{\mathfrak{n}}$ of elements which are K_p-finite for some (resp. any) compact open $K_p \subset U(\mathbb{Q}_p)$.

We note that this subspace is stable under the action of $\mathbb{T}$.

For any character $\chi^S : \mathbb{T}^S \longrightarrow L$, we let $\Pi[\chi^S]$ (resp. $S^\dagger(K^p)[\chi^S]$, resp. $S^{\mathrm{cl}}(K^p)[\chi^S]$) denote the subspace of χ^S-eigenvectors for $\mathbb{T}^S$ in Π (resp. $S^\dagger(K^p)$, resp. $S^{\mathrm{cl}}(K^p)$). If $\delta : T \to L$ is a character of T (defining a character of $\mathcal{A}(p)$) and if $\chi = \chi^S \otimes \delta$ is the corresponding character of $\mathbb{T} = \mathbb{T}^S \otimes_{\mathbb{Z}} \mathcal{A}(p)$, we write $S^\dagger(K^p)[\chi]$ etc. for the corresponding eigenspace.

Let $\mathfrak{m}$ be a maximal ideal in $\mathbb{T}^S$. We then define

$$\Pi_{\mathfrak{m}} \coloneqq \Pi^\circ_{\mathfrak{m}} \otimes_{\mathcal{O}_L} L, \quad \text{where} \quad \Pi^\circ_{\mathfrak{m}} \coloneqq \varprojlim_n (\Pi^\circ/\pi_L^n \Pi^\circ)_{\mathfrak{m}}.$$

As there are only finitely many maximal ideals $\mathfrak{m}$ of $\mathbb{T}^S$ such that $(\Pi^\circ/\pi_L\Pi^\circ)_{\mathfrak{m}}$ is nonzero, the space $\Pi_{\mathfrak{m}}$ is a topological direct summand of Π stable under the actions of $U(\mathbb{Q}_p)$ and $\mathbb{T}$.

Recall that if $\mathfrak{m}$ is a maximal ideal (whose residue field is assumed to equal k_L) such that $\Pi_{\mathfrak{m}}$ is non zero, then we may associate to $\mathfrak{m}$ a continuous representation $\overline{\rho} : \mathrm{Gal}_E \longrightarrow \mathrm{GL}_n(k_L)$ which is conjugate autodual, and unramified away from S. Such representations $\overline{\rho}$ are called modular (see for example [BHS17b, §2.4]).

5.2 Patching the completed cohomology

We fix a maximal ideal $\mathfrak{m} \subset \mathbb{T}^S$ such that $\Pi_{\mathfrak{m}} \neq 0$ is non-zero and denote by $\overline{\rho} : \mathrm{Gal}_E \longrightarrow \mathrm{GL}_n(k_L)$ the corresponding modular Galois representation. For each place v of F which splits in E we write

$$\overline{\rho}_v \coloneqq \overline{\rho}_{|\,\mathrm{Gal}_{E_{\widetilde{v}}}},$$

for a choice of $\widetilde{v}|v$ of E. From now on we assume that, for $v \in S$, the place v splits in E/F, we make a fixed choice $\widetilde{v}|v$ as before such that $\widetilde{v} \in \Sigma$ if $v|\ p$, and denote $\widetilde{S} = \{\widetilde{v}|v \in S\}$ so that $\widetilde{S}$ is in bijection with S and contains Σ. For $v \in S$ we write $R^\square_{\overline{\rho}_v}$ for the universal lifting (i.e. framed deformation) ring of $\overline{\rho}_v$ and define

$$R^\square_{\overline{\rho}_v} \twoheadrightarrow \overline{R}^\square_{\overline{\rho}_v}$$

to be the maximal reduced $\mathbb{Z}_p$-flat quotient.

Remark 5.8. If $v|p$ we have in fact, by the main results of [BIP23], $\overline{R}^\square_{\overline{\rho}_v} = R^\square_{\overline{\rho}_v}$. Using the main result of [DHKM24] we find that the same applies to places $v \nmid p$, as the deformation rings $R^\square_{\overline{\rho}_v}$ may be identified with versal rings to the moduli space of L-parameters. We still keep the notations introduced above in order to be consistent with the notations from the references for the patching construction below.

We denote by $R_{\overline{\rho},\mathcal{S}}$ the quotient of $R_{\overline{\rho}}$ corresponding to the deformation problem

$$\mathcal{S} = (E/F, S, \widetilde{S}, \mathcal{O}_L, \overline{\rho}, \varepsilon^{1-n}\delta^n_{E/F}, \{\overline{R}^\square_{\overline{\rho}_v}\}_{v\in S})$$

in the notations of [CHT08, §2.3], where $\delta_{E/F} : \mathrm{Gal}_F \longrightarrow \{\pm 1\}$ is the quadratic character associated to E/F, and

$$R^{\mathrm{loc}} \coloneqq \widehat{\bigotimes_{v \in S}} \overline{R}^{\square}_{\overline{\rho}_v}.$$

There is an action of $R_{\overline{\rho},\mathcal{S}}$ on $\Pi_{\mathfrak{m}}$ by continuous $\mathbb{T}^S$-linear endomorphisms (see [BHS17b, §2.4]). If $\chi^S : \mathbb{T}^S_{\mathfrak{m}} \to L$ is a character such that $\Pi[\chi^S] \neq 0$, then the action of $R_{\overline{\rho},\mathcal{S}}$ on $\Pi[\chi^S]$ factors through the character $R_{\overline{\rho},\mathcal{S}} \to L$ corresponding the unique lift $\rho : \mathrm{Gal}_E \to \mathrm{GL}_n(L)$ of $\overline{\rho}$ which is said to be *associated to* χ^S.

Remark 5.9. If π is an automorphic representation of $U(\mathbb{A}_F)$ such that $\pi^{K^p} \neq 0$. Let $\psi^S : \mathbb{T}^S \to \mathbb{C}$ be the character of $\mathbb{T}^S$ on π^{K^p}. If we fix an isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}_p}$ and if L is big enough so that $\chi^S = \iota \circ \psi^S$ takes values in L and $\mathrm{Ker}(\chi^S) \subset \mathfrak{m}$, then $\Pi^{\mathrm{cl}}_{\mathfrak{m}}[\chi^S] \neq 0$ and ρ is the Galois representation associated to π and ι.

We assume the following (strong) Taylor–Wiles hypothesis on p, E/F, U, K^p and $\overline{\rho}$.

Hypothesis 5.10. 1. $p > 2$;

2. the extension E/F is unramified and E does not contain a (non-trivial) p-th root ζ_p of 1 ;

3. the group U is quasi-split at all finite places of $\mathbb{Q}$;

4. the level K^p is chosen such that K_v is hyperspecial whenever the finite place v of F is inert in E ;

5. the representation $\overline{\rho}_{|\,\mathrm{Gal}_{E(\zeta_p)}}$ is adequate.

By [CEG$^+$16] sections 2.7,2.8, (see also [BHS17b, Théorème 3.5]), we have the following data.

Proposition 5.11. *There exist*

1. *an integer* $g \geqslant 1$ *;*
2. *a continuous, admissible, unitary* R_∞*-representation* Π_∞ *of* $U(\mathbb{Q}_p)$ *over* L*, where*

$$R_\infty \coloneqq R^{\mathrm{loc}}[[x_1, \dots, x_g]];$$

3. *a local map of local rings* $S_\infty \coloneqq \mathcal{O}_L[[y_1, \dots, y_t]] \longrightarrow R_\infty$ *with*

$$t = g + \dim R^{loc} - [F^+ : \mathbb{Q}]\frac{n(n+1)}{2}$$

such that

(i) there exists an $\mathcal{O}_L$-lattice $\Pi_\infty^0 \subset \Pi_\infty$ stable by $U(\mathbb{Q}_p)$ and R_∞ such that

$$(\Pi_\infty^0)' = \mathrm{Hom}_{\mathcal{O}_L}(\Pi_\infty^0, \mathcal{O}_L),$$

is a projective $S_\infty[[K_p]]$-module of finite type (via $S_\infty \longrightarrow R_\infty$) for some (equivalently all) compact open subgroup $K_p \subset U(\mathbb{Q}_p)$;

(ii) there exists a surjective map of local R^{loc}-algebras $R_\infty/\mathfrak{a}R_\infty \twoheadrightarrow R_{\bar{\rho},\mathcal{S}}$ and an isomorphism of continuous admissible unitary $R_\infty/\mathfrak{a}R_\infty$-representations of $U(\mathbb{Q}_p)$ on L

$$\Pi_\infty[\mathfrak{a}] \simeq \Pi_{\mathfrak{m}},$$

where $\mathfrak{a} = (y_1, \dots, y_t)$ denotes the augmentation ideal of S_∞,

It is a direct consequence of this proposition that the R_∞-representation Π_∞ of $U(\mathbb{Q}_p)$ satisfies Hypothesis 3.15. We note that the same applies to a slightly more general context:

Lemma 5.12. *Let V be a finite dimensional algebraic representation of $U(\mathbb{Q}_p)$ over L. Then the R_∞-Banach representation $\Pi_\infty \otimes_L V$ satisfies Hypothesis 3.15.*

Proof. As Π_∞ satisfies Hypothesis 3.15, for any open pro-p-subgroup H of $U(\mathbb{Q}_p)$ there exists an isomorphism of $\mathbb{Z}_p^t \times H$-representations $\Pi_{\infty|\mathbb{Z}_p^t\times H} \simeq \mathcal{C}(\mathbb{Z}_p^t \times H, L)^m$ for some $m \geqslant 1$. But then

$$(\Pi_\infty \otimes_L V)_{|\mathbb{Z}_p^t\times H} \simeq \mathcal{C}(\mathbb{Z}_p^t \times H, V)^m \simeq \mathcal{C}(\mathbb{Z}_p^t \times H, L)^{m \dim_L V}. \qquad \square$$

In the reminder of this paper we will use the following notations: we set

$$\mathcal{X}^p \coloneqq \mathrm{Spf}(\widehat{\bigotimes}_{v\in S\setminus S_p} \overline{R}_{\bar{\rho}_v}^{\square})^{\mathrm{rig}} \simeq \prod_{v\in S\setminus S_p} \mathrm{Spf}(\overline{R}_{\bar{\rho}_v}^{\square})^{\mathrm{rig}},$$

where $\mathbb{U}^g \coloneqq \mathrm{Spf}(\mathcal{O}_L[[x_1, \dots, x_g]])^{\mathrm{rig}}$ is an open polydisc. Moreover, we set

$$\begin{aligned} \mathcal{X}_{\bar{\rho}_p} &\coloneqq \mathrm{Spf}(\widehat{\bigotimes}_{v\in S_p} R_{\bar{\rho}_v}^{\square})^{\mathrm{rig}}, \\ \mathcal{X}_\infty &\coloneqq \mathrm{Spf}(R_\infty)^{\mathrm{rig}} \simeq \mathcal{X}^p \times \mathcal{X}_{\bar{\rho}_p} \times \mathbb{U}^g. \end{aligned}$$

By construction the space $\mathcal{X}_\infty$ contains $\mathcal{X}_{\bar{\rho},\mathcal{S}} = (\mathrm{Spf}\, R_{\bar{\rho},\mathcal{S}})^{\mathrm{rig}}$ as a closed subspace. For a point $x = (x^p, x_p, z) \in \mathcal{X}_\infty(L)$ and a place v of F dividing p, we denote by $\rho_{x,v}$ the framed representation $\mathrm{Gal}_{F_v} \to \mathrm{GL}_n(L)$ associated to x. Finally we write $\rho_{x,p}$ for the the family of representations $(\rho_{x,v})_{v|p}$.

6 Patching functors

In this section, we keep notations and conventions of section 5. In particular, we have $\underline{G} \simeq \prod_{v \in S_p}(L \times_{\mathbb{Q}_p} \mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_{n,F_v})$ which is an algebraic group over L and we consider the associated categories $\mathcal{O}, \mathcal{O}^{\infty}_{\mathrm{alg}}$ and $\widetilde{\mathcal{O}}_{\mathrm{alg}}$ as in section 2 (for the choice of the upper triangular Borel subgroup $\underline{B}$).

We fix once and for all a point $x \in \mathcal{X}_\infty(L)$ such that x maps to the origin in $(\mathrm{Spf}\ S_\infty)^{\mathrm{rig}}$ (i.e. the point defined by the augmentation ideal of S_∞) and we denote by $\widehat{R}_{\infty,x}$ the completed local ring of $\mathcal{X}_\infty$ at x.

6.1 Locally analytic patching functors

We fix a smooth and unramified character $\varepsilon : \underline{T}(\mathbb{Q}_p) \to L^\times$ and consider ε as a point of $\widehat{T}$.

By Lemma 5.12, we can apply Corollary 3.18 to the admissible locally analytic representation Π^{la}_∞, and obtain a functor

$$\begin{aligned} \mathcal{O}^{\infty}_{\mathrm{alg}} &\to \mathrm{Coh}(\mathcal{X}_\infty \times \widehat{T}) \\ M &\mapsto \mathcal{M}_{\Pi_\infty}(M). \end{aligned}$$

Definition 6.1. For $M \in \mathcal{O}^{\infty}_{\mathrm{alg}}$ we define

$$\mathcal{M}_{\infty,x,\varepsilon}(M) \coloneqq \mathcal{M}_{\Pi_\infty}(M)_{x,\varepsilon}$$

to be the stalk of $\mathcal{M}_{\Pi_\infty}(M)$ at (x, ε).

It follows from Proposition 3.17 that $\mathcal{M}_{\infty,x,\varepsilon}(M)$ is a Cohen–Macaulay $\widehat{R}_{\infty,x}$-module and it follows from Theorem 3.14 that the functor $M \mapsto \mathcal{M}_{\infty,x,\varepsilon}(M)$ is exact.

Remark 6.2. We also have the following description:

$$\mathcal{M}_{\infty,x,\varepsilon}(M) \simeq \Big(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi^{\mathrm{la}}_\infty[\mathfrak{m}_x^\infty])^{N_0}[\mathfrak{m}_\varepsilon^\infty]\Big)'$$

where $\mathfrak{m}_\varepsilon$ is the maximal ideal of $\mathcal{A}(p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{Q}_p[\underline{T}(\mathbb{Q}_p)^+]$ corresponding to the character ε and $\mathfrak{m}_x$ is the maximal ideal of $R_\infty[1/p]$ corresponding to x.

Remark 6.3. Note that we have two $U(\mathfrak{t})$-module structures on $\mathcal{M}_{\infty,x,\varepsilon}(M)$: The first one comes from the nilpotent $U(\mathfrak{t})$-module structure on M as in section 2.2. The second one comes from the action of $U(\mathfrak{t})$ induced from the locally analytic T-structure on Π^{la}_∞. It is a tautological consequence of the construction, but we point out that these two actions coincide.

Definition 6.4. Let $I \subset \Delta$ be a finite subset of simple roots and let M be an object of $\widetilde{\mathcal{O}}^I_{\mathrm{alg}}$. Then we define

$$\mathcal{M}_{\infty,x,\varepsilon}(M) \coloneqq \varprojlim_n \mathcal{M}_{\infty,x,\varepsilon}(M/\mathfrak{m}_I^n).$$

Proposition 6.5. *The functor $M \mapsto \mathcal{M}_{\infty,x,\varepsilon}(M)$ is exact on $\widetilde{\mathcal{O}}^I_{\mathrm{alg}}$ and for each $M \in \widetilde{\mathcal{O}}^I_{\mathrm{alg}}$ the $\widehat{R}_{\infty,x}$-module $\mathcal{M}_{\infty,x,\varepsilon}(M)$ is finitely generated and Cohen–Macaulay of dimension $t + \dim_K \mathfrak{z}_I$. Moreover $\mathcal{M}_{\infty,x,\varepsilon}(M)$ is flat over $U(\mathfrak{z}_I)$.*

Proof. Let $\widehat{S}_\infty$ be the completion of $S_\infty[1/p]$ along the maximal ideal generated by the augmentation ideal $\mathfrak{a}$ of S_∞. Moreover, we write $\widehat{U}_I$ for the completion of $U(\mathfrak{z}_I)$ at the maximal ideal $\mathfrak{m}_I$.

By exactness of the functor $\mathcal{M}_{\infty,x,\varepsilon}$, we have

$$\mathcal{M}_{\infty,x,\varepsilon}(M/\mathfrak{m}_I^{n+1})/\mathfrak{m}_I^n \simeq \mathcal{M}_{\infty,x,\varepsilon}(M/\mathfrak{m}_I^n)$$

for any $n \geqslant 1$. It follows from Theorem 3.17 that $\mathcal{M}_{\infty,x,\varepsilon}(M/\mathfrak{m}_I)$ is a finite projective $\widehat{S}_\infty$-module. We denote its rank by $r \geqslant 0$. The exactness of $\mathcal{M}_{\infty,x,\varepsilon}$ implies that $\mathcal{M}_{\infty,x,\varepsilon}(M/\mathfrak{m}_I^n)$ is a finite projective $\widehat{S}_\infty \otimes_L U(\mathfrak{z}_I)/\mathfrak{m}_I^n$-module of rank r and it follows that $\mathcal{M}_{\infty,x,\varepsilon}(M)$ is a finite projective $\widehat{S}_\infty \hat{\otimes}_L \widehat{U}_I$-module of rank r. As the action of $\widehat{S}_\infty \hat{\otimes}_L \widehat{U}_I$ factors through $\widehat{R}_{\infty,x}$ we deduce the result. The exactness of the functor $\mathcal{M}_{\infty,x,\varepsilon}$ is a consequence of the exactness of $\mathcal{M}_{\infty,x,\varepsilon}$ restricted to $\mathcal{O}^\infty_{\mathrm{alg}}$ and the fact that each system $(\mathcal{M}_{\infty,x,\varepsilon}(M/\mathfrak{m}_I^n))_n$ satisfies the Mittag-Leffler condition.

Let $\underline{t} = (t_1, \dots, t_m)$ be a regular sequence generating the maximal ideal of $U(\mathfrak{z}_I)_{\mathfrak{m}_I}$. This is also a regular sequence generating the maximal ideal of the completion $\widehat{U}_I$. By exactness of the functor $\widehat{S}_\infty \otimes_L -$ on strict exact sequences of Fréchet L-algebras, the sequence $\underline{t}$ is $\widehat{S}_\infty \hat{\otimes}_L \widehat{U}_I$-regular. As $\mathcal{M}_{\infty,x,\varepsilon}(M)$ is a finite free $\widehat{S}_\infty \hat{\otimes}_L \widehat{U}_I$-module, the sequence $\underline{t}$ is $\mathcal{M}_{\infty,x,\varepsilon}(M)$-regular. This is equivalent to flatness over $U(\mathfrak{z}_I)_{\mathfrak{m}_I}$. □

6.2 A factorization property

We use the spaces and notations introduced in section 4. A point $x \in \mathcal{X}_\infty(L)$ is said to be crystalline φ-generic and Hodge–Tate regular if for all $v|p$ the representation $\rho_{x,v}$ is crystalline φ-generic and Hodge–Tate regular. Let $x = (\rho^p, \rho_p, z) \in \mathcal{X}_\infty(L)$ be such a φ-generic Hodge–Tate regular point. We fix a refinement $\mathcal{R}$ of ρ_p.

Recall that $\underline{G} \simeq \prod_{v \in S_p}(L \times_{\mathbb{Q}_p} \mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_{n,F_v})$. If I is a set of simple roots of $\underline{G}$, we set

$$\mathcal{X}^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}} := \widehat{\mathcal{X}^p}_{\rho^p} \times \mathcal{X}^{I-\mathrm{qtri}}_{\rho_p,\mathcal{R}} \times \widehat{\mathbb{U}^g},$$
$$\overline{\mathcal{X}}^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}} := \widehat{\mathcal{X}^p}_{\rho^p} \times \overline{\mathcal{X}}^{I-\mathrm{qtri}}_{\rho_p,\mathcal{R}} \times \widehat{\mathbb{U}^g}.$$

This is a closed subscheme of $(\widehat{\mathcal{X}_\infty})_x$ and we write $\widehat{R}_{\infty,x} \twoheadrightarrow R^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ the corresponding quotient map. Moreover, for $w \in W$, we set

$$\mathcal{X}^{I-\mathrm{qtri},w}_{\infty,x,\mathcal{R}} := \widehat{\mathcal{X}^p}_{\rho^p} \times \mathcal{X}^{I-\mathrm{qtri},w}_{\rho_p,\mathcal{R}} \times \widehat{\mathbb{U}^g},$$
$$\overline{\mathcal{X}}^{I-\mathrm{qtri},w}_{\infty,x,\mathcal{R}} := \widehat{\mathcal{X}^p}_{\rho^p} \times \overline{\mathcal{X}}^{I-\mathrm{qtri},w}_{\rho_p,\mathcal{R}} \times \widehat{\mathbb{U}^g}.$$

If $\mathcal{R} = (\varphi_{1,v}, \dots, \varphi_{n,v})_{v|p} \in \prod_{v|p}(L^\times)^n$, we define $\delta_{\mathcal{R}}$ to be the smooth unramified character of T defined by

$$(x_{1,v}, \dots, x_{n,v})_{v|p} \mapsto \prod_{v|p}\prod_{i}(\varphi_{i,v}^{v_{F_v}(x_{i,v})} q_v^{i-n})$$

where q_n denotes the cardinality of the residue field of F_v. We use the notation $\mathcal{M}_{\infty,x,\mathcal{R}} \coloneqq \mathcal{M}_{\infty,x,\delta_{\mathcal{R}}}$. The goal of this section is to prove the following result.

Theorem 6.6. *Let $x \in \mathcal{X}_\infty(L)$ be a φ-generic Hodge–Tate regular crystalline point and let $\mathcal{R}$ be a refinement of x. Then, for any $M \in \mathcal{O}_{\mathrm{alg}}^{I,\infty}$, the $\widehat{R}_{\infty,x}$-module $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ is killed by the kernel of the map $\widehat{R}_{\infty,x} \twoheadrightarrow R_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri}}$. Equivalently its support is contained in $\mathcal{X}_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri}}$.*

Proof. This is a consequence of Proposition 3.20, Proposition 2.14 and Corollary 6.11 which will be proved below. ☐

We will prove the auxiliary statements in (the proof of) this theorem by making use of variants of the construction of eigenvarieties. More precisely, for a subset $I \subset \Delta$, a character $\lambda \in X^*(\underline{T})_I^+$ (dominant with respect to $\underline{P}_I$) and an algebraic representation V of $\underline{G}$ we will consider the scheme-theoretic supports

$$\mathcal{E}_\infty^I(\lambda) = \mathrm{supp}(\mathcal{M}_{\Pi_\infty}^{I,\lambda}) \subset \mathcal{X}_\infty \times \widehat{T}$$
$$\mathcal{E}_\infty^I(\lambda, V) = \mathrm{supp}(\mathcal{M}_{\Pi_\infty}^{I,\lambda,V}) \subset \mathcal{X}_\infty \times \widehat{T},$$

where $\mathcal{M}_{\Pi_\infty}^{I,\lambda}$ respectively $\mathcal{M}_{\Pi_\infty}^{I,\lambda,V}$ are the coherent sheaves associated to $J_{I,\lambda}(\Pi_\infty^{\mathrm{la}})'$ respectively to $J_{I,\lambda}((\Pi_\infty \otimes_L V)^{\mathrm{la}})'$ (see section 3.4 for the notation). We will link the completions of $\mathcal{E}_\infty^I(\lambda)$ resp. $\mathcal{E}_\infty^I(\lambda, V)$ at points $(x, \delta) \in \mathcal{X}_\infty \times \widehat{T}$ to the quasi-trianguline deformation rings of section 4. This is done in two steps: we first show that the set-theoretic support of $\mathcal{M}_{\Pi_\infty}^{I,\lambda}$ resp. of $\mathcal{M}_{\Pi_\infty}^{I,\lambda,V}$ is contained in the (quasi-)trianguline locus (see the proof of Proposition 6.7). We then prove that $\mathcal{E}_\infty^I(\lambda)$ resp. $\mathcal{E}_\infty^I(\lambda, V)$ is reduced (see the proof of Proposition 6.9). The proof of the latter statement follows the usual argument in the case of eigenvarieties, see e.g. [BHS17b, Corollaire 3.12 and Corollaire 3.20]: the general properties of eigenvarieties (deduced from the fact that the sheaves $\mathcal{M}_{\Pi_\infty}^{I,\lambda}$ resp. $\mathcal{M}_{\Pi_\infty}^{I,\lambda,V}$ are locally finite projective over $(\mathrm{Spf}\, S_\infty)^{\mathrm{rig}} \times \widehat{T}_0$ imply that $\mathcal{E}_\infty^I(\lambda)$ resp. $\mathcal{E}_\infty^I(\lambda, V)$ have no embedded components. Hence it is enough to produce on each of their irreducible components a point y such that $\mathcal{E}_\infty^I(\lambda)$ resp. $\mathcal{E}_\infty^I(\lambda, V)$ are reduced in a neighborhood of y. By the same projectivity argument as above, the point y can be chosen so that the weight map to $\widehat{T}_0$ is smooth at this point. Reducedness then boils down to checking that the Hecke operators (that generate the local ring of $\mathcal{E}_\infty^I(\lambda)$ resp. $\mathcal{E}_\infty^I(\lambda, V)$ at y) act semi-simply on the fiber of $\mathcal{M}_{\Pi_\infty}^{I,\lambda}$ resp. $\mathcal{M}_{\Pi_\infty}^{I,\lambda,V}$ over $\widehat{T}_0$ which in turn follows from the fact that Hecke-operators act semi-simply on spaces of classical automorphic forms. We now give the details of these arguments.

Let $\delta = (\delta_{1,v}, \dots, \delta_{n,v})_{v|p} \in \hat{T}(L)$ be a parameter for a quasi-triangulation of x at p, i.e. the trianguline filtration of the (φ, Γ)-module $D^{\dagger}_{\mathrm{rig}}(\rho_v)[1/t]$ over $\mathcal{R}_{K,L}[1/t]$ has graded pieces $\mathcal{R}_{K,L}(\delta_{i,v})[1/t]$. As x is Hodge–Tate regular, there is a natural map

$$\omega_\delta : \mathcal{X}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}} \longrightarrow \widehat{T}^{\wedge}_{\delta},$$

mapping a deformation at p of the (φ, Γ)-module $D^{\dagger}_{\mathrm{rig}}(\rho_v)[1/t]$, equipped with its trianguline filtration, to its parameter (see e.g. [BHS19, eq (3.15)]). If δ is locally algebraic of the form $\delta = \lambda\delta_{\mathcal{R}}$ for $\lambda \in X^*(T)$ and some smooth character $\delta_{\mathcal{R}} \in \widehat{T}(L)$, we shift the previous map to get

$$\omega = t_{-\lambda}\omega_\delta : \mathcal{X}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}} \longrightarrow \widehat{T}^{\wedge}_{\delta_{\mathcal{R}}}$$

which only depends on the chosen refinement. This induces a map

$$i \times \omega : \mathcal{X}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}} \longrightarrow \widehat{\mathcal{X}_{\infty,x}} \times \widehat{T}^{\wedge}_{\delta_{\mathcal{R}}},$$

or equivalently, a homomorphism $\widehat{R}_{\infty,x} \otimes \mathcal{O}^{\wedge}_{\hat{T},\delta_{\mathcal{R}}} \longrightarrow R^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}}$.

Proposition 6.7. *Let $\lambda \in X^*(\underline{T})^+_I$ be a weight dominant with respect to $\underline{P}_I$. The $\widehat{R}_{\infty,x}$-module $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(\lambda))$ is annihilated by the kernel of $\widehat{R}_{\infty,x} \to R^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}}$. More precisely, $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(\lambda))$ is an $\widehat{R}_{\infty,x} \otimes \mathcal{O}^{\wedge}_{\hat{T},\delta_{\mathcal{R}}}$-module and annihilated by the kernel of*

$$\widehat{R}_{\infty,x} \otimes \mathcal{O}^{\wedge}_{\hat{T},\delta_{\mathcal{R}}} \longrightarrow R^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}}.$$

Proof. It follows from Proposition 3.20 and the definition of $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(\lambda))$ that

$$\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(\lambda)) = (t^*_\lambda \mathcal{M}^{I,\lambda}_{\Pi_\infty})^{\wedge}_{(x,\delta_{\mathcal{R}})}$$

as an $\widehat{R}_{\infty,x} \otimes \mathcal{O}^{\wedge}_{\hat{T},\delta_{\mathcal{R}}}$-module. It is thus enough to show that the completion of $\mathcal{M}^{I,\lambda}_{\Pi_\infty}$ at the point $(x, \lambda\delta_{\mathcal{R}}) \in \mathcal{X}_\infty(L) \times \widehat{T}(L)$ is supported at the closed subspace

$$i \times w_\delta : \mathcal{X}^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}} \longrightarrow \widehat{\mathcal{X}_{\infty,x}} \times \widehat{T}^{\wedge}_{\delta_\lambda \delta_{\mathcal{R}}}.$$

We closely follow the proof of [Wu, Prop. 5.13]. Let us write $\mathcal{E}_\infty \subset \mathcal{X} \times \widehat{T}$ for the scheme-theoretic support of the coherent sheaf defined by $J_B(\Pi^{\mathrm{la}}_\infty)'$. By [Wu, 5.4] this contains $\mathcal{E}^I_\infty(\lambda)$ as a closed subspace. As in the proof of [Wu, Prop. 5.13] we consider a proper birational map $f : \mathcal{E}'_\infty \to \mathcal{E}_\infty$ such that the universal (φ, Γ)-module over $\mathcal{E}'_\infty$ has a quasi-triangulation, and write $\mathcal{E}''_\infty$ for the preimage of $\mathcal{E}^I_\infty(\lambda)$ in $\mathcal{E}'_\infty$. Let $Y \subset \mathcal{E}''_\infty$ be the Zariski closed reduced subspace of $\mathcal{E}''_\infty$ whose points are exactly the points of $\mathcal{E}''_\infty$ where the universal filtered (φ, Γ)-module over $\mathcal{R}[1/t]$ is P_I-de Rham. As in [Wu], the existence of Y is a consequence of [Wu, Prop. A.10]. It follows that for any $y \in Y$ lying above $(x, \delta_{\mathcal{R}})$ the map

$$\widehat{Y}_y \to \mathcal{X}_\infty \times \widehat{T}$$

factors through $\mathcal{X}^{I-\mathrm{qtri}}_{\infty,y,\mathcal{R}_y}$. Let $U \subset \mathcal{E}^I_\infty(\lambda)$ be an affinoid open subset containing x and a Zariski dense subset of points which are de Rham (and in particular P_I-de Rham) and trianguline with parameter given by $\mathcal{E}^I_\infty(\lambda) \longrightarrow \widehat{T}$. Such a neighborhood exists by [Wu, Prop. 5.11 & 5.12]. We deduce that $f(Y) \supset U$ and hence $f^{-1}(U) \subset Y$ and we conclude as in the proof of [BHS19, Prop. 3.7.2] (see the erratum in [BD]) that the map

$$\widehat{U}_{x,\lambda\delta_{\mathcal{R}}} \to \mathcal{X}_\infty \times \widehat{T}$$

factors through $\mathcal{X}^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}}$. $\square$

Corollary 6.8. *Let V be an algebraic representation of $\underline{G}$, then*

$$\mathcal{M} = \mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(\lambda) \otimes_L V)$$

is annihilated by some power of the kernel of $\widehat{R}_{\infty,x} \otimes \mathcal{O}^\wedge_{\hat{T},\delta_{\mathcal{R}}} \longrightarrow R^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}}$.

Proof. We recall that

$$\widetilde{M}_I(\lambda) \otimes_L V = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} (L_I(\lambda) \otimes_L A_I) \otimes_L V \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} (L_I(\lambda) \otimes V_{|P_I} \otimes A_I)$$

and that $V_{|P_I}$ is an extension of algebraic irreducible representations of $\underline{L}_I$. Exactness of $\mathcal{M}_{\infty,x,\mathcal{R}}$ (see Proposition 6.5) implies that the $\widehat{R}_{\infty,x}$-module $\mathcal{M}$ is an extension of $\widehat{R}_{\infty,x,\mathcal{R}}$-module of the form $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(\mu))$ for $\mu \in X^*(\underline{T})^+_I$. We deduce the result from Proposition 6.7. $\square$

Proposition 6.9. *Let V be an algebraic representation of $\underline{G}$. Then the schematic support $\mathcal{E}^I_\infty(\lambda, V)$ of the coherent sheaf associated to $J_{I,\lambda}((\Pi_\infty \otimes_L V)^{\mathrm{la}})'$ is reduced.*

Proof. We follow closely the proof of [BHS17b, Cor. 3.20] replacing, where it is nedeed, some arguments by results of [Wu]. To simplify notations we just write $\mathcal{E} = \mathcal{E}^I_\infty(\lambda, V)$ and $\mathcal{M} = \mathcal{M}^{I,\lambda,V}_{\Pi_\infty}$ for the reminder of this proof.

Let $\mathcal{N}$ be the radical ideal of $\mathcal{O}_{\mathcal{E}}$. Assume that $\mathcal{N} \neq 0$ and let $x \in \mathcal{E}$ be a point in the support of $\mathcal{N}$. Let $\widehat{T}^\circ_\lambda$ be the preimage of $\lambda_{|\mathfrak{t}\cap\mathfrak{l}^{\mathrm{ss}}_I} \in (\mathfrak{t} \cap \mathfrak{l}^{\mathrm{ss}}_I)^*$ under the map

$$\widehat{T} \to \mathfrak{t}^* \to (\mathfrak{t} \cap \mathfrak{l}^{\mathrm{ss}}_I)^*,$$

where the first map is the weight map (5). According to [Wu, §5.4] there exists an open affinoid neighborhood U of x and an open affinoid subset $W \subset \widehat{T}^\circ_\lambda \times \mathrm{Spf}(S_\infty)^{\mathrm{rig}}$ such that $\Gamma(U, \mathcal{M})$ is a finite free $\mathcal{O}(W)$-module (such a data exists according to the results of [Wu, §5.4]). Then $\Gamma(U, \mathcal{N})$ is the radical ideal of $\mathcal{O}(U)$. Moreover, as $\mathcal{O}(U) = \Gamma(U, \mathcal{O}_{\mathcal{E}})$ is a sub-$\mathcal{O}(W)$-module of $\mathrm{End}(\Gamma(U, \mathcal{M}))$ (by the same argument as in the proof of Theorem 3.17 respectively of [BHS17b, Prop. 3.11]), the same is true for $\Gamma(U, \mathcal{N})$. Therefore $\Gamma(U, \mathcal{N})$ is a torsion free $\mathcal{O}(W)$-module and its support has the same dimension as W and hence contains an irreducible component U_0 of U. As a consequence the support of $\mathcal{N}$ contains an admissible open subset of $\mathcal{E}$. As the support of $\mathcal{N}$ is also a closed analytic subset of $\mathcal{E}$, it follows from [Con99, Lemm. 2.2.3] that the support of $\mathcal{N}$ contains an irreducible component of $\mathcal{E}$. It hence suffices to produce on each irreducible component of $\mathcal{E}$ a point y such that $\mathcal{E}$ is reduced in a neighborhood of y.

By [Wu, Prop. 5.11] every irreducible component of $\mathcal{E}$ contains a point with algebraic weight.

Therefore we fix a point $x \in \mathcal{E}(L)$ with integral weight $\lambda' \in \widehat{T}^{\circ}_{\lambda}$. Let U be an open affinoid neighborhood of x and $W \subset \widehat{T}^{\circ}_{\lambda} \times \mathrm{Spf}(S_\infty)^{\mathrm{rig}}$ an open affinoid open subset such that $M = \Gamma(U, \mathcal{M})$ is a direct factor of $\mathcal{O}(W)\hat{\otimes}_L J_{B_I}(J_{P_I}(\Pi_\infty^{\mathrm{la}} \otimes_L V)_\lambda)'$. Let $A = \mathcal{O}(W)$ and $B = \mathcal{O}(U)$. Then M is a finitely generated B-module and a finite projective A-module. Let $C > 0$ and $C' > 0$ as in the proof of [Wu, Prop. 5.11]. We set $Z \subset W$ be the subset of algebraic character $\delta_{\lambda'}$ such that, for any simple root $\alpha \notin I$, $\langle \lambda' + \nu, \alpha \rangle > C'$ for any ν weight of $V^\vee$. This is a Zariski dense subset of W. Then for $z = \delta_{\lambda'}\delta_{\mathrm{sm}}$ with δ_{sm} a smooth character, using Proposition 3.20, we see that the B-module $M_z = M \otimes k(z)$ is a direct factor of $J_B(\mathrm{Hom}(M_I(\lambda'), \Pi_\infty^{\mathrm{la}} \otimes_L V))'$. Let $(x, \delta) \in U$ be a point above z, i.e. $\delta = \delta_{\lambda'}\delta_{\mathrm{sm}}$, then arguing as in *loc. cit.*, we have $\mathrm{Hom}_G(\mathcal{F}_{\overline{B}}^{G}(N \otimes_L V^\vee, \delta_{\mathrm{sm}}\delta_B^{-1}), \Pi_\infty^{\mathrm{la}}[\mathfrak{p}_x]) = 0$ for any subquotient N of $M_I(\lambda')$ different from $L(\lambda')$. This implies that M_z is actually a quotient of $J_B(\mathrm{Hom}_{U(\mathfrak{g})}(L(\lambda') \otimes_L V^\vee, \Pi_\infty^{\mathrm{la}}))$ which is isomorphic to a finite direct sum of $J_B(\mathrm{Hom}_{U(\mathfrak{g})}(L(\mu), \Pi_\infty^{\mathrm{la}}))$ with μ dominant. The proof of [BHS17b, Cor. 3.20] shows that the global sections of the coherent sheaf associated to each $J_B(\mathrm{Hom}_{U(\mathfrak{g})}(L(\mu), \Pi_\infty^{\mathrm{la}}))'$ on $U \cap \kappa^{-1}(\{\delta_{\lambda'}\})$ is a semisimple B-module. This concludes the proof. $\square$

Corollary 6.10. *The rigid analytic space $\mathcal{E}^I_\infty(\lambda)$ is reduced.*

Proof. This is Proposition 6.9 with V the trivial representation. $\square$

Corollary 6.11. *Let V be an irreducible algebraic representation of $\underline{G}$. Then the $\widehat{R}_{\infty,x} \otimes \mathcal{O}_{\widehat{T}^\wedge_{\delta_{\mathcal{R}}}}$-module $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(\lambda) \otimes_L V)$ is killed by the kernel of the map*

$$\widehat{R}_{\infty,x} \otimes \mathcal{O}_{\widehat{T}^\wedge_{\delta_{\mathcal{R}}}} \to R^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}}.$$

Proof. By Proposition 6.9, the support of the module $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(\lambda) \otimes_L V)$ is reduced for any $\lambda \in X^*(\underline{T})$ dominant with respect to $\underline{P}_I$ and any algebraic representation V of $\underline{G}$. Therefore the result follows from Corollary 6.8. $\square$

6.3 Bi-module structure on the patched functor

Let M be an object of $\mathcal{O}^\infty_{\mathrm{alg}}$ or $\widetilde{\mathcal{O}}^I_{\mathrm{alg}}$ for some $I \subset \Delta$. As seen in section 2.2, there is a natural structure of $A = U(\mathfrak{t})_{\mathfrak{m}}$-module on M which provides, by functoriality, the structure of an A-module on $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$. This A-module structure extends to an action of the completion $\widehat{A}$ of A with respect to the maximal ideal $\mathfrak{m}$. We recall from Remark 6.3 that this action coincides with the structure of an $\widehat{A}$-module on $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ induced from the T-action on Π_∞.

On the other hand, the ring $R^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ also carries a structure of an $\widehat{A}$-module induced from the map κ_1 defined in section 4.1. This gives a further structure of an $\hat{A}$-module on the $R^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$-module $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$. We will show that these $\hat{A}$-module structures agree.

For $a \in \widehat{A}$, we denote by a (resp. $\widetilde{a}$) the endomorphism of $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ defined by the first (resp. second) action. Note that if M is an object of $\widetilde{\mathcal{O}}_{\mathrm{alg}}$, then $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ is a finite free $\widehat{A}\widehat{\otimes}_L\widehat{S}_\infty$-module for the first $\hat{A}$-module structure by the proof of Proposition 6.5. Thus it is A-torsion free (since $\widehat{A}$ is domain).

Lemma 6.12. *For any $a \in \widehat{A}$ and any M in $\mathcal{O}^{\infty}_{\mathrm{alg}}$ or $\widetilde{\mathcal{O}}^{I}_{\mathrm{alg}}$, there is an equality*

$$a = \widetilde{a} \in \mathrm{End}(\mathcal{M}_{\infty,x,\mathcal{R}}(M)).$$

Proof. If $M = \widetilde{M}(\mu) \otimes_{U(\mathfrak{t})} U(\mathfrak{t})/m^n$ for some $\mu \in X^*(\underline{T})$, this is a consequence of [BHS17b, Thm. 3.21], the commutative diagram [BHS19, (3.30)] and Remark 6.3. This implies that for any $\mu \in X^*(\underline{T})$, we have $a = \widetilde{a}$ on $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}(\mu))$.

Now we consider the general case. By definition of $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$, it is enough to treat the case of $M \in \mathcal{O}^{\infty}_{\mathrm{alg}}$. It follows from Proposition 2.14 that it is sufficient to prove the equality $\widetilde{a} = a$ when $M = \widetilde{M}(\mu) \otimes_L V$ for $\mu \in X^*(\underline{T})$ dominant and V a finite dimensional $U(\mathfrak{g})$-module. Let (Fil_i) be an increasing filtration of $\widetilde{M}(\mu) \otimes_L V$ such that $\mathrm{Fil}_i / \mathrm{Fil}_{i-1} \simeq \widetilde{M}(\mu_i)$ where $\mu_1, \dots, \mu_d \in X^*(\underline{T})$ and $d = \dim_L V$ (such a filtration exists by [Soe92, Lem. 8]. Let K denote the fraction field of A. It follows from Proposition 2.12 that we have a decomposition of $U(\mathfrak{g})_K$-modules

$$(\widetilde{M}(\mu) \otimes_L V) \otimes_A K \simeq \bigoplus_{i=1}^{d} \widetilde{M}(\mu_i)_K$$

splitting the filtration $(\mathrm{Fil}_i \otimes_A K)$. Let $p_i \in \mathrm{End}_{U(\mathfrak{g})_K}((\widetilde{M}(\mu) \otimes_L V) \otimes_A K)$ be the projector on $\widetilde{M}(\mu_i)_K$. As

$$\mathrm{End}_{U(\mathfrak{g})_K}((\widetilde{M}(\mu) \otimes_L V) \otimes_A K) \simeq \mathrm{End}_{U(\mathfrak{g})}((\widetilde{M}(\mu) \otimes_L V)) \otimes_A K$$

by [Soe92, Thm. 5], there exists, for each $1 \leqslant i \leqslant d$, a nonzero element $q_i \in A$ such that $q_i p_i$ actually restricts to an endomorphism of $\widetilde{M}(\mu) \otimes_L V$. We set $q = q_1 \cdots q_r$ and $\alpha_i = q p_i$. Then the α_i are endomorphisms of $\widetilde{M}(\mu) \otimes_L V$ that stabilize the filtration $\mathrm{Fil}_\bullet$. As each $\mathrm{Fil}_i / \mathrm{Fil}_{i-1}$ is a free A-module, the endomorphisms α_i induce the zero endomorphism of Fil_{i-1} and $\widetilde{M}(\mu) \otimes_L V / \mathrm{Fil}_i$ and the multiplication by q on $\mathrm{Fil}_i / \mathrm{Fil}_{i-1}$.

In order to simplify notations we set

$$\begin{aligned} M_\infty &= \mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}(\mu) \otimes_L V), \\ \mathrm{Fil}_i M_\infty &= \mathcal{M}_{\infty,x,\mathcal{R}}(\mathrm{Fil}_i). \end{aligned}$$

By construction, for each i the endomorphism α_i induces an $R_{\infty,x}$-linear endomorphisms of $\mathrm{Fil}_j M_\infty$ for all j. By exactness of $\mathcal{M}_{\infty,x,\mathcal{R}}$, the family $(\mathrm{Fil}_i M_\infty)$ is a filtration of

M_∞ and $\mathrm{Fil}_i M_\infty / \mathrm{Fil}_{i-1} M_\infty \simeq \mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}(\mu_i))$ for any i, so that a and $\widetilde{a}$ induces the same endomorphism of $\mathrm{Fil}_i M_\infty / \mathrm{Fil}_{i-1} M_\infty$. Finally, for $1 \leqslant i \leqslant d$, we denote by $M_\infty^{(i)} = \alpha_i(\mathrm{Fil}_i M_\infty)$ the image of the i-th filtration step under α_i. It follows from the properties of α_i that

- $M_\infty^{(i)} \subset \mathrm{Fil}_i M_\infty$;
- the quotient $\mathrm{Fil}_i M_\infty / (\mathrm{Fil}_{i-1} M_\infty + M_\infty^{(i)})$ is killed by q;
- $M_\infty^{(i)}$ is isomorphic to a quotient of $\mathrm{Fil}_i M_\infty / \mathrm{Fil}_{i-1} M_\infty$.

Therefore, we have $\widetilde{a} = a$ on $M_\infty^{(i)}$ for any $a \in \widehat{A}$ and the quotient of M_∞ by the sum of the $M_\infty^{(i)}$ is killed by q^d. As M_∞ is A-torsion free it follows that $\widetilde{a} = a$. $\square$

Let $\xi : Z(\mathfrak{g}) \to U(\mathfrak{t})$ be the Harish-Chandra map as recalled in section 2.4. As in loc.cit. we write t_ν for the unique endomorphism of $U(\mathfrak{t})$ mapping $x \in \mathfrak{t}$ to $t_\nu(x) = x + \nu(x)$.

Let $h = (h_{1,\tau,v} < \cdots < h_{n,\tau,v})_{\tau,v} \in X^*(\underline{T})$ be the weight corresponding to the Hodge–Tate weights of $\rho_x = (\rho_v)_{v|p}$ and let $\delta'_G = (0, -1, -2, \dots, 1-n)_{\tau,v} \in X^*(\underline{T})$ be fixed central shift of the half sum of the positive roots $\delta_G \in X^*(\underline{T}) \otimes \mathbb{Q}$. We have a map

$$\kappa_2 : \widehat{A} = \widehat{U(\mathfrak{t})_{\mathfrak{m}}} \to R^{\mathrm{qtri}}_{\rho_p,\mathcal{R}}$$

induced from the map κ_2 of section 4.1 and we define the L-algebra homomorphism

$$\alpha = \kappa_2 \circ t_{h-\delta'_G} \circ \xi : Z(\mathfrak{g}) \to R^{\mathrm{qtri}}_{\rho_p,\mathcal{R}}.$$

As in [DPS, Def. 4.23], we define, for any $v|p$, an L-algebra homomorphism

$$\zeta^C_{\widetilde{\rho}^C_v} : Z(\mathrm{Lie}(\mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_n)) \longrightarrow R^{\square,\mathrm{rig}}_{\overline{\rho}_v}$$

where $\widetilde{\rho}_v$ is the universal family of Galois representations over $R^{\square,\mathrm{rig}}_{\overline{\rho}_v}$. After completion at ρ_v and taking the tensor product over all $v|p$, we obtain an L-algebra homomorphism

$$\zeta^C : Z(\mathfrak{g}) = \bigotimes_{v|p} Z(\mathrm{Lie}(\mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_n)) \longrightarrow R^{\square}_{\rho_p} \twoheadrightarrow R^{\mathrm{qtri}}_{\rho_p,\mathcal{R}}.$$

Note that the definition of $\zeta^C_{\widetilde{\rho}^C_v}$ from ρ_v depends on a choice of a central shift of δ_G (see the discussion ending [DPS, §4.7]). We choose it equal to δ'_G. More concretely ζ^C is characterized by the following property. This is the unique continuous homomorphism such that, for any local artinian L-algebra and any local homomorphism $f : R^{\mathrm{qtri}}_{\rho_p,\mathcal{R}} \to A$, corresponding to $\rho_A = (\rho_{A,v} : \mathrm{Gal}_{F_v} \to \mathrm{GL}_n(A))_{v|p}$, the composition map $Z(\mathfrak{g}) \xrightarrow{\zeta^C} R^{\mathrm{qtri}}_{\rho_p,\mathcal{R}} \to A$ is $Z(\mathfrak{g}) \xrightarrow{\xi} U(\mathfrak{t}) \xrightarrow{t_{\nu-\delta'_G}} A$ where

$$\nu \in \mathrm{Hom}_L(U(\mathfrak{t})^W, A) \simeq \mathrm{Hom}_L(U(\mathfrak{t}^*)^W, A) \simeq \mathrm{Hom}_L(U(\mathfrak{g}^*)^{\underline{G}_L}, A)$$

is the map induced by the conjugacy class of the Sen operators

$$(\Theta_{\mathrm{Sen},\rho_{A,v}})_{v|p} \in (\mathfrak{g} \otimes_L A).$$

Proposition 6.13. *The homomorphisms ζ^C and α defined above coincide.*

Proof. It is sufficient to prove that for any local artinian L-algebra A and any map $f : R^{\mathrm{qtri}}_{\rho_p,\mathcal{R}} \to A$, we have $f \circ \zeta^C = f \circ \alpha$. Note that the map f gives rise to a family $(\rho_{A,v})_{v|p}$ of local Galois representations. It follows from [BHS19, Lem. 3.7.5] that, for any embedding $\tau : F_v \hookrightarrow L$, the τ-part of the Sen polynomial of ρ_v is $\prod_{i=1}^n (X - (h_{i,\tau} + \nu_{i,\tau}))$ where $(\nu_{i,\tau}) \in \mathrm{Hom}_L(\mathfrak{t}, A)$ corresponds to $f \circ \kappa_2 : U(\mathfrak{t}) \to A$. The result is then a direct comparison of the definitions of α and ζ^C. □

For each element M of the category $\mathcal{O}^{I,\infty}_{\mathrm{alg}}$ or $\widetilde{\mathcal{O}}^I_{\mathrm{alg}}$, there is a natural homomorphism of L-algebras $Z(\mathfrak{g}) \to \mathrm{End}(M)$. By functoriality of $\mathcal{M}_{\infty,x,\mathcal{R}}$, this gives a map

$$z : Z(\mathfrak{g}) \to \mathrm{End}_{\widehat{R}_{\infty,x}}(\mathcal{M}_{\infty,x,\mathcal{R}}(M)).$$

The following result tells us that this map factors through $R^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}}$.

Corollary 6.14. *For any $x \in Z(\mathfrak{g})$, the element $z(x)$ is the multiplication by $\alpha(x) \otimes 1 \in R^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}}$.*

Proof. This is a consequence of Proposition 6.13 and of [DPS, Thm. 9.27]. □

Remark 6.15. Recall that $h = (h_{1,\tau,v} < \cdots < h_{n,\tau,v})_{\tau,v}$ denotes the weight corresponding to the Hodge–Tate weights of ρ. Let $\lambda \coloneqq w_0(h) - \delta'_G \in X^*(\underline{T})$, which is still a dominant character. Recall that $t_{-\delta_G} \circ \xi$ has image contained in $U(\mathfrak{t})^W$. Hence we have

$$t_{h-\delta'_G} \circ \xi = t_h \circ \mathrm{Ad}(w_0) \circ t_{-\delta'_G} \circ \xi = \mathrm{Ad}(w_0) \circ t_{w_0(h)} \circ t_{-\delta'_G} \circ \xi = \mathrm{Ad}(w_0) \circ t_\lambda \circ \xi.$$

Therefore

$$\mathrm{Id} \otimes \alpha = (\mathrm{Id} \otimes \mathrm{Ad}(w_0)) \circ h_\lambda : A \otimes_L Z(\mathfrak{g}) \to A \otimes_{A^W} A,$$

where h_λ is the map defined in section 2.4.

6.4 Computation of a support

Now we can prove our main result of this section concerning the support of the patched functor applied to a generalized Verma module respectively applied to its dual.

Theorem 6.16. *Let $x \in \mathcal{X}_\infty(L)$ be a point whose associated Galois representation is crystalline, φ-generic and Hodge–Tate regular. Let $\mathcal{R}$ be a refinement of x. Let $h = (h_{1,\tau} < \cdots < h_{n,\tau})_{\tau:F\hookrightarrow L} \in X^*(\underline{T})$ be the character given by the Hodge–Tate weights of ρ_x. Let $\delta'_G = \det^{\frac{1-n}{2}} \delta_G = (0, -1, \ldots, 1-n)_{\tau:F\hookrightarrow L} \in X^*(\underline{T})$, where δ_G is the half sum of the positive roots, and define $\lambda \coloneqq w_0(h) - \delta'_G \in X^*(\underline{T})^+$.*

Then, for $I \subset \Delta$ and $w \in W$, the schematic supports of $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w^{\min}\cdot\lambda))$ and $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w^{\min}\cdot\lambda)^\vee)$ are either $\mathcal{X}_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri},w^{\min}w_0}$ or empty.

Proof. Let M be $\widetilde{M}_I(w^{\min}\cdot\lambda)$ or $\widetilde{M}_I(w^{\min}\cdot\lambda)^\vee$. As $R_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri}}$ is generically reduced and equi-dimensional by Lemma 4.4 and as $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ is Cohen–Macaulay of dimension $\dim R_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri}}$, its schematic support is reduced and is a union of irreducible components of $\operatorname{Spec} R_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri}}$, i.e. it is a union of $\operatorname{Spec} R_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri},w'}$ for some $w' \in W$.

By Proposition 2.15, the module M is annihilated by $I_w \subset A_I \otimes_L Z(\mathfrak{g})$. This implies in particular that the action of $A_I \otimes_L Z(\mathfrak{g})$ on M factors through h_λ. By functoriality, this gives rise to a structure of an $A_I \otimes_{A^W} A$-module on $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$. Note that the map (κ_1,κ_2) of section 4.1 provides a morphism of L-algebras $A_I \otimes_{A^W} A \to R_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri}}$ and, using Theorem 6.6, a second structure of an $A_I \otimes_{A^W} A$-module on $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$. It follows from Lemma 6.12, Corollary 6.14 and Remark 6.15 that this two actions coincide up to composition with $\mathrm{Id}\otimes\mathrm{Ad}(w_0)$. We deduce that $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ is killed by the ideal of $R_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri}}$ defining the inverse image of $T_{I,ww_0} \subset \mathfrak{z}_I \times_{\mathfrak{t}/W} \mathfrak{t}$. Therefore Lemma 4.5 (see also Remark 2.16) implies that the action of $R_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri}}$ factors through $R_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri},ww_0}$ so that the schematic support of $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ is $\operatorname{Spec} R_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri},ww_0}$. □

The following corollary is also a direct consequence of Theorem 6.6 and Lemma 6.12.

Corollary 6.17. *Let $x \in \mathcal{X}_\infty(L)$ be a φ-generic Hodge–Tate regular crystalline point and let $\mathcal{R}$ be a refinement of x. Then, for any $M \in \mathcal{O}_{\mathrm{alg}}^I$, the schematic support of the $\widehat{R}_{\infty,x}$-module $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ is contained in $\overline{\mathcal{X}}_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri}}$.*

7 Main results

We keep the assumptions of section 5 and 6, in particular that Hypothesis 5.10 is satisfied. In this section we assume that our point $x = (\rho_p, \rho^p, z) \in \mathcal{X}_\infty(L)$ fixed in section 6 corresponds to a classical automorphic form of tame level K^p. This means that $x \in \mathcal{X}_{\overline{\rho},\mathcal{S}}(L) \subset \mathcal{X}_\infty(L)$ and that there exists an automorphic representation π of $U(\mathbb{A}_\mathbb{Q})$ such that $\Pi[\mathfrak{m}_x] = \Pi[\chi^S]$ where χ^S is the character of $\mathbb{T}^S$ acting on $\Pi^{K^p} \otimes_{\mathbb{C},\iota} \overline{\mathbb{Q}_p}$ for some isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}_p}$ (see Remark 5.9). Let $\rho : \mathrm{Gal}_E \to \mathrm{GL}_n(L)$ be the Galois representation corresponding to x so that $\rho_p = (\rho_{|\mathrm{Gal}_{F_v}})_{v|p}$. Moreover, we assume that (the Galois representation defined by) x is crystalline, Hodge–Tate regular and φ-generic (see section 6.2) at p. In particular the automorphic representation π is unramified, and thus finite slope, at p. It follows from the proof of [BHS17a, Cor. 3.12] that the image ρ^p of x in $\mathrm{Spf}(\bigotimes_{v\in S, p\nmid v} \overline{R}_{\overline{\rho}_v}^\square)^{\mathrm{rig}}$ lies in the smooth locus.

We fix a refinement $\mathcal{R} = (\varphi_{1,v},\dots,\varphi_{n,v})_v$ of x. Let us denote the τ-Hodge–Tate weights of $\rho_{x,v}$ for $v|p$ in F and $\tau : F_v \hookrightarrow L$ by $h_{v,\tau} \coloneqq (h_{1,v,\tau} < \cdots < h_{1,v,\tau})$. Given this collection of Hodge–Tate weights we write $h = (h_{v,\tau})_{v,\tau}$ and $h_v = (h_{v,\tau})_\tau$. We then define $R_{\rho_v}^{\mathrm{cris},h_v}$ to be the crystalline deformation ring of ρ_v of labelled Hodge–Tate weight h_v and set

$$R_{\rho_p}^{\mathrm{cris},h} = \widehat{\bigotimes}_{v|p} R_{\rho_v}^{\mathrm{cris},h_v}.$$

We further define

$$\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{cris},h} = \widehat{\mathcal{X}^p}_{\rho^p} \times (\mathrm{Spf}\, R_{\rho_v}^{\mathrm{cris},h_v})^{\mathrm{rig}} \times \widehat{\mathbb{U}^g}.$$

Note that is follows from the definitions that $\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{cris},h}$ embeds into $\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{qtri},w_0}$ for any choice of a refinement $\mathcal{R}$.

We set

$$\mu_{v,\tau} = (h_{1,v,\tau}, h_{2,v,\tau}+1, \dots, h_{n,v,\tau}+(n-1)) = h_{v,\tau} - \delta'_{G,v,\tau},$$

and $\mu = (\mu_{v,\tau})_{v,\tau}$, which is thus antidominant (for the upper Borel), and $\lambda = w_0(h) - \delta'_G = w_0 \cdot \mu \in X^*(\underline{T})^+$. For all $v|p$ in F, we denote by W_v the Weyl group of $\mathrm{GL}_n(F_v)$, which we identify with $\mathfrak{S}_n$ and denote by $s_{1,v}, \dots, s_{n-1,v}$ the simple reflections with respect to the choice of the upper Borel $B_v \subset \mathrm{GL}_{n,F_v}$. Moreover, $w_{0,v} = s_{n-1,v} \dots s_{2,v} s_{1,v} s_{2,v} \dots s_{n-1,v}$ will denote the longest element of W_v. We then write $W = \prod_v W_v$ the Weyl group of $\underline{G}_{\mathbb{Q}_p} \simeq \prod_{v|p} \mathrm{GL}_{n,F_v}$ with respect to the Borel $B = \prod_{v|p} B_v$. Because of the product structure, we will sometimes abuse notations and simply write s_i for the simple reflections and w_0 for the longest element.

For a scheme X of dimension d we write $Z^0(X) = Z_d(X)$ for the free abelian group on the irreducible components of X. Moreover, for $d' \leqslant d$ we write $Z_{d'}(X)$ for the free abelian group on the irreducible and reduced closed subschemes of dimension d'. We recall that a coherent sheaf $\mathcal{F}$ on X with d'-dimensional support defines a class $[\mathcal{F}] \in Z_{d'}(X)$, see e.g. [BHS19, Equation (2.13)].

7.1 Sheaves and supports.

Let $\lambda = w_0 \cdot \mu \in X^*(\underline{T})^+$ dominant, integral. We moreover write

$$m_x = \dim \mathcal{M}_{\infty,x,\mathcal{R}}(L(\lambda)) \otimes k(x). \tag{8}$$

It follows from [BHS19, Thm. 5.1.3] that $m_x \geqslant 1$ and that m_x does not depend on the choice of a refinement $\mathcal{R}$. Indeed,

$$m_x = \dim \mathrm{Hom}_{U(\mathfrak{g})}(L(\lambda), \Pi_\infty^{\mathrm{la}}[\mathfrak{m}_\rho])^{N_0}[\mathfrak{m}_{\delta_\mathcal{R}}] = \dim \mathrm{Hom}_G(\mathrm{Ind}_{\overline{B}}^G(\delta_\lambda \delta_\mathcal{R} \delta_B^{-1})^{\mathrm{lalg}}, \Pi_\infty^{\mathrm{la}}[\mathfrak{m}_\rho]),$$

coincides with the multiplicity of the locally algebraic vectors associated to ρ in Π^{la} and those do not depend on the choice of $\mathcal{R}$. We refer to the discussion before the Corollary 7.29 below for the notation and a justification of these facts. To x and $\mathcal{R}$ we associate a permutation

$$w_{x,\mathcal{R}} = (w_{x,\mathcal{R}_v})_{v \in \Sigma} = (w_{x,\mathcal{R}_v,\tau})_{v,\tau} \in W$$

defined as in [HMS, § 3.7]. We recall that these permutations encode the relative position of the Hodge–Tate flag with respect to the full flag corresponding to the refinement $\mathcal{R}$. We recall that, for any object M of $\mathcal{O}^{\infty}_{\mathrm{alg}}$ (resp. $\widetilde{\mathcal{O}}^{I}_{\mathrm{alg}}$), the sheaf $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ is zero or Cohen–Macaulay of dimension $t = \dim \overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ (resp. $t + \dim_K \mathfrak{z}_I = \dim \mathcal{X}^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}}$).

Lemma 7.1. *Let R be a Cohen–Macaulay noetherian local ring of dimension d' and let M and M' be two finitely generated Cohen–Macaulay modules. Let $(t_1, \dots, t_m)$ be a regular sequence of elements of the maximal ideal of R which is also M and M'-regular. Assume that $[M] = [M']$ in $Z_{d'}(\operatorname{Spec} R)$. Then*

$$[M/(t_1, \dots, t_m)M] = [M'/(t_1, \dots, t_m)M'] \in Z_{d'-m}(R).$$

Proof. By induction it is sufficient to prove the result when $m = 1$. Set $t = t_1$. Let $\mathfrak{p}$ be a prime ideal of R which is a generic point of $\operatorname{Supp}(M)$ or $\operatorname{Supp}(M')$. It is sufficient to prove that $[M_{\mathfrak{p}}/tM_{\mathfrak{p}}] = [M'_{\mathfrak{p}}/tM'_{\mathfrak{p}}]$ in $Z_{d'-1}(\operatorname{Spec} R_{\mathfrak{p}}/(t))$, i.e. that $M_{\mathfrak{p}}/tM_{\mathfrak{p}}$ and $M'_{\mathfrak{p}}/tM'_{\mathfrak{p}}$ are two $R_{\mathfrak{p}}/(t)$-modules of the same length. This is a consequence of [Sta24, Lemma 02QG]. $\square$

Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone and let $\widetilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution. Similarly to the definition of the closed subschemes $X_w \subset X$ in 4.1 we define

$$Z_w \subset \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} \subset X$$

to be the Zariski closure of preimage under $\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} \to \underline{G}_L/\underline{B} \times \underline{G}_L/\underline{B}$ of the orbit $\underline{G}_L(1, w) \subset \underline{G}_L/\underline{B} \times \underline{G}_L/\underline{B}$. Set

$$\mathcal{Z}_w = g(f^{-1}(Z_w \cap \widehat{X}_{I,w,x_{\mathrm{dR}}})) \times \widehat{\mathcal{X}}^{p}_{\rho^p} \times \widehat{\mathbb{U}}^g \subset \overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}},$$

where f and g are the maps from Theorem 4.7.

In the following we will make use of the following abusive notation for (local) formal schemes: Let $\operatorname{Spf} R$ be a (local) affine formal scheme. Then we will say that $\operatorname{Spf} R$ is reduced, if R is reduced. Moreover, we will say that $\operatorname{Spf} R$ is irreducible if $\operatorname{Spec} R$ is irreducible. More generally, for a given irreducible component $\operatorname{Spec} R/\mathfrak{a} \subset \operatorname{Spec} R$, we will refer to the formal subscheme $\operatorname{Spf} R/\mathfrak{a} \subset \operatorname{Spf} R$ as an irreducible component of $\operatorname{Spf} R$. Similarly, we will write $Z^0(\operatorname{Spf} R) = Z^0(\operatorname{Spec} R)$ for the free abelian group on the irreducible components of $\operatorname{Spf} R$ to which we also refer as the irreducible components of $\operatorname{Spf} R$, etc.

Proposition 7.2. *Let $w \in W$. Then the following properties hold:*

1) For all $I \subset \Delta$ and all $\overline{w} \in W_I \backslash W$ satisfying $w^{\min} w_0 \geqslant w_{x,\mathcal{R}}$, the formal subscheme $\mathcal{X}^{I-\mathrm{qtri},ww_0}_{\infty,x,\mathcal{R}}$ is reduced and irreducible and coincides with an irreducible component of $\mathcal{X}^{I-\mathrm{qtri}}_{\infty,x,\mathcal{R}}$.

2) The schematic supports of $\mathcal{M}_{\infty,x,\mathcal{R}}(M(w\cdot\lambda))$ and $\mathcal{M}_{\infty,x,\mathcal{R}}(M(w\cdot\lambda)^\vee)$, for $w\in W$, are contained in $\overline{\mathcal{X}}^{\mathrm{qtri},ww_0}_{\infty,x,\mathcal{R}}$ if $ww_0\geqslant w_{x,\mathcal{R}}$, and this sheaf is zero otherwise. Moreover,

$$[\mathcal{M}_{\infty,x,\mathcal{R}}(M(w\cdot\lambda))]=m_x[\overline{\mathcal{X}}^{\mathrm{qtri},ww_0}_{\infty,x,\mathcal{R}}]\in Z^0(\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}})$$
$$[\mathcal{M}_{\infty,x,\mathcal{R}}(M(w\cdot\lambda)^\vee)]=m_x[\overline{\mathcal{X}}^{\mathrm{qtri},ww_0}_{\infty,x,\mathcal{R}}]\in Z^0(\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}})$$

for $ww_0\geqslant w_{x,\mathcal{R}}$, where m_x is the integer defined by (8).

3) There is an equality

$$[\mathcal{M}_{\infty,x,\mathcal{R}}(L(ww_0\cdot\lambda))]=m_x\sum_{w'\leqslant w}a_{w,w'}[\mathcal{Z}_w]\in Z^0(\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}})$$

where the $a_{w,w'}\in\mathbb{N}$ are the integers defined in [BHS19, Thm. 2.4.7]. In particular $a_{w,w}=1$.

4) For all $I\subset\Delta$, the sheaves

$$\mathcal{M}_{\infty,x,\mathcal{R}}(M_I(w^{\min}\cdot\lambda))\ \textit{and}\ \mathcal{M}_{\infty,x,\mathcal{R}}(M_I(w^{\min}\cdot\lambda)^\vee)$$

are non zero if and only if $w^{\min}w_0\geqslant w_{x,\mathcal{R}}$.

5) For all $I\subset\Delta$, the support of

$$\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w^{\min}\cdot\lambda))\ \textit{and}\ \mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w^{\min}\cdot\lambda)^\vee),$$

for $\overline{w}\in W_I\backslash W$, is $\mathcal{X}^{I-\mathrm{qtri},w^{\min}w_0}_{\infty,x,\mathcal{R}}$ if $w^{\min}w_0\geqslant w_{x,\mathcal{R}}$ and these sheaves are zero otherwise.

6) The module $\mathcal{M}_{\infty,x,\mathcal{R}}(L(\lambda))$ is free of rank m_x over $\mathcal{X}^{\mathrm{cris},h}_{\infty,x,\mathcal{R}}\subset\mathcal{X}^{\mathrm{qtri},w_0}_{\infty,x,\mathcal{R}}$.

7) For any $I\subset\Delta$ and any $w\in W$, the sheaves

$$\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w^{\min}\cdot\lambda))\ \textit{and}\ \mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w^{\min}\cdot\lambda)^\vee)$$

are generically free of rank m_x over their support.

Proof. We first prove point 1)). As $\mathcal{X}^p$ is smooth at ρ^p (as recalled in the begining of this section), the formal completion $\widehat{\mathcal{X}^p}_{\rho^p}$ is formally smooth. As $\widehat{\mathbb{U}^g}$ is also formally smooth, the claim follows from the fact that

$$\mathcal{X}^{I_v-\mathrm{qtri},\square}_{r_v,\mathcal{R}_v}\longrightarrow\mathcal{X}^{I_v-\mathrm{qtri}}_{r_v,\mathcal{R}_v}\text{ and }\mathcal{X}^{I_v-\mathrm{qtri},\square}_{r_v,\mathcal{R}_v}\longrightarrow\widehat{X_{I,x_{\mathrm{pdR}}}}$$

are formally smooth and that $X_{I,w,x_{\mathrm{pdR}}}$ is an irreducible component of $\widehat{X_{I,x_{\mathrm{pdR}}}}$.

By Theorem 6.16, the schematic support of the Cohen-Macaulay sheaves

$$\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w\cdot\lambda))\text{ and }\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w\cdot\lambda)^\vee)$$

is contained in $\mathcal{X}_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri},w}$ which is irreducible. By Proposition 6.5, as the sheaves are Cohen–Macaulay of dimension $t+\dim \mathfrak{z}_I = \dim \mathcal{X}_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri}}$ (e.g. [BHS19, equation (5.8)] and Proposition 3.20), we deduce that, if non empty, their schematic support is all $\mathcal{X}_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri},w}$.

By Remark 6.3 we deduce also that

$$\mathrm{supp}(\mathcal{M}_{\infty,x,\mathcal{R}}(M_I(w\cdot\lambda))) \subset \overline{\mathcal{X}}_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri},w^{\min}w_0}$$

for $w \in {}^IW$. Note that the Jordan–Hölder factors of $M_I(w\cdot\lambda)$ are among the the $L(w'\cdot\lambda)$ with $w' \geqslant w$ and that $L(w\cdot\lambda)$ is the cosocle of $M_I(w\cdot\lambda)$. Therefore $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w\cdot\lambda)) \neq 0$ if and only if $\mathcal{M}_{\infty,x,\mathcal{R}}(M_I(w\cdot\lambda)) \neq 0$ if and only if $\mathcal{M}_{\infty,x,\mathcal{R}}(L(w'\cdot\lambda)) \neq 0$ for some $w' \geqslant w$. Therefore the non nullity assertions in 4) and 5) follow from the exactness of $\mathcal{M}_{\infty,x,\mathcal{R}}$ (Proposition 6.5) and from [BHS19, Thm. 5.3.3] (and this theorem implies that the non-vanishing is actually also equivalent to $\mathcal{M}_{\infty,x,\mathcal{R}}(L(w\cdot\lambda)) \neq 0$). This proves 4) and 5)

We prove point 6). By [BHS19, Remark 4.3.1 and Proof of Theorem 5.3.3, Step 7], the schematic support of $\mathcal{M}_{\infty,x,\mathcal{R}}(L(\lambda))$ is contained in the crystalline locus $\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{cris},h} \subset \mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{qtri}}$, which is smooth and irreducible of the same dimension as the support of $\mathcal{M}_{\infty,x,\mathcal{R}}(L(\lambda))$. Thus these coincide and $\mathcal{M}_{\infty,x,\mathcal{R}}(L(\lambda))$ is free of rank m_x over the crystalline locus.

Now we prove point 2). The first assertion has already been proved with 4) and 5) (together with Lemma 6.12), therefore it remains to prove the assertion on the cycle. Let us fix w so that $ww_0 \geqslant w_{x,\mathcal{R}}$. As $M(w\cdot\lambda)$ and $M(w\cdot\lambda)^\vee$ have the same Jordan–Hölder constituent (with multiplicity), it is sufficient to prove the result for $M(w\cdot\lambda)$. We know from point 5) that the schematic support of $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}(w\cdot\lambda))$ is $\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{qtri},ww_0}$ and it follows from Step 9 (ii) in the proof of [BHS19, Thm. 5.3.3] that $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}(w\cdot\lambda))$ is generically free of rank m_x over $\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{qtri},ww_0}$. Indeed, Proposition 3.20 identifies $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}(w\cdot\lambda))$ with the localisation of $\mathcal{M}_\infty$ of loc.cit. at $x_{\mathcal{R},ww_0}$, the point corresponding to x, refinement $\mathcal{R}$ and Hodge-Tate weights determined by ww_0 (see [BHS19, §5.3]). As $\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{qtri},ww_0}$ is Cohen–Macaulay, the result is a consequence of point 5) and of Lemma 7.1 applied with

$$M = \mathcal{O}^{m_x}_{\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{qtri},ww_0}} \text{ and } M' = \mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}(w\cdot\lambda))$$

and to a regular sequence generating the maximal ideal of $U(\mathfrak{t})_{\mathfrak{m}}$. This sequence is M'-regular by Proposition 6.5.

We deduce 3) from 2) together with formulas (5.23) and (5.24) of [BHS19] and the fact that the Verma modules form a basis of the Grothendieck group of the category $\mathcal{O}_{\chi_\lambda}$.

We prove point 7). As $\mathcal{X}_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri},w'}$ is generically smooth for any w', the module $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ is generically free, say of rank r, over its support where

$$M \in \{\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w^{\min}\cdot\lambda)), \mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w^{\min}\cdot\lambda)^\vee)\}.$$

Now we claim that there exists an open an subset U in the regular locus of $\mathrm{Spec}(R_{\infty,x,\mathcal{R}}^{I-\mathrm{qtri},w^{\min}w_0})$ such that U intersects the support of $\mathcal{M}_{\infty,x,\mathcal{R}}(L(w^{\min}\cdot\lambda))$. The claim then implies $r=m_x$. Indeed, the restriction of $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w^{\min}\cdot\lambda))$ to U is locally free since U is regular. Therefore $\mathcal{M}_{\infty,x,\mathcal{R}}(M_I(w^{\min}\cdot\lambda))$ is locally free of rank r over its support intersected with U. It follows from the point 3) that $\mathcal{M}_{\infty,x,\mathcal{R}}(L(w'\cdot\lambda))$ is not supported at the generic point of $\mathcal{Z}_{w^{\min}w_0}$ for $w'>w^{\min}$ and that $\mathcal{M}_{\infty,x,\mathcal{R}}(L(w^{\min}\cdot\lambda))$ has length m_x at the generic point of $\mathcal{Z}_{w^{\min}w_0}$. As $L(w^{\min}\cdot\lambda)$ appears with multiplicity one in $M_I(w^{\min}\cdot\lambda)$ and all other subquotient are of the form $L(w'\cdot\lambda)$ with $w'>w^{\min}$, we have $r=m_x$. We now construct an open subset U with the claimed properties. We set

$$U=g(f^{-1}(V_{w^{\min}w_0}\cap\widehat{X}_{I,w^{\min}w_0,x_{\mathrm{dR}}}))\times\widehat{\mathcal{X}}_{\rho^p}^{p}\times\widehat{\mathbb{U}}^g,$$

where f and g are the maps of Theorem 4.7 and $V_{w^{\min}w_0}$ is the preimage of the Schubert cell $\underline{G}_L(1,w^{\min}w_0)\subset\underline{G}_L/\underline{B}\times\underline{G}_L/\underline{B}$ in $X_{I,w^{\min}w_0}$. This is an open and smooth subset of $X_{I,w^{\min}w_0}$: indeed, the maps f and g are formally smooth, the formal scheme $\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{qtri}}\longrightarrow\mathcal{X}_{\rho_p,\mathcal{R}}^{\mathrm{qtri}}$ is formally smooth and the point as ρ^p lies in the smooth locus of $\mathcal{X}^p$. $\square$

Remark 7.3. We would like to emphasize that Proposition 7.2 is the only place where we need to work with deformed objects to study the patching functors. Moreover the equalities in 2) and 3) where essentially proved in [BHS19] (in the proof of Thm. 5.3.3) but only at points which are in the smooth locus of the support of $\mathcal{M}_{\infty,x,\mathcal{R}}(M(w\cdot\lambda))$.

Proposition 7.4. *Assume that x_{pdR} is a smooth point of X_{ww_0}. Then*

$$\mathcal{M}_{\infty,x,\mathcal{R}}(M(w\cdot\lambda))\ \text{and}\ \mathcal{M}_{\infty,x,\mathcal{R}}(M(w\cdot\lambda)^{\vee})$$

are finite free $\mathcal{O}_{\overline{\mathcal{X}}_{\infty,x,\mathcal{R}}^{\mathrm{qtri},ww_0}}$-modules.

Proof. By Remark 6.3, the two $U(\mathfrak{t})$-module structures on $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}(w\cdot\lambda))$ coming from the $U(\mathfrak{t})$-action on $\widetilde{M}(w\cdot\lambda)$ and the one coming from the derivative of the locally analytic action, coincide. Thus we have the equality between $\mathcal{M}_{\infty,x,\mathcal{R}}(M(w\cdot\lambda))$ and the localisation

$$\mathcal{M}_{\infty,x,\mathcal{R}}(M(w\cdot\lambda))\simeq i_*i^*\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}(w\cdot\lambda)),$$

where $i:\widehat{T}^{\mathrm{sm}}\longrightarrow\widehat{T}$ denotes the inclusion of the closed subspace of smooth characters. A similar remark applies to the dual Verma module. In particular, it is enough to show that the $\mathcal{O}_{\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{qtri},ww_0}}$-modules

$$\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}(w\cdot\lambda))\ \text{and}\ \mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}^{\vee}(w\cdot\lambda))$$

are finite free. But these modules are Cohen-Macaulay with support the localization at x of $\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{qtri},ww_0}$, which is smooth. $\square$

7.2 Recollection on Bezrukavnikov's functor

The aim of this section (or even of the paper) is to identify the patching functor that takes objects in $\mathcal{O}_{\mathrm{alg}}$ (or more generally in $\mathcal{O}^{\infty}_{\mathrm{alg}}$) to Cohen-Macaulay modules on certain Galois deformation rings with a functor constructed by Bezrukavnikov in geometric representation theory (more precisely: with the pullback from our local models to the Galois deformation rings). Before doing so, we will need to recall the result of Bezrukavnikov.

Recall that $X = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ where $\mathfrak{g}$ is the Lie algebra of $\underline{G}_L = \prod_{v \in \Sigma}(L \times_{\mathbb{Q}_p} \mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_n)$ as in section 4.1 and denote by $X^\wedge$ the completion of X along the preimage of $\{(0,0)\} \in \mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}$ in X. Moreover, we write $\overline{X} = X \times_{\mathfrak{t}} \{0\}$, where the fiber product is taken with respect to the map $\kappa_1 : X \longrightarrow \mathfrak{t}$ of 4.1 that maps $(g\underline{B}, h\underline{B}, N)$ to $\mathrm{ad}(g^{-1})(N)$ $(\mathrm{mod}\ \mathfrak{n}) \in \mathfrak{t}$. As in the preceding sections we fix the shift

$$\delta'_{\underline{G}} = \det^{\frac{1-n}{2}} \delta_{\underline{G}} \in X^*(\underline{T})$$

of the half sum of the positive roots $\delta_{\underline{G}}$. As $\underline{G}_L$ is a product of (split) groups $\underline{G}_\tau$ isomorphic to $\mathrm{GL}_{n,L}$, the Lie algebra $\mathfrak{g} = \prod_{\tau \in \Sigma_F} \mathfrak{g}_\tau$, $X = \prod_{\tau \in \Sigma_F}$ and $\mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}$ splits accordingly. In particular, this applies also the the category $\mathcal{O}$. Let $\lambda = (\lambda_\tau)_{\tau \in \Sigma_F} \in X^*(T)$. The category $\mathcal{O}_{\chi_\lambda}$ identifies canonically with the Deligne's tensor product $\boxtimes_{\tau \in \Sigma_F} \mathcal{O}_{\chi_{\lambda_\tau}}$ of the categories $\mathcal{O}_{\chi_{\lambda_\tau}}$ where $\mathcal{O}_{\chi_{\lambda_\tau}}$ is the χ_{λ_τ}-block of the category $\mathcal{O}^{\mathfrak{g}_\tau, \mathfrak{b}_\tau}_{\mathrm{alg}}$. Given an object $M_\tau \in \mathcal{O}_{\chi_{\lambda_\tau}}$ for each $\tau \in \Sigma_F$, then the object $\boxtimes_{\tau \in \Sigma_F} M_\tau$ in the Deligne tensor products identifies to the tensor product over L of the M_τ's viewed as a $U(\mathfrak{g}_L) = \bigotimes_{L, \tau \in \Sigma_F} U(\mathfrak{g}_\tau)$-module.

Theorem 7.5 (Bezrukavnikov)**.** *Let $\lambda \in X^*(\underline{T})$ be a dominant character. There exists an exact functor*

$$\mathcal{B} : \mathcal{O}_{\chi_\lambda} \longrightarrow \mathrm{Coh}^{\underline{G}_L}(X^\wedge),$$

such that

1) for all $M \in \mathcal{O}_{\chi_\lambda}$ the sheaf $\mathcal{B}(M)$ is a Cohen-Macaulay sheaf,

2) for all $w \in W$ there is an isomorphism $\mathcal{B}(M(ww_0 \cdot \lambda)^\vee) \simeq \mathcal{O}_{\overline{X_w}}$,

3) for all $w \in W$ there is an isomorphism $\mathcal{B}(M(ww_0 \cdot \lambda)) \simeq \omega_{\overline{X_w}}$,

4) the image $\mathcal{B}(P(w_0 \cdot \lambda))$ of the anti-dominant projective $P(w_0 \cdot \lambda)$ is the structure sheaf $\mathcal{O}_{\overline{X}}$,

5) the image $\mathcal{B}(L(\lambda))$ of the algebraic representation $L(\lambda)$ is the line bundle $\mathcal{O}(-\delta'_{\underline{G}}) \boxtimes \mathcal{O}(-\delta'_{\underline{G}})$ on $\underline{G}_L/\underline{B} \times \underline{G}_L/\underline{B}$ which is viewed as a closed subscheme of $X^\wedge$ via

$$(g\underline{B}, h\underline{B}) \mapsto (g\underline{B}, h\underline{B}, 0).$$

6) For all $M = \boxtimes_{\tau \in \Sigma_F} M_\tau \in \mathcal{O}_{\chi_\lambda}$, we have

$$\mathcal{B}(M) = \boxtimes_\tau \mathcal{B}_\tau(M_\tau),$$

where $\mathcal{B}_\tau : \mathcal{O}_{\chi_{\lambda_\tau}} \longrightarrow \mathrm{Coh}^{\underline{G}_\tau}(X_\tau^\wedge)$ is Bezrukavnikov's functor for the group $\underline{G}_\tau$ and infinitesimal character χ_{λ_τ}. Here $\boxtimes_\tau$ denotes the exterior tensor product for (equivariant) coherent sheaves on each X_τ.

This result is (a small part of a result) due to Bezrukavnikov and his collaborators whose proof is spread out through the papers [Bez16, BR12, BL23, BR22]). For the convenience of the reader, we explain how to get the result in the previous form.

Proof. We actually construct directly $\mathcal{B}$ as a product satisfying 6) using [EGNO15, Proposition 1.11.2] and each $\mathcal{B}_\tau$. The resulting functor is exact as each $\mathcal{B}_\tau$ will be. As each object in points 1)-5) is of the form $\boxtimes_\tau M_\tau$, we can assume that we have fixed one τ and $\underline{G}$ is one of the $\underline{G}_\tau$, which we do until the end of this proof.

By the main result of [Bez16], there are reverse equivalence of categories

$$\Psi : D_{I^0,I^0} \leftrightarrow D^b(\mathrm{Coh}^{\underline{G}}_{\mathcal{N}}(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}})) : \Phi_{I^0,I^0},$$

where $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone and we have $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} = X$. Up to use translation functors, we can focus on the case $\lambda = 0$. By [Bez16, Corollary 42] the functor Ψ in fact takes values in ($\underline{G}$-equivariant) coherent sheaves on $\overline{X}$, when restricted to perverse sheaves $F \in \mathrm{Perv}_{\underline{N}}(\underline{G}/\underline{B})$. Moreover, the Beillinson–Bernstein localization theorem, more precisely by [BG99] Localization Theorem 2.2, and the Riemann–Hilbert correspondence provide an exact fully faithfull embedding of categories

$$\mathcal{O}_{\chi_0} \longrightarrow \mathrm{Perv}_{\underline{N}}(\underline{G}/\underline{B}).$$

Composing the Beillinson–Bernstein equivalence with Bezrukavnikov's functor (noting that the blocks $\mathcal{O}_{\chi_0}$ and $\mathcal{O}_{\chi_\lambda}$ are equivalent) we get the exact functor $\mathcal{B}$.

The properties 2), 3) and 4) follow from [Bez16, Ex. 57]. Denote $\mu = w_0 \cdot \lambda$ denote the antidominant weight in the dot-orbit of λ. Now the proof of [BL23, Proposition 5.8] implies that $\mathcal{B}(M(s \cdot \mu)^\vee) = \mathcal{O}_{\overline{X_s}}$ for all simple reflection s and $\mathcal{B}(P(\mu)) = \mathcal{O}_{\overline{X}}$. Bezrukavnikov's main result [Bez16, Theorem 1] implies that Ψ (hence $\mathcal{B}$) intertwines the convolutions on both sides. Here the convolution on the category $\mathcal{O}_{\chi_\lambda} \simeq \mathcal{O}_{\chi_0}$ is inherited from the convolution in $\mathrm{Perv}_{\underline{N}}(\underline{G}/\underline{N})$ (by pullback from $\mathrm{Perv}_{\underline{N}}(\underline{G}/\underline{B})$) defined as in [BR22, 7.]. We write $w = s_1 \dots s_r$ and compute convolutions on both sides. By [BR12, Theorem 2.2.1] we have

$$\mathcal{O}_{\overline{X_w}} = \mathcal{O}_{\overline{X_{s_1}}} \star \cdots \star \mathcal{O}_{\overline{X_{s_r}}}.$$

By [BR22, Lemma 7.7] we have $M(w \cdot \mu)^\vee = M(s_1 \cdot \mu)^\vee \star \cdots \star M(s_r \cdot \mu)^\vee$ and hence $\mathcal{B}(M(w \cdot \mu)^\vee) = \mathcal{O}_{\overline{X_w}}$. Moreover, by [BR12, Theorem 2.2.1] again, the dualizing sheaf of $\overline{X_w}$ is given by the convolution

$$\omega_{\overline{X_w}} = \omega_{\overline{X_{s_1}}} \star \cdots \star \omega_{\overline{X_{s_r}}}.$$

But [BR12, Proposition 1.10.3] implies that the inverse of $\mathcal{O}_{\overline{X_s}}$ for the convolution is $\omega_{\overline{X_s}}$, and as $\mathcal{B}$ is compatible with convolution, and as the inverse of $M(s\cdot\mu)^\vee$ is $M(s\cdot\mu)$ (again using [BR22, Lemma 7.7] for example), we deduce $\omega_{\overline{X_s}} = \mathcal{B}(M(s\cdot\mu))$. The point 5) is a consequence of [BL23, Lemma 6.7] (with $P = \underline{G}$).
Finally we prove 1). By points, 2), 3), [BHS19, Prop. 2.3.3](which follows from [BR12, Thm. 2.2.1] and the fact that any injective object of $\mathcal{O}_\lambda$ is a successive extension of dual Verma modules, we deduce that $\mathcal{B}(M)$ is Cohen-Macaulay of dimension $\dim \overline{X}$ for M injective. Each object of $\mathcal{O}_\lambda$ has a finite injective resolution. Therefore by exactness of $\mathcal{B}$ we conclude that for any object M in $\mathcal{O}_\lambda$, $\mathcal{B}(M)$ has a finite resolution by Cohen–Macaulay coherent sheaves having all the same dimension. It is easy to check that a coherent sheaf $\mathcal{F}$ on $\overline{X}$ having a finite resolution $\mathcal{F}[0] \longrightarrow \mathcal{R}^\bullet$ by objects $\mathcal{R}^n$, in degree $n \geqslant 0$, which are all Cohen–Macaulay of the same dimension is also Cohen–Macaulay of this dimension. Namely this is direct if the resolution has length 2 and we conclude by induction on the length of the resolution by cohomological shifting. □

Remark 7.6. Instead of constructing $\mathcal{B}$ as a product as above, we could also directly apply the previous results of Bezrukavnikov directly to the split group $\underline{G}_L$. The resulting functor, say $\mathcal{B}'$, will satisfy exactly the same Theorem 7.5 (with the same proof) except maybe point 6). Surely, $\mathcal{B}'$ should also satisfy 6), and thus $\mathcal{B} = \mathcal{B}'$, but we couldn't find a reference for this fact and this is beyond the scope of the present article.

Recall that we have fixed a point $x \in \mathcal{X}_\infty$ associated which we have defined the positive integer m_x in (8).

Corollary 7.7. *The functor $\mathcal{B}$ induces an exact functor*

$$\mathcal{B}_x : \mathcal{O}_{\chi_\lambda} \longrightarrow \mathrm{Coh}(\mathcal{X}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}})$$

such that, for all $M \in \mathcal{O}_{\chi_\lambda}$ the sheaf $\mathcal{B}_x(M)$ is a Cohen-Macaulay sheaf and such that

$$[\mathcal{M}_{\infty,x,\mathcal{R}}(M)] = m_x[\mathcal{B}_x(M)] \in Z^0(\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}).$$

Proof. Let $\underline{G}_1$ be the completion of $\underline{G}$ at the unit element. As the representations $(\rho_v)_{v|p}$ defined by the point x are crystalline and hence de Rham we may choose a basis α of $W(x) = \prod_{v\in\Sigma} W_{\mathrm{dR}}(D_{\mathrm{rig}}(\rho_{x,v})[1/t])$ and define a point x_{pdR} associated to x (or rather to the representations $(\rho_v)_{v|p}$) as in (6). For all $M \in \mathcal{O}_{\chi_\lambda}$, the sheaf $\mathcal{B}(M)$ is a $\underline{G}_L$-equivariant sheaf on $X^\wedge$ and hence gives rise to a $\underline{G}_1$-equivariant sheaf on $\widehat{X}_{x_{\mathrm{pdR}}}$. Now by [BHS19, Theorem 3.4.4. and Corollary 3.5.8], see also Theorem 4.7 above, we have a diagram

$$\begin{tikzcd} & \mathcal{X}^{\mathrm{qtri},\square}_{\infty,x,\mathcal{R}} \arrow[dl, "\pi"'] \arrow[dr, "W"] & \\ \mathcal{X}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}} & & X^\wedge_{x_{\mathrm{pdR}}}. \end{tikzcd}$$

More precisely, the map π forgets the deformation of the fixed basis α, and hence it is a $\underline{G}_1$-torsor. Moreover, W formally smooth and $\underline{G}_1$-equivariant for the natural left actions $g\cdot\widetilde{\alpha} := \widetilde{\alpha}\circ g^{-1}$ on the source (acting only on the deformation of the isomorphisms $\alpha_v : L\otimes_{\mathbb{Q}_p} F_v \xrightarrow{\sim} W_v$) and $g\cdot(kB, hB, N) = (gkB, ghB, g^{-1}Ng)$ on the target of W.

It follows that the pullback of $\mathcal{B}(M)^{\wedge}_{x_{\mathrm{pdR}}}$ at $\widehat{X}_{x_{\mathrm{pdR}}}$ along W is a $\underline{G}_1$-equivariant sheaf and hence descends to a coherent sheaf

$$\mathcal{B}_x(M)\in \mathrm{Coh}(\mathcal{X}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}).$$

It follows from the construction that $M\mapsto \mathcal{B}_x(M)$ and that $\mathcal{B}_x(M)$ is Cohen-Macaulay, as $\mathcal{B}(M)$ is. Moreover, $\mathcal{B}_x$ is exact, as W is formally smooth and hence flat.

It remains to check the assertion on cycles. But as taking cycles is additive and $\mathcal{B}_x$ is exact, we only need to check this equality on a generating set of the Grothendieck group of $\mathcal{O}_{\chi_\lambda}$, such as the Verma modules $M(w\cdot\mu)$. Hence the desired equality follows from the previous result on Bezrukavnikov's functor together with Proposition 7.2. $\square$

7.3 A detailed study of local models when $n = 3$

From now on we assume $n = 3$ until the end of section 7, so that the group $\underline{G}_L$ is

$$\underline{G}_L \simeq (\mathrm{Res}_{F\otimes_{\mathbb{Q}}\mathbb{Q}_p/\mathbb{Q}_p}\mathrm{GL}_3)\times_{\mathbb{Q}_p} L \simeq \prod_{v\in S_p}(L\times_{\mathbb{Q}_p}\mathrm{Res}_{F_v/\mathbb{Q}_p}\mathrm{GL}_{3,F_v}) \simeq \prod_{\tau\in\Sigma_F}\mathrm{GL}_{3,L}\,.$$

We identify the previous local Weyl group W with $\prod_\tau W_\tau$ and each W_τ with $W_{\mathrm{GL}_3}\simeq\mathfrak{S}_3$ and denote $s_{1,\tau}, s_{2,\tau}$ the two simple reflection corresponding to the choice of the upper Borel, and $w_{0,\tau} = s_{1,\tau}s_{2,\tau}s_{1,\tau}$ the longuest element in W_τ. If τ is understood, we often omit it from the notation.

As in section 4.1 we denote by X the Steinberg variety for the group

$$\underline{G} = \mathrm{Res}_{F\otimes_{\mathbb{Q}}\mathbb{Q}_p/\mathbb{Q}_p}\mathrm{GL}_3,$$

over L. As L is assumed to contain all Galois conjugates of F we have $X\simeq\prod_{\tau\in\Sigma_F}X_3$ (see Remark 4.6 for the notation X_3). The Steinberg variety X (resp. X_3) has dimension $9^{|\Sigma_F|}$ (resp. 9) and $6^{|\Sigma_F|}$ (resp. 6) irreducible components $X_w, w\in W$ (resp. $X_{3,w}, w\in\mathfrak{S}_3$), see e.g. [BHS19, Proposition 2.2.5].

Proposition 7.8. *For $w = (w_\tau)_{\tau\in\Sigma_F}$, let $s = |\{\tau\in\Sigma_F \mid w_\tau = w_0\}|$. Then the component X_w is smooth if and only if $s = 0$. Moreover, if $s\neq 0$, then the component X_w is Cohen–Macaulay but not Gorenstein. More precisely, let*

$$x_{\mathrm{pdR}} = (g\underline{B}, h\underline{B}, N) = (g_\tau\underline{B}_\tau, N_\tau, h_\tau\underline{B}_\tau)\in X_w(L) = \prod_{\tau\in\Sigma_F}X_{3,w_\tau}(L),$$

and assume that $N_\tau = 0$ when $w_\tau = w_0$. Then

$$\dim_L\omega_{X_w}\otimes k(x_{\mathrm{pdR}}) = 2^r,$$

where $r := |\{\tau \mid w_\tau = w_0,\ \text{and}\ g_\tau\underline{B}_\tau = h_\tau\underline{B}_\tau\}|$.

Proof. The smoothness is a consequence of Proposition 4.1. As $X = \prod_{\tau\in\Sigma_F} X_3$, it is enough to prove the analogous result for X_3 only. Indeed, by base change and composition of upper shriek functors, the dualizing sheaf of X is a derived tensor product $\bigotimes_\tau^{\mathbb{L}} p_\tau^*\omega_{X_3}$, where $p_\tau : X \longrightarrow X_3$ is projection to the τ-component. But as the product $X = \prod_\tau X_3$ is a product over a field, we find

$$\omega_X = \bigotimes_\tau p_\tau^*\omega_{X_3}.$$

Thus from now on we denote X_3 simply by X.

It is thus enough to prove that the fiber of $\omega_{X_{w_0}}$, is 2-dimensional at a point of the form $(g\underline{B}, 0, g\underline{B})$. Let $q : \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ denote the Grothendieck resolution, then $X \simeq \underline{G}_L \times^{\underline{B}} q^{-1}(\mathfrak{b})$. Moreover, $Y := q^{-1}(\mathfrak{b})$ decomposes into irreducible components $Y = \bigcup_{w\in W} Y_w$ such that $X_w \simeq \underline{G}_L \times^{\underline{B}} Y_w$. Hence it is enough to prove that $\omega_{Y_{w_0}}$ has fiber dimension 2 at the point $y_{\mathrm{pdR}} = (\underline{B}, 0)$. As X_{w_0} is Cohen-Macaulay and flat over $\mathfrak{t}$ (cf [BHS19, Proposition 2.2.3]), we have the base change formula $\omega_{X_{w_0}} \otimes_X \overline{X} \simeq \omega_{\overline{X_{w_0}}}$. We are thus reduced to compute the dualizing sheaf $\omega_{\overline{Y_{w_0}}}$ of the irreducible component

$$\overline{Y_{w_0}} = Y_{w_0} \times_{\mathfrak{t}} \{0\}$$

of $\overline{Y} = q^{-1}(\mathfrak{n})$. This scheme now has dimension 3 and we can use explicit computations.

A point of $\overline{Y}(L)$ is of the form $(g\underline{B}, N) \in (\underline{G}/\underline{B} \times \mathfrak{g})(L)$. We use the embedding $\underline{G}/\underline{B} \hookrightarrow \mathbb{P}^2_L \times (\mathbb{P}^2_L)^\vee$ that sents a full flag $(0 \subset \mathcal{L} \subset \mathcal{P} \subset k^3)$ to $(\mathcal{L} \subset k^3, \mathcal{P} \subset k^3)$. In homogeneous coordinates $([x_0 : x_1 : x_2], [y_0 : y_1 : y_2])$ the condition $\mathcal{L} \subset \mathcal{P}$ is given by $x_0y_0 + x_1y_1 + x_2y_2 = 0$. Let $\overline{Y}^0 \subset \overline{Y}$ denote the open subset defined by the condition $x_0 = y_2 = 1$. It is enough to compute on this open subset, as this is a neighborhood of the point $y_{\mathrm{pdR}} = (\underline{B}, 0) = ([1 : 0 : 0], [0 : 0 : 1])$. On $\overline{Y}^0$ we can thus remove y_0 from our equations. Let us write

$$N = \begin{pmatrix} 0 & u_{12} & u_{13} \\ & 0 & u_{23} \\ & & 0 \end{pmatrix}$$

for the universal matrix over $\overline{Y}^0$. The ideal defining

$$\overline{Y}^0_{w_0} \subset Z := \mathrm{Spec}(k[x_1, x_2, y_1, u_{12}, u_{23}, u_{13}])$$

is then given by

$$I_{w_0} = (u_{23}x_2, u_{12}(x_2 + x_1y_1), u_{12}x_1 + u_{13}x_2, u_{23}y_1 - u_{13}(x_2 + x_1y_1)).$$

We remark that we can replace $u_{12}(x_2+x_1y_1)$ by $u_{12}x_2-x_{13}x_2y_1$ using the third equation, and that automatically $y_1u_{12}u_{23} = 0$ using our new equation and $u_{23}y_1-u_{13}(x_2+x_1y_1) = 0$. We then check (e.g. using Macaulay2) that

$$0 \longrightarrow \mathcal{O}_Z^2 \xrightarrow{A'} \mathcal{O}_Z^6 \xrightarrow{A} \mathcal{O}_Z^5 \xrightarrow{A''} \mathcal{O}_Z$$

is a resolution of $\mathcal{O}_Z/I_{w_0}$, where

$$A' = \begin{pmatrix} y_1 & y_1u_{13}-u_{12} \\ -x_2 & 0 \\ x_1 & u_{23} \\ 0 & -u_{12}u_{23} \\ 0 & -x_2u_{23} \\ 0 & x_1u_{12}+x_2u_{12} \end{pmatrix}, \ A'' = \begin{pmatrix} x_1u_{12}+x_2u_{13} \\ x_2u_{23} \\ y_1u_{12}u_{23} \\ x_1y_1u_{13}-y_1u_{23}+x_2u_{13} \\ x_2y_1u_{13}-x_2u_{12} \end{pmatrix}^t$$

$$A = \begin{pmatrix} -x_2u_{23} & -y_1u_{23} & 0 & x_2 & -y_1u_{13} & 0 \\ x_1u_{12}+x_2u_{13} & y_1u_{13} & -y_1u_{12} & -y_1 & 0 & -y_1u_{13}+u_{12} \\ 0 & x_1 & x_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -x_2 & u_{12} & 0 \\ 0 & 0 & 0 & x_1 & u_{13} & u_{23} \end{pmatrix}.$$

Let $i : \overline{Y}^0_{w_0} \hookrightarrow Z$ denote the canonical closed embedding. Then the dualizing sheaf can be computed as $\omega_{\overline{Y}^0_{w_0}} = i^*\mathrm{Ext}^3_{\mathcal{O}_Z}(\mathcal{O}_{\overline{Y^0_{w_0}}}, \mathcal{O}_Z)$ which is given by

$$\omega_{\overline{Y}^0_{w_0}} \simeq \mathcal{O}_Z^2/ < (y_1, y_1u_{13}-u_{12}), (x_2, 0), (x_1, u_{12}), (0, u_{12}u_{23}) >,$$

as $x_2u_{23} = x_2u_{12} + x_2u_{12} = 0$ on $\overline{Y}^0_{w_0}$. It follows that the fiber of $\omega_{\overline{Y}^0_{w_0}}$ at y_{pdR} is 2-dimensional. □

Lemma 7.9. *Let $J \subset \Delta_{\mathrm{GL}_3}$.*

1. For $w \in W(\mathrm{GL}_3) \simeq \mathfrak{S}_3$ the component $X_{3,w}$ is smooth if $w \neq w_0$.

2. If $x_{\mathrm{pdR}} = (g\underline{B}_3, h\underline{B}_3, 0) \in X_{3,w_0}(L)$, with $g\underline{B}_3 \neq h\underline{B}_3$, then x_{pdR} is a smooth point of X_{3,w_0}.

3. For $\emptyset \neq J \subset \Delta_{\mathrm{GL}_3} = \{s_1, s_2\}$ the component $X_{3,J,\overline{w}}$ is smooth for any $\overline{w} \in W_J\backslash W_{\mathrm{GL}_3}$.

Proof. Point 1 is Proposition 7.8. For the point 2, denote w' the index of the Schubert stratum in which x_{pdR} lies. By [BHS19, Proposition 2.5.3(ii)] it is thus enough (as $\overline{U_{w_0}} = \mathrm{GL}_3/\underline{B}_3 \times \mathrm{GL}_3/\underline{B}_3$ is smooth) to prove that $\mathrm{codim}_{\mathfrak{t}}(\mathfrak{t}^{w_0w'^{-1}}) = \lg(w_0) - \lg(w')$. But this codimension is what we have denoted $\ell(w_0w'^{-1})$ in the proof of Proposition 4.1. As $w' \neq 1$ and $n = 3$, $w_0w'^{-1}$ is a product of distinct simple reflections thus $\ell(w_0w'^{-1}) = \lg(w_0w'^{-1}) = \lg(w_0) - \lg(w')$. For point 3, as $n = 3$ we have that $J = \{s_1\}, \{s_2\}$ or $J = \{s_1, s_2\}$. Denote $\underline{P} = \underline{P}_J$. In the case $J = \{s_1, s_2\}$, then $\underline{P}_J = \mathrm{GL}_3$ and $X_{3,J} = \widetilde{\mathfrak{g}}$ is smooth. It is sufficient to prove the case of $J = \{s_1\}$ (the other case is exactly the same), where an explicite computation gives the smoothness (alternatively, when w^{min} has length $\leqslant 1$, [BD, Corollary 5.3.4] also implies smoothness). □

Corollary 7.10. *Let $w = (w_\tau)_\tau \in W$ and let $I = \coprod_\tau I_\tau \subset \Delta$. Let $x_{\mathrm{pdR}} = (x_{\mathrm{pdR},\tau})_\tau = (g_\tau \underline{B}_\tau, h_\tau \underline{B}_\tau, N_\tau)$ be a point such that $N_\tau = 0$ whenever $I_\tau = \emptyset, w_\tau = 1$. If*

$$\mathcal{M}_{\infty,x,\mathcal{R}}(M_I(w^{\min} \cdot \lambda)) \quad \textit{(resp. } \mathcal{M}_{\infty,x,\mathcal{R}}(M_I(w^{\min} \cdot \lambda)^\vee)),$$

is not a finite free $\overline{\mathcal{X}}^{I-\mathrm{qtri},w^{\min}w_0}_{\infty,x,\mathcal{R}}$-module, then there exists an embedding τ such that $I_\tau = \emptyset$, $w_\tau = 1$ and $w_{x,\mathcal{R},\tau} = 1$.

Proof. Assume that there is no τ such that $I_\tau = \emptyset$ and $w_\tau = w_{x,\mathcal{R},\tau} = 1$. Lemma 7.9, then shows that the local model X_I is smooth at x_{pdR}. By 7.2 the support

$$\mathcal{X}^{I-\mathrm{qtri},ww_0}_{\infty,x,\mathcal{R}} = \mathrm{supp}\, \mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w \cdot \lambda))$$

is smooth. Thus $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w \cdot \lambda))$ is a free of rank m_x over $\mathcal{X}^{I-\mathrm{qtri},ww_0}_{\infty,x,\mathcal{R}}$. By Remark 6.3 its follows that $\mathcal{M}_{\infty,x,\mathcal{R}}(M_I(w \cdot \lambda))$ is a free of rank m_x over $\overline{\mathcal{X}}^{I-\mathrm{qtri},ww_0}_{\infty,x,\mathcal{R}}$.

The same argument also applies to $\mathcal{M}_{\infty,x,\mathcal{R}}(\widetilde{M}_I(w \cdot \lambda))$. □

Proposition 7.11. *For all $w \in W$ the sheaf $\mathcal{B}_x(L(w \cdot \lambda))$ is cyclic (we recall we are in the case $n = 3$). Moreover, for all $w \in W$ such that $ww_0 \geqslant w_{x,\mathcal{R}}$ the sheaf $\mathcal{M}_\infty(L(w \cdot \lambda))$ is free of rank m_x over its support.*

Proof. Recall that, for $w \in W$, Z_w is the closure in $\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$ of the preimage V_w of the Bruhat Cell $U_w = \underline{G}_L(1, w) \subset \underline{G}_L/\underline{B} \times \underline{G}_L/\underline{B}$. By [CG10, Prop. 3.3.4], V_w can be identified with the conormal bundle of U_w in $\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}} \simeq T^*(\underline{G}_L/\underline{B} \times \underline{G}_L/\underline{B})$. As $\mathfrak{g}$ is isomorphic to direct sum of copies of $\mathfrak{gl}_3$, the closure $\overline{U_w}$ of U_w in $\underline{G}_L/\underline{B} \times \underline{G}_L/\underline{B}$ is smooth, hence a local complete intersection. This proves that the conormal bundle of $\overline{U_w}$ is a closed smooth subscheme of $\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$ containing V_w as an open dense subset so that it coincides with Z_w and Z_w is smooth. This implies that $\mathfrak{Z}_w$ is a smooth. As $\mathcal{M}_{\infty,x,\mathcal{R}}(L(ww_0 \cdot \lambda))$ is Cohen–Macaulay, it follows from Proposition 7.2 3) and from the fact that $a_{w,w'} = 0$ for $w \neq w'$ (see [BHS19, Rk. 2.4.5]) that the sheaf $\mathcal{M}_{\infty,x,\mathcal{R}}(L(ww_0 \cdot \lambda))$ is locally free over its support. For the same reason, the support of the Cohen-Macaulay sheaf $\mathcal{B}_x(L(w \cdot \lambda))$ is Z_w, which is smooth, and thus the sheaf $\mathcal{B}_x(L(w \cdot \lambda))$ is free of rank 1 over its support (i.e. cyclic). □

7.4 The case of dual Vermas

For later use, let us recall the following Lemma.

Lemma 7.12. *Let R be a commutative local ring and let $I \subset J$ two ideals of R. Let $m \geqslant 1$ and $\pi : (R/I)^m \longrightarrow (R/J)^m$ a surjective R-linear map. Then there exist isomorphisms*

$$\varphi : (R/J)^m \longrightarrow (R/J)^m, \quad \psi : (R/I)^m \longrightarrow (R/I)^m$$

such that $\varphi \circ \pi = \pi \circ \psi = \mathrm{can}^{\oplus m}$ where $\mathrm{can} : R/I \longrightarrow R/J$ is the quotient map.

Proof. Let $(e_1, \dots, e_m)$ be the standard basis of $(R/I)^m$ as an (R/I)-module and $(f_1, \dots, f_m)$ the standard basis of $(R/J)^m$. Then $(\pi(e_1), \dots, \pi(e_m))$ is a generating family of $(R/J)^m$. As any generating family of cardinal m of a finite free module of rank m over a local ring is a basis (see Cor. to Prop. 6 in [Bou, Ch. 2§3.]), we see that $(\pi(e_1), \dots, \pi(e_m))$ is also a basis of $(R/J)^m$. Therefore we can define φ by the formula $\varphi(\pi(e_i)) = f_i$. Now, for any $1 \leqslant i \leqslant m$, let $f_i' \in (R/I)^m$ such that $\pi(f_i') = f_i$. By Nakayama Lemma the family $(f_1', \dots, f_m')$ generates $(R/I)^m$ and so is a basis of $(R/I)^m$. We can therefore define ψ by the formula $\psi(e_i) = f_i'$. $\square$

We will use the previous Corollary 7.10 to start a devissage which will be assured by the following Lemma. Note that in Lemma 7.13 below, we don't need to assume that $n = 3$.

Lemma 7.13. *Let M be an object of $\mathcal{O}_{\chi_\lambda}$ and let $Q_1, \dots, Q_r$ be quotients of M. Let Q be the smallest quotient of M dominating all the Q_i, i.e. $Q = M/(M_1 \cap \dots \cap M_r)$ where $M_i = \mathrm{Ker}(M \to Q_i)$ for $1 \leqslant i \leqslant r$. We assume that*

(i) for any $1 \leqslant i \leqslant r$, the sheaf $\mathcal{M}_{\infty,x,\mathcal{R}}(Q_i)$ is free of rank m_x over it support;

(ii) for any $1 \leqslant i \leqslant r$, the sheaf $\mathcal{B}_x(Q_i)$ is cyclic (generated by one element);

(iii) for any $1 \leqslant i \leqslant r$, $\mathrm{Supp}\,\mathcal{M}_{\infty,x,\mathcal{R}}(Q_i) = \mathrm{Supp}\,\mathcal{B}_x(Q_i)$;

(iv) the sheaf $\mathcal{B}_x(Q)$ is cyclic.

Then the sheaf $\mathcal{M}_{\infty,x,\mathcal{R}}(Q)$ is free of rank m_x over its support and

$$\mathrm{Supp}(\mathcal{M}_{x,\infty,\mathcal{R}}(Q) = \mathrm{Supp}(\mathcal{B}_x(Q)).$$

Proof. To ease notation we note $m = m_x$. Let's prove the result when $r = 2$. Let $A = \overline{R}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ be the ring of global sections of $\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ and let $I_i = \mathrm{Ann}(\mathcal{B}_x(Q_i))$ for $i \in \{1, 2\}$. Define Q_0 the largest common quotient of Q_1 and Q_2, i.e. $Q_0 = M/(M_1+M_2)$. Then we have a short exact sequence

$$0 \longrightarrow Q \longrightarrow Q_1 \oplus Q_2 \longrightarrow Q_0 \longrightarrow 0,$$

where the map $Q_1 \oplus Q_2 \to Q_0$ is given by $(x, y) \mapsto x - y$. By exactness of $\mathcal{M}_{\infty,x,\mathcal{R}}$, we have a short exact sequence

$$0 \longrightarrow \mathcal{M}_{\infty,x,\mathcal{R}}(Q) \longrightarrow \mathcal{M}_{\infty,x,\mathcal{R}}(Q_1) \oplus \mathcal{M}_{\infty,x,\mathcal{R}}(Q_2) \longrightarrow \mathcal{M}_{\infty,x,\mathcal{R}}(Q_0) \longrightarrow 0.$$

We fix isomorphisms $(A/I_i)^m \xrightarrow{\sim} \mathcal{M}_{\infty,x,\mathcal{R}}(Q_i)$ for $i \in \{1, 2\}$. As Q_0 is a quotient of both Q_1 and Q_2, we have surjective maps

$$(A/I_i)^m \longrightarrow \mathcal{M}_{\infty,x,\mathcal{R}}(Q_i) \longrightarrow \mathcal{M}_{\infty,x,\mathcal{R}}(Q_0),$$

whose composite factors through $(A/(I_1 + I_2))^m$. Using Lemma 7.12 we can choose the previous isomorphisms such that the following diagram commutes

$$\begin{tikzcd}
(A/I_1)^m \oplus (A/I_2)^m \arrow[r, "(x,y)\mapsto x-y"] \arrow[d, "\simeq"] & A/(I_1+I_2)^m \arrow[r] \arrow[d, two heads] & 0 \\
\mathcal{M}_{\infty,x,\mathcal{R}}(Q_1) \oplus \mathcal{M}_{\infty,x,\mathcal{R}}(Q_2) \arrow[r] & \mathcal{M}_{\infty,x,\mathcal{R}}(Q_0) \arrow[r] & 0.
\end{tikzcd} \tag{9}$$

As the kernel of the upper horizontal map is isomorphic to $(A/(I_1 \cap I_2))^m$, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (A/(I_1\cap I_2))^m \arrow[r] \arrow[d, hook] & (A/I_1)^m \oplus (A/I_2)^m \arrow[r] \arrow[d, "\simeq"] & A/(I_1+I_2)^m \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathcal{M}_{\infty,x,\mathcal{R}}(Q) \arrow[r] & \mathcal{M}_{\infty,x,\mathcal{R}}(Q_1) \oplus \mathcal{M}_{\infty,x,\mathcal{R}}(Q_2) \arrow[r] & \mathcal{M}_{\infty,x,\mathcal{R}}(Q_0) \arrow[r] & 0.
\end{tikzcd} \tag{10}$$

As $\mathrm{Ann}(\mathcal{B}_x(Q)) = I_1 \cap I_2$ and $\mathcal{B}_x(Q)$ is cyclic, there exists an isomorphism $\mathcal{B}_x(Q) \simeq A/(I_1 \cap I_2)$. Moreover, by hypothesis, we have $\mathrm{Supp}(\mathcal{B}_x(Q_i)) = \mathrm{Spec}(A/I_i)$ so that the maps $A/(I_1 \cap I_2) \simeq \mathcal{B}_x(Q) \twoheadrightarrow \mathcal{B}_x(Q_i)$ factors through isomorphisms $A/I_i \simeq \mathcal{B}_x(Q_i)$. Therefore, by exactness of $\mathcal{B}_x$, we also have a commutatif diagram

$$\begin{tikzcd}
0 \arrow[r] & (A/(I_1\cap I_2)) \arrow[r, "x\mapsto(x,x)"] \arrow[d, "\simeq"] & (A/I_1) \oplus (A/I_2) \arrow[d, "\simeq"] & & \\
0 \arrow[r] & \mathcal{B}_x(Q) \arrow[r] & \mathcal{B}_x(Q_1) \oplus \mathcal{B}_x(Q_2) \arrow[r] & \mathcal{B}_x(Q_0) \arrow[r] & 0.
\end{tikzcd}$$

This implies that we have an isomorphism $A/(I_1 + I_2) \simeq \mathcal{B}_x(Q_0)$. As $\mathcal{B}_x(Q_0)$ is Cohen–Macaulay, so is $A/(I_1 + I_2)$. As the ring $A/(I_1 + I_2)$ is Cohen–Macaulay, the vertical right arrow of diagram (9) is a surjective map $(A/(I_1 + I_2))^m \twoheadrightarrow \mathcal{M}_{\infty,x,\mathcal{R}}(Q_0)$ between two Cohen–Macaulay modules with the same cycle by Corollary 7.7. It is therefore an isomorphism and the Snake Lemma allows us to conclude that the left vertical arrow in (10) is an isomorphism.

Assume that the result is proved for some integer $r \geqslant 2$. Let $Q_1, \dots, Q_{r+1}$ be quotients of M satisfying the hypotheses of the Lemma. Let Q' be the smallest quotient of M dominating all the Q_i for $1 \leqslant i \leqslant r$. Note that $\mathcal{B}_x(Q')$ is a quotient of $\mathcal{B}_x(Q)$ and is therefore cyclic. By induction, $\mathcal{M}_{\infty,x,\mathcal{R}}(Q')$ is free of rank m over its support and $\mathrm{Supp}\,\mathcal{M}_{\infty,x,\mathcal{R}}(Q') = \mathrm{Supp}\,\mathcal{B}_x(Q')$. The quotient Q is now the smallest quotient of M dominating Q' and Q_{r+1}. Therefore the case $r = 2$ implies that $\mathcal{M}_{\infty,x,\mathcal{R}}(Q)$ is free of rank m over its support and $\mathrm{Supp}\,\mathcal{M}_{\infty,x,\mathcal{R}}(Q) = \mathrm{Supp}\,\mathcal{B}_x(Q)$, which concludes the induction. □

Proposition 7.14. *The coherent sheaf $\mathcal{M}_{\infty,x,\mathcal{R}}(M(\lambda)^\vee)$ is locally free of rank m_x over its support. We recall that the hypothesis $n = 3$ is in force in this statement.*

Proof. Recall that $W = \prod_{\tau: F \hookrightarrow L} W_\tau$ and write $w_{x,\mathcal{R}} = (w_{x,\tau})$. Let $J \subset \mathrm{Hom}(F, L)$ be the set embeddings such that $w_{x,\tau} = 1$. Let E be the set of elements $w = (w_v) \in W$ such that $w_\tau \in \{s_1, s_2\}$ if $\tau \in J$ and $w_\tau = 1$ if $\tau \notin J$. By Corollary 7.10 and Theorem 7.5 2), for $w \in E$, the module $\mathcal{M}_{\infty,x,\mathcal{R}}(M(w \cdot \lambda)^\vee)$ is free of rank m_x over its support and $\mathcal{M}_{\infty,x,\mathcal{R}}(M(w \cdot \lambda)^\vee) = \mathcal{B}_x(M(w \cdot \lambda)^\vee)^{m_x}$. Let Q be the smallest quotient of $M(\lambda)^\vee$ dominating all the $M(w \cdot \lambda)^\vee$ for $w \in E$. Lemma 7.13 implies that $\mathcal{M}_{\infty,x,\mathcal{R}}(Q)$ is free of rank m_x over its support and $\mathcal{M}_{\infty,x,\mathcal{R}}(Q) = \mathcal{B}_x(Q)^{m_x}$. Let N be the kernel of the map $M \twoheadrightarrow Q$.

Let I of the form $\prod_{\tau \in J}\{s_{i_\tau}\}$ where $i_\tau \in \{1, 2\}$. Then the image of the map $M_I(\lambda)^\vee \hookrightarrow M(\lambda)^\vee \twoheadrightarrow Q$ is $Q_I \coloneqq \boxtimes_{\tau \in J} L(s_{3-i_\tau} \cdot \lambda_\tau) \boxtimes_{\tau \notin J} M(\lambda_\tau)^\vee$. By Corollary 7.10, the module $\mathcal{M}_{\infty,x,\mathcal{R}}(M_I(\lambda)^\vee)$ is free of rank m_x over its support. Thus $\mathcal{M}_{\infty,x,\mathcal{R}}(Q_I)$ is generated by m_x elements, and the quotient

$$L_I \coloneqq \bigboxtimes_{\tau \in J} L(s_{3-i_\tau} \cdot \lambda_\tau) \boxtimes \bigboxtimes_{\tau \notin J} M(w_{x,\tau} w_0 \cdot \lambda_\tau)^\vee,$$

of Q_I satisfies

$$\mathcal{M}_{\infty,x,\mathcal{R}}(L_I) = \mathcal{M}_{\infty,x,\mathcal{R}}\Big(\bigboxtimes_{\tau \in J} L(s_{3-i_\tau} \cdot \lambda_\tau) \boxtimes \bigboxtimes_{\tau \notin J} L(w_{x,\tau} w_0 \cdot \lambda_\tau)\Big).$$

by Proposition 7.2. Moreover, by Proposition 7.11, this module is free of rank m_x over its support so that its fiber at x has dimension m_x. This implies that the following surjective maps are all isomorphisms

$$\begin{aligned} k(x)^{m_x} \simeq \mathcal{M}_{\infty,x,\mathcal{R}}(M_I(\lambda)^\vee) \otimes k(x) &\xrightarrow{\sim} \mathcal{M}_{\infty,x,\mathcal{R}}(Q_I) \otimes k(x) \\ &\xrightarrow{\sim} \mathcal{M}_{\infty,x,\mathcal{R}}(L_I) \otimes k(x) \simeq k(x)^{m_x}. \end{aligned}$$

As moreover $\mathrm{Ker}(M_I(\lambda)^\vee \to Q_I) = N \cap M_I(\lambda)^\vee$, we see that the map

$$\mathcal{M}_{\infty,x,\mathcal{R}}(N \cap M_I(\lambda)^\vee) \otimes k(x) \longrightarrow \mathcal{M}_{\infty,x,\mathcal{R}}(M(\lambda)^\vee) \otimes k(x)$$

is zero. As $M(\lambda)^\vee$ is multiplicity-free, we have $N = \sum_I (N \cap M_I(\lambda)^\vee)$ and we conclude that the map

$$\mathcal{M}_{\infty,x,\mathcal{R}}(N) \otimes k(x) \longrightarrow \mathcal{M}_{\infty,x,\mathcal{R}}(M(\lambda)^\vee) \otimes k(x)$$

is zero. Therefore $\mathcal{M}_{\infty,x,\mathcal{R}}(M(\lambda)^\vee) \otimes k(x) \simeq \mathcal{M}_{\infty,x,\mathcal{R}}(Q) \otimes k(x) \simeq k(x)^{m_x}$. It follows from Nakayama Lemma and the first part of Proposition 7.2 2) that we have a surjection

$$\mathcal{O}^{m_x}_{\overline{\mathcal{X}}^{\mathrm{qtri},w_0}_{\infty,x,\mathcal{R}}} \twoheadrightarrow \mathcal{M}_{\infty,x,\mathcal{R}}(M(\lambda)^\vee).$$

These modules are both Cohen–Macaulay of the same dimension with identical associated maximal cycle by the last assertion in Proposition 7.2 2). Therefore this map is an isomorphism. $\square$

7.5 The isomorphism between the two functors

We recall that we assumed that $n = 3$.

Theorem 7.15. *There is an isomorphism of coherent sheaves $\mathcal{O}^{m_x}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}} \simeq \mathcal{M}_{\infty,x,\mathcal{R}}(P(w_0\cdot\lambda))$.*

Proof. Recall that $A = U(\mathfrak{t})_{\mathfrak{m}}$ and set $D := L\otimes_{A^W} A$. By Proposition 2.17, the action of $Z(\mathfrak{g})$ on $P(w_0\cdot\lambda)$ induces a structure of D-module on $P(w_0\cdot\lambda)$. As $M(\lambda)^\vee$ is an injective object, it follows from [Soe90, Prop. 6], that $M(\lambda)^\vee \simeq P(w_0\cdot\lambda)\otimes_D (D/\mathfrak{m}_D)$, where $\mathfrak{m}_D$ is the maximal ideal of D. We have also a local map of local algebras $\alpha : D \to \mathcal{O}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}}$ defined in section 6.3. It follows from Corollary 6.14 that these define the same action of D on $\mathcal{M}_{\infty,x,\mathcal{R}}(P(w_0\cdot\lambda))$. As moreover the functor $\mathcal{M}_{\infty,x,\mathcal{R}}$ is exact, we have an isomorphism $\mathcal{M}_{\infty,x,\mathcal{R}}(M(\lambda)^\vee) \simeq \mathcal{M}_{\infty,x,\mathcal{R}}(P(w_0\cdot\lambda))\otimes_D (D/\mathfrak{m}_D)$. As moreover the map $A\otimes_{A^W} A \to \mathcal{O}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}}$ is a local map of local rings, we have an isomorphism $\mathcal{M}_{\infty,x,\mathcal{R}}(P(w_0\cdot\lambda))\otimes k(x) \xrightarrow{\sim} \mathcal{M}_{\infty,x,\mathcal{R}}(M(\lambda)^\vee)\otimes k(x)$ and thus $\dim_L \mathcal{M}_{\infty,x,\mathcal{R}}(P(w_0\cdot\lambda))\otimes k(x) = m_x$ by Proposition 7.14.

It follows from Corollary 6.17 that we have a surjection

$$\mathcal{O}^{m_x}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}} \twoheadrightarrow \mathcal{M}_{\infty,x,\mathcal{R}}(P(w_0\cdot\lambda)^\vee).$$

These modules are both Cohen–Macaulay of the same dimension with identical associated maximal cycle by Corollary 7.7. Therefore this map is an isomorphism. □

Recall that the map (κ_1,κ_2) of section 4.1 provides a map $A\otimes_{A^W} A \to R^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ and thus a map $C := L\otimes_{A^W} A \to \overline{R}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$. We use this map to see $\overline{R}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ as a local C-algebra. Recall from Proposition 2.17 that the map

$$(\mathrm{Id}\otimes\mathrm{Ad}(w_0))\circ t_\lambda\circ\xi : Z(\mathfrak{g}) \longrightarrow L\otimes_{U(\mathfrak{t})^W_{\mathfrak{m}}} U(\mathfrak{t})_{\mathfrak{m}} \simeq C$$

is surjective and that its kernel of $\widetilde{\xi}$ coincides with the kernel of the natural map $Z(\mathfrak{g}) \to \mathrm{End}(P(w_0\cdot\lambda))$. For any object M of $\mathcal{O}_{\chi_\lambda}$, this provides a structure of C-module on $\mathbb{V}(M) := \mathrm{Hom}_{\mathcal{O}_{\mathrm{alg}}}(P(w_0\cdot\lambda), M)$ (note that the twist by $\mathrm{Ad}(w_0)$ compared to [Soe90] comes from Remark 6.15).

Proposition 7.16. *There is an isomorphism of functors $\mathcal{B}^{m_x}_x \simeq \mathcal{M}_{\infty,x,\mathcal{R}}$ on the full subcategory of $\mathcal{O}_{\chi_\lambda}$ whose objects are the injective objects of $\mathcal{O}_{\chi_\lambda}$.*

Proof. Let M be an injective object of the category $\mathcal{O}_{\chi_\lambda}$. By [Soe90, Prop. 6], the canonical map $P(w_0\cdot\lambda)\otimes_{Z(\mathfrak{g})}\mathbb{V}(M) \to M$ is an isomorphism. If $\mathcal{F}$ is $\mathcal{M}_{\infty,x,\mathcal{R}}$ or $\mathcal{B}_x$, there is therefore an isomorphism of functors on the subcategory of injective objects of $\mathcal{O}_{\chi_\lambda}$, $\mathcal{F}(-) \simeq \mathcal{F}(P(w_0\cdot\lambda))\otimes_C \mathbb{V}(-)$. This follows from the exactness of $\mathcal{F}$ and Corollary 6.14 for $\mathcal{F} = \mathcal{M}_{\infty,x,\mathcal{R}}$ and from [Bez16, Prop. 23] for $\mathcal{F} = \mathcal{B}_x$. Therefore the result follows from Theorem 7.15 and Theorem 7.5 4). □

Corollary 7.17. *There exists an isomorphism of functors $\mathcal{B}_x^{m_x} \simeq \mathcal{M}_{\infty,x,\mathcal{R}}$.*

Proof. As the functors $\mathcal{B}_x^{m_x}$ and $\mathcal{M}_{\infty,x,\mathcal{R}}$ are both exact, this follows from Proposition 7.16 and [Gab62, Cor. 2 to Prop. 14]. □

Corollary 7.18. *Let Q be a quotient of the anti-dominant projective $P(w_0 \cdot \lambda)$ in the category $\mathcal{O}_{\chi_\lambda}$. If $\mathcal{M}_{\infty,x,\mathcal{R}}(Q) \neq 0$, then it is finite free of rank m_x over its support and its support is Cohen–Macaulay.*

Proof. As $\mathcal{B}_x(P(w_0 \cdot \lambda))$ is generated by one element so is $\mathcal{B}_x(Q)$ which is thus free of rank 1 over its support. It follows from Corollary 7.17, that $\mathcal{M}_{\infty,x,\mathcal{R}}(Q)$ is free of rank m_x over its support. □

Corollary 7.19. *For all $w \in W$, the coherent sheaf*

$$\mathcal{M}_{\infty,x,\mathcal{R}}(P(w \cdot \lambda)^\vee),$$

is free of rank m_x over its support.

Proof. By Corollary 7.18, it is sufficient to prove that $\mathcal{M}_{\infty,x,\mathcal{R}}(P(w \cdot \lambda)^\vee)$ is non zero and that there exists a surjective map

$$P(w_0 \cdot \lambda) \longrightarrow P(w \cdot \lambda)^\vee.$$

As $P(w_0 \cdot \lambda)$ is the projective envelope of $L(w_0 \cdot \lambda)$, this is equivalent to showing that the socle of $P(w \cdot \lambda)$ is isomorphic to $L(w_0 \cdot \lambda)$. By [Str03, Thm. 8.1], the socle of $P(w \cdot \lambda)$ is isomorphic to $L(w_0 \cdot \lambda)^m$ with $m = [P(w \cdot \lambda) : M(\lambda)] = [M(\lambda) : L(w \cdot \lambda)]$ by [Hum08, Thm. 3.9]. As $\mathfrak{g}$ is isomorphic to a direct sum of copies of $\mathfrak{gl}_{3,L}$, we have $[M(\lambda) : L(w \cdot \lambda)] = 1$ for any $w \in W$.

Moreover, as $[M(\lambda) : L(\lambda)] = 1$, we have

$$[P(w \cdot \lambda)^\vee : L(\lambda)] = [P(w \cdot \lambda) : L(\lambda)] = 1.$$

As $\mathcal{M}_{\infty,x,\mathcal{R}}(L(\lambda)) \neq 0$, we have $\mathcal{M}_{\infty,x,\mathcal{R}}(P(w \cdot \lambda)^\vee) \neq 0$. □

Theorem 7.20. *For all $w \in W$, with $ww_0 \geqslant w_{x,\mathcal{R}}$, the coherent sheaf $\mathcal{M}_{\infty,x,\mathcal{R}}(M(w \cdot \lambda)^\vee)$ is isomorphic to $\mathcal{O}^{\oplus m_x}_{\overline{\mathcal{X}}^{\mathrm{qtri},ww_0}_{\infty,x,\mathcal{R}}}$. For all $w \in W$, with $ww_0 \geqslant w_{x,\mathcal{R}}$, the coherent sheaf $\mathcal{M}_{\infty,x,\mathcal{R}}(M(w \cdot \lambda))$ is isomorphic to*

$$\left(\omega_{\overline{\mathcal{X}}^{\mathrm{qtri},ww_0}_{\infty,x,\mathcal{R}}}\right)^{\oplus m_x}.$$

Proof. This is direct consequence of Corollary 7.17 and Theorem 7.5 2) and 3). □

7.6 Duality

We recall that we assumed that $n = 3$.

For a Cohen–Macaulay sheaf $\mathcal{F}$ on $\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ of dimension $\dim \overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$, we write $\omega^{\bullet}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}}$ for the dualizing complex and the complex

$$\mathbb{D}(\mathcal{F}) := R\operatorname{Hom}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}}(\mathcal{F}, \omega^{\bullet}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}})[-\dim \overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}].$$

is concentrated in degree 0. We denote

$$\mathcal{F}^{\vee} = H^0(\mathbb{D}(\mathcal{F})) = \operatorname{Hom}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}}(\mathcal{F}, \omega_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}})$$

the degree 0 coherent sheaf to which we refer as the *shifted* Serre dual of $\mathcal{F}$, where

$$\omega_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}} := H^{-\dim \overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}}(\omega^{\bullet}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}}) = \mathcal{O}^{\vee}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}}.$$

Lemma 7.21. *Let $\mathcal{F}$ be a maximal Cohen–Macaulay coherent sheaf over $\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$. Then $[\mathcal{F}^{\vee}] = [\mathcal{F}]$. As a consequence if $\mathcal{Y} \subset \overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ is a maximal Cohen–Macaulay closed subscheme (i.e. a closed subscheme whose structure sheaf is a maximal Cohen–Macaulay coherent sheaf), we have $[\omega_{\mathcal{Y}}] = [\mathcal{Y}]$.*

Proof. Let R be a local complete regular ring such that $\mathcal{O}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}}$ is isomorphic to a quotient of R. Then we can compute $\mathcal{F}^{\vee}$ by the formula $\mathcal{F}^{\vee} = \operatorname{Ext}^d_R(\mathcal{F}, R)$ where d is the codimension of $\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ in $\operatorname{Spec}(R)$. By definition, we have $[\mathcal{F}] = \sum_z a(z)z$ where the sum is over all maximal points in $\operatorname{Supp}(\mathcal{F})$ and $a(z)$ is the length of the finite length R_z-module $\mathcal{F}_z$. Let $z \in \operatorname{Spec}(R)$ be a maximal point of the support of $\mathcal{F}$. The localization R_z of R at z is a local regular ring and we have $\mathcal{F}_z^{\vee} \simeq \operatorname{Ext}^d_{R_z}(\mathcal{F}_z, R_z)$. As $\operatorname{Ext}^d_{R_z}(-, R_z)$ is a an exact functor on the subcategory of finite length R_z-modules and $\dim_{k(z)} \operatorname{Ext}^d_{R_z}(k(z), R_z) = 1$, the length of the R_z-module $\operatorname{Ext}^d_{R_z}(\mathcal{F}_z, R_z)$ is $a(z)$. So we have proved the claim. □

Proposition 7.22. *Let M be a subobject of the anti-dominant projective $P(w_0 \cdot \lambda)$. Assume that $\mathcal{M}_{\infty,x,\mathcal{R}}(M) \neq 0$ and let $\mathcal{Y}$ be the support of $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$. Then $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ is isomorphic to $\omega_{\mathcal{Y}}^{m_x}$ and $\mathcal{Y}$ is Cohen–Macaulay.*

Proof. Let Q be the quotient of $P(w_0 \cdot \lambda)$ by M. If $\mathcal{M}_{\infty,x,\mathcal{R}}(Q) = 0$, then Theorem 7.15 implies the result. So we can assume that $\mathcal{M}_{\infty,x,\mathcal{R}}(M) \neq 0$ and $\mathcal{M}_{\infty,x,\mathcal{R}}(Q) \neq 0$. By Corollary 7.18, $\mathcal{M}_{\infty,x,\mathcal{R}}(Q)$ is isomorphic to $\mathcal{O}_{\mathcal{Z}}^{m_x}$ for $\mathcal{Z} \subset \overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ maximal Cohen–Macaulay. Using Lemma 7.12, we can construct a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}_{\infty,x,\mathcal{R}}(M) \arrow[r] \arrow[d, "\simeq"] & \mathcal{M}_{\infty,x,\mathcal{R}}(P(w_0 \cdot \lambda)) \arrow[r] \arrow[d, "\simeq"] & \mathcal{M}_{\infty,x,\mathcal{R}}(Q) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathrm{Ker} \arrow[r] & \mathcal{O}^{m_x}_{\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}} \arrow[r, "\mathrm{can}^{m_x}"] & \mathcal{O}_{\mathcal{Z}}^{m_x} \arrow[r] & 0.
\end{tikzcd}$$

Let I be the ideal defining $\mathcal{Z}$ so that $\mathrm{Ker} \simeq I^{m_x}$, so we can assume $m_x = 1$. As $\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ and $\mathcal{Z}$ are Cohen–Macaulay of the same dimension, then I is also Cohen-Macaulay of the same dimension. In particular its support is determined by its cycle and Lemma 7.21 gives $[I^\vee] = [I]$. But as $\overline{\mathcal{X}}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}$ is a complete intersection, the dual of the previous bottom sequence gives

$$I^\vee \simeq \mathcal{O}_{\mathcal{Y}},$$

with $\mathcal{Y} = \mathrm{Supp}(I^\vee) = \mathrm{Supp}(I)$, which is thus Cohen-Macaulay, and thus $I = \omega_{\mathcal{Y}}$. $\square$

Remark 7.23. Actually Proposition 7.22 is also true with the functors $\mathcal{B}$ and $\mathcal{B}_x$ of Theorem 7.5 and Corollary 7.7, with the same proof. For $\mathcal{B}$, this uses that the dualizing sheaf of $\overline{X}$ is the structure sheaf. This is true as the closed immersion given as the composite $\overline{X} \hookrightarrow \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} \hookrightarrow \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$ is a global relative complete intersection and the latter has a trivial dualizing module, by [MvdK92, Lemma 2.3], and [Sta24, Lemma 0AA2, 0AA3].

We choose for all λ dominant weight, and all $w \in W$ a surjective map $\pi_w : P(w_0 \cdot \lambda) \longrightarrow P(w \cdot \lambda)^\vee$ (see proof of Corollary 7.19).

Lemma 7.24. *For all map $f_{w,w'} : P(w \cdot \lambda)^\vee \longrightarrow P(w' \cdot \lambda)^\vee$ there exists a map $\widetilde{f}_{w,w'} : P(w_0 \cdot \lambda) \longrightarrow P(w_0 \cdot \lambda)$ such that the following diagram commutes*

$$\begin{tikzcd} P(w_0 \cdot \lambda) \arrow[r, "\widetilde{f}_{w,w'}"] \arrow[d, "\pi_w"] & P(w_0 \cdot \lambda) \arrow[d, "\pi_{w'}"] \\ P(w \cdot \lambda)^\vee \arrow[r, "f_{w,w'}"] & P(w' \cdot \lambda)^\vee \end{tikzcd} \tag{11}$$

Proof. As $\pi_{w'} : P(w_0 \cdot \lambda) \longrightarrow P(w' \cdot \lambda)^\vee$ is surjective and $P(w_0 \cdot \lambda)$ is projective, the map $\mathrm{Hom}(P(w_0 \cdot \lambda), P(w_0 \cdot \lambda)) \longrightarrow \mathrm{Hom}(P(w_0 \cdot \lambda), P(w' \cdot \lambda)^\vee)$ is surjective, thus there exists $\widetilde{f}_{w,w'}$ mapping to $f_{w,w'} \circ \pi_w$. This proves the claim. $\square$

Lemma 7.25. *Let $\mathcal{F}$ be either $\mathcal{B}_x$ or $\mathcal{M}_{\infty,x,\mathcal{R}}$. There exists a family of isomorphisms indexed by $w \in W$*

$$\Psi_w : \mathcal{F}(P(w \cdot \lambda)^\vee) \xrightarrow{\sim} \mathcal{F}(P(w \cdot \lambda))^\vee.$$

such that for any $w, w' \in W$ and any if $f_{w,w'} : P(w \cdot \lambda)^\vee \longrightarrow P(w' \cdot \lambda)^\vee$, the following diagram commutes

$$\begin{tikzcd} \mathcal{F}(P(w \cdot \lambda)^\vee) \arrow[r, "\mathcal{F}(f_{w,w'})"] \arrow[d, "\Psi_w"] & \mathcal{F}(P(w' \cdot \lambda)^\vee) \arrow[d, "\Psi_{w'}"] \\ \mathcal{F}(P(w \cdot \lambda))^\vee \arrow[r, "\mathcal{F}(f^\vee_{w,w'})^\vee"] & \mathcal{F}(P(w' \cdot \lambda))^\vee \end{tikzcd} \tag{12}$$

where we denote by the same symbol $(\cdot)^\vee$ the duality in $\mathcal{O}$ and Serre duality on coherent sheaves.

Proof. Let $w \in W$. The sheaves $\mathcal{F}(P(w\cdot\lambda)^\vee)$ and $\mathcal{F}(P(w\cdot\lambda))^\vee$ are isomorphic to the same quotient of $\mathcal{F}(P(w_0\cdot\lambda))$ by Theorem 7.5 for $\mathcal{B}_x$ and Corollary 7.19 and Proposition 7.22 for $\mathcal{M}_{\infty,x,\mathcal{R}}$. This implies that there exists an isomorphism $\Psi_w : \mathcal{F}(P(w\cdot\lambda)^\vee) \xrightarrow{\sim} \mathcal{F}(P(w\cdot\lambda))^\vee$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}(P(w_0\cdot\lambda)) \arrow[r, "\Psi_{w_0}"] \arrow[d, "\mathcal{F}(\pi_w)"] & \mathcal{F}(P(w_0\cdot\lambda))^\vee \arrow[d, "\mathcal{F}(\pi_w^\vee)^\vee"] \\
\mathcal{F}(P(w\cdot\lambda)^\vee) \arrow[r, "\Psi_w"] & \mathcal{F}(P(w\cdot\lambda))^\vee
\end{tikzcd} \tag{13}$$

Fix w, w' and let's show that the diagram (12) is commutative. Let $f_{w,w'} \in \mathrm{Hom}(P(w\cdot\lambda)^\vee, P(w'\cdot\lambda)^\vee)$. By Lemma 7.24, there exists a map $\widetilde{f}_{w,w'} \in \mathrm{End}(P(w_0\cdot\lambda))$ such that the diagram (11) is commutative. We first consider the following diagram

$$\begin{tikzcd}
\mathcal{F}(P(w_0\cdot\lambda)) \arrow[r, "\Psi_{w_0}"] \arrow[d, "\mathcal{F}(\widetilde{f}_{w,w'})"] & \mathcal{F}(P(w_0\cdot\lambda))^\vee \arrow[d, "\mathcal{F}(\widetilde{f}^\vee_{w,w'})^\vee"] \\
\mathcal{F}(P(w_0\cdot\lambda)) \arrow[r, "\Psi_{w_0}"] & \mathcal{F}(P(w_0\cdot\lambda))^\vee
\end{tikzcd} \tag{14}$$

But as $\widetilde{f}_{w,w'} \in \mathrm{End}_{\mathcal{O}}(P(w_0\cdot\lambda), P(w_0\cdot\lambda)) \simeq D = L \otimes_{A^W} A$, it follows from Corollary 6.14 for $\mathcal{F} = \mathcal{M}_{\infty,x,\mathcal{R}}$ and [Bez16, Prop. 23] for $\mathcal{F} = \mathcal{B}_x$, and the fact that Ψ_{w_0} is $\mathcal{O}_{\mathcal{X}^{\mathrm{qtri}}_{\infty,x,\mathcal{R}}}$-linear, that this diagram commutes. Now consider the diagram

$$\begin{tikzcd}
 & \mathcal{F}(P(w_0\cdot\lambda))^\vee \arrow[rr, "\mathcal{F}(\pi_{w'}^\vee)^\vee"] & & \mathcal{F}(P(w'\cdot\lambda))^\vee \\
\mathcal{F}(P(w_0\cdot\lambda)) \arrow[ur, "\Psi_{w_0}"] \arrow[rr, "\mathcal{F}(\pi_{w'})"] & & \mathcal{F}(P(w'\cdot\lambda)^\vee) \arrow[ur, "\Psi_{w'}"] & \\
 & \mathcal{F}(P(w_0\cdot\lambda))^\vee \arrow[uu, "\mathcal{F}(\widetilde{f}^\vee_{w,w'})^\vee"] \arrow[rr, "\mathcal{F}(\pi_w^\vee)^\vee"] & & \mathcal{F}(P(w\cdot\lambda))^\vee \arrow[uu, "\mathcal{F}(f^\vee_{w,w'})^\vee"'] \\
\mathcal{F}(P(w_0\cdot\lambda)) \arrow[uu, "\mathcal{F}(\widetilde{f}_{w,w'})"] \arrow[ur, "\Psi_{w_0}"] \arrow[rr, "\mathcal{F}(\pi_w)"] & & \mathcal{F}(P(w\cdot\lambda)^\vee) \arrow[uu, "\mathcal{F}(f_{w,w'})"] \arrow[ur, "\Psi_w"'] &
\end{tikzcd}$$

All faces, except maybe the right hand one (which is the one of the statement), of this cube are commutative diagrams by functoriality and diagrams (11), (13), (14). Moreover $\mathcal{F}(\pi_w)$, $\mathcal{F}(\pi_w^\vee)^\vee$, $\mathcal{F}(\pi_{w'})$, $\mathcal{F}(\pi_{w'}^\vee)^\vee$ are surjective, thus the last right hand face also commutes. $\square$

Corollary 7.26. *For any $M \in \mathcal{O}_{\mathrm{alg}}$, there is a compatible choice of isomorphisms*

$$\Psi_M : \mathcal{F}(M^\vee) \xrightarrow{\sim} \mathcal{F}(M)^\vee,$$

where $\mathcal{F}$ is either the functor $\mathcal{B}, \mathcal{B}_x$ or $\mathcal{M}_{\infty,x,\mathcal{R}}$. In particular, $\mathcal{F}$ is compatible with duality.

Proof. By Lemma 7.25, we have an isomorphism $\mathcal{F}((-)^\vee) \simeq \mathcal{F}(-)^\vee$ on the full subcategory of $\mathcal{O}_{\chi_\lambda}$ whose objects are the injective one, when $\mathcal{F}$ is $\mathcal{M}_{x,\infty,\mathcal{R}}, \mathcal{B}_x$ or $\mathcal{B}$, see Remark 7.23. The result follows again from [Gab62, Cor. 2 to Prop. 14]. □

7.7 Consequences

We recall that the point $x = (\rho_p, \rho^p, z)$ is associated to an automorphic representation π with associated Galois representation ρ and that $n = 3$. We write $\mathfrak{m}_\rho$ for the ideal of $R_{\overline{\rho},\mathcal{S}}$ corresponding to ρ, see Remark 5.9 and just before. In particular $\mathfrak{m}_x$ is the pullback of $\mathfrak{m}_\rho$ by $R_\infty \longrightarrow R_{\overline{\rho},\mathcal{S}}$. We recall that we assumed that ρ_p is φ-generic. In this section we keep the setting introduced in subsection 7.3. In particular $n = 3$.

Lemma 7.27. *For all $M \in \mathcal{O}_{\chi_\lambda}$,*

$$\mathcal{M}_{\infty,x,\mathcal{R}}(M) \otimes k(x) \simeq \left(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi_\infty^{\mathrm{la}}[\mathfrak{m}_x])^{N_0}[\mathfrak{m}_{\delta_\mathcal{R}}]\right)' \simeq \left(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi^{\mathrm{la}}[\mathfrak{m}_\rho])^{N_0}[\mathfrak{m}_{\delta_\mathcal{R}}]\right)'.$$

Proof. By construction (see Remark 6.2), we have

$$\mathcal{M}_{\infty,x,\mathcal{R}}(M) \simeq \left(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi_\infty^{\mathrm{la}}[\mathfrak{m}_x^\infty])^{N_0}[\mathfrak{m}_{\delta_\mathcal{R}}^\infty]\right)'.$$

Moreover, as x corresponds to ρ, which is classical, we have $\Pi_\infty[\mathfrak{m}_x] = \Pi[\mathfrak{m}_x]$ (Proposition 5.11). By Corollary 6.11, the $\mathcal{X}_\infty \times \widehat{T}$-structure on the sheaf $\mathcal{M}_{\infty,x,\mathcal{R}}(M)$ factors through $\mathcal{X}_{\infty,x,\mathcal{R}}^{\mathrm{qtri}} \longrightarrow \mathcal{X}_\infty \times \widehat{T}$. Thus,

$$\mathcal{M}_{\infty,x,\mathcal{R}}(M) \otimes k(x) \simeq \left(\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi_\infty^{\mathrm{la}}[\mathfrak{m}_x])^{N_0}[\mathfrak{m}_{\delta_\mathcal{R}}]\right)'. \qquad \square$$

Corollary 7.28. *Let $\delta = \delta_\lambda \delta_{\mathrm{sm}} : T \to L^\times$ be a continuous character with $\lambda \in X^*(T)^+$ and $\delta_{\mathrm{sm}} : T \longrightarrow L^\times$ smooth, and let $\chi^S : \mathbb{T}^S \to L$ be a character such that $S^\dagger(K^p)[\chi^S \otimes \delta] \neq \{0\}$. Assume that the Galois representation ρ associated to χ^S is crystalline Hodge-Tate regular and φ-generic at p satisfying (1) and let $x = (\rho_p, \rho^p, z)$ associated to ρ as before. Let $r = |\{\tau \in \Sigma_F \mid \omega_{x,\mathcal{R},\tau} = 1\}|$. Then*

$$\dim S^\dagger(K^p)[\chi^S \otimes \delta] = 2^r \dim S^{\mathrm{cl}}(K^p)[\chi^S \otimes \delta] \neq 0.$$

Proof. The assumptions imply $\delta_{\mathrm{sm}} = \delta_\mathcal{R}$ for a refinement $\mathcal{R}$ of ρ_p. In particular δ_{sm} is unramified. By Breuil's adjunction formula [Bre15, Théorème 4.3] (see also [BHS19, eq. (5.5)]) and [BHS19, Lemma 5.2.3] we have

$$S^\dagger(K^p)[\chi^S \otimes \delta] = \mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), \Pi_\infty^{\mathrm{la}}[\chi^S])^{N_0}[\mathfrak{m}_{\delta_\mathcal{R}}],$$
$$S^{\mathrm{cl}}(K^p)[\chi^S \otimes \delta] = \mathrm{Hom}_{U(\mathfrak{g})}(L(\lambda), \Pi_\infty^{\mathrm{la}}[\chi^S])^{N_0}[\mathfrak{m}_{\delta_\mathcal{R}}].$$

In particular, by Lemma 7.27, these spaces are indentified with the dual vector spaces of the fiber of $\mathcal{M}_{\infty,x,\mathcal{R}}(M(\lambda))$ resp. of $\mathcal{M}_{\infty,x,\mathcal{R}}(L(\lambda))$ at $k(x)$. Thus, as $m_x = \dim \mathcal{M}_{\infty,x,\mathcal{R}}(L(\lambda)) \otimes k(x)$, the result is a direct corollary of Theorem 7.20 (and Proposition 7.8). □

We can also deduce the following corollaries on the structure of the completed cohomology Π (see Definition 5.2), which is a representation of $G \coloneqq U(\mathbb{Q}_p)$. Recall that $\mathfrak{g} = \mathrm{Lie}(\underline{G}_L)$.

If M is a $U(\mathfrak{g})$-module, we denote $\mathrm{Hom}_L(M, L)$ the $U(\mathfrak{g})$-module with underlying vector space $\mathrm{Hom}_L(M, L)$ and action of $\mathfrak{r} \in U(\mathfrak{g})$ given by

$$(\mathfrak{r} \cdot \phi)(m) \coloneqq \phi(\dot{\mathfrak{r}} m), \quad \phi \in \mathrm{Hom}_L(M, L), m \in M,$$

where $\mathfrak{r} \mapsto \dot{\mathfrak{r}}$ is the anti-involution of $U(\mathfrak{g})$ extending the multiplication by -1 on $\mathfrak{g}$. We denote $\overline{\underline{B}}$ the Borel opposite to $\underline{B}$ with respect to $\underline{T}$, $\overline{\mathfrak{b}}$ its Lie algebra and $\overline{\mathfrak{n}}$ the nilpotent radical of $\overline{\mathfrak{n}}$. We then denote $B = \underline{B}(\mathbb{Q}_p), \overline{B} = \overline{\underline{B}}(\mathbb{Q}_p)$ and δ_B the modulus character of B. We then denote $M' \coloneqq \mathrm{Hom}_L(M, L)^{\overline{\mathfrak{n}}^\infty}$ the vectors which are killed by some finite power of the augmentation ideal of $U(\overline{\mathfrak{n}})$. If $M = \bigoplus_{\lambda \in X^*(\underline{T})_L} M_\lambda \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{g}, \mathfrak{b}}$, then $M' \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{g}, \overline{\mathfrak{b}}}$. Finally recall that if $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{g}, \overline{\mathfrak{b}}}$ and δ is a smooth character of $\underline{T}(\mathbb{Q}_p)$ with values in $L^\times$, then Orlik-Strauch constructed (see [OS10] or also [Bre16])

$$\mathcal{F}_{\overline{B}}^G(M, \delta),$$

which is a locally analytic representation of G. In particular, locally analytic principal series are of this form : if $\lambda \in X^*(\underline{T})$, then we have $(M(\lambda)^\vee)' \simeq U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} (-\lambda) \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{g}, \overline{\mathfrak{b}}}$, and

$$\mathcal{F}_{\overline{B}}^G((M(\lambda)^\vee)', \delta) = \mathrm{Ind}_{\overline{B}}^G(\delta_\lambda \delta)^{\mathrm{la}} \tag{15}$$

where $\mathrm{Ind}_{\overline{B}}^G(\delta_\lambda \delta)^{\mathrm{la}}$ denotes the locally analytic induced representation of $\delta_\lambda \delta$ from $\overline{B}$ to G. When λ is dominant, it contains the (locally) algebraic induction $\mathrm{Ind}_{\overline{B}}^G(\delta_\lambda \delta)^{\mathrm{lalg}}$.

Recall that $\rho : \mathrm{Gal}_E \longrightarrow \mathrm{GL}_n(L)$ associated to our point $x \in \mathcal{X}_\infty(L)$ is crystalline, Hodge-Tate regular and φ-generic. Let $\mathcal{R}$ a choice of refinement and $\delta_{\mathcal{R}}$ the associated unramified character. Denote $\lambda = (\lambda_\tau)_\tau \coloneqq \mathrm{HT}(\rho) - \delta_G \in X^*(T)^+$ the (dominant) algebraic character associated to ρ as before, where $\mathrm{HT}(\rho) = (h_{1,\tau} < \cdots < h_{n,\tau})_{\tau \in \Sigma_F} \in X^*(\underline{T})$ gives the Hodge-Tate weights of ρ. Recall also $w_{x,\mathcal{R}} = (w_{x,\mathcal{R},\tau})_{\tau \in \Sigma_F}$ and $m_x \geqslant 1$ as in Section 7.1. More directly

$$m_x = \dim \mathrm{Hom}_G(\mathrm{Ind}_{\overline{B}}^G(\delta_\lambda \delta_{\mathcal{R}} \delta_B^{-1})^{\mathrm{lalg}}, \Pi^{\mathrm{la}}[\mathfrak{m}_\rho]),$$

and is independant of $\mathcal{R}$ as $\mathrm{Ind}_{\overline{B}}^G(\delta_\lambda \delta_{\mathcal{R}} \delta_B^{-1})^{\mathrm{lalg}}$ is, by [Bre16, Lemme 6.2].

Corollary 7.29. *Let $x, \rho, \lambda, \mathcal{R}$ as above. For all $w \leqslant w_{x,\mathcal{R}} w_0$, we have*

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_{\overline{B}}^G(\delta_{w \cdot \lambda} \delta_{\mathcal{R}} \delta_B^{-1})^{\mathrm{la}}, \Pi^{\mathrm{la}}[\mathfrak{m}_\rho]) = m_x.$$

Proof. By [Bre15, Proposition 4.2] and [BHS19, Lemma 5.2.3], we have, for all $M \in \mathcal{O}$

$$\mathrm{Hom}_{U(\mathfrak{g})}(M, \Pi^{\mathrm{la}}[\mathfrak{m}_\rho])^{N_0}[\mathfrak{m}_{\delta_{\mathcal{R}}}] \simeq \mathrm{Hom}_G(\mathcal{F}_{\overline{B}}^G(M', \delta_{\mathcal{R}} \delta_B^{-1}), \Pi^{\mathrm{la}}[\mathfrak{m}_\rho]). \tag{16}$$

Thus, using $\Pi[\mathfrak{m}_\rho] = \Pi_\infty[\mathfrak{m}_x]$, equation (15) and Lemma 7.27 we deduce that the statement is equivalent to

$$\dim \mathrm{Hom}_{U(\mathfrak{g})}(M(w\cdot\lambda)^\vee, \Pi^{\mathrm{la}}[\mathfrak{m}_\rho])[\mathfrak{m}_\delta] \ = \dim \mathcal{M}_{\infty,x,\mathcal{R}}(M(w\cdot\lambda)^\vee)\otimes k(x) = m_x,$$

which is Theorem 7.20. $\square$

Let $\mathfrak{gl}_3$ be the Lie algebra (over L) of the group GL_3. For a dominant λ for $\mathfrak{gl}_3$ we consider the extension

$$N(\lambda) = [L(s_1\cdot\lambda)\oplus L(s_2\cdot\lambda) - \ L(\lambda)] \in \mathrm{Ext}^1_{\mathcal{O}}(L(\lambda), L(s_1\cdot\lambda)\oplus L(s_2\cdot\lambda)),$$

which is non trivial when mapped in each of $\mathrm{Ext}^1_{\mathcal{O}}(L(\lambda), L(s_i\cdot\lambda))$, for $i = 1, 2$. This extension can be realized as quotient of the Verma module $M(\lambda)$ by $M(s_1s_2\cdot\lambda)+M(s_2s_1\cdot\lambda)$. As before we consider the Lie algebra

$$\mathfrak{g} = \mathrm{Lie}(\underline{G}_L) \simeq \mathrm{Lie}((\mathrm{Res}_{F\otimes_{\mathbb{Q}}\mathbb{Q}_p/\mathbb{Q}_p}\mathrm{GL}_3)\times_{\mathbb{Q}_p} \ L) \simeq \prod_{\tau\in\Sigma_F}\mathfrak{gl}_3.$$

We have a decomposition with $\mathfrak{b}\simeq\prod_\tau \mathfrak{b}_\tau$ where $\mathfrak{b}_\tau$ is the projection of $\mathfrak{b}$ on the $\mathfrak{gl}_3$-factor of $\mathfrak{g}$ indexed by $\tau\in\Sigma_F$. Associated to a dominant weight $\lambda = (\lambda_\tau)_\tau \in X^*(\underline{T})^+$ and $w = (w_\tau)_{\tau\in\Sigma_F}\in W$ we define the objects

$$N(\lambda, w) = \left(\boxtimes_{\tau: w_\tau\neq 1} L(\lambda_\tau)\boxtimes\boxtimes_{\tau: w_\tau = 1} N(\lambda_\tau)\right) \qquad \text{and} \qquad S(\lambda, w) = \boxtimes_{\tau\in\Sigma_F} S(\lambda_\tau, w_\tau)$$

of the category $\mathcal{O}_{\chi_\lambda}$, where

$$S(\lambda_\tau, w_\tau) = \begin{cases} \bigoplus_{w'\leqslant w_\tau w_0} L(w'\cdot\lambda_\tau) & \text{if } w_\tau\neq 1 \\ \bigoplus_{\ell(w')\neq 1} L(w'\cdot\lambda_\tau)\bigoplus N(\lambda_\tau) & \text{if } w_\tau = 1\end{cases},$$

so that $N(\lambda, w)\subset S(\lambda, w)$ and moreover $S(\lambda, w) = \bigoplus_{w'\leqslant ww_0} L(w'\cdot\lambda)$ if $w_\tau\neq 1$ for all τ. The motivation for defining $N(\lambda, w)$ in this way is that $\mathcal{F}_{\overline{B}}^G(N(\lambda, w_{x,\mathcal{R}})', \delta_{\mathcal{R}}\delta_B^{-1})$ (resp. $\mathcal{F}_{\overline{B}}^G(S(\lambda, w_{x,\mathcal{R}})', \delta_{\mathcal{R}}\delta_B^{-1})$) should be the largest subrepresentation of $\Pi^{\mathrm{la}}[\rho]$ which can be realized as a quotient of $\mathcal{F}_{\overline{B}}^G(M(\lambda)', \delta_{\mathcal{R}}\delta_B^{-1})$ (resp. through which all maps from $\mathcal{F}_{\overline{B}}^G((M(w\cdot\lambda))', \delta_{\mathcal{R}}\delta_B^{-1})$, $w\in W$, to $\Pi^{\mathrm{la}}[\rho]$ factors).

Theorem 7.30. *For $x, \rho, \lambda, \mathcal{R}$ as before, we have an injection of G-representations*

$$\mathcal{F}_{\overline{B}}^G(S(\lambda, w_{x,\mathcal{R}})', \delta_{\mathcal{R}}\delta_B^{-1})^{\oplus m_x}\hookrightarrow\Pi^{\mathrm{la}}[\mathfrak{m}_\rho].$$

Moreover, $\forall w\in W$, each map from $\mathcal{F}_{\overline{B}}^G(M(w\cdot\lambda)', \delta_{\mathcal{R}}\delta_B^{-1})$ to $\Pi^{\mathrm{la}}[\mathfrak{m}_\rho]$ factors through a map from $\mathcal{F}_{\overline{B}}^G(S(\lambda, w_{x,\mathcal{R}})', \delta_{\mathcal{R}}\delta_B^{-1})$ to $\Pi^{\mathrm{la}}[\mathfrak{m}_\rho]$.

Proof. In order to simplify notations, in all this proof we will use notation $\mathcal{F}(M) = \mathcal{F}_{\overline{B}}^{G}(M', \delta_{\mathcal{R}}\delta_B^{-1})$ for any object M in $\mathcal{O}_{\chi_\lambda}$. In particular note that $\mathcal{F}$ is covariant.

Let $J := \{w \in W \mid w \leqslant w_{x,\mathcal{R}}w_0\}$ and let $\sim$ be the equivalence relation on J generated by $w \sim w'$ if and only if there exists $\tau \in \Sigma$ such that $w_{x,\mathcal{R},\tau} = 1$ with $w_\tau, w'_\tau \in \{s_1, s_2\}$ and $w_{\tau'} = w'_{\tau'}$ for all $\tau' \neq \tau$. Note that we have

$$S(\lambda, w_{x,\mathcal{R}}) \simeq \bigoplus_{\overline{w} \in J/\sim} F(\overline{w}) \tag{17}$$

where, given $\overline{w}$, let $I_w = \{\tau \in \Sigma_F \mid w_{x,\mathcal{R},\tau} = 1 \text{ and } w_\tau \in \{s_1, s_2\}\}$ for any choice of a representative w of $\overline{w}$, and

$$F(\overline{w}) := \boxtimes_{\tau \in \Sigma_F \setminus I_w} L(w_\tau \cdot \lambda_\tau) \boxtimes \boxtimes_{\tau \in I_w} N(\lambda_\tau).$$

For later reasons, we introduce the following notation. Let $w \in W$ and $I \subset \{\tau \in \Sigma_F \mid w_{x,\mathcal{R},\tau} = 1 = w_\tau\}$. Then we set

$$F_I(w) := \boxtimes_{\tau \in \Sigma_F \setminus I} L(w_\tau \cdot \lambda_\tau) \boxtimes \boxtimes_{\tau \in I} N(\lambda_\tau).$$

Then $F_I(w)$ is a quotient of $M(w \cdot \lambda)$. Remark that $F(\overline{w}) = F_{I_w}(w')$ where $w' = (w'_\tau)_\tau$ with $w'_\tau = w_\tau$ if $\tau \notin I_w$ and $w'_\tau = 1$ if $\tau \in I_w$. Conversely not all $F_I(w)$ are some $F(\overline{w}')$ but is a subobject of a unique $F(\overline{w}')$, namely $F(\overline{w})$.

We first prove the second claim, i.e. that each map from $\mathcal{F}(M(w \cdot \lambda))$ to $\Pi^{\mathrm{la}}[\mathfrak{m}_\rho]$ factors through a map from $\mathcal{F}(S(\lambda, w_{x,\mathcal{R}}))$ to $\Pi^{\mathrm{la}}[\mathfrak{m}_\rho]$. This will be a consequence of the fact that the map

$$\mathcal{M}_{\infty,x,\mathcal{R}}(M(w \cdot \lambda)) \otimes k(x) \to \mathcal{M}_{\infty,x,\mathcal{R}}(F_I(w)) \otimes k(x)$$

is an isomorphism when

$$I = \{\tau \in \Sigma_F \mid w_{x,\mathcal{R},\tau} = w_\tau = 1\}.$$

Let's prove this claim. By Corollary 7.17, it is equivalent to prove the same claim with $\mathcal{M}_{\infty,x,\mathcal{R}}$ replaced with $\mathcal{B}_x$. By Theorem 7.5 6), it is sufficient to prove that for each $\tau \in \Sigma_F$, the maps

$$\mathcal{B}_\tau(M(w_\tau \cdot \lambda_\tau))_{x_{\mathrm{pdR},\tau}} \otimes k(x_{\mathrm{pdR},\tau}) \to \mathcal{B}_\tau(L(w_\tau \cdot \lambda_\tau))_{x_{\mathrm{pdR},\tau}} \otimes k(x_{\mathrm{pdR},\tau}) \text{ if } \tau \notin I$$
$$\mathcal{B}_\tau(M(\lambda_\tau))_{x_{\mathrm{pdR},\tau}} \otimes k(x_{\mathrm{pdR},\tau}) \to \mathcal{B}_\tau(N(\lambda_\tau))_{x_{\mathrm{pdR},\tau}} \otimes k(x_{\mathrm{pdR},\tau}) \text{ if } \tau \in I$$

are isomorphisms. For the case $\tau \notin I$, this is a consequence of the fact that $\mathcal{B}_\tau(M(w_\tau \cdot \lambda_\tau))_{x_{\mathrm{pdR},\tau}}$ is free of rank one over its support (using Theorem 7.5 and the fact that the support is a complete intersection by Proposition 7.8). Therefore it is sufficient to prove that the map

$$\mathcal{B}_\tau(M(\lambda_\tau))_{x_{\mathrm{pdR},\tau}} \otimes k(x_{\mathrm{pdR},\tau}) \to \mathcal{B}_\tau(N(\lambda_\tau))_{x_{\mathrm{pdR},\tau}} \otimes k(x_{\mathrm{pdR},\tau})$$

is an isomorphism when $\tau \in I$. As $N(\lambda_\tau)$ is the cokernel of a map

$$M(s_1s_2 \cdot \lambda_\tau) \oplus M(s_2s_1 \cdot \lambda_\tau) \to M(\lambda_\tau),$$

it is sufficient to prove that the unique (up to scalar) non trivial map $M(s_1s_2\cdot\lambda) \to M(\lambda_\tau)$ induces the zero map after application of the functor $\mathcal{B}_\tau(-)_{x_{\mathrm{pdR},\tau}} \otimes k(x_{\mathrm{pdR},\tau})$ (the case of $M(s_2s_1 \cdot \lambda_\tau)$ is similar). As the map $M(s_1s_2 \cdot \lambda) \to M(\lambda_\tau)$ factors through the unique non trivial map $M(s_1 \cdot \lambda_\tau) \to M(\lambda_\tau)$, it is sufficient to prove that any map $M(s_1s_2\cdot\lambda) \to M(s_1\cdot\lambda_\tau)$ induces the zero map after application of $\mathcal{B}_\tau(-)_{x_{\mathrm{pdR},\tau}}\otimes k(x_{\mathrm{pdR},\tau})$. As the composition $M(s_1s_2 \cdot \lambda_\tau) \to M(s_1 \cdot \lambda_\tau) \to L(s_1 \cdot \lambda_\tau)$ is zero, it is sufficient to prove that the map $M(s_1 \cdot \lambda_\tau) \to L(s_1 \cdot \lambda)$ induces an isomorphism after application of the functor $\mathcal{B}_\tau(-)_{x_{\mathrm{pdR},\tau}} \otimes k(x_{\mathrm{pdR},\tau})$. This is again a direct consquence of the fact that $\mathcal{B}_\tau(M(s_1 \cdot \lambda))_{x_{\mathrm{pdR},\tau}}$ is free of rank one over its support.

Note that this proves that

$$\dim_L \mathcal{B}_\tau(N(\lambda_\tau))_{x_{\mathrm{pdR},\tau}} \otimes k(x_{\mathrm{pdR},\tau}) = \dim_L \mathcal{B}_\tau(M(\lambda_\tau))_{x_{\mathrm{pdR},\tau}} \otimes k(x_{\mathrm{pdR},\tau}) = 2$$

for any τ such that $w_{x,\mathcal{R},\tau} = w_\tau = 1$. Using Corollary 7.17, Theorem 7.5 6) and Proposition 7.8, we deduce that

$$\dim_L \mathcal{M}_{\infty,x,\mathcal{R}}(F_I(w)) \otimes k(x) = 2^{|I|}m_x \tag{18}$$

for any w and $I \subset \{\tau \in \Sigma_F \mid w_{x,\mathcal{R},\tau} = w_\tau = 1\}$. When $I = \{\tau \in \Sigma_F \mid w_{x,\mathcal{R},\tau} = w_\tau = 1\}$ this coincides with $\dim_L \mathcal{M}_{\infty,x,\mathcal{R}}(M(w \cdot \lambda)) \otimes k(x)$ proving the equality and thus the second part of the statement.

We now focus on the first part of the statement, i.e. that there is an injection of G-representations

$$\mathcal{F}(S(\lambda, w_{x,\mathcal{R}}))^{\oplus m_x} \hookrightarrow \Pi^{\mathrm{la}}[\mathfrak{m}_\rho].$$

We first show that the kernel of the map

$$\mathcal{M}_{\infty,x,\mathcal{R}}(F(\overline{w})) \otimes k(x) \longrightarrow \bigoplus_Q \mathcal{M}_{\infty,x,\mathcal{R}}(Q) \otimes k(x) \tag{19}$$

has dimension m_x for any $\overline{w} \in J/\sim$, where the sum is taken over all strict quotients Q of $S(\lambda, w_{x,\mathcal{R}})$.

We claim that if $\tau \in I_w$, then

$$\dim_L \mathcal{B}_\tau(N(\lambda_\tau)/L(s_i \cdot \lambda_\tau))_{x_{\mathrm{pdR},\tau},\tau} \otimes k(x_{\mathrm{pdR},\tau}) = 1 \tag{20}$$

for any $i \in \{1,2\}$. As $N(\lambda_\tau)/L(s_i \cdot \lambda_\tau) \simeq M_{\{s_i\}}(\lambda)$, it is sufficient to check that $\mathcal{B}_\tau(M_{\{s_i\}}(\lambda))$ is free of rank 1 over its support. However this is a consequence of Theorem 7.5 and the fact that its support is a complete intersection as follows from the explicit computations of section 7.3.

Now each quotient of $F(\overline{w})$ is of the form

$$F_{J_1,J_2}(\overline{w}) := \boxtimes_{\tau\in(\Sigma_F\setminus I_w)} L(w_\tau\cdot\lambda_\tau)\boxtimes\boxtimes_{\tau\in J_1} L(\lambda_\tau)\boxtimes\boxtimes_{\tau\in J_2} N(\lambda_\tau)/L(s_{i_\tau,\tau}\cdot\lambda_\tau)\boxtimes\boxtimes_{\tau\in I_w\setminus(J_1\cup J_2)} N(\lambda_\tau),$$

for some $J = J_1\coprod J_2 \subset I_w$, $i_\tau\in\{1,2\}$ for $\tau\in J_2$. Using again Corollary 7.17, Theorem 7.5 6) and (20), we deduce that

$$\dim_L \mathcal{M}_{\infty,x,\mathcal{R}}(F_{J_1,J_2}(\overline{w}))\otimes k(x) = 2^{|I_w|-|J_1\cup J_2|}m_x.$$

Moreover, we have the natural quotient map

$$F_{J_1,J_2}(\overline{w}) \longrightarrow F_{J_1\cup J_2,\emptyset}(\overline{w}),$$

and thus, because of equality of dimensions,

$$\mathcal{M}_{\infty,x,\mathcal{R}}(F_{J_1,J_2}(\overline{w}))\otimes k(x) = \mathcal{M}_{\infty,x,\mathcal{R}}(F_{J_1\cup J_2,\emptyset}(\overline{w}))\otimes k(x).$$

Therefore the kernel of the map (19) is equal to the kernel of the map

$$\mathcal{M}_{\infty,x,\mathcal{R}}(F(\overline{w}))\otimes k(x) \longrightarrow \bigoplus_{\emptyset\neq K\subset I_w} \mathcal{M}_{\infty,x,\mathcal{R}}(F_{K,\emptyset}(\overline{w}))\otimes k(x).$$

For $K\subset I_w$, set $G(K) := \mathrm{Ker}(\mathcal{M}_{\infty,x,\mathcal{R}}(F(\overline{w}))\otimes k(x) \longrightarrow \mathcal{M}_{\infty,x,\mathcal{R}}(F_{K,\emptyset}(\overline{w}))\otimes k(x))$. Note that if $K, K'\subset I_w$, $F_{K\cap K',\emptyset}(\overline{w})$ is the smallest quotient of $F(\overline{w})$ dominating $F_{K,\emptyset}(\overline{w})$ and $F_{K',}(\overline{w})$. As $\mathcal{M}_{\infty,x,\mathcal{R}}(-)\otimes k(x)$ is right exact, this implies that $G(K\cap K') = G(K)\cap G(K')$.

By Grassmann's formula the dimension of the kernel of the map (19) is thus

$$D(\overline{w}) := \dim G(\emptyset) - \sum_{\emptyset\neq K\subset I_w} \dim(\cap_{k\in K} G(\{k\})).$$

Now for each $K\subset I_w$, we have $\cap_{k\in K}G(\{k\}) = G(K)$ which has dimension $2^{|I_w|-|K|}m_x$ so that

$$D(\overline{w}) = 2^{|I_w|}m_x - \sum_{\emptyset\neq K\subset I_w} (-1)^{|K|+1}2^{|I_w|-|J|}m_x = m_x.$$

Now we prove that, for any $\overline{w}\in J/\sim$,

$$\mathrm{soc}_G(\mathcal{F}(F(\overline{w}))) = \mathcal{F}(\mathrm{soc}_{\mathcal{O}_{\chi_\lambda}}(F(\overline{w}))). \tag{21}$$

As any $F(\overline{w})$ is of the form $F_I(w)$ for some $w\in J$ and $I\subset\{\tau\in\Sigma_F \mid w_{x,\mathcal{R},\tau} = w_\tau = 1\}$, it is sufficient to prove, more generally, that for any $w\in J$ and $I\subset\{\tau\in\Sigma_F \mid w_{x,\mathcal{R},\tau} = w_\tau = 1\}$

$$\mathrm{soc}_G(\mathcal{F}(F_I(w))) = \mathcal{F}(\mathrm{soc}_{\mathcal{O}_{\chi_\lambda}}(F_I(w))).$$

First we remark that $F_I(w)$ is multiplicity free and that all its simple subquotients are isomorphic to $L(w'\cdot\lambda)$ for some $w'\in J$ so that $\dim(\mathcal{M}_{\infty,x,\mathcal{R}}(L(w'\cdot\lambda))\otimes k(x)) =$

m_x by Proposition 7.11. Moreover it follows from [OS15, Theorem 1.1] and [Bre16, Thm. 2.3] that each $\mathcal{F}(L(w'\cdot\lambda))$ is topologically irreducible. By exactness of $\mathcal{F}(-)$, we have $\mathrm{soc}_G(\mathcal{F}(F_I(w))) \supset \mathcal{F}(\mathrm{soc}_{\mathcal{O}_{\chi_\lambda}}(F_I(w)))$. We prove the converse : let Q be a simple subquotient of $F_I(w)$ such that $\mathcal{F}(Q) \subset \mathrm{soc}_G(\mathcal{F}(F_I(w)))$, then $Q \subset \mathrm{soc}_{\mathcal{O}_{\chi_\lambda}} F_I(w)$. We prove the result by induction on the cardinal of I. If $I = \emptyset$, there is nothing to prove as $F_I(w)$ is simple. So we assume $I \neq \emptyset$ and let Q be a simple subquotient Q of $F_I(w)$ such that $\mathcal{F}(Q) \subset \mathcal{F}(F_I(w))$. Then Q is of the form $L(w'\cdot\lambda)$, for some w' such that $w_\tau = w'_\tau$ for all $\tau \notin I$, $w'_\tau \in \{1, s_1, s_2\}$. Let $I' = \{\tau \in I \mid w'_\tau = 1\}$. Then

$$F_{I'}(w') := \bigboxtimes_{\tau\notin I'} L(w'_\tau \cdot \lambda_\tau) \boxtimes \bigboxtimes_{\tau\in I'} N(\lambda_\tau),$$

is a subrepresentation of $F_I(w)$ (as $L(s_i\cdot\lambda_\tau)$ is a subrepresentation of $N(\lambda_\tau)$) and Q is the cosocle of $F_{I'}(w')$. By [Bre16, Cor. 2.7], the representation $\mathcal{F}(F_I(w))$ has multiplicity one so that $\mathcal{F}(Q) \subset \mathcal{F}(F_{I'}(w'))$. If $I' \subsetneq I$, our induction hypothesis implies that $Q \subset \mathrm{soc}_{\mathcal{O}_{\chi_\lambda}} F_{I'}(w') \subset \mathrm{soc}_{\mathcal{O}_{\chi_\lambda}}(F_I(w))$. So we can assume that $I = I'$ and $Q = L(w\cdot\lambda)$ is actually the cosocle of $F_I(w)$. The exactness of $\mathcal{F}(-)$ and our hypothesis assure that

$$\mathcal{F}(F_I(w))) \simeq \mathcal{F}(Q) \oplus \mathcal{F}_{\overline{B}}^{G}(N),$$

where $N = \mathrm{Ker}(F_I(w) \longrightarrow L(w\cdot\lambda))$. Now, N has $2^{|I|}$ distincts simple objects in its cosocle which are of the form $L(w'\cdot\lambda)$ for $w' \in J$ so that, using the exactness of $\mathcal{M}_{\infty,x,\mathcal{R}}$ and (18),

$$\begin{aligned}
2^{|I|}m_x &= \dim_L \mathrm{Hom}_G(\mathcal{F}_{\overline{B}}^{G}(F_I(w)), \Pi^{\mathrm{la}}[\mathfrak{m}_\rho]) \\
&= \dim_L \mathrm{Hom}_G(\mathcal{F}(Q), \Pi^{\mathrm{la}}[\mathfrak{m}_\rho]) + \dim_L \mathrm{Hom}_G(\mathcal{F}(N), \Pi^{\mathrm{la}}[\mathfrak{m}_\rho]) \\
&\geqslant \dim_L \mathcal{M}_{\infty,x,\mathcal{R}}(Q) \otimes k(x) + \dim_L \mathcal{M}_{\infty,x,\mathcal{R}}(\mathrm{cosoc}_{\mathcal{O}_{\chi_\lambda}}(N)) \otimes k(x) \\
&= m_x + 2^{|I|}m_x > 2^{|I|}m_x.
\end{aligned}$$

This gives a contradiction and finishes the induction.

Finally we prove the existence of an injection

$$\mathcal{F}(S(\lambda, w_{x,\mathcal{R}}))^{\oplus m_x} \hookrightarrow \Pi^{\mathrm{la}}[\mathfrak{m}_\rho].$$

Let $\overline{w} \in J/\sim$. Dualizing (19), we see that the cokernel of the map

$$\bigoplus_Q \mathrm{Hom}_G(\mathcal{F}(Q), \Pi^{\mathrm{la}}[\mathfrak{m}_\rho]) \longrightarrow \mathrm{Hom}_G(\mathcal{F}(F(\overline{w})), \Pi^{\mathrm{la}}[\mathfrak{m}_\rho]) \tag{22}$$

has dimension m_x where the sum is taken over the strict quotients Q of $F(\overline{w})$. We choose m_x maps $f_1, \dots, f_{m_x}$ in $\mathrm{Hom}_G(\mathcal{F}(F(\overline{w})), \Pi^{\mathrm{la}}[\mathfrak{m}_\rho])$ whose images in this cokernel are linearly independant. We claim that the map $f_{\overline{w}} = (f_1, \dots, f_{m_x})$ is injective. Namely, if it is not, a linear combination of these maps is zero on some simple constituant of the socle of $\mathcal{F}(F(\overline{w}))$ and thus on some $\mathcal{F}(Q)$ for $Q \subset F(\overline{w})$ by (21). This implies that this linear combination is zero in the cokernel of (22), that is false.

Finally the decomposition (17), the fact that the socles of the various $F(\overline{w})$ are two by two distincts, (21) and [Bre16, Cor. 2.7] again impliy that the map $(f_{\overline{w}})_{\overline{w}\in J/\sim}$ provides the desired injection. $\square$

Remark 7.31. Assume for simplicity that Σ_F is reduced to one element τ so that $G = U(\mathbb{Q}_p) \simeq \mathrm{GL}_3(\mathbb{Q}_p)$. Assume moreover that $w_{\mathcal{R}} = 1$ so that the Galois representation ρ_p is completely reducible and semisimple, i.e. a direct sum of three characters. Let

$$\mathrm{LALG} := \mathrm{Ind}_{\overline{B}}^{G}(\delta_\lambda \delta_{\mathcal{R}} \delta_B^{-1})^{\mathrm{lalg}} = L(\lambda) \otimes_L \mathrm{Ind}_{\overline{B}}^{G}(\delta_{\mathcal{R}} \delta_B^{-1})^{\mathrm{sm}}.$$

Then the subspace of locally analytic vectors of $\Pi[\mathfrak{m}_\rho]^{\mathrm{la}}$ is isomorphic to LALG^{m_x}. Moreover it is expected that $\Pi[\mathfrak{m}_\rho]$ is isomorphic to the direct sum of (m_x copies of) 6 continuous unitary principal series and an other direct factor SC which should be a kind of "supercuspidal" representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ (see [BHH$^+$] for a similar and precise conjecture in the mod p case). We deduce from Theorem 7.30 the existence of an injection of the following locally analytic representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ in $\Pi[\mathfrak{m}_\rho]$:

$$\mathcal{F}_{\overline{B}}^{G}(N(\lambda)', \delta_{\mathcal{R}} \delta_B^{-1}) = [(\mathrm{LA}_{s_1} \oplus \mathrm{LA}_{s_2}) - \mathrm{LALG}],$$

which is a non split extension of LALG by the direct sum of two topologically irreducible locally analytic representations without locally algebraic vectors. More precisely, for $s \in \mathfrak{S}_3$,

$$\mathrm{LA}_s := \mathcal{F}_{\overline{B}}^{G}(L(s\cdot\lambda)', \delta_{\mathcal{R}} \delta_B^{-1}) \tag{23}$$

is the socle of the locally analytic principal series $\mathrm{Ind}_{\overline{B}}^{G}(\delta_{s\cdot\lambda} \delta_{\mathcal{R}} \delta_B^{-1})^{\mathrm{la}}$. If our expectation holds true, the representation (23) appears as a subspace of locally analytic vectors of $\mathrm{SC}^{\mathrm{la}}$ showing that this representation $\mathrm{SC}^{\mathrm{la}}$ has to contain non trivial locally algebraic vectors in subquotient but not in its socle.

8 Existence of very critical classical modular forms

In this section we show the existence of a classical form f satisfying the hypothesis of Theorem 1.2. The main difficulty is to find a form satisfying the Taylor-Wiles hypothesis, which is moreover completely critical at p (i.e. $w_{\rho_f,\mathcal{R}} = 1$).

For a finite extension F of $\mathbb{Q}_p$, we denote by $\mathrm{rec}_F : F^\times \to \mathrm{Gal}_F^{\mathrm{ab}}$ the local reciprocity map sending a uniformizer of F on a geometric Frobenius. If K is a number field we denote by Art_K the Artin reciprocity map $\mathbb{A}_K^\times/K^\times \to \mathrm{Gal}_K^{\mathrm{ab}}$ such that, for any finite place v of K the precomposition of Art_K with the inclusion $K_v^\times \hookrightarrow \mathbb{A}_K^\times$ is rec_{K_v}. If Ψ is a character of $\mathbb{A}_K^\times/K^\times$ and v is a finite place of K such that Ψ_v is unramified, we write $\Psi(v)$ for the evaluation of Ψ_v at an uniformizer of $F_v^\times$. First, we remark the following,

Lemma 8.1. *Let $K/\mathbb{Q}_p$ be a finite extension and let $\rho_p : \mathrm{Gal}_K \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ be a crystalline representation with regular Hodge–Tate weights such that there exists a refinement $F_\bullet \subset D_{\mathrm{cris}}(\rho_p)$ which contains the Hodge filtration. We moreover assume that* *the eigenvalues of the linearization of the crystalline Frobenius on $D_{\mathrm{cris}}(\rho_p)$ are pairwise distinct. Then ρ_p is a split sum of characters.*

Proof. This is a simple application of weak admissibility. Up to extending scalars, we can assume that $D = D_{\mathrm{cris}}(\rho_p) = \bigoplus_\tau D_\tau$ is split, and is a filtered φ-module. We consider the linearization φ_τ^f of the Frobenius on D_τ, where $f = [K_0 : \mathbb{Q}_p]$. We write $\mathrm{Fil}^\bullet D_\tau$ for the filtration on D_τ induced by the Hodge-filtration on D. The assumption is that the Hodge filtration on D is φ-stable i.e. there is a full flag of $K \otimes \overline{\mathbb{Q}}_p$-modules $F_\bullet$, stable under φ, such that, for all τ, if $k_1^\tau \leqslant \ldots \leqslant k_n^\tau$ are the (opposite) τ-Hodge-Tate weights (with multiplicities) then $F_{i,\tau} \subset \mathrm{Fil}^{k_{n-i+1}} D_\tau$. Denote the eigenvalues of φ^f on $F_{i,\tau}$ by $(\varphi_1, \ldots, \varphi_i)$. Thus by weak admissibility,

$$\frac{1}{f}(v(\varphi_1) + \cdots + v(\varphi_i)) \geqslant \sum_\tau \sum_{k=1}^{i} k_{n+1-k}^\tau.$$

Now, if G_i is a complementary φ-stable subspace of F_i in D (which exists due to the assumptions on the eigenvalues of φ^f), then we see directly that the τ-Hodge-Tate weights of G_i are $k_1^\tau, \ldots, k_{n-i}^\tau$. Thus by weak admissibility again,

$$\frac{1}{f}(v(\varphi_{i+1}) + \cdots + v(\varphi_n)) \geqslant \sum_\tau \sum_{k=1}^{n-i} k_k^\tau.$$

But by weak admissibility of D, the endpoints of both polygons gives

$$\frac{1}{f}(v(\varphi_1) + \cdots + v(\varphi_n)) = \sum_\tau \sum_i k_i^\tau.$$

Thus both G_i and F_i are weakly admissible, thus admissible, thus ρ_p splits accordingly. As this is true for all i, we get the Lemma. □

It follows that, when $n = 3$, an eigenform f as in Theorem 1.2 has a split representation at p. In the case of modular forms, it was asked by Greenberg (see the work of Ghate and Vatsal [Gha04], [GV04]) if a cuspform whose representation is split at p is necessarily a CM form. The natural generalization of this question to GL_3 would suggest that we cannot find a form f to apply Theorem 1.2 with very large image. Fortunately, we can construct an analog of a CM form for GL_3 (more precisely for $U(3)$) which still has adequate image modulo p.

8.1 Choosing a Hecke character

Let E be a CM field with totally real subfield $E^+ = F$ and let F' be a totally real field disjoint from E, such that $F'/\mathbb{Q}$ is Galois and such that $[F' : \mathbb{Q}] = 3$. Set $K = EF'$. This is a CM field. We moreover assume that all the ramified primes of K/E lie above split primes in E/E^+. Choose two distinct primes p and ℓ such that ℓ is totally split in $K = EF'$ and primes above p in $E^+ = F$ are totally split in K. Moreover assume $p > 8 (= 2(n+1)$ when $n = 3)$ and $\zeta_p \notin E$.

Example 8.2. 1. The easiest choice is $F' = \mathbb{Q}(\zeta_7)^+$ and $E = \mathbb{Q}(i\sqrt{3})$ so that 7 is split in E. For this F', we can also choose $E = \mathbb{Q}(i, \sqrt{3})$, with maximal totally real subfield $E^+ = \mathbb{Q}(\sqrt{3})$ so that E/E^+ is unramified everywhere.

2. The second easiest choice for F' is $F' = \mathbb{Q}(\zeta_9)^+$. In this case we can choose $E = \mathbb{Q}(i\sqrt{5})$.

3. If $E = \mathbb{Q}(i)$, we can choose $F' = \mathbb{Q}(\alpha)$ with α a root of $X^3 - X^2 - 4X - 1$, which has discriminant 13^2.

4. If $F' = \mathbb{Q}(\alpha)$ and $E = \mathbb{Q}(i)$, we can choose any prime $p > 8, \ell$ congruent to $1, 5, 21, 25 \pmod{52}$, like $5, 53, 73, \ldots.$ In particular in that case we better should exclude $p = 13$ as in the early version [Bel10] (who knows?).

5. If $F' = \mathbb{Q}(\zeta_7)^+$ and $E = \mathbb{Q}(i\sqrt{3})$, we can choose any prime congruent to $1, 13 \pmod{21}$ like $13, 43, 97\ldots.$

6. If $F' = \mathbb{Q}(\zeta_7)^+$ and $E = \mathbb{Q}(i, \sqrt{3})$, we can take any prime $\ell \equiv 1, 13 \pmod{84}$ like $13, 97, 169\ldots$ and $p \equiv 1, 13 \pmod{21}$ like $13, 43, 97\ldots.$

7. If we really want to use $p = 13$ and that $p = 13$ is inert in $F = E^+$, and if we want moreover E/E^+ to be unramified everywhere, we can choose $E = \mathbb{Q}(i, \sqrt{7})$ with $F' = \mathbb{Q}(\beta) \subset \mathbb{Q}(\zeta_{43})$ as 43 is split in $\mathbb{Q}(i, \sqrt{7})/\mathbb{Q}(\sqrt{7})$, with β a root of $X^3 - X^2 - 14X - 8$.

In the following we say that a weight $\underline{k} \in \mathbb{Z}^{\mathrm{Hom}(K, \mathbb{C})}$ is *very regular* if, for $\tau_1 \neq \tau_2$ in $\mathrm{Hom}(K, \mathbb{C})$, we have $|k_{\tau_1} - k_{\tau_2}| \geqslant 2$.

Let Ψ be an algebraic Hecke character of $\mathbb{A}_K^\times$ with algebraic very regular weight $\underline{k} = (k_v)_{v|\infty}$, such that $\Psi^c = \Psi^\vee$ and such that Ψ is unramified both at p and ℓ. Choose an isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}_p}$. We moreover assume that

(Ψ, p) if $\mathfrak{p}|p$ in E, we have $\Psi(v)\Psi(v')^{-1} \notin \{1, p\}$ for $v \neq v'$ places of K dividing $\mathfrak{p}$.

(Ψ, ℓ) There exist $\lambda|\ell$ in E, and $\lambda'|\lambda$ in $E(\zeta_p)$, such that for all $v_1 \neq v_2$ places of K dividing λ, if v_1', v_2' are the corresponding places above λ' in $K(\zeta_p)$, $\iota(\Psi(v_1')) \pmod{\mathfrak{m}_{\overline{\mathbb{Q}}_p}} \neq \iota(\Psi(v_2')) \pmod{\mathfrak{m}_{\overline{\mathbb{Q}}_p}}$.

Consider moreover the following hypothesis on Ψ :

(Ψ, Ram) If v is a place of K such that Ψ is ramified at v, then v divides a prime which is totally split in $K/\mathbb{Q}$.

Let $\Psi_p : \mathbb{A}_K^\times \longrightarrow \overline{\mathbb{Q}}_p^\times$ be the p-adic realization of Ψ and ι, and $\psi_p : \mathrm{Gal}_K \to \overline{\mathbb{Q}}_p^\times$ such that $\psi_p = \Psi_p \circ \mathrm{Art}_K$. It is a Galois representation satisfying $\psi_p^\vee = \psi_p^c$.

8.2 Galois induction

Definition 8.3. We denote by ρ the induced Galois representation

$$\rho = \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_E} \psi_p = \{f : \mathrm{Gal}_E \longrightarrow \overline{\mathbb{Z}_p}^{\times} \mid f(gk) = \psi_p^{-1}(k) f(g) \forall g \in \mathrm{Gal}_E, k \in \mathrm{Gal}_K\},$$

where the action of $g \in \mathrm{Gal}_E$ is given by $(g \cdot f)(x) = f(g^{-1}x)$.

Then ρ is a three dimensional Galois representation since $[K : E]$ is Galois of degree 3. We claim the following

Lemma 8.4. *1. The representation $\overline{\rho} := \rho \otimes \overline{\mathbb{F}_p}$ is absolutely irreducible, in particular ρ is absolutely irreducible.*

2. The representation $\overline{\rho}(\mathrm{Gal}_{E(\zeta_p)})$ is adequate.

3. The representation ρ is polarized, i.e. $\rho^c \simeq \rho^\vee$.

4. The representation $\rho_{\mathrm{Gal}_{E_v}}$ is split, φ-generic, Hodge–Tate regular for any $v|p$ in E,

5. If v is a place of E such that ρ is ramified at v, then $\mathrm{Hom}_{\mathrm{Gal}_{E_v}}(\rho_v, \rho_v(1)) = 0$.

Proof. We will actually prove that $\overline{\rho}(\mathrm{Gal}_{E(\zeta_p)})$ acts absolutely irreducibly, which will imply point 1 and point 2 will follow by [Tho12] Lemma 2.4. To prove point 1, remark that if we denote by $\sigma \in \mathrm{Gal}_E$ a lift of a generator of the Galois group $\mathrm{Gal}(K/E) =< \overline{\sigma} >= \mathbb{Z}/3\mathbb{Z}$, then ρ has a basis given by $f, \sigma \cdot f, \sigma^2 \cdot f$, where f is the function

$$f : \mathrm{Gal}_E = \mathrm{Gal}_K \coprod \sigma \mathrm{Gal}_K \coprod \sigma^2 \mathrm{Gal}_K \longrightarrow \overline{\mathbb{Z}}_p^{\times}, k \in \mathrm{Gal}_K \mapsto \psi_p^{-1}(k), \sigma k, \sigma^2 k \mapsto 0.$$

Then $\sigma^3 \cdot f = \psi_p(\sigma^3) f$. Thus, after restricting to Gal_K, there is an isomorphism $\rho_{|\mathrm{Gal}_K} \simeq \psi_p \oplus \psi_p^\sigma \oplus \psi_p^{\sigma^2}$, where $\psi_p^\sigma = \psi_p(\sigma^{-1} \cdot \sigma)$. We reduce mod p, where we have a similar reduction after restricting to Gal_K. Because of the hypothesis (Ψ, ℓ) away from p, we have that $\overline{\rho}_{\mathrm{Gal}_{E(\zeta_p)_{\lambda'}}}$, for $\lambda'|\ell$, is the sum of three distinct characters. Moreover the group Gal_E acts transitively on these three eigenspaces. Therefore this representation is absolutely irreducible. To prove point 3, we compute $\rho^\vee$. By [CR81, Prop. 10.28], we have an isomorphism

$$\rho^\vee \simeq \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_E} \psi_p^{-1} = \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_E} \psi_p^c \simeq \rho^c.$$

Let us prove 4. As p is totally split in K/F, we have for $v|p$ in E, $\mathrm{Gal}_{E_v} \subset \mathrm{Gal}_K$ so that $\rho_{|\mathrm{Gal}_{E_v}} \simeq \psi_{p,v} \oplus \psi_{p,v}^\sigma \oplus \psi_{p,v}^{\sigma^2}$. As the group Gal_E acts transitively on the three places of K over v, we have $\rho_{|\mathrm{Gal}_{E_v}} \simeq \bigoplus_{v'|v} \psi_{p,v'}$. Therefore $\rho_{|\mathrm{Gal}_{E_v}}$ is crystalline and the eigenvalues of the Frobenius endomorphism of $D_{\mathrm{cris}}(\rho_{|\mathrm{Gal}_{E_v}})$ are the $\Psi(v')$ for $v'|v$ in K. It follows from hypothesis (Ψ, p) that $\rho_{|\mathrm{Gal}_{E_v}}$ is φ-generic. Moreover the Hodge–Tate weights of $\rho_{|\mathrm{Gal}_{E_v}}$ corresponds to the algebraic (infinitesimal) weight of Ψ, which was assumed regular so that $\rho_{\mathrm{Gal}_{E_v}}$ is Hodge–Tate regular.

Finally we prove 5. Let v be a place of E such that ρ_v is ramified. Then either v is ramified in K/E or Ψ_v is ramified. Assume in a first time that Ψ_v is ramified. Then (Ψ, Ram) implies that v divides a prime of $\mathbb{Q}$ which is totally split in K. In particular, v is split in K/E. As above, we have $\rho_v \simeq \bigoplus_{v'|v} \psi_{p,v'}$ with $\psi_{p,v'} = \Psi_{v'} \circ \mathrm{rec}_{K_{v'}}^{-1}$ as $v' \nmid p$. Therefore it follows from Lemma 8.5 below that $\mathrm{Hom}_{\mathrm{Gal}_{E_v}}(\rho_v, \rho_v(1)) = 0$.

Now assume that v is non split in K. As K/E is Galois there is a unique place w of K over v and $\rho_v \simeq \mathrm{Ind}_{\mathrm{Gal}_{K_w}}^{\mathrm{Gal}_{E_v}} \psi_{p,w}$. By Frobenius reciprocity, we have

$$\mathrm{Hom}_{\mathrm{Gal}_{E_v}}(\rho_v, \rho_v(1)) \simeq \mathrm{Hom}_{\mathrm{Gal}_{K_w}}(\psi_{p,w} \oplus \psi_{p,w}^{\sigma} \oplus \psi_{p,w}^{\sigma^2}, \psi_{p,w}\chi_{\mathrm{cyc}|K_w}).$$

Assume that $\psi_{p,w} = \psi_{p,w}^{\sigma}\chi_{\mathrm{cyc}|K_w}$. As $\chi_{\mathrm{cyc}|K_w} = \chi_{\mathrm{cyc}|K_w}^{\sigma}$, we deduce $\psi_{p,w}^{\sigma} = \psi_{p,w}^{\sigma^2}\chi_{\mathrm{cyc}|K_w}$ and $\psi_{p,w}^{\sigma^2} = \psi_{p,w}^{\sigma^3}\chi_{\mathrm{cyc}|K_w} = \psi_{p,w}\chi_{\mathrm{cyc}|K_w}$ so that $\psi_{p,w} = \psi_{p,w}\chi_{\mathrm{cyc}|K_w}^3$ which is false. We prove similarly than $\psi_{p,w} \neq \psi_{p,w}^{\sigma^2}\chi_{\mathrm{cyc}|K_w}$ and deduce $\mathrm{Hom}_{\mathrm{Gal}_{E_v}}(\rho_v, \rho_v(1)) = 0$. If $\Psi_w \neq \Psi_w \circ \sigma$, then the characters $\Psi_w, \Psi_w \circ \sigma, \Psi \circ \sigma^2$ are pairwise distinct and ρ_v is irreducible so that $\mathrm{Hom}_{\mathrm{Gal}_{E_v}}(\rho_v, \rho_v(1)) = 0$. If $\Psi_w = \Psi_w \circ \sigma$, then ρ_v is not irreducible, but clearly $\mathrm{Hom}_{\mathrm{Gal}_{E_v}}(\rho_v, \rho_v(1)) = 0$ (as $\psi_{p|\mathrm{Gal}_{E_v}} \neq \psi_{p|\mathrm{Gal}_{E_v}}(1)$). $\square$

Lemma 8.5. *Let $\Psi : \mathbb{A}_K^{\times}/K^{\times} \longrightarrow \mathbb{C}^{\times}$ be an algebraic Hecke character of very regular weight $\underline{k}$. Then, if ℓ is a prime number which is totally split in K, then $\Psi_v \neq \Psi_w|\cdot|_w$ for all places v, w of K dividing ℓ.*

Proof. Let Ψ and ℓ be as in the statement. Fix $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ and let $|\cdot|_\ell$ be the unique absolute value on $\overline{\mathbb{Q}}_\ell$ extending the one on $\mathbb{Q}_\ell$. Let Ψ_ι be the continuous character $\mathbb{A}_K^{\times}/K^{\times}K_\infty^{\times} \to \overline{\mathbb{Q}}_\ell^{\times}$ defined by

$$\Psi_{\iota,w}(x_w) = \begin{cases} \Psi_w(x_w) & \text{if } w \not|\ell, w \not|\infty \\ 1 & \text{if } w|\infty \\ \iota(\Psi_w(w_w)) \prod_{\tau \in \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_\ell), \tau|w} \tau(x_w)^{k_{\iota^{-1}\tau}} & \text{if } w|\ell, \end{cases}$$

where $\tau|w$ means that $|.|_\ell \circ \tau$ extends the absolute value given by w on K, and $(k_\sigma)_{\sigma \in \mathrm{Hom}(K,\mathbb{C})}$ is the weight of Ψ. As the group $\mathbb{A}_K^{\times}/K^{\times}K_\infty^{\times}$ is compact, we have $\mathrm{Im}(\Psi_\iota) \subset \overline{\mathbb{Z}}_\ell^{\times}$. As ℓ is totally split ι induces a bijection between $\{v|\ell\}$ and $\mathrm{Hom}(K, \mathbb{C})$. Let v be a place of K dividing ℓ corresponding to τ (i.e. $|.|_\ell \circ \iota^{-1}\tau$ extends $|.|_v$) and denote $k_v \coloneqq k_{\iota^{-1}\tau}$. We have

$$|\iota\Psi_v(\ell)\tau(\ell)^{k_v}|_\ell = 1$$

so that $|\iota(\Psi_v(\ell))| = l^{k_v}$. As ℓ is a uniformizer of K_v, for any $v|\ell$, the result follows. $\square$

8.3 Construction of an explicit set of Hecke characters

In this subsection we explain one way to find a Ψ as before, satisfying hypothesis $(\Psi, p), (\Psi, \ell), (\Psi, Ram)$. Fix E a CM extension, with $E^+ = F$ its maximal totally real subfield, so that $[E : E^+] = 2$. Fix also F' disjoint from E, a totally real degree 3 Galois extension of $\mathbb{Q}$. Choose p, ℓ two primes with are totally split in $K \coloneqq EF'$ such that $p > 8$. The following Lemma is a more precise version of [CHT08, Lem. 4.1.1].

Lemma 8.6. *Let F be a number field. Let S be a finite set of places of F. Let χ_S be an unramified continuous character $F_S^\times \coloneqq \prod_{v\in S} F_v^\times \to \mathbb{C}^\times$ of finite order. Let T be a set of finite places of F, disjoint from S and of Dirichlet density 1. Then there exists a continuous character $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ of finite order such that $\chi_{|F_S^\times} = \chi_S$ and the ramification places of χ are in T.*

Proof. Let U^S be the product of the $\mathcal{O}_{F_v}^\times$ for $v \notin S$. Then $F^\times \cap U^S$ is a finitely generated subgroup of $F^\times$. Let us write m for the order of the finite cyclic group $\chi_S(F^\times \cap U^S)$. It follows from the proof of Theorem 1 in [Che51] that we can find finitely many places $w_1, \ldots, w_r$ in T such that the subgroup of $F^\times \cap U^S$ congruent to 1 modulo $\mathfrak{p}_{w_1}, \ldots, \mathfrak{p}_{w_r}$ is contained in $(F^\times \cap U^S)^m$. We conclude as in the proof of [CHT08, Lem. 4.1.1] choosing for U the product of the U_v for v not in S nor $\{w_1, \ldots, w_r\}$ and a small enough subgroup at $w_1, \ldots, w_r$. □

Lemma 8.7. *Let K be an (imaginary) CM field with totally real subfield K^+ and complex conjugacy c. Denote $\psi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ be a continuous character. Assume that there exists a finite set S of places of K which are split in K/K^+ and such that $\psi_v^{-1} = \psi_{cv}$ for $v \in S$. Moreover, assume that S contains the Archimedean places. Let T be a finite set of places of K that contains S and is stable under c, such that ψ is unramified outside of T. Then there exists a Hecke character $\widetilde{\psi} : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ such that $\widetilde{\psi}^{-1} = \widetilde{\psi}^c$ and $\widetilde{\psi}_v = \psi_v$ for $v \in S$ and such that $\widetilde{\psi}_v$ is unramified outside of T.*

Proof. Let $\theta = \psi \circ N_{K/K^+}$. As S contains the Archimedean places, the character θ is trivial at Archimedean places and is therefore a character of finite order. Let $U_T \subset \prod_{v\in T\smallsetminus S} K_v^\times$ be a compact open subgroup such that $\theta_{|U_T}$ is trivial and such that $c(U_T) = U_T$. Let

$$U = \Big(\prod_{v\notin T} \mathcal{O}_{K_v}^\times\Big) \cdot U_T \cdot \Big(\prod_{v\in S} K_v^\times\Big).$$

We have an injection of compact groups

$$N_{K/K^+}(\mathbb{A}_K^\times)/(N_{K/K^+}(\mathbb{A}_K^\times) \cap K^\times U) \hookrightarrow \mathbb{A}_K^\times/K^\times U.$$

Under our hypothesis, the character $\psi_{|N_{K/K^+}(\mathbb{A}_K^\times)}$ is trivial on $(N_{K/K^+}(\mathbb{A}_K^\times) \cap K^\times U)$. Therefore it extends to a character α of finite order of $\mathbb{A}_K^\times$ trivial on $K^\times U$. We thus have $\psi \circ N_{K/K^+} = \alpha \circ N_{K/K^+}$. It is easy to check that the character $\widetilde{\psi} = \psi\alpha^{-1}$ satisfies our requirements. □

Proposition 8.8. *For each choice of fields E and F' and places p and ℓ and very regular weight $\underline{k}$ as above there exists a Hecke character $\Psi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ satisfying (Ψ, p), (Ψ, ℓ) and (Ψ, Ram) and such that $\Psi^{-1} = \Psi^c$.*

Proof. Let $\underline{k}$ be a very regular weight. It follows from [Sch88], Section 0.3, that there exists a Hecke character Ψ_0 of $\mathbb{A}_K^\times/K^\times$ with weight $\underline{k}$. Using Lemma 8.6, we can construct a Hecke character θ of finite order such that, setting $\Psi_1 = \Psi_0\theta$, we have

- the character Ψ_1 satisfies (Ψ_1, p) and (Ψ_1, ℓ) ;
- there exists finitely many primes $\ell_1, \ldots, \ell_r$, different from p and ℓ, which are totally split in K and such that Ψ_1 is only ramified at places dividing $\ell_1, \ldots, \ell_r$;
- we have $\Psi_{1,w}^{-1} = \Psi_{1,cw}$ for any place w of K dividing ℓ or p.

Now it follows from Lemma 8.7 that there exists a Hecke character Ψ of $\mathbb{A}_K^\times/K^\times$ such that

- $\Psi^{-1} = \Psi^c$;
- $\Psi_v = \Psi_{1,v}$ if v is a place of K dividing p or ℓ ;
- Ψ is ramified only at places dividing $\ell_1, \ldots, \ell_r$. □

8.4 Automorphic Induction and base change

Let Ψ and ρ as in subsection 8.1 and let U denote the unitary group in three variables for E/E^+ that is compact at infinity and quasi-split at all finite places. We need to find an automorphic form for U whose associated Galois representation is induced representation ρ from 8.3.

Proposition 8.9. *There exists an automorphic representation Π of $\mathrm{GL}_{3,E}$, cuspidal, cohomological at infinity, unramified at ℓ and p, polarized, whose associated Galois representation is given by ρ.*

Proof. This is the content of [Hen12] Théorème 3 (as K/E cyclic of degree 3) for the existence of the automorphic representation, Théorème 5 for the compatibility with the local correspondence at ℓ and p and at infinity (cf. the following remark of [Hen12]). Polarization can be checked after base change of the automorphic induction to K, where it follows as $\Psi^c = \Psi^\vee$, and as $\Psi \neq \Psi^\sigma$ for $\sigma \in \mathrm{Gal}(K/E)$ such that $\sigma \neq 1$. Moreover, the automorphic induction is also cuspidal (Theorem 2 of [Hen12]). □

Conjecture 8.10. *There exists a cohomological, cuspidal, automorphic representation π of U whose base change to $\mathrm{GL}_{3,E}$ is Π.*

Proposition 8.11. *If E/E^+ is everywhere unramified (e.g. for $E = \mathbb{Q}(i, \sqrt{3})$ or $\mathbb{Q}(i, \sqrt{7})$), then the previous conjecture is true.*

Proof. This is [Lab11] Theorem 5.4. □

Proposition 8.12. *If E is quadratic imaginary, then the previous conjecture is true.*

Proof. By [Mor10] Corollary 8.5.3 (ii), there exists π' an automorphic representation for $GU(3)$ associated to $\Pi \times 1$, which is automorphic for $\mathrm{GL}_3 \times \mathrm{GL}_1$. By [HS22] Lemma A.7 (based on [HT01]), there exists π, an automorphic representation of $U(3)$ associated to π'. $\square$

Corollary 8.13. *If E is quadratic imaginary or if E/E^+ is everywhere unramified, then there exists a classical form on $U(3)$ satisfying the hypothesis of Theorem 1.2.*

Proof. Let π be the automorphic representation of U considered above, and let $f \in \pi$ be an eigenform for the Hecke operators away from a set S of bad places of π. Then $\rho_f = \rho_\pi = \rho$ is crystalline at p and φ-generic. In particular it has $3! = 6$ refinements which are automorphic and split at p. Hence there exists an automorphic refinement $\mathcal{R}$ of f with relative position $w_{\mathcal{R}} = 1$ with respect to the Hodge filtration. In particular, for this choice of a refinement, there exists a refined classical modular form f' satisfying all hypothesis of Theorem 1.2. But, by Lemma 8.4(5) we know that f gives, for all $v \in S \backslash S_p$, a point of $\mathcal{X}_{\rho_v}^{\square}$ which satisfies $\mathrm{Hom}_{\mathrm{Gal}_{E_v}}(\rho_v, \rho_v(1)) = 0$. When v splits in E/E^+, such a point is a smooth point by [All16] Prop 1.2.2. $\square$

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