Stability an discretization for some elliptic inverse parameter problems from internal data - application to elastography

Laurent Seppecher
École Centrale de Lyon

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One World Imagine Seminars

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Find μ such that

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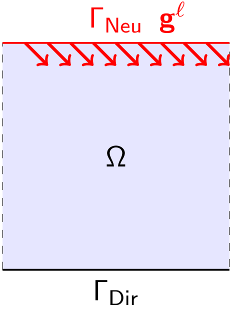
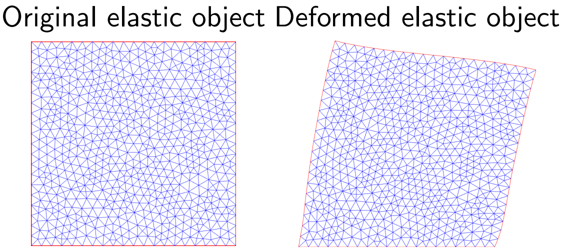
$$-\nabla \cdot (\mu S) = \mathbf{f} \quad \text{in } \Omega,$$

- Ω is a Lipschitz domain of $\mathbb{R}^d$
- $S \in L^\infty(\Omega, \mathbb{R}^{d \times d})$ is given
- $\mathbf{f}$ is a given vector field (can be zero)
- μ is the unknown parameter function

Joint work with E. Bretin, P. Millien

Elastography from internal data





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- step 1: Record the displacement field $\mathbf{u}(x)$ inside the domain

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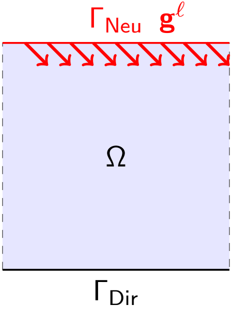

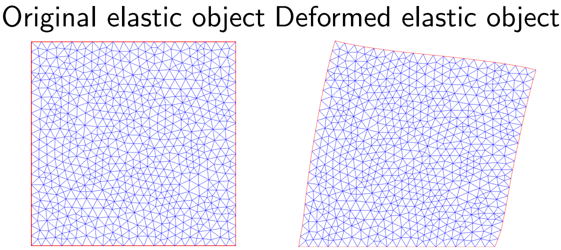

Inverse problem in two steps

- step 1: Record the displacement field $\mathbf{u}(x)$ inside the domain
- step 2: Reconstruct the elastic properties of the medium

Medical elastography

Goal

Measure the elastic parameters of soft biological tissues

Advantages:

- High contrast (for the shear modulus)
- Good discrimination between pathological states

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Difficulties:

- High contrast (multiple scattering of waves)
- High wavelength

Medical elastography

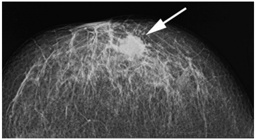

Figure: X-rays image of breast-tumor

Detect an characterize tumoral and pre-tumoral tissues

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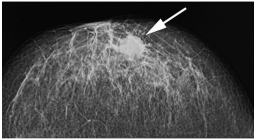

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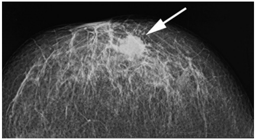

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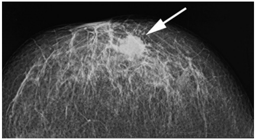

Figure: X-rays image of breast-tumor

Detect an characterize tumoral and pre-tumoral tissues

- Scanner (X-rays imaging) is harmful and expensive (poor discrimination),
- Untrasound imaging fails (no contrast),
- Shear modulus $\mu(x)$ is very high in tumoral tissues.

Quasi-static deformation of a phantom

Quasi-static elastography

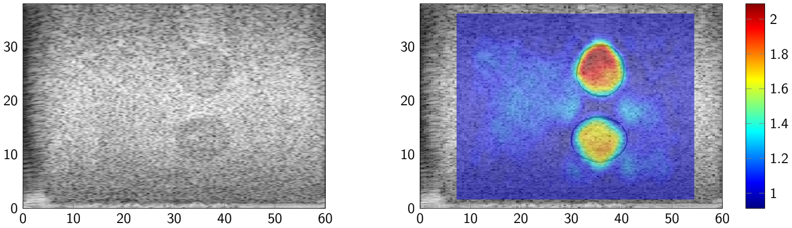

Figure: Shear modulus image of phantom from quasi-static data (data from E. Brusseau and L. Pretrusca - CREATIS/INSA)

Shear wave imaging by fast-ultrasound

Reconstruct the corresponding shear modulus

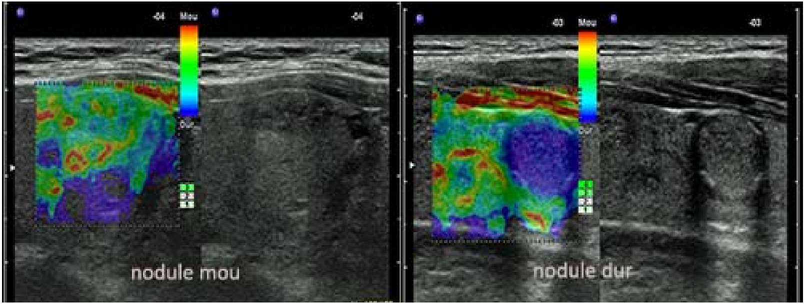

Figure: Thyroid nodules image by UF Ultrasound elatography (soft/hard)

An hybrid (multi-physics) imaging method

General idea

Mecanically perturbate a medium and track the response using a high resolution imaging modality. Hopefully get some info from the reaction of the medium.

- MRI Elastography

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- US Elastography
- OCT Elastography

Different types of perturbations : static, dynamic, harmonic.

Inversion step 2 : recover the shear modulus

Linear elasticity:

$$\begin{cases} -\nabla \cdot (2\mu\mathcal{E}(\mathbf{u})) - \nabla(\lambda\nabla\cdot\mathbf{u}) = \mathbf{f} & \Omega \\ BC & \partial\Omega \end{cases}$$

with $\mathbf{u} \in \mathbb{R}^d$ the displacement field, $\mathcal{E}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ and (λ, μ) are the Lamé coefficients.

Inverse problem

Recover (λ, μ) from the knowledge of $\mathbf{u}$ in Ω.

Remark

In soft tissues, $\lambda(x) \sim \lambda_0$ and assumed known.

Available inversion algorithms (1)

$$-\nabla \cdot (\mu \mathcal{E}(\mathbf{u})) = \mathbf{f}$$

Solving a first order transport equation in μ

INSTITUTE OF PHYSICS PUBLISHING INVERSE PROBLEMS
Inverse Problems **20** (2004) 1–24 PII: S0266-5611(04)62168-X

Recovery of the Lamé parameter μ in biological tissues

Lin Ji and Joyce McLaughlin

Department of Mathematics, Rensselaer Polytechnic Institute, Troy, NY 12180, USA

IOP Publishing Inverse Problems
Inverse Problems **30** (2014) 125004 (22pp) doi:10.1088/0266-5611/30/12/125004

Reconstruction of constitutive parameters in isotropic linear elasticity from noisy full-field measurements

Guillaume Bal[1], Cédric Bellis[2], Sébastien Imperiale[3] and François Monard[4]

Inversion step 2 : recover the shear modulus

Transport : Assume that μ is smooth and known near $\partial\Omega$ and remark that

$$\nabla \cdot (\mu \nabla^s \mathbf{u}) = \nabla^s \mathbf{u} \nabla \mu + \mu \nabla \cdot \nabla^s \mathbf{u}$$

assume that $\nabla^s \mathbf{u}$ is a.e. invertible μ is solution of the transport problem,

$$\nabla \mu + \mu (\nabla^s \mathbf{u})^{-1} \nabla \cdot \nabla^s \mathbf{u} = -(\nabla^s \mathbf{u})^{-1} \mathbf{f}$$
$$\nabla \mu + \mu \mathbf{b} = -\tilde{\mathbf{f}}$$

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(Proof of uniqueness and stability with several measurements and strong smoothness hypothesis and boundary data)

Available inversion algorithms (2)

Least squares : Assume knowledge of **g** the surface density of force outside of Ω and define

$$F : \mu \mapsto \mathbf{u}[\mu] : \begin{cases} -\nabla \cdot (\mu \nabla^s \mathbf{u}) = \mathbf{f} \quad (\partial_{tt}\mathbf{u}), & \text{in } \Omega, \\ \mu \nabla^s \mathbf{u} \cdot \nu = \mathbf{g} & \text{on } \partial\Omega, \end{cases}$$

defining $F : L^\infty(\Omega, [\mu_0, +\infty)) \to H^1(\Omega, \mathbb{R}^d)$ fréchet differentiable. Then minimize

$$J[\mu] = \|F[\mu] - \mathbf{u}_{mes}\|^2_{H^1(\Omega)} + \text{ reg. term}$$

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- Very slow (flat problem)
- needs knowledge of $\mathbf{g}$ and μ on the boundary

Available inversion algorithms (3)

Wave front traking : assuming that μ is piecewise constant,

$$\partial_{tt}\mathbf{u} - \mu\nabla \cdot \nabla^s\mathbf{u} \approx \mathbf{0}, \quad \textit{a.e.},$$

the wave speed is $c = \sqrt{\mu}$.

Available inversion algorithms (4)

Algebraic inversion :

IOP PUBLISHING — PHYSICS IN MEDICINE AND BIOLOGY
Phys. Med. Biol. **52** (2007) 1577–1593 — doi:10.1088/0031-9155/52/6/003

Elastic modulus imaging: some exact solutions of the compressible elastography inverse problem

Paul E Barbone[1] and Assad A Oberai[2]

426 — IEEE TRANSACTIONS ON ULTRASONICS, FERROELECTRICS, AND FREQUENCY CONTROL, VOL. 49, NO. 4, APRIL 2002

Shear Modulus Imaging with 2-D Transient Elastography

Laurent Sandrin, Mickaël Tanter, Stefan Catheline, and Mathias Fink

Example

If μ is constant then $\mu \Delta \mathbf{u} = \rho \partial_{tt} \mathbf{u} \Longrightarrow \mu \approx \rho \frac{|\partial_{tt} \mathbf{u}|}{|\Delta \mathbf{u}|}$

Current chalenges for medical elastography

- Increase the resolution

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- Be more quantitative

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- Be more stable

Current chalenges for medical elastography

- Increase the resolution
- Be more quantitative
- Be more stable
- Be more practical (quasi-static with acoustic probe ?)

A general equation

The problem takes the general form

Reduced elastography problem

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in the cases

- λ is known: $S := 2\mathcal{E}(\mathbf{u})$

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But also for conductivity equation with two internal data:

$$-\nabla \cdot (\sigma[\nabla u_1 \ \nabla u_2]) = \mathbf{f}$$

And other problems. . .

The Reverse Weak Formulation

Define the operator

$$\begin{aligned} T : L^\infty(\Omega) \subset L^2(\Omega) &\to H^{-1}(\Omega, \mathbb{R}^d) \\ \mu &\mapsto -\nabla \cdot (\mu S) \end{aligned}$$

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or by the equivalent variational formulation

$$a(\mu, \mathbf{v}) := \langle T\mu, \mathbf{v} \rangle_{H^{-1}, H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d \times d})$$

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- No boundary data used
- Only smoothness hypothesis: $S \in L^\infty(\Omega, \mathbb{R}^{d\times d})$
- "Easy" to discretize through the Galerkin method

Reverse Weak Formulation: discretization

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Find $\mu_h \in M_h$ s.t.

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where

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where

- (M_h, V_h) approaches $(L^2(\Omega), H_0^1(\Omega, \mathbb{R}^d))$
- T_h approaches T
- $\mathbf{f}_h$ approaches $\mathbf{f}$

Approximation of the spaces

Let M be a Hilbert and $M_h \subset M$ a sub-Hilbert space and $\pi_h : M \to M_h$ the orthogonal projection.

Definition

The sequence $(M_h)_{h>0}$ approaches M if for any $\mu \in M$,

$$\lim_{h \to 0} \|\pi_h \mu - \mu\|_M = 0.$$

For any non zero $\mu \in M$, we define its relative error of interpolation onto M_h by

$$\varepsilon_h^{\text{int}}(\mu) := \frac{\|\pi_h \mu - \mu\|_M}{\|\mu\|_M}.$$

Approximation of the operator

The operator $T : L^2 \to H^{-1}$ given by

$$\langle T\mu, \mathbf{v} \rangle_{H^{-1},H_0^1} := \int_\Omega \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d\times d})$$

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Hence

$$\begin{aligned} \langle (T_h - T)\mu, \mathbf{v} \rangle_{V_h', V_h} &= \int_\Omega \mu (S_h - S) : \nabla \mathbf{v} \\ &\leq \|\mu\|_{L^\infty} \|S_h - S\|_{L^2(\Omega)} \|\mathbf{v}\|_{H_0^1} \end{aligned}$$

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The error $T_h - T$ is small for the $\mathcal{L}(L^\infty, V_h')$ topology weaker than the $\mathcal{L}(L^2, V_h')$) topology!

Approximation of the operator

Definition

The interpolation error $\varepsilon_h^{\mathrm{op}}$ between T and T_h is defined by

$$\varepsilon_h^{\mathrm{op}} := \| T_h - T \|_{L^\infty, V_h'} := \sup_{\mu \in M_h} \sup_{\mathbf{v} \in V_h} \frac{\langle (T_h - T)\mu, \mathbf{v} \rangle_{V_h', V_h}}{\|\mu\|_{L^\infty} \|\mathbf{v}\|_{H_0^1}}.$$

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- This error contains both the data noise and the interpolation error over (M_h, V_h).
- This particular norm does not allow us to use directly the sensitivity analysis and discretization analysis for the Moore-Penrose generalized inverse of T when T is a closed range operator

Approximation of the right-hand side

Definition

The relative error of interpolation $\varepsilon_h^{\text{rhs}}$ between $\mathbf{f} \neq \mathbf{0}$ and $\mathbf{f}_h$ is defined by

$$\varepsilon_h^{\text{rhs}} := \frac{1}{\|\mathbf{f}\|_{V'}} \sup_{\mathbf{v} \in V_h} \frac{\langle \mathbf{f}_h - \mathbf{f}, \mathbf{v} \rangle_{V_h', V_h}}{\|\mathbf{v}\|_V} := \frac{\|\mathbf{f}_h - \mathbf{f}\|_{V_h'}}{\|\mathbf{f}\|_{V'}}$$

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- Is $T\mu = \mathbf{f}$ invertible with stability ?

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- Is the solution μ_h close to μ in $L^2(\Omega)$?

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Proposition

If Ω is Lipschitz, then $\nabla : L^2 \to H^{-1}$ has closed range. i.e. there exists $C > 0$ s.t.

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Operator ∇ satisfies the inf-sup condition. It is invertible with stability in $L^2(\Omega) \cap N(\nabla)^\perp$.

A model problem: discretization

Problem: the constant β may not behave well in finite element spaces!

Take $M_h \subset L^2_0(\Omega)$ and $V_h \subset H^1_0(\Omega, \mathbb{R}^d)$ the discrete *inf-sup* constant

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Pairs of finite element spaces that satisfy the discrete *inf-sup* condition are known as *inf-sup* stable elements and play an important role in the stability of the Galerkin approximation for the Stokes problem.

Inf-sup constant for the operator T

Theoretical study of $T\mu := -\nabla \cdot (\mu S)$, with Ammari, Bretin and Millien (2020):

If $S \in W^{1,p}$ $p > d$ and $|\det S(x)| \geq c > 0$ a.e, we have

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- At worst T is a "gradient type" operator
- works for S "piecewise" $W^{1,p}$
- minimal assumption on S to have closed range property is an open question (as far as we know)

Generalized inf-sup constant

M, V two Hilbert spaces and $T \in \mathcal{L}(M, V')$,

Definition (classic constants)

$$\alpha(T) := \inf_{\mu \in M} \sup_{\mathbf{v} \in V} \frac{\langle T\mu, \mathbf{v} \rangle_{V',V}}{\|\mu\|_M \|\mathbf{v}\|_V} \quad \text{and} \quad \rho(T) := \sup_{\mu \in M} \sup_{\mathbf{v} \in V} \frac{\langle T\mu, \mathbf{v} \rangle_{V',V}}{\|\mu\|_M \|\mathbf{v}\|_V}.$$

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The generalized *inf-sup* constant red$\beta(T)$ is built as follows:

$$\beta_e(T) := \inf_{\substack{\mu \in M \\ \mu \perp e}} \sup_{\mathbf{v} \in V} \frac{\langle T\mu, \mathbf{v} \rangle_{V',V}}{\|\mu\|_M \|\mathbf{v}\|_V} \quad \beta(T) := \sup_{\substack{e \in M \\ \|e\|_M = 1}} \beta_e(T).$$

Correspondance

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If $N(T) \neq \{0\}$, consider any $z \in N(T)$ such that $\|z\|_M = 1$. Then we have $\beta(T) = \beta_z(T)$.

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Proposition

If there exists $z \in M$ such that $\|z\|_M = 1$ and $\|Tz\|_{V'} = \alpha(T)$, Then we have $\beta(T) = \beta_z(T)$.

True for any finite rank (and finite dimensional) operator

Discrete inf-sup constant

Definition (Discrete inf-sup constant)

$$\beta(T_h) := \inf_{\substack{\mu \in M_h \\ \mu \perp z_h}} \sup_{\mathbf{v} \in V_h} \frac{\langle T_h \mu, \mathbf{v} \rangle_{V_h', V_h}}{\|\mu\|_M \|\mathbf{v}\|_V}.$$

where

$$z_h = \arg\min_{z \in M_h} \sup_{\mathbf{v} \in V_h} \frac{\langle T_h \mu, \mathbf{v} \rangle_{V_h', V_h}}{\|z\|_M \|\mathbf{v}\|_V}.$$

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What is the behavior of $\beta(T_h)$ with respect to $\beta(T)$?

Upper semi-continuity of the inf-sup constant

Theorem

If $\varepsilon_h^{op} \to 0$ *when* $h \to 0$, *then*

$$\limsup_{h\to 0} \alpha(T_h) \le \alpha(T).$$

Moreover, if the problem $Tz = \mathbf{0}$ *admits a non zero solution* $z \in L^\infty(\Omega)$ *and if the sequence* $(T_h)_{h>0}$ *satisfies the discrete inf-sup condition, then*

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- $\beta(T_h)$ is not asymptotically better than $\beta(T)$.
- It might be a possible way to show that T as closed range.

Discrete stability estimate (case $\mathbf{f} = \mathbf{0}$)

Theorem (1)

Let $z \in L^\infty(\Omega)$ *be a solution of* $T\,z = \mathbf{0}$ *with* $\|z\|_M = 1$.. *Fix* $r \geq \|z\|_\infty$ *and consider* $z_h \in M_h$ *a solution of*

$$\|T_h z_h\|_{V_h'} = \alpha(T_h) \quad \text{with} \quad \|z_h\|_M = 1 \quad \text{and} \quad \langle z_h, z\rangle_M \geq 0. \quad (3)$$

If $\beta(T_h) > 0$ *we have*

$$\|z_h - \pi_h z\|_{L^2(\Omega)} \leq \frac{4}{\beta(T_h)}\big(\sqrt{2}\, r\, \varepsilon_h^{op} + 2\rho(T)\varepsilon_h^{int}(z)\big).$$

Moreover, if $\beta(T_h) \geq \beta^* > 0$ *and if* $\varepsilon_h^{op} \to 0$, *then* $z_h \to z$.

Discrete stability estimate general case

Theorem (2)

Consider $\mu \in L^\infty(\Omega)$ a solution of $T\mu = \mathbf{f}$. Fix $r > 0$ such that $\|\mu\|_{L^\infty} \leq r\,\|\mu\|_{L^2}$. Consider $z_h \in M_h$ a solution of

$$\|T_h z_h\|_{V_h'} = \alpha(T_h) \quad \textit{with} \quad \|z_h\|_M = 1.$$

Consider now $\mu_h \in M_h$ a solution of $\mu_h = \underset{\substack{m \in M_h \\ m \perp z_h}}{\arg\min} \|T_h m - \mathbf{f}_h\|_{V_h'}$.

If $\beta(T_h) > 0$, there exits $t \in \mathbb{R}$ such that $\mu_{h,t} := t z_h + \mu_h$ satisfies

$$\frac{\|\mu_{h,t} - \pi_h \mu\|_{L^2}}{\|\pi_h \mu\|_{L^2}} \leq \frac{4}{\beta\,(T_h)} \left[r\,\varepsilon_h^{op} + \rho(T) \left(\varepsilon_h^{rhs} + \varepsilon_h^{int}(\mu) \right) + \frac{\alpha\,(T_h)}{2} \right].$$

honeycomb finite element

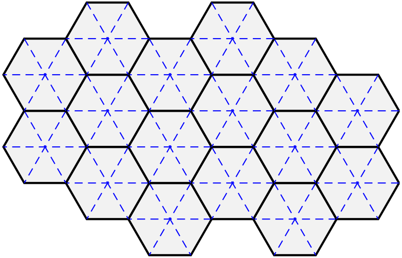


Figure: Honeycomb space discretization. In plain black, the hexagonal subdivision and in dashed blue, the triangular subdivision.

$$M_h := \mathbb{P}^0\left(\Omega_h^{\text{hex}}\right) = \left\{\mu \in L^2(\Omega_h) \mid \forall j\ \mu|_{\Omega_{h,j}^{\text{hex}}} \text{ is constant}\right\}.$$

$$V_h := \mathbb{P}_0^1\left(\Omega_h^{\text{tri}}, \mathbb{R}^2\right) = \left\{\mathbf{v} \in H_0^1(\Omega_h, \mathbb{R}^d) \mid \forall k\ \mathbf{v}|_{\Omega_{h,k}^{\text{tri}}} \text{ is linear}\right\}.$$

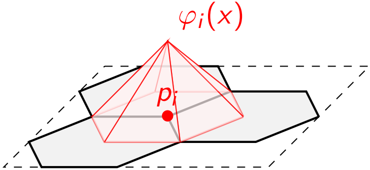
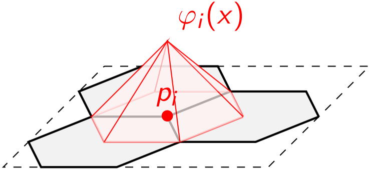


Figure: Support and graph of basis test function φ_i.

Why does it work ?



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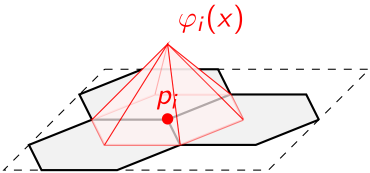

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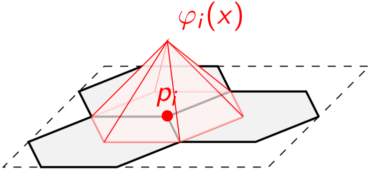

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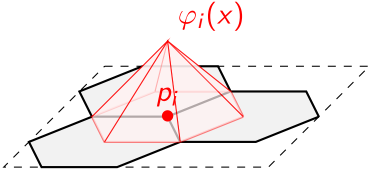

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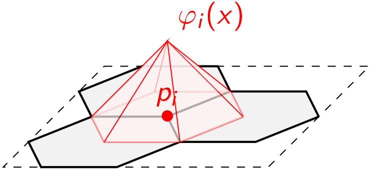

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One value is given $\Rightarrow$ all the other are fixed. $\Rightarrow$ null-space is at most of dimension 1 $\Rightarrow \beta(T_h) > 0$

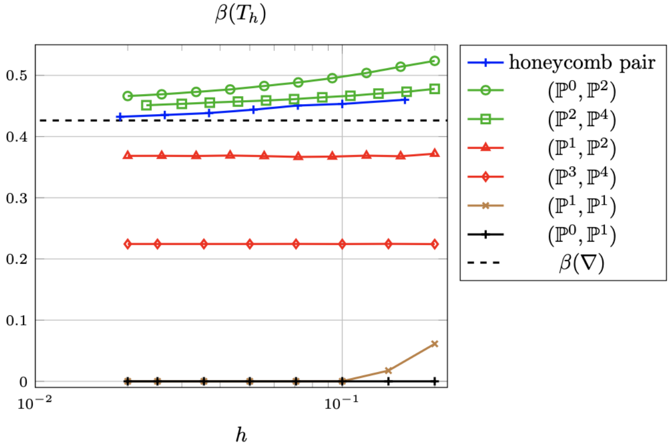
Inverse gradient problem

In $\Omega = (0,1)^2$ we approach $-\nabla\mu = \mathbf{f}$. Here $T_h := -\nabla|_{M_h}$ and then $\varepsilon_h^{\mathsf{op}} = 0$. Moreover $\rho(\nabla) \leq 1$. In the absence of noise, the result of Theorem 2 reads,

$$\frac{\|\mu_h - \pi_h\mu\|_{L^2}}{\|\pi_h\mu\|_{L^2}} \leq \frac{4}{\beta\left(T_h\right)}\left(\frac{\|\mathbf{f} - \mathbf{f}_h\|_{V_h'}}{\|\mathbf{f}\|_{H^{-1}}} + \frac{\|\mu - \pi_h\mu\|_{L^2}}{\|\mu\|_{L^2}}\right).$$

Note that we know $\beta(\nabla) = \sqrt{1/2 - 1/\pi}$ as a conjecture.

Inverse gradient problem: behavior of $\beta(T_h)$



Inverse gradient problem: result

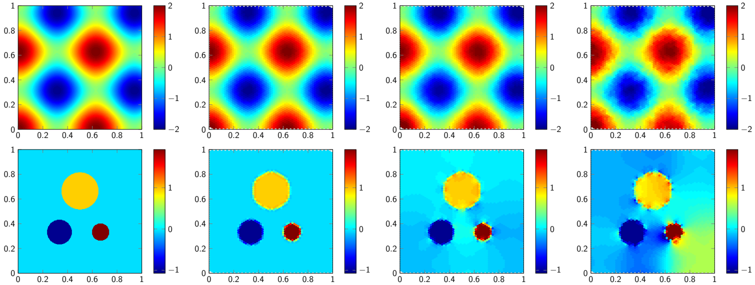


Figure: Numerical stability of the reconstruction of maps μ_1 and μ_2 using method given by Theorem 2 with resolution $h = 0.01$. From left to right: column 1: exact map to recover, 2. reconstruction with no noise, column 3: reconstruction with noise level $\sigma = 1$, column 4: reconstruction with noise level $\sigma = 2$.

Quasistatic elastography

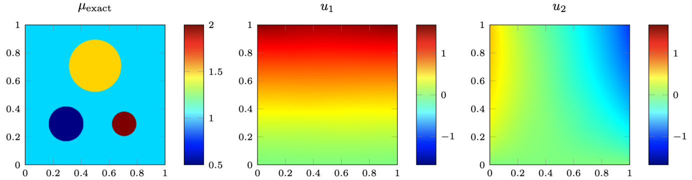


FIG. 5. *First line, from left to right: The exact map μ_{exact}, the two components of the data field $\boldsymbol{u} = (u_1, u_2)$ computed via (5.6), the only data used to inverse the problem.*

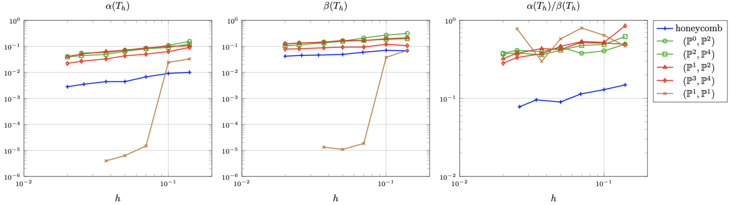


FIG. 6. *Behavior of the contants $\alpha(T_h)$, $\beta(T_h)$ and the ratio $\alpha(T_h)/\beta(T_h)$ for the inverse static elastography problem in the unit square $\Omega := (0,1)^2$, for various choices of pair of discretization spaces.*

Algorithm

Write T_h as a matrix $\mathcal{T}$ in the basis of the chosen M_h and V_h. Define the matrix

$$\mathcal{M} := \mathcal{B}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1}$$

where $\mathcal{B}_M$ and $\mathcal{B}_V$ are the basis matrix of M_h and V_h. Then

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- $\alpha(T_h)$ is the smallest singular value of $\mathcal{M}$
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- μ is the first singular vector of $\mathcal{M}$.

Reconstruction for the honeycomb

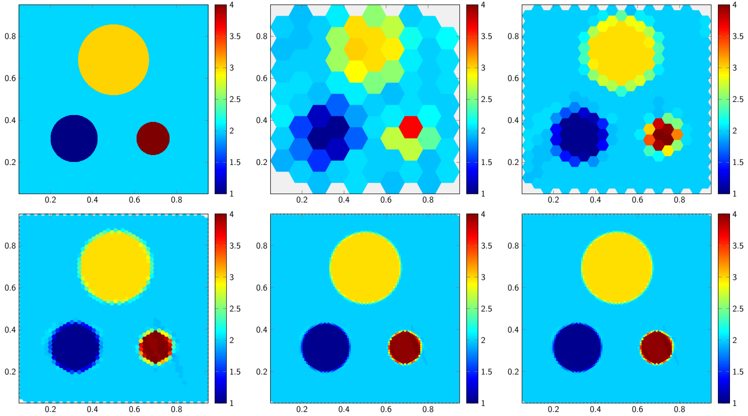

Figure: Reconstruction of the shear modulus map μ using the honeycomb pair.

Reconstruction for various pairs of spaces

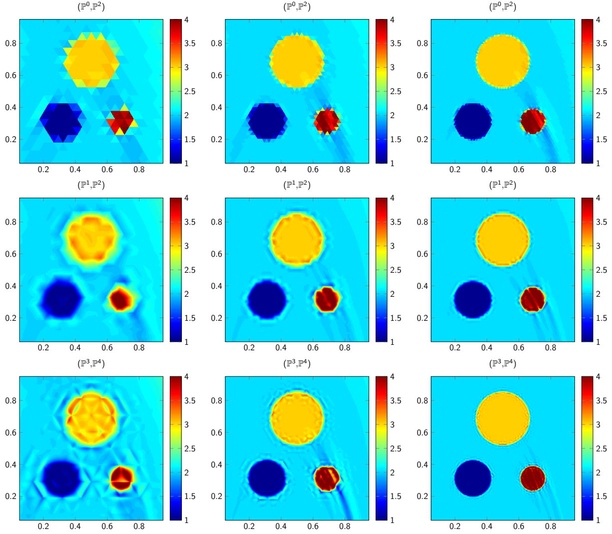

Figure: Reconstruction of the shear modulus map μ using various pairs of finite element spaces in the subdomain of interest $(0.1, 0.9)^2$.

Quasi-static elastography

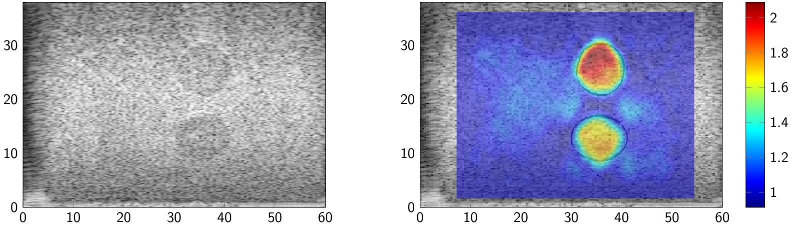

Figure: Shear modulus image of phantom from quasi-static data (data from E. Brusseau and L. Pretrusca - CREATIS/INSA)

In vivo quasistatic elastography

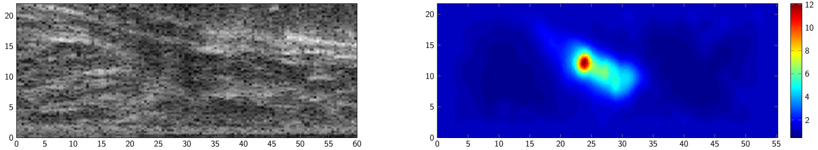

Figure: Reconstruction of the shear modulus of *in-vivo* malignant breast tumor from quasi-static elastography (data from E. Brusseau - INSA/CREATIS) $h = 0.7$ mm.

Thank you for your attention