Regularization for seismic sources inversion from interferometric data

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joint work with Laurent Demanet (MIT).

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ÉCOLE CENTRALE LYON

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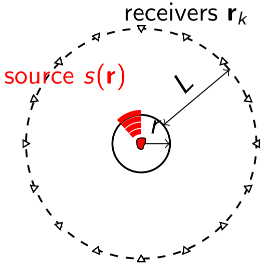
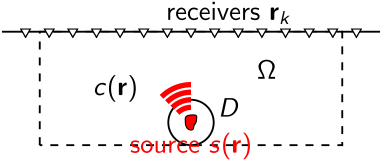
General model





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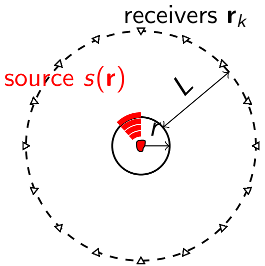

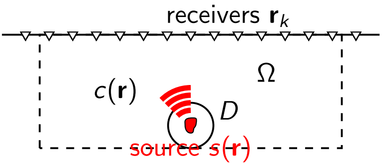

Time

$$\begin{cases} -\Delta U(\mathbf{r},t) + c^{-2}(\mathbf{r})\partial_{tt}U(\mathbf{r},t) = s(\mathbf{r})f(t), \ \mathbf{r} \in \mathbb{R}^d, \ t \in [0,+\infty) \\ \text{outgoing condition in } \mathbb{R}^d \end{cases}$$

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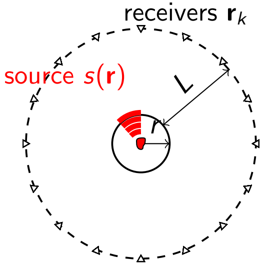

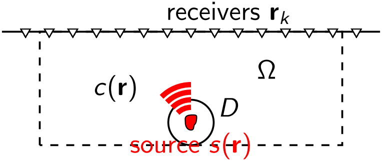

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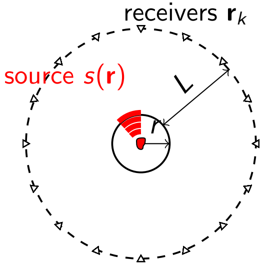
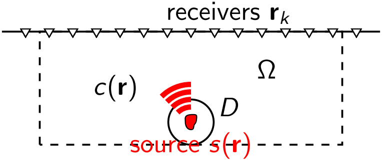
Frequency

$$\begin{cases} \Delta u(\mathbf{r},\omega) + \dfrac{\omega^2}{c^2(\mathbf{r})}u(\mathbf{r},\omega) = -s(\mathbf{r}), \quad \mathbf{r}\in\mathbb{R}^d, \quad t\in[0,+\infty) \\ \text{outgoing condition in } \mathbb{R}^d \end{cases}$$

Interferomtric data (time)





Linear Data

$d_k(\omega) := u(\mathbf{r}_k, \omega)$, $\quad \mathbf{r}_k$ receivers positions, $\quad \omega \in [0, +\infty)$.

Linear inversion problem:

Find the source $s(\mathbf{r})$ from the knowledge of all $d_k(\omega)$.

Interferometric data

Time Interferometric Data (cross-correlations)

$$D_{k\ell}(\tau) := \int_{\mathbb{R}} U(\mathbf{r}_k, t)U(\mathbf{r}_\ell, t-\tau)\mathrm{d}t, \quad t \in [0, +\infty)$$

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Frequency Interferometric Data

$$D_{k\ell}(\omega) := d_k(\omega)\overline{d_\ell}(\omega),\ \omega \in [0, +\infty)$$

and more generally

$$D_{k\ell}(\omega, \omega') := d_k(\omega)\overline{d_\ell}(\omega'), \quad \omega, \omega' \in [0, +\infty).$$

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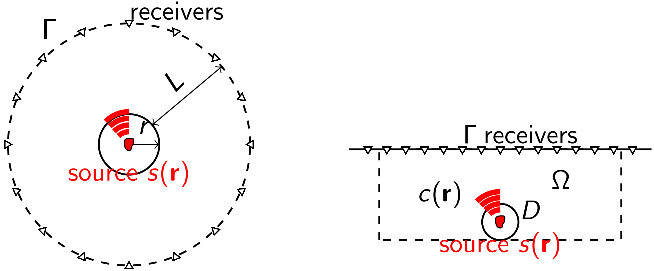
How these data can be used to increase stability?

Interferomtric source inversion problem



Interferometric inversion problem:

Find the source $s(\mathbf{r})$
from the knowledge of some $D_{k\ell}(\omega,\omega') := d_k(\omega)\overline{d_\ell}(\omega')$.

Contents

- Advantage of interferometric data
- Coherent Interferometric imaging (back propagation)
- Non convex interferometric inversion
- Regularization is needed
- Algorithm for non convex descent
- Numerical exemples

Phase shift uncertainty

Under far field approximation, and with constant wave speed c_0

$$d_k^{\text{ex}}(\omega) := u^{\text{ex}}(\mathbf{r}_k, \omega) = G_\omega^{\text{ex}}(\mathbf{r}_k)\hat{s}\left(\frac{\omega}{c_0}\frac{\mathbf{r}_k}{|\mathbf{r}_k|}\right)$$

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$$G_\omega^{\text{ex}}(\mathbf{r}_k) := \frac{e^{\frac{i\omega}{c_0}|\mathbf{r}_k|}}{4\pi|\mathbf{r}_k|}.$$

Smooth uncertainties over wavespeed leads at first order to an error on the travel time:

$$\tau_k := \frac{1}{c_0}|\mathbf{r}_k| + \varphi_k.$$

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In general,

- φ_k is not small (to compare to $2\pi/\omega$) $\Rightarrow$ linear inversion fails!
- φ_k is smooth ($\varphi_k \approx \varphi_{k+1}$).

Phase uncertainties

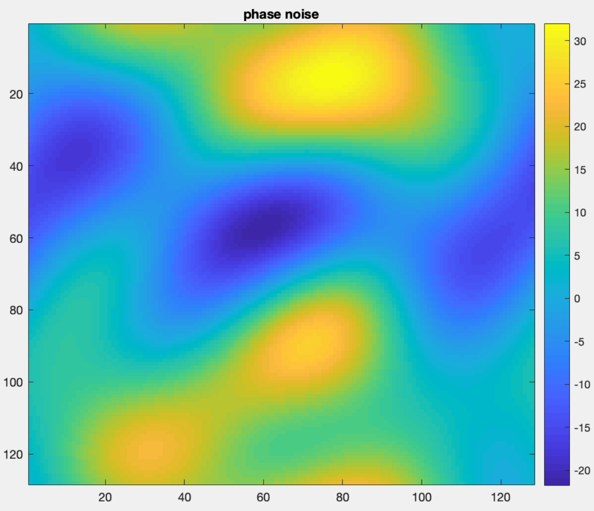

Figure: Phase uncertainty on the receivers/frequencies domain.

Why interferometric data are interesting?

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$$d_k(\omega)\overline{d_\ell}(\omega') = d_k^{\text{ex}}(\omega)\overline{d_\ell^{\text{ex}}}(\omega') e^{i(\omega\varphi_k - \omega'\varphi_\ell)}.$$

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One can choose receivers k, ℓ and frequencies ω and ω' such that

$$|\omega\varphi_k - \omega'\varphi_\ell| \leq \varepsilon$$

for ε small. Then

$$d_k(\omega)\overline{d_\ell}(\omega') \approx d_k^{\text{ex}}(\omega)\overline{d_\ell^{\text{ex}}}(\omega').$$

Coherent Interferometric Imaging

Borcea, Garnier, Papanicolaou, Tsogka:

$$I^{CINT}_{\Delta_\omega,\Delta_x}(\mathbf{r}) := \int_{|\omega-\omega'|<\Delta_\omega} \sum_{|\mathbf{r}_k-\mathbf{r}_\ell|<\Delta_x} \overline{G_\omega}(\mathbf{r}-\mathbf{r}_k) d_k(\omega) G_{\omega'}(\mathbf{r}-\mathbf{r}_\ell)\overline{d_\ell}(\omega')\mathrm{d}\omega\mathrm{d}\omega'.$$

Imaging formed by back propagation applied on interferometric data for close-by frequencies and receivers.

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Imaging formed by back propagation applied on interferometric data for close-by frequencies and receivers.

- It forms a smoothed version of the exact source.
- Resistant to random medium noise (wavespeed uncertainties)
- Difficult setting of Δ_ω, Δ_x. To small $\Rightarrow$ to smooth, To large $\Rightarrow$ defocusing.

Classic linear inversion

After discretization of the source $x \in \mathbb{C}^n$, and the frequencies,

Discrete linear problem

$$\text{Find } x \in \mathbb{C}^n \quad \text{s.t.} \quad Ax = b \quad b \in \mathbb{C}^p.$$

where the forward operator matrix $A \in \mathbb{C}^{pn}$ is such that

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- A Fourier-type matrix (well-conditioned), one can take $n = p$.
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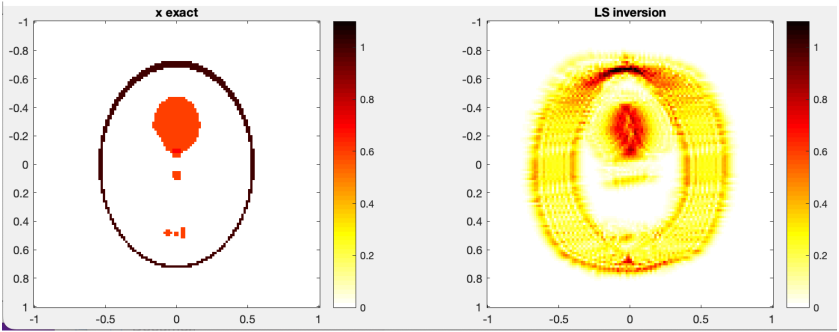
$\Rightarrow$ solution $x^{LS} = A^{-1}b$ is a bad solution ($x^{LS} - x^{\text{ex}}$ is large).

Numerics LS



Interferometric inversion problem

For all $x \in \mathbb{C}^n$, call $x^* := \overline{x}^T$.

$$Ax = b \quad \Rightarrow \quad Ax(Ax)^* = bb^*.$$

bb^* is the inferometric data matrix. We only want to consider products $b_k \overline{b_\ell}$ with $|k - \ell|$ small.

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- Case $E = I$ is eq. to $|Ax| = |b|$. (amplitude inversion pb.)
- Case $E = 1$ is eq. to $Ax = b$. (up to a multiplication by $e^{i\theta}$).

Interferometric inversion problem

Least squares cost functional

$$x^{LS} := \arg\min J^{LS}(x), \qquad J^{LS}(x) := \|Ax - b\|_2^2 \,.$$

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$$x^{\text{int}} \in \arg\min J_E^{\text{int}}(x), \qquad J_E^{\text{int}}(x) := |Ax(Ax)^* - bb^*|_E^2 .$$

where $|M|_E^2 := \sum_{k\ell} E_{k\ell} |M_{k\ell}|^2$.

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- Are these two problems equivalent ?
- What's the influence of the selector matrix E on the solutions?
- Is the second problem numerically solvable?

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- Are these two problems equivalent ?
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- If x^{int} is a solution, then $e^{i\alpha}x^{\text{int}}$ is solution.
- x^{LS} is a global minimizer of J_E^{int}.

Graph laplacian an phase recovery 1

Graph laplacian matrix

Take $E \in \mathcal{S}_n(\{0,1\})$, the graph laplatian matrix of E is the matrix

$$(\Delta_E)_{ii} := \sum_{j \neq i} E_{ij} \quad \text{and} \quad (\Delta_E)_{ij} := -E_{ij} \quad \text{for } i \neq j.$$

Δ_E is symetric positive semi-definite and $\lambda_1(\Delta_E) = 0$ and $\lambda_2(\Delta_E)$ measures the connection of the graph E.

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Theorem (Phase recovery from cross-products)

Consider $x \in \mathbb{C}^n$ such that $|x_k| = 1$ for all k. For any $x' \in \mathbb{C}^n$ satisfying $|x'_k| = 1$, there exists $\alpha \in [0, 2\pi)$ such that

$$\left\|x - e^{i\alpha} x'\right\|_2 \leq \frac{\pi\sqrt{2}}{4\sqrt{\lambda_2(\Delta_E)}} \left|xx^* - x'x'^*\right|_E .$$

Graph laplacian an phase recovery 2

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- If E is a connected graph, then $E_{k\ell} x_k \overline{x_\ell}$ contains enough phase differences to recover all phases up to a constant phase shift.
- The more E is a connected, the more this phase recovery is stable.

Data-graph laplacian and vector recovery 1

If the general case $x \in \mathbb{C}^n$, a similar result is possible.
Problem: if $x_i = 0$ for some i, that can kill the connectivity between the phases.

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Problem: if $x_i = 0$ for some i, that can kill the connectivity between the phases.

Data-graph laplacian

Take $E \in \mathcal{S}_n(\{0,1\})$ and $x \in \mathbb{C}^n \backslash \{0\}$, the data-graph laplacian matrix of E is the matrix

$$\Delta_{E,|x|} := \mathrm{diag}(d) - S \qquad \text{where} \quad S_{ij} := E_{ij} \frac{|x_i|^2 |x_j|^2}{|xx^*|_E^2}.$$

and $d_i := \sum_j S_{ij}$. This matrix is also symmetric positive semi-definite.

$\Delta_{E,|x|}$: is a weighted-graph laplacian.

Data-graph laplacian and vector recovery 2

Theorem (Vector recovery from cross-products)

Consider $x \in \mathbb{C}^n$ and assume that for some $\eta > 0$, we have

$$\min_k(|x_k|) \geq \eta \|x\|_2 .$$

For any $x' \in \mathbb{C}^n$ satisfying, there exists $\alpha \in [0, 2\pi)$ such that

$$\frac{\left\|x - e^{i\alpha}x'\right\|_2}{\|x\|_2} \leq \left(\frac{\pi}{\sqrt{\lambda_2(\Delta_{E,|x|})}} + \frac{\sqrt{2}}{\eta} \right) \frac{|xx^* - x'x'^*|_E}{|xx^*|_E}.$$

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Consequence on the linear system $Ax = b$:

$$\left\|e^{i\alpha}Ax - b\right\|_2^2 \leq 2\left(\frac{\pi^2}{\lambda_2(\Delta_{E,|b|})} + \frac{2}{\eta^2}\right) |(Ax)(Ax)^* - bb^*|_E^2.$$

Consequence on objective functions

Corollary

If b satisfies $\min_k(|b_k|) \geq \eta \|b\|_2$, *then for all* $x \in \mathbb{C}^n$, $\exists \alpha \in [0, 2\pi)$

$$J^{LS}(e^{i\alpha}x) \leq \frac{2}{\|x\|_2^2} \left(\frac{\pi^2}{\lambda_2(\Delta_{E,|b|})} + \frac{2}{\eta^2} \right) J_E^{int}(x).$$

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Consequence: Under this hypothesis, the interferometric inversion is equivalent to the least squares inversion! Remember: that x^{LS} minimizes J_E^{int}.

What can be done?

We have the following situation :

Vectors x^{ex} and x^{LS} are very different and

$$J^{LS}(x^{LS}) = 0 \qquad J_E^{\text{int}}(x^{LS}) = 0$$
$$J^{LS}(x^{ex}) \text{ large} \qquad J_E^{\text{int}}(x^{ex}) < \varepsilon.$$

$\Rightarrow$ minimizing J_E^{int} is ill-posed.

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ℓ^1-penalized interferometric inversion

$$x^{\text{int}} \in \arg\min |Ax(Ax)^* - bb^*|_E^2 + \lambda \|x\|_1 .$$

Algorithm

ℓ^1-penalized interferometric inversion

$$x^{\text{int}} \in \arg\min |Ax(Ax)^* - bb^*|_E^2 + \lambda \|x\|_1 .$$

We use optimal step descent: $\nabla J_E^{\text{int}}(x) = A^* E(Axx^*A^* - bb^*)Ax$

- Initialize $x \in \mathbb{C}^n \setminus \{0\}$.
- Compute $g = \nabla J_E^{\text{int}}(x) + \lambda(\frac{x_i}{|x_i|})_{i=1}^n$
- Compute The order 4, polynomial $P(t) = J_E^{\text{int}}(x + tg)$.
- Solve
$$t^* = \arg\min_t P(t) + \lambda \|x + tg\|_1$$
.
- Iterate $x = x + t^* g$
- Threshold $x_i = 0$ if $|x_i| < \varepsilon$.
- Loop.

Numerics 1

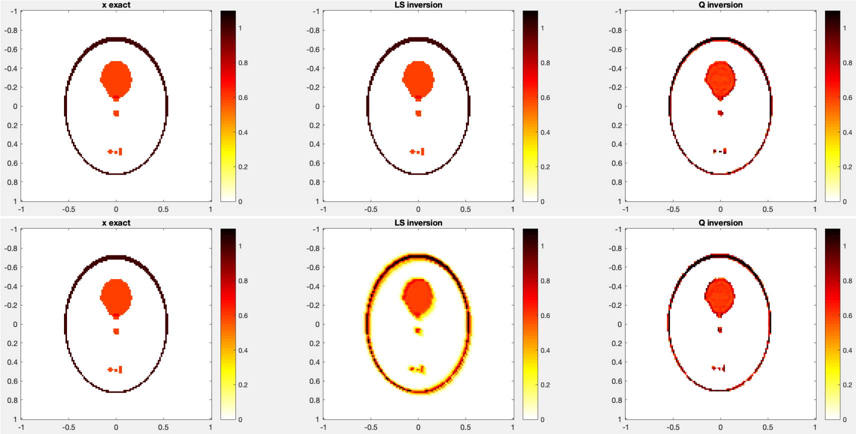

Figure: Line 1: no noise, Line 2: max noise amplitude = 2.

Numerics 2

We choose E tri diagonal.

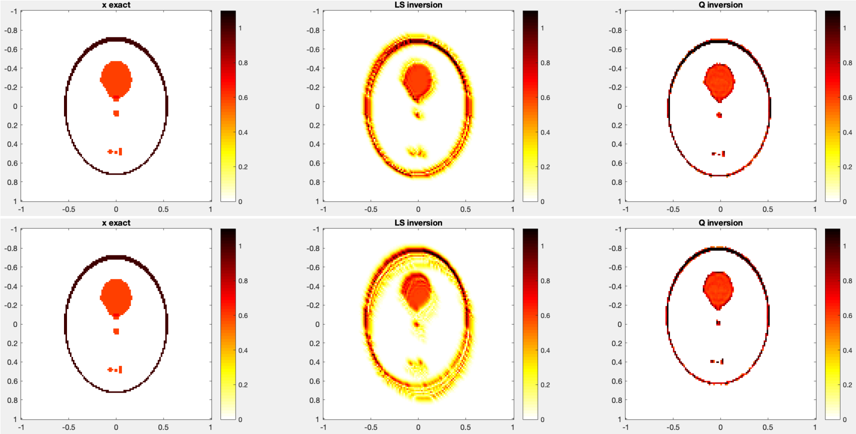

Figure: Line 1: max noise amplitude $= 4$, Line 2: max noise amplitude $= 8$

Numerics 3

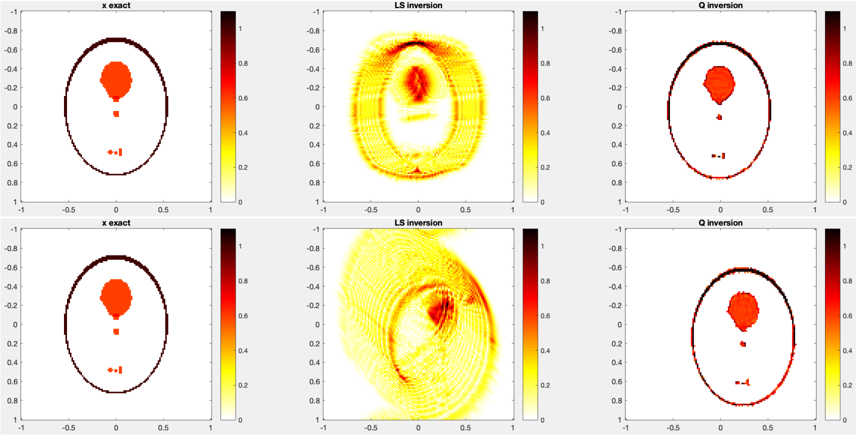

Figure: Line 1: max noise amplitude = 16, Line 2: max noise amplitude = 32.

Numerics 4

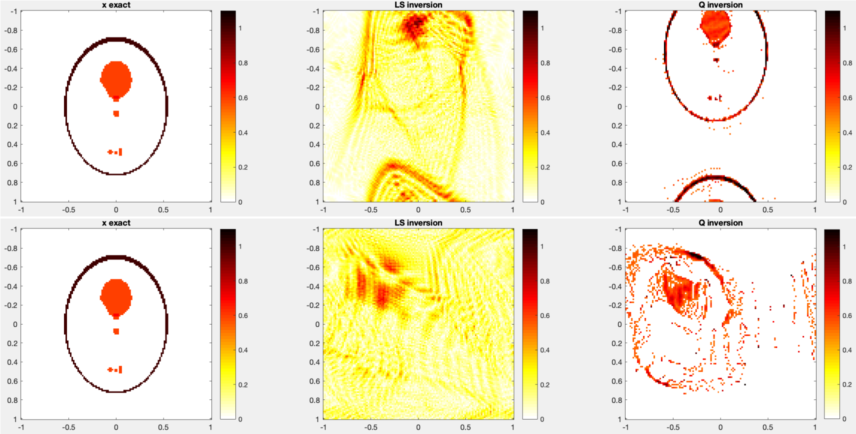

Figure: Line 1: max noise amplitude = 64, Line 2: max noise amplitude = 128.

Thank you for your attention!