Entanglement and information in classical and quantum systems

Jean-Marie Stéphan[1]

[1]MPIPKS (Dresden)

Condensed Matter Theory Seminar — Köln – 24/10/2014

[JMS, Inglis, Fendley & Melko, PRL (2014)]

[JMS, PRB (2014)]

Outline

1. Introduction
   - Basic concepts
   - Classic results
2. Classical systems
   - Mutual information
   - Replica picture
   - Universal behavior
   - Numerics
3. Quantum systems
   - Quantum entanglement and line entropies
   - Transitions
   - Ansatz wave functions from field-theory

Entropies

For some probability distribution $\{p_i\}$, define the Renyi entropy

$$S_n = \frac{1}{1-n} \log \left(\sum_i p_i^n \right)$$

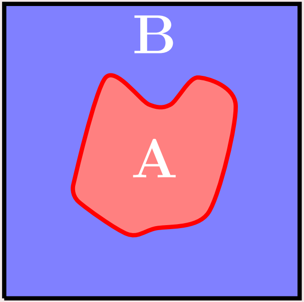
Generalizes the usual Shannon entropy

$$S = \lim_{n \to 1} S_n = -\sum_i p_i \log p_i$$

- Classical system: probabilities given by Boltzmann weights $p_i \propto e^{-\beta E(i)}$
- Quantum: the p_i label the eigenvalues of the reduced density matrix $\rho_A = \mathrm{Tr}_B \langle \psi | \psi \rangle$.

Quantum entanglement entropy

Bipartition

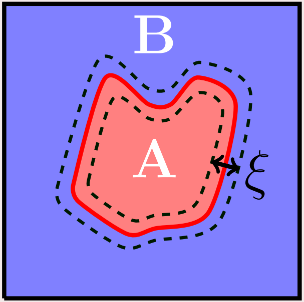


[von Neumann, 1955]

- $|\psi\rangle$ e.g. ground state of $H_{A\cup B}$
- $\rho_A = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$
- $S_n = \frac{1}{1-n} \log\left(\mathrm{Tr}\, \rho_A^n\right)$

Quantum entanglement entropy

Bipartition



[von Neumann, 1955]

- $|\psi\rangle$ e.g. ground state of $H_{A\cup B}$
- $\rho_A = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$
- $S_n = \frac{1}{1-n} \log\left(\mathrm{Tr}\, \rho_A^n\right)$

Extended quantum system: *Boundary law*

- correlation length ξ, dimension d.
- $S_n(L) = a_n L^{d-1} + o(L^{d-1})$

Entanglement entropy (2/2)

Why studying this quantity?

- How to store efficiently quantum states in a computer?
- Tool to distinguich between subtly different phases of matter.
- Replica trick: Twist, Swap.

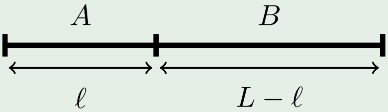
Entanglement entropy (2/2)

Why studying this quantity?

- How to store efficiently quantum states in a computer?
- Tool to distinguich between subtly different phases of matter.
- Replica trick: Twist, Swap.

Classic results

- $1d$ critical systems: $S_n \sim \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left[\frac{L}{\pi}\sin\frac{\pi\ell}{L}\right]$ [Holzhey et al, NPB 1994 — Vidal et al, PRL 2003 — Calabrese & Cardy, JSM 2004]



Entanglement entropy (2/2)

Why studying this quantity?

- How to store efficiently quantum states in a computer?
- Tool to distinguich between subtly different phases of matter.
- Replica trick: Twist, Swap.

Classic results

- $1d$ critical systems: $S_n \sim \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left[\frac{L}{\pi}\sin\frac{\pi\ell}{L}\right]$ [Holzhey et al, NPB 1994 — Vidal et al, PRL 2003 — Calabrese & Cardy, JSM 2004]
- Topological order in gapped systems: $S_n = aL + S_{\text{topo}} + o(1)$ [Kitaev & Preskill, PRL 2006 — Levin & Wen, PRL 2006]

Entanglement entropy (2/2)

Why studying this quantity?

- How to store efficiently quantum states in a computer?
- Tool to distinguich between subtly different phases of matter.
- Replica trick: Twist, Swap.

Classic results

- $1d$ critical systems: $S_n \sim \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left[\frac{L}{\pi}\sin\frac{\pi\ell}{L}\right]$ [Holzhey et al, NPB 1994 — Vidal et al, PRL 2003 — Calabrese & Cardy, JSM 2004]
- Topological order in gapped systems: $S_n = aL + S_{\text{topo}} + o(1)$ [Kitaev & Preskill, PRL 2006 — Levin & Wen, PRL 2006]

Issues

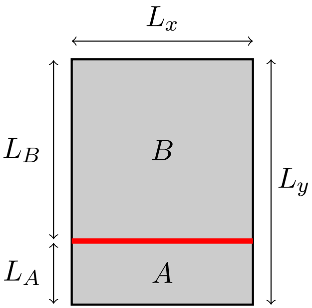
- Sometimes difficult to compute in dimension $d > 1$
- What about experiments?

1. Introduction
   - Basic concepts
   - Classic results
2. Classical systems
   - Mutual information
   - Replica picture
   - Universal behavior
   - Numerics
3. Quantum systems
   - Quantum entanglement and line entropies
   - Transitions
   - Ansatz wave functions from field-theory

Mutual information

Question: can we do the same in 2d classical systems?

Mutual information



$$S_n(A) = S_n(\{p_{i_A}\}) \qquad p_{i_A} = \sum_{i_B} p_{i_A, i_B}$$

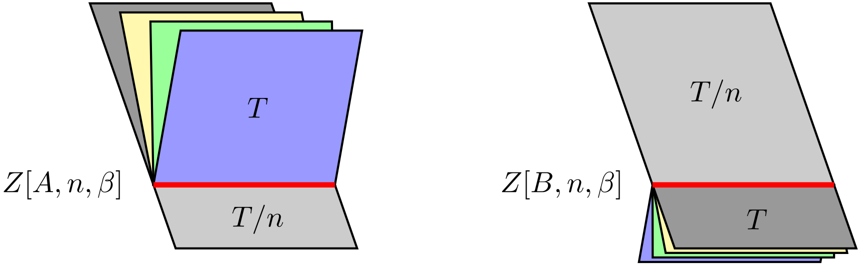
$$I_n(A,B) = S_n(A) + S_n(B) - S_n(A \cup B)$$

Replica picture

$$I_n(A,B) = \frac{1}{1-n}\log\left(\frac{Z[A,n,\beta]Z[B,n,\beta]}{Z(\beta)^n Z(n\beta)}\right),$$

$$Z[A,n,\beta] = \sum_{i_A}\sum_{i_{B_1},\dots,i_{B_n}} e^{-\beta\sum_{k=1}^{n} E_{i_A,i_{B_k}}}$$

Replica picture



Off critical/topological behavior [Castelnovo & Chamon, PRB 2007 — Iaconis, Inglis, Kallin & Melko, PRB 2013 — Hermanns & Trebst, PRB 2014]

$$I_n(A, B) = a_n L + \mathcal{G}_n + o(1)$$

Here: $\mathcal{G}_n$ potentially universal at criticality

Critical behavior

Systems with a critical point separating two gapped phases: critical behavior at both $T = T_c$ and $T = nT_c$.

- $T = T_c$: n critical systems coupled to one ordered system.

$$\mathcal{G}_n(T_c) = \frac{1}{1-n} \log \left(d \times \left[\frac{\mathcal{Z}_A^{\text{fix}} \mathcal{Z}_B^{\text{fix}}}{\mathcal{Z}_{A \cup B}} \right]^n \right)$$

Critical behavior

Systems with a critical point separating two gapped phases: critical behavior at both $T = T_c$ and $T = nT_c$.

- $T = T_c$: n critical systems coupled to one ordered system.

$$\mathcal{G}_n(T_c) = \frac{1}{1-n} \log \left(d \times \left[\frac{\mathcal{Z}_A^{\mathrm{fix}} \mathcal{Z}_B^{\mathrm{fix}}}{\mathcal{Z}_{A \cup B}} \right]^n \right)$$

- $T = nT_c$: one critical system coupled to n disordered systems.

$$\mathcal{G}_n(nT_c) = \frac{1}{1-n} \log \left(\frac{\mathcal{Z}_A^{\mathrm{free}} \mathcal{Z}_B^{\mathrm{free}}}{\mathcal{Z}_{A \cup B}} \right)$$

Critical behavior

Systems with a critical point separating two gapped phases: critical behavior at both $T = T_c$ and $T = nT_c$.

- $T = T_c$: n critical systems coupled to one ordered system.

$$\mathcal{G}_n(T_c) = \frac{1}{1-n} \log \left(d \times \left[\frac{\mathcal{Z}_A^{\text{fix}} \mathcal{Z}_B^{\text{fix}}}{\mathcal{Z}_{A \cup B}} \right]^n \right)$$

- $T = nT_c$: one critical system coupled to n disordered systems.

$$\mathcal{G}_n(nT_c) = \frac{1}{1-n} \log \left(\frac{\mathcal{Z}_A^{\text{free}} \mathcal{Z}_B^{\text{free}}}{\mathcal{Z}_{A \cup B}} \right)$$

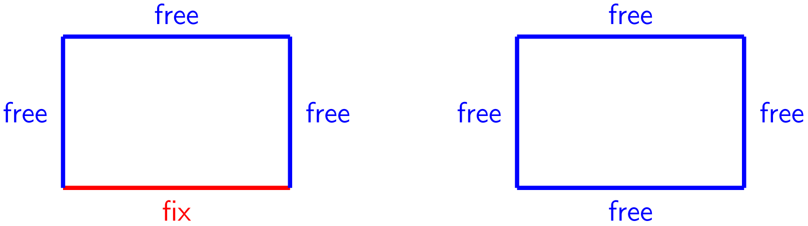
This is true in any geometry in any dimension.

Universal shape from conformal field theory (rectangle I)

$\mathcal{Z}$ is the universal part in the partition function at criticality:

$$Z_{\text{lattice}}(L_x, L_y) = A^{L_x L_y} B^{L_x + L_y} \mathcal{Z} \left(1 + o(1)\right)$$

Important point: lattice boundary conditions will renormalize to conformally invariant boundary conditions.



$T = T_c$ $T = nT_c$

Universal shape from conformal field theory (rectangle II)

$$\mathcal{Z}(L_x, L_y) = L_x^{c/4-4h} \left[f(L_y/L_x)\right]^{16h-c/2} \left[f(2L_y/L_x)\right]^{-8h}$$

with

$$f(u) = e^{-\pi u/12} \prod_{k=1}^{\infty} \left(1 - e^{-2\pi k u}\right)$$

[Kleban & Vassileva, J. Phys. A (1992)]

Universal shape from conformal field theory (rectangle II)

$$\mathcal{Z}(L_x, L_y) = L_x^{c/4-4h} \left[f(L_y/L_x)\right]^{16h-c/2} \left[f(2L_y/L_x)\right]^{-8h}$$

with

$$f(u) = e^{-\pi u/12} \prod_{k=1}^{\infty} \left(1 - e^{-2\pi k u}\right)$$

[Kleban & Vassileva, J. Phys. A (1992)]

Universal data:

- $c =$ central charge of the CFT.
- $h =$ dimension of the boundary condition changing operator.

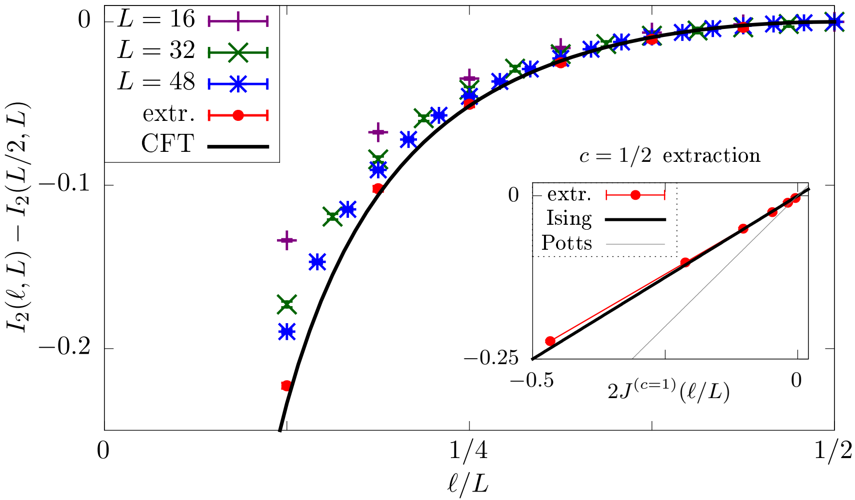
Numerical checks at $T = nT_c$

I_n computed in Monte Carlo through a transfer matrix ratio trick.

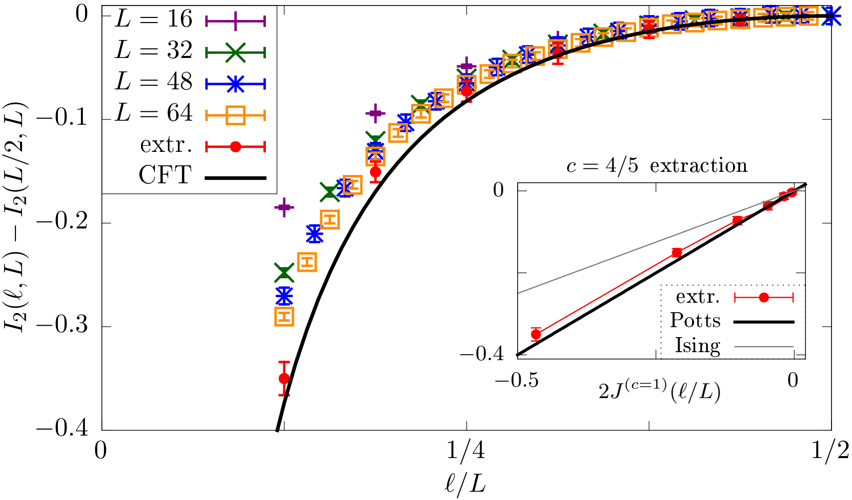
$$\frac{Z[A,n,\beta]}{Z(\beta)^n} = \prod_{i=0}^{N-1} \frac{Z[A_{i+1},n,\beta]}{Z[A_i,n,\beta]}$$

Main examples: Ising ($c = 1/2$) and $Q = 3$ Potts ($c = 4/5$).

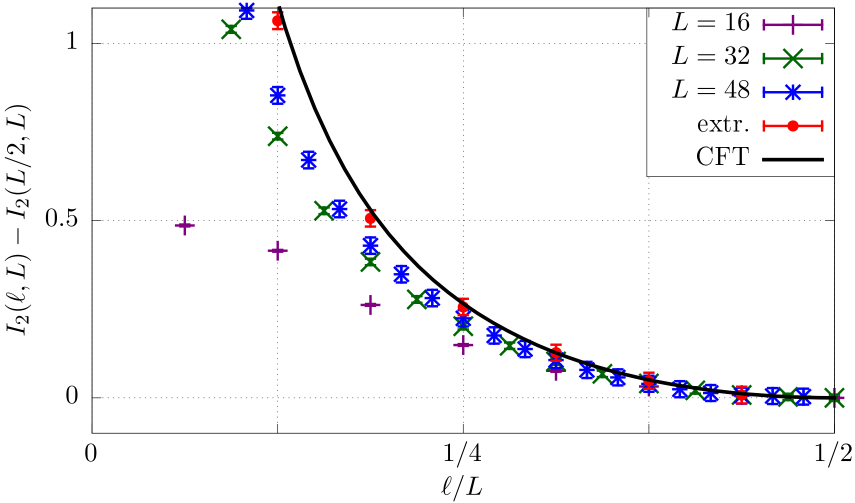
Numerical checks at $T = nT_c$



Numerical checks at $T = nT_c$



Numerical checks at $T = T_c$



Partial conclusion

- central charge extraction from Entropy and Information in 2d Stat. Mech
- Other geometries (torus, cylinder) have been checked too.
- XY model: gluing of CFTs with different radii (or Luttinger parameters).
- Can be generalized to any $n > 1$ or $n < 1$.
- Shannon limit $(n \to 1)$ sometimes highly non-trivial. For Ising, leading term appears to be $(\log L)^2$.

1 Introduction
- Basic concepts
- Classic results

2 Classical systems
- Mutual information
- Replica picture
- Universal behavior
- Numerics

3 Quantum systems
- Quantum entanglement and line entropies
- Transitions
- Ansatz wave functions from field-theory

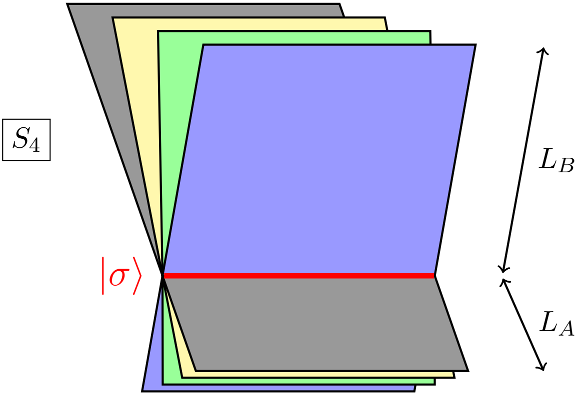
Entropy of a line

$$Z[A, n, \beta] = \sum_{\sigma} Z_A^{\sigma}(n\beta)\,[Z_B^{\sigma}(\beta)]^n$$

Using this, $I_1(A, B) = 2S_1(\text{line of spins})$

Entropy of a line

Renyi entropy of a line [JMS, Misguich & Pasquier, PRB 2011]



$$S_n = \frac{1}{1-n} \log \left[\frac{\mathcal{Z}_{\text{book}}}{\left(\mathcal{Z}_{\text{sheet}}\right)^n} \right]$$

Entanglement in Rokhsar-Kivelson wave function

- Take some classical statistical model $Z = \sum_c e^{-\beta E(c)}$
- Construct some Hilbert space $|c\rangle$.
- Orthogonality $\langle c|c'\rangle = \delta_{c,c'}$
- $|\psi\rangle = \frac{1}{\sqrt{Z}} \sum_c e^{-\beta E(c)/2} |c\rangle$

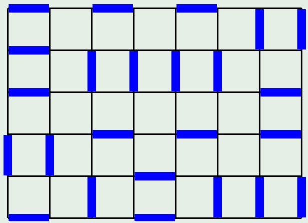
[Rokhsar & Kivelson, PRL 1988] [Henley, J. Phys. Cond. Mat 2004]

Entanglement in Rokhsar-Kivelson wave function

- Take some classical statistical model $Z = \sum_c e^{-\beta E(c)}$
- Construct some Hilbert space $|c\rangle$.
- Orthogonality $\langle c|c'\rangle = \delta_{c,c'}$
- $|\psi\rangle = \frac{1}{\sqrt{Z}} \sum_c e^{-\beta E(c)/2} |c\rangle$

[Rokhsar & Kivelson, PRL 1988] [Henley, J. Phys. Cond. Mat 2004]

A possible choice of classical model



Connections between (too many) entropies

- $S_n(\text{entanglement RK}) = S_n(\text{classical line})$ [JMS, Furukawa, Misguich & Pasquier Phys. Rev. B 2009] (weaker version holds also for RVB [JMS, Ju, Fendley & Melko New. J. Phys 2013])

- $S_1(\text{entanglement RK}) = S_1(\text{classical line}) = \frac{1}{2} I_1(\text{classical})$

Infinite strip/cylinder limit

$$e^{-\lambda H}|s\rangle \sim e^{-\lambda E_0}|\psi\rangle\langle\psi|s\rangle \qquad , |\psi\rangle \text{ ground state of } H.$$

Infinite strip/cylinder limit

$$e^{-\lambda H}|s\rangle \sim e^{-\lambda E_0}|\psi\rangle\langle\psi|s\rangle \qquad ,|\psi\rangle \text{ ground state of } H.$$

$$I_n = \frac{2}{1-n}\log\left(\sum_\sigma \psi_\sigma(\beta)^n \psi_\sigma(n\beta)\right)$$

$$S_n^{\text{line}} = \frac{1}{1-n}\log\left(\sum_\sigma \psi_\sigma(\beta)^{2n}\right)$$

with

$$|\psi(\beta)\rangle = \sum_\sigma \psi_\sigma(\beta)|\sigma\rangle$$

ground-state of a corresponding spin chain (Ising, XXZ, . . .)

Free bosonic theory (Luttinger liquid)

$$\mathcal{A} = \frac{g}{4\pi} \int_0^L dx \int_{-\infty}^{\infty} d\tau \left[(\nabla\varphi)^2 + A_1 \cos\left(\frac{\varphi}{r}\right) + A_2 \cos\left(\frac{2\varphi}{r}\right) + \ldots \right]$$

Free bosonic theory (Luttinger liquid)

$$\mathrm{Tr}\,\rho^n = \sum_{\phi} p(\phi)^n$$

$$p(\phi)^n \propto \exp(-S_g(\phi))^n = \exp(-nS_g(\phi)) = \exp(-S_{ng}(\phi))$$

$$\boxed{p_g(\phi)^n \propto p_{ng}(\phi)}$$

Free bosonic theory (Luttinger liquid)

$$\mathrm{Tr}\,\rho^n = \sum_\phi p(\phi)^n$$

$$p(\phi)^n \propto \exp(-S_g(\phi))^n = \exp(-nS_g(\phi)) = \exp(-S_{ng}(\phi))$$

$$\boxed{p_g(\phi)^n \propto p_{ng}(\phi)}$$

Close to the boundary, the stiffness is modified to $g \longrightarrow ng$. We get:

$$s_n = \frac{1}{1-n}\left[\log\left(\frac{\mathcal{Z}_{ng}}{Z^D_{ng}}\right) - n\log\left(\frac{\mathcal{Z}_g}{\mathcal{Z}^D_g}\right)\right]$$

Boundary phase transition

- Vertex operators in the action (d integer)

$$V_d = \cos\left(\frac{\pi d}{2} h\right)$$

- Irrelevant if $d^2 > 2g$. Otherwise locks the field to a flat configuration with degeneracy d. [Coleman, PRB 1975]

Boundary phase transition

- Vertex operators in the action (d integer)

$$V_d = \cos\left(\frac{\pi d}{2} h\right)$$

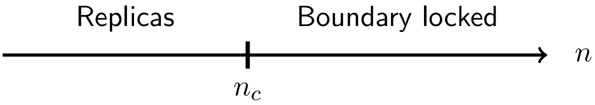
- Irrelevant if $d^2 > 2g$. Otherwise locks the field to a flat configuration with degeneracy d. [Coleman, PRB 1975]

However, $g \to ng$ near the boundary in the book.

$$\Rightarrow \text{Phase transition at } n_c = d_{\min}^2/(2g)$$

Phase transition (2/2)

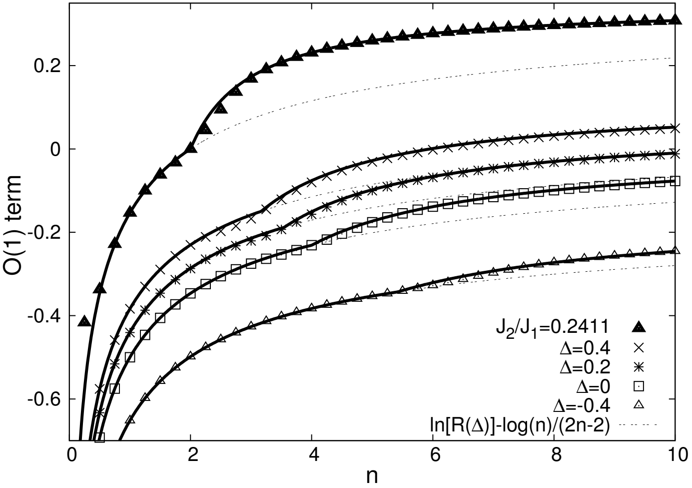
- XXZ, half-filling $\longrightarrow\ d = 2$



In the locked phase, we have $2n$ “half-sheets”.

$$s_n = \frac{n}{1-n} \log \left(\frac{\mathcal{Z}(L_A)\mathcal{Z}(L_B)}{\mathcal{Z}(L_A + L_B)} \right)$$

Numerical checks (XXZ)



CFT discretizations (1/2)

Constructing lattice wave functions from CFT correlators

[Cirac & Sierra, PRB 2009], [Nielsen, Cirac & Sierra, JSM 2011]

[Tu, Nielsen & Sierra, NPB 2014] [Bondesan & Quella, NPB 2014], . . .

CFT discretizations (1/2)

Constructing lattice wave functions from CFT correlators

[Cirac & Sierra, PRB 2009], [Nielsen, Cirac & Sierra, JSM 2011]

[Tu, Nielsen & Sierra, NPB 2014] [Bondesan & Quella, NPB 2014],. . .

Simplest example of such a construction

$$|\Psi_n\rangle = \sum_{x_1,\dots,x_{L/2}} \left(\prod_{j<i}^{L/2} \sin \frac{\pi(x_i - x_j)}{L} \right)^n |x_1, \dots, x_{L/2}\rangle$$

CFT discretizations (2/2)

- Good ansatz for the XXZ spin chain. $n = 1$ is exact (XX chain)

CFT discretizations (2/2)

- Good ansatz for the XXZ spin chain. $n = 1$ is exact (XX chain)

- The norm $\langle \Psi_n | \Psi_n \rangle$ is the line entropy of the XX chain.

CFT discretizations (2/2)

- Good ansatz for the XXZ spin chain. $n = 1$ is exact (XX chain)

- The norm $\langle \Psi_n | \Psi_n \rangle$ is the line entropy of the XX chain.

- We know there is a transition in the norm at $n = 4$, so these states are gapped for $n > 4$!

Conclusion

- Universal terms in entropies at criticality. Comparison CFT/numerics.
- Renyi index n distinguishes between competing orders.
- What about models in higher dimensions?

Conclusion

Thanks you!