Free fermions at the edge of interacting systems

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Entangled Interacting Quantum Matter, ENIQMA, Lyon 2018

Outline

1. Free fermions in traps
2. Interacting systems in traps

$$H = \int_{\mathbb{R}} dx\, \psi^\dagger(x) \left[-\frac{\partial^2}{\partial x^2} + x^2 \right] \psi(x)$$

$$\{\psi(x)^\dagger, \psi(y)\} = \delta(x-y) \quad , \quad \{\psi(x)^\dagger, \psi(y)^\dagger\} = \{\psi(x), \psi(y)\} = 0$$

- Well known model (maps to Tonks-Girardeau gas) in a confining potential, written here in the language of second quantization.
- Exactly solvable, free fermions.
- Single particle states can be obtained in explicit form [Hermite].

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Introduce the modes $\chi^\dagger(\lambda)=\int dx\,u(\lambda,x)\,\psi^\dagger(x)$.

Then

$$H=\int_{\mathbb{R}} d\lambda\epsilon(\lambda)\chi^\dagger(\lambda)\chi(\lambda)$$

provided

$$\left[-\frac{\partial^2}{\partial x^2}+x\right]u(\lambda,x)=\epsilon(\lambda)u(\lambda,x)$$

The single particle wave functions are Airy functions

$$u(\lambda,x)=\mathrm{Ai}(x+\lambda)\quad,\quad\epsilon(\lambda)=-\lambda\quad,\quad\mathrm{Ai}(x)=\int_{\mathbb{R}}\frac{dq}{2\pi}e^{i(qx+q^3/3)}.$$

Full counting statistics in the ground state

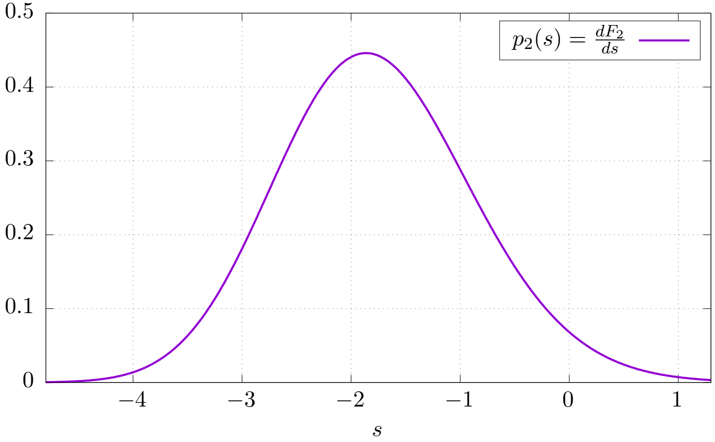
The ground state is a Dirac sea obtained by filling all single particle states with negative energy.

$$\langle \psi^\dagger(x)\psi(y)\rangle = \int_0^\infty d\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$

What is the probability $E(s)$ that an interval $[s,\infty)$ contains no particles? Obviously $E(-\infty)=0$ and $E(\infty)=1$. This probability can be computed exactly.

The probability density $p(s)=\frac{dE(s)}{ds}$ is known, it is nothing but the Tracy-Widom distribution. [Spohn]

Tracy-Widom distribution



Back to the harmonic trap: semiclassical solution (=LDA)

$$H = \int_{\mathbb{R}} dx\, \psi^\dagger(x) \left[-\frac{\partial^2}{\partial x^2} + x^2 - \mu \right] \psi(x)$$

Assume separation of scales. Around some point x_0, the single particle ground state looks like the projector onto

$$-\frac{d^2}{(d\delta x)^2} < \mu - x_0^2 \qquad \longrightarrow \qquad k^2 < \mu - x_0^2 = k(x_0)$$

This is a disk in phase (k, x_0) space. Thinking in Fourier, the projection acts as a pure low-pass filter, with response function

$$\langle \psi^\dagger(x_0 + \delta x)\psi(x_0 + \delta y) \rangle = \frac{\sin\left[k(x_0)(\delta x - \delta y)\right]}{\pi(\delta x - \delta y)}$$

Remark: connection to random matrices [Vandermonde]

$$|\varphi(x_1,\ldots,x_N)|^2 \propto \prod_{i<j}(x_i-x_j)^2 e^{-\sum_j x_j^2}$$

which is the joint eigenvalue pdf for GUE random matrices.
Exploited in [Eisler 2013, Calabrese, Majumdar & Le Doussal 2014, ...]

Density $\langle\psi^\dagger(x_0)\psi(x_0)\rangle = \frac{k(x_0)}{\pi} = \frac{1}{\pi}\sqrt{\mu^2 - x_0^2}$ [Wigner semicircle law].
Here $\mu = \sqrt{2N}$, where N is particle number.

What about the edge scaling close to $\mu = \sqrt{2N}$? From LDA (or semiclassics [Praehoffer & Spohn 2000; Spohn 2006; Tao lecture notes 2012]):

$$k^2 + (\mu + x)^2 < \mu^2$$

$$\Rightarrow\ k^2 + 2\mu x \cancel{+ x^2} < 0$$

$$-\frac{d^2}{dx^2} + \mu x < 0$$

With $x = (2\mu)^{-1/3} y$, we get

$$-\frac{d^2}{dy^2} + y < 0$$

Hence, Airy kernel scaling at the edge, and Tracy-Widom follows.

Consistency check: this occurs on a scale $\mu^{-1/3} \sim N^{-1/6}$. LDA is still valid, since mean distance between particles is $N^{-1/2}$.

What about interactions?

Lieb Liniger in a harmonic trap (repulsive)

$$H = \int dx \Psi^\dagger(x) \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \mu + x^2 \right] \Psi(x) + \hbar g \Psi^{\dagger 2}(x) \Psi(x)^2$$

Look at an $\hbar \to 0$ limit (still interacting).

The trapping potential breaks integrability, but assuming LDA, the density profile may be computed using thermodynamic Bethe ansatz (TBA). [Dunjko, Laurent & Olshanii 2001; Gangardt & Shlyapnikov 2003; Brun & Dubail 2018]

TBA for the homogeneous ground state ($V(x) = 0$):

$$\begin{aligned} \rho(k,\mu) - \int_{-k_F}^{k_F} \frac{dq}{2\pi} V(k,q)\rho(q,\mu) &= \frac{1}{2\pi} \\ \epsilon(k,\mu) - \int_{-k_F}^{k_F} \frac{dq}{2\pi} V(k,q)\epsilon(q,\mu) &= k^2 - \mu \end{aligned}$$

Ask $\epsilon(k_F, \mu) = 0$. The kernel V is known explicitly, and vanishes for free fermions, in which case we recover $k_F(\mu) = \sqrt{\mu}$.

LDA just tells us to replace $\mu \to \mu(x) = \mu - x^2$.

Edge scaling

Claim: can look at the edge scaling using TBA considerations. In that case interactions only provide extra subleading corrections, compared to free fermions.

So TW scaling still holds, we find $-\frac{d^2}{dx^2}+\mu x<0$, which exactly the same result irrespective of interactions for the harmonic trap.

Physically, interacting particles are diluted near the edge, they just renormalize to free fermions. In terms of Luttinger parameter, $K(x)$ varies with position, but $K(x_{\mathrm{e}})=1$.

XXZ in a varying magnetic field

$$\sum_{x\in\mathbb{Z}+1/2}\left(S^{\mathrm{x}}_x S^{\mathrm{x}}_{x+1} + S^{\mathrm{y}}_x S^{\mathrm{y}}_{x+1} + \Delta S^{\mathrm{z}}_x S^{\mathrm{z}}_{x+1} + h(x/R) S^{\mathrm{z}}_x\right)$$

The edge is at $x_{\mathrm{e}} = Rh^{-1}(1)$. We find the edge scaling

$$-\frac{d^2}{dx^2} + 2\frac{h'(h^{-1}(1+\Delta))}{R}x < 0$$

Tracy-Widom scaling occurs on a scale $\ell_\Delta^{1/3}$ now, with

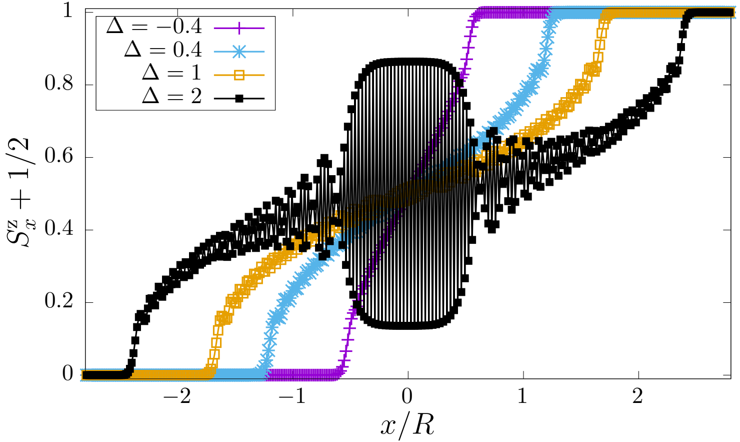
$$\ell_\Delta = \frac{R}{2h'(h^{-1}(1+\Delta))}$$

Variance now depends on interactions, through Δ.

Numerical checks

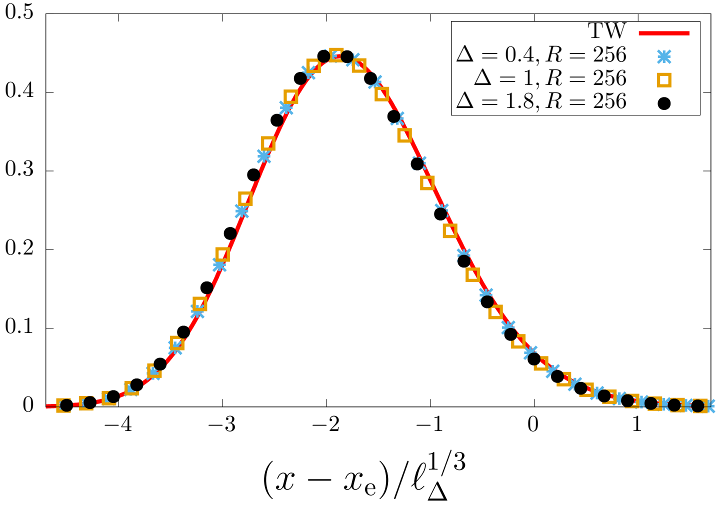
Widely available DMRG codes in C++ [ITensor] and Python [TeNPy].

Density profile



Numerical checks

Distribution of the rightmost down spin



An exception: Calogero-Sutherland

$$H = \sum_{j=1}^{N} \left(-\frac{\partial^2}{\partial x_j^2} + x_j^2 \right) + \sum_{i \neq j} \frac{\beta(\beta/2 - 1)}{(x_i - x_j)^2}$$

Luttinger parameter is known to be $K = 2/\beta$, constant, everywhere in the domain. So the edge is not free for $\beta \neq 2$.

Ground state wave function

$$|\varphi(x_1, \dots, x_N)|^2 \propto \prod_{i<j} (x_i - x_j)^\beta e^{-\sum_j x_j^2}$$

which is exactly the eigenvalue pdf for β-ensemble in random matrix theory, which leads to β-deformed Tracy-Widom.

Out of equilibrium setups

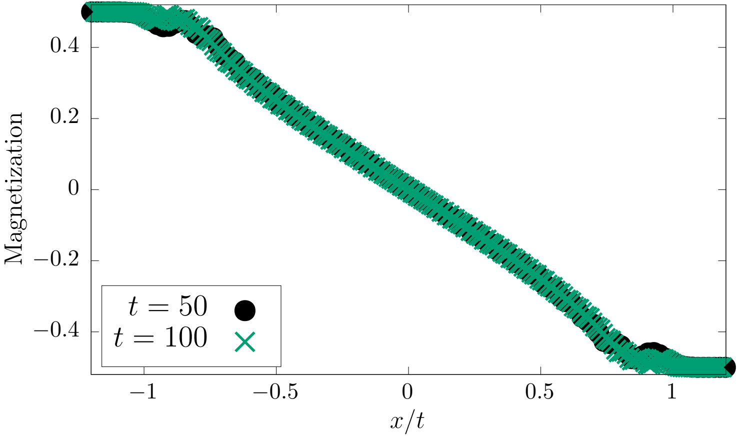
$$|\Psi(t)\rangle = e^{-itH}\,|\Psi_0\rangle \qquad , \qquad |\Psi_0\rangle = |\ldots \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \ldots\rangle$$

$$H = \sum_{x\in\mathbb{Z}} \left(S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2 + \Delta S_x^3 S_{x+1}^3\right)$$

Free fermion case ($\Delta = 0$) [Antal, Rácz, Rákos, and Schütz, 1999]
Interactions: Numerics [Gobert, Kollath, Schollwöck & Schütz 2005]

Generalized hydrodynamics (GHD) framework ($|\Delta| < 1$, ballistic)
[Castro-Alvaredo, Doyon, Yoshimura 2016] [Bertini, Collura, De Nardis, Fagotti 2016]

This quench: light cone $x_{\mathrm{e}}(t) = t\sqrt{1-\Delta^2}$ [JMS, 2017]
Full density profile from GHD [De Luca, Collura, Viti 2017]



Contrary to previous situations density profile is linear near the light cone, compared to previous root behavior.

Tracy-Widom edge for $\Delta = 0$ (free fermions) [Eisler & Racz 2013]

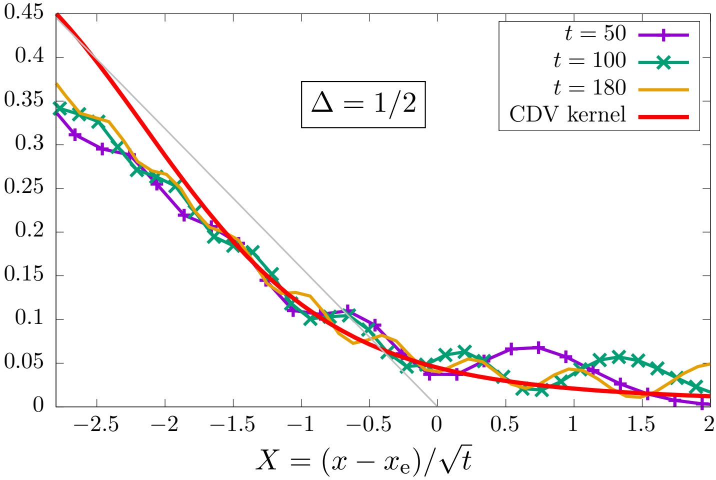
From GHD considerations in the bulk, [De Luca, Collura & Viti 2017] guessed a new (diffusive) kernel for the edge. In our language, this reads free fermions $k + x/t < 0$.

Fastest quasi-particle goes as $x \simeq t$ [Sabetta & Misguich 2013]. [Bulchandani & Karrasch 2018] observed $t^{1/3}$ scaling near this other front, which they interpreted as signature of Tracy-Widom.

Subleading corrections to GHD are generically diffusive in the bulk [De Nardis, Bernard & Doyon 2018]

Rescaled density profiles

$X = \frac{x - t\sqrt{1-\Delta^2}}{\sqrt{t}}$ Rescaled density decays as $\rho(X \gg 1) \sim \frac{1}{4\pi^2 X}$



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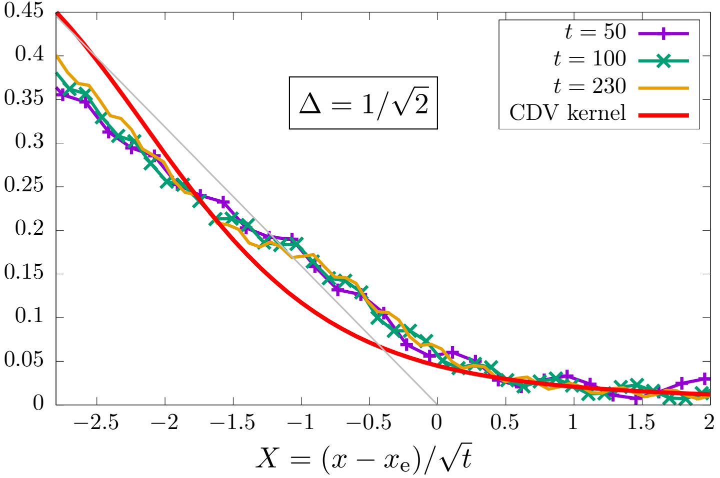
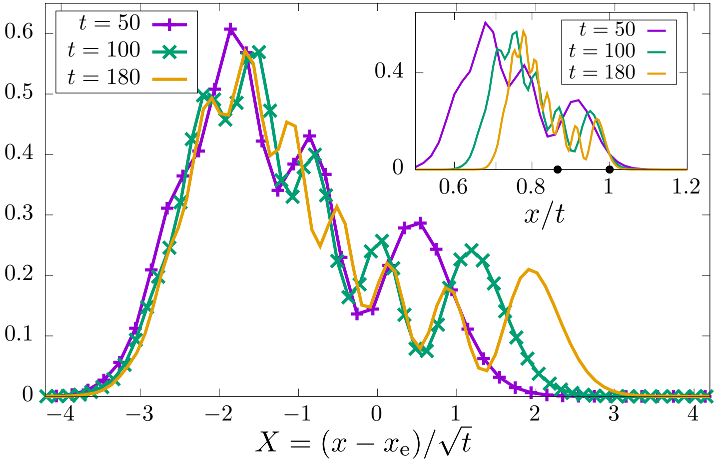


Distribution of the last particle

$\boxed{\Delta = 1/2}$



Distribution of the last particle

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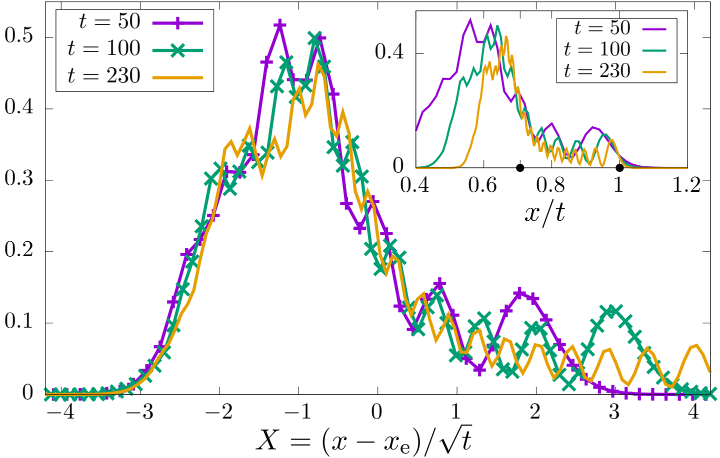

- Not clear if it's free fermions at the edge, and if the proposed diffusive kernel is correct ot not.

- The diffusive tail around $x_{\mathrm{e}} = t\sqrt{1-\Delta^2}$ appears to not be square integrable. This means skewness and diffusion constant might diverge.

- One needs a hard cutoff at $x_{\mathrm{LR}} = t$, due to Lieb-Robinson type bounds. $t^{1/3}$ behavior near $x = t$ is just the right tail of the quantum delocalization of the rightmost particle.

Summary

- LDA works even at the edge in interacting systems.
- The edge of several inhomogeneous interacting systems renormalizes to free fermions. The most typical example is Tracy-Widom.
- Numerical checks.
- Quantum quench problems provide us with new edge universality classes, which are worth exploring.

Thank you!