Strange effects of integrability in a simple out of equilibrium problem

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Entanglement, Integrability and Topology in Many-Body Systems, CRM Montreal 2018

[JMS, J. Stat. Mech 2017]
[Dubail, JMS, Viti & Calabrese, Scipost 2017]
[Allegra, Dubail, JMS & Viti, J. Stat. Mech. 2016]

Integrable systems

Statistical mechanics $2 + 0d$: universality

Out of equilibrium $1 + 1d$: peculiar thermalization properties

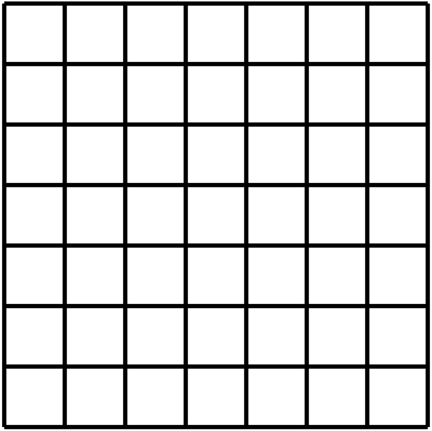
Go from one to the other: “just” perform the Wick rotation $\tau = it$.

This talk

There is a precise way to compute out of equilibrium quantities in 1d quantum systems, starting from the underlying 2d stat mech model.

I will illustrate this on an explicit example, where things can be worked out in considerable detail, and give exact formulas valid at all time.

Statistical mechanics



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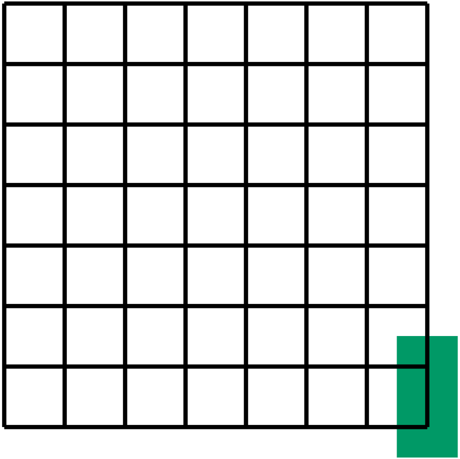

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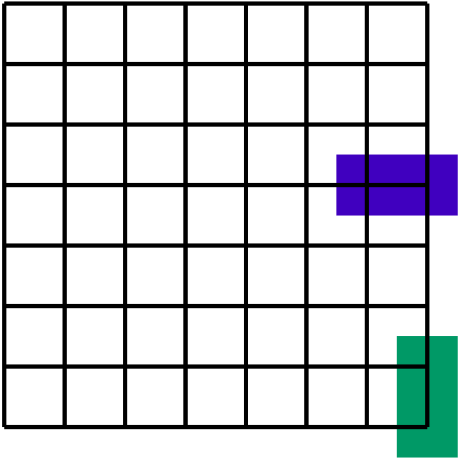

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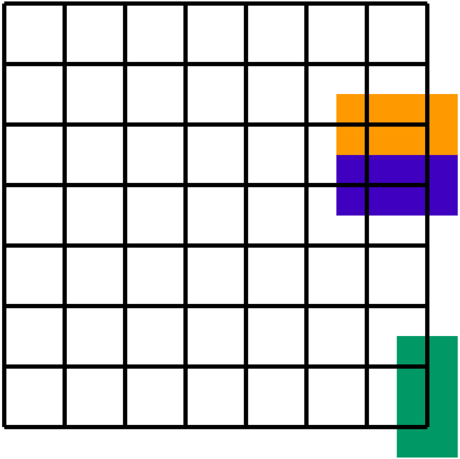

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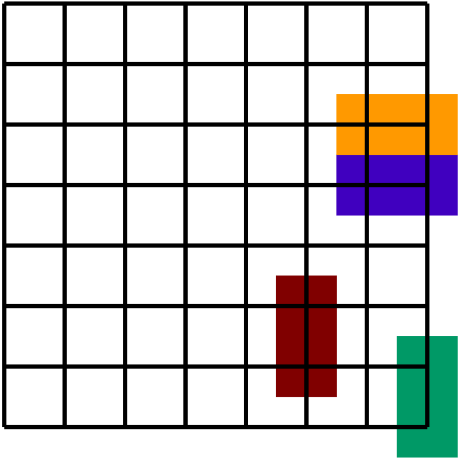

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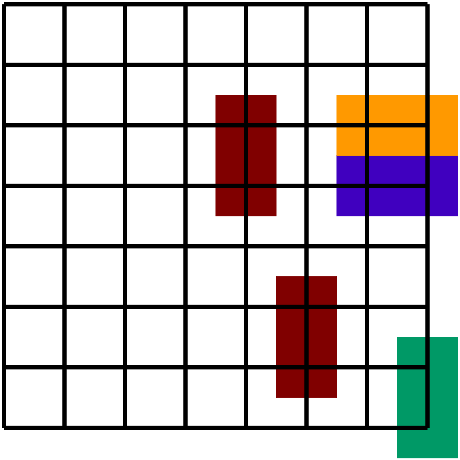

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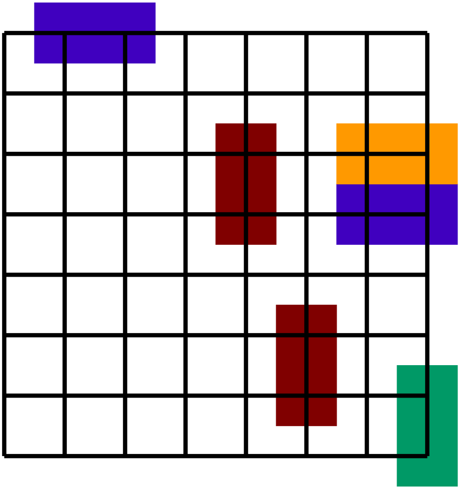

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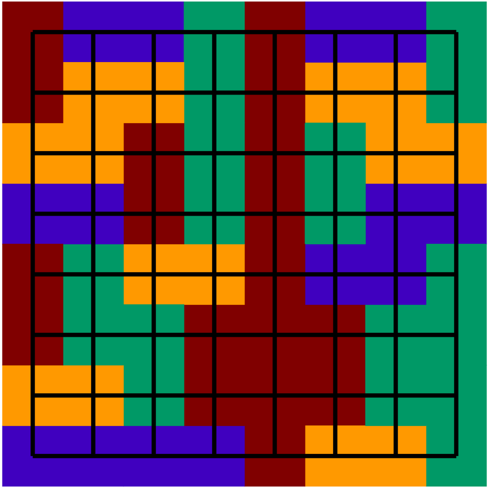

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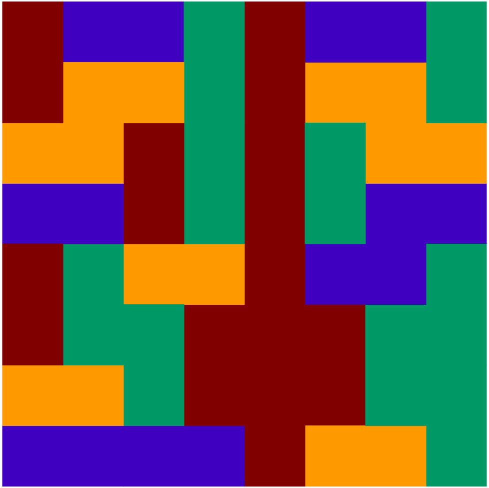

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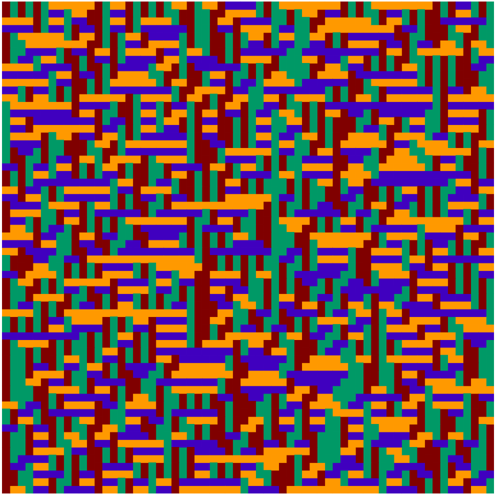

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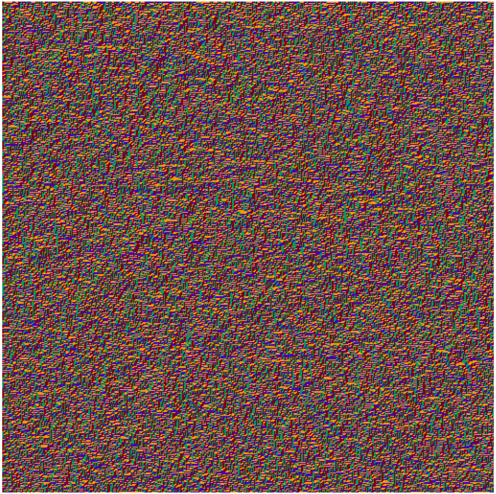
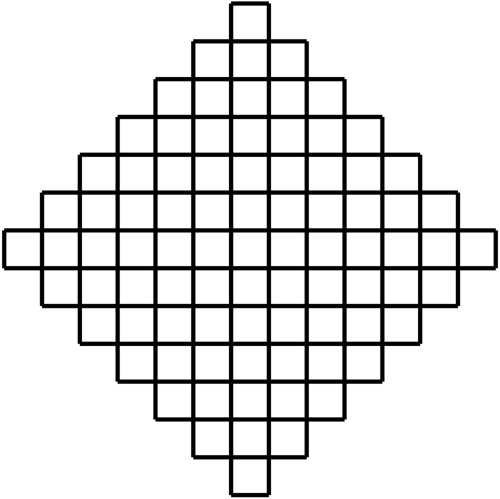
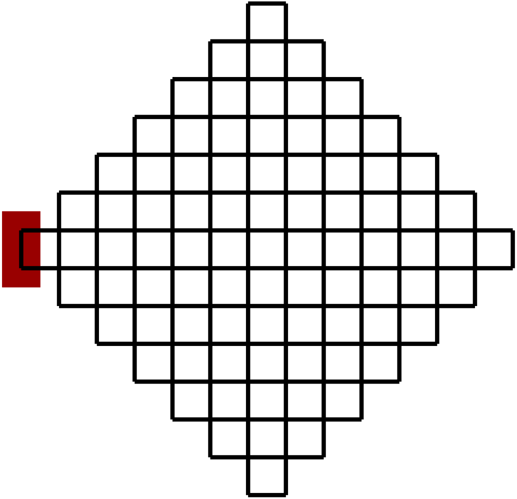


Long range correlations: gaussian free field, or coulomb gas, or free compact boson CFT ($c = 1$), or euclidean Luttinger liquid.

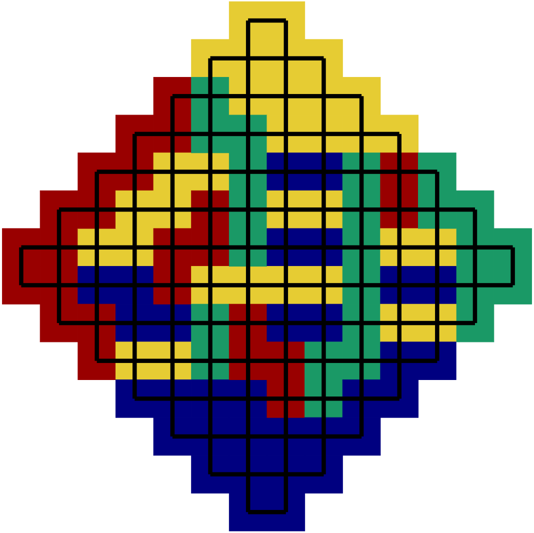
Dimer coverings on the Aztec diamond



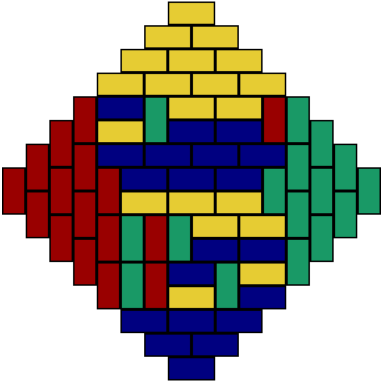
Dimer coverings on the Aztec diamond



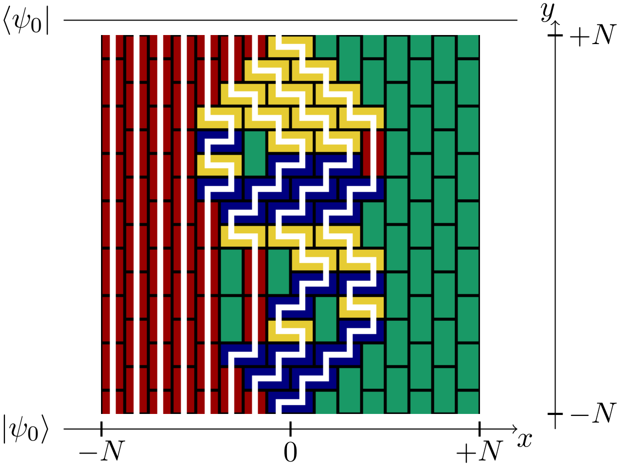
Dimer coverings on the Aztec diamond

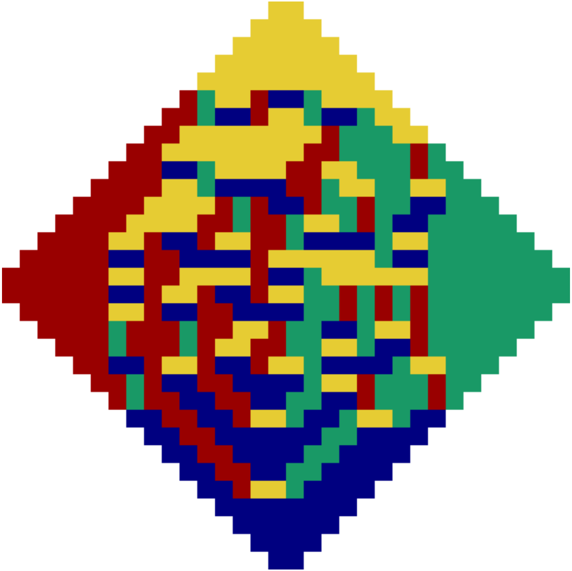


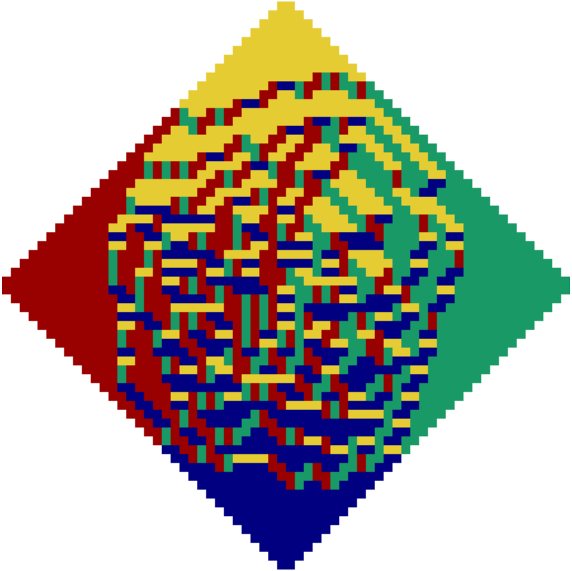
Mapping to free fermions

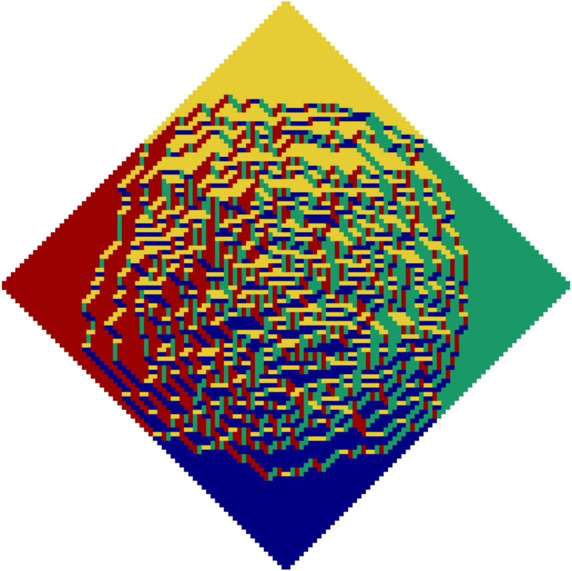


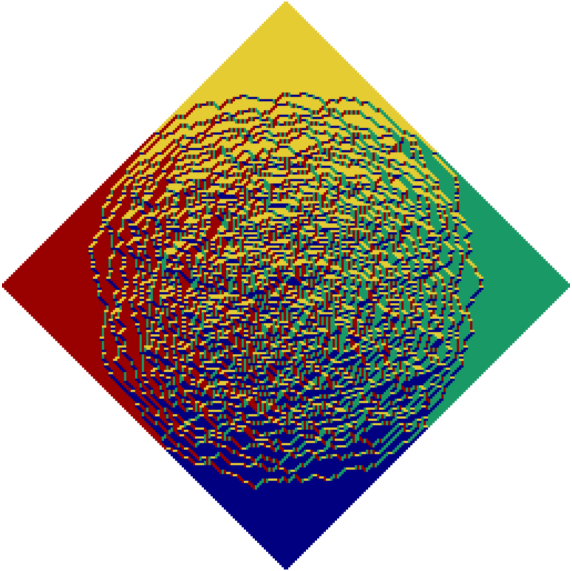
Mapping to free fermions

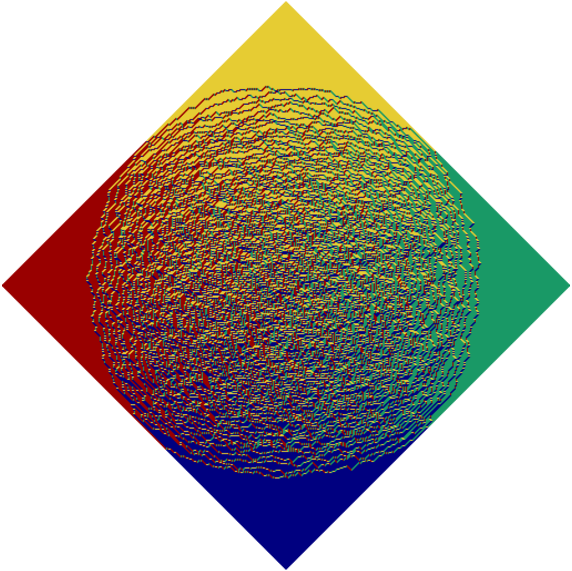


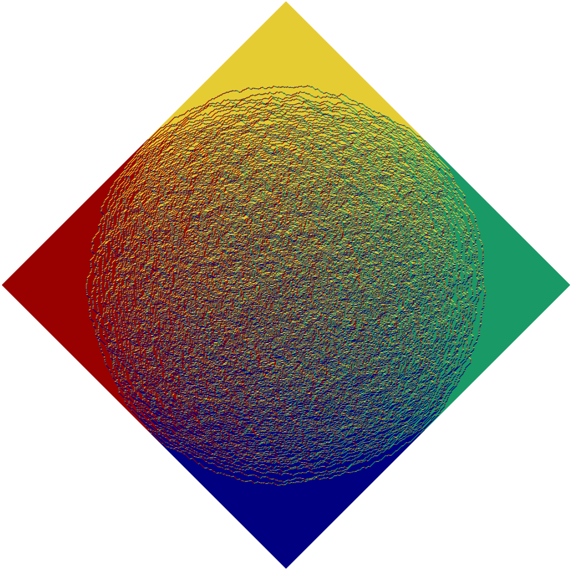


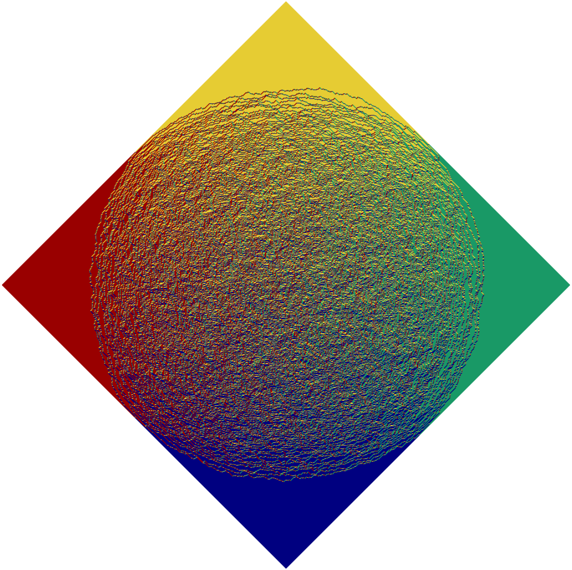




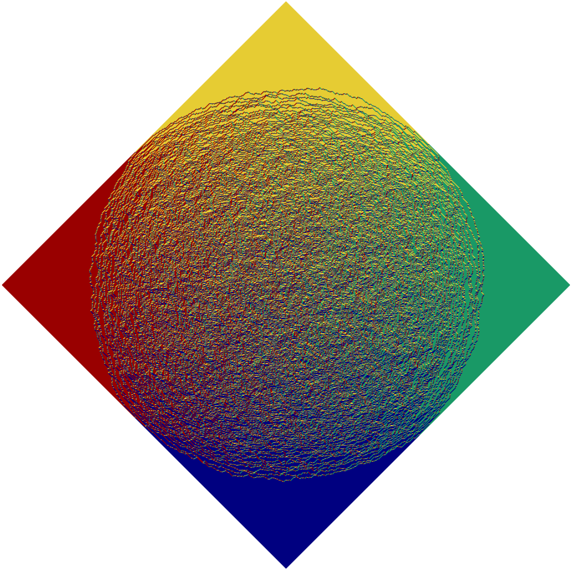




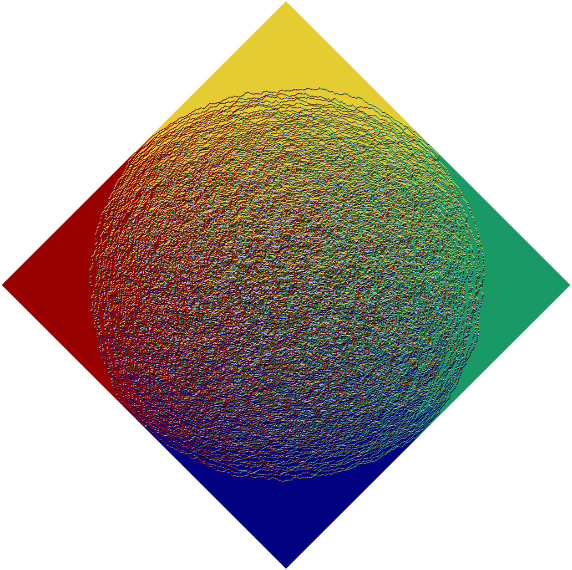




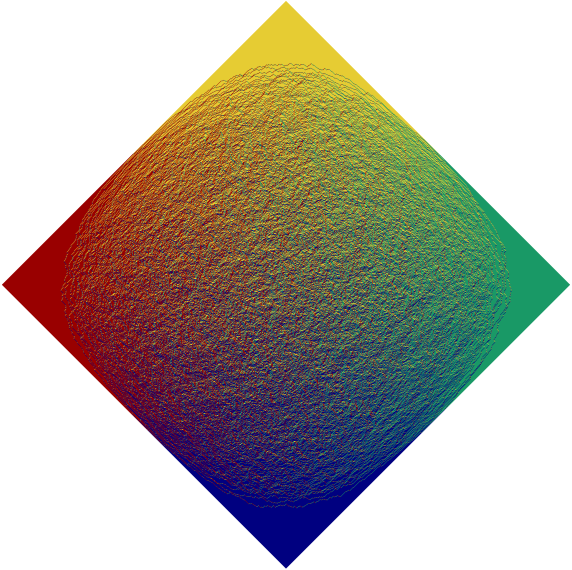
Arctic circle theorem [Jockusch, Propp and Shor 1998]



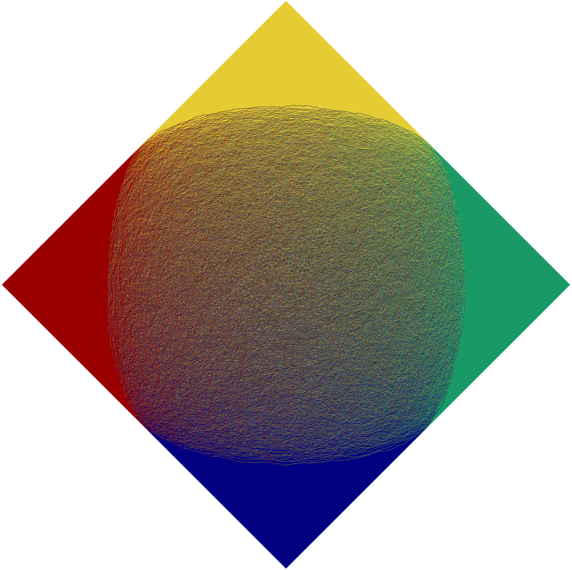
Inhomogeneous CFT [Allegra, Dubail, JMS, Viti 2016]



$N^{1/3}$ scaling near the edge: rightmost particle follows the Tracy-Widom distribution [Johansson 2005]

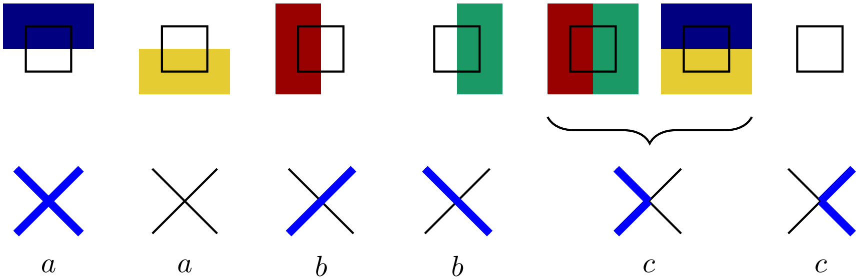


Repulsive interactions



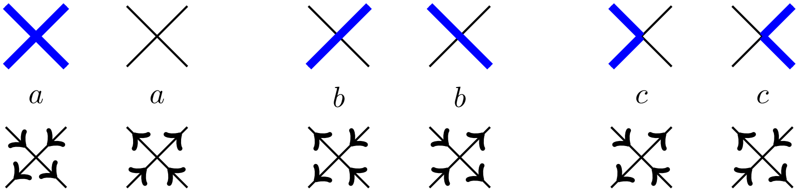
Attractive interactions

Can also do six vertex model



$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} = 1 - e^{\lambda}$$

Six-vertex model



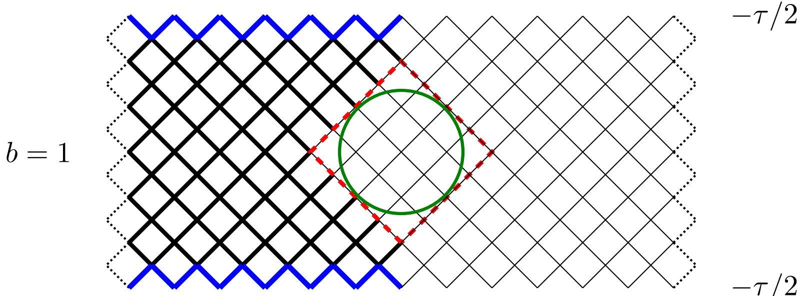
$$a = d\sin(\gamma+\epsilon) \quad , \quad b = d\sin\epsilon \quad , \quad c = d\sin\gamma$$

$$\Delta = \frac{a^2+b^2-c^2}{2ab} = \cos\gamma.$$

Disclaimer: in the following $a = 1$, and Δ is fixed to some value.

An Observation

[JMS 2014] [Allegra, Dubail, JMS, Viti 2016]

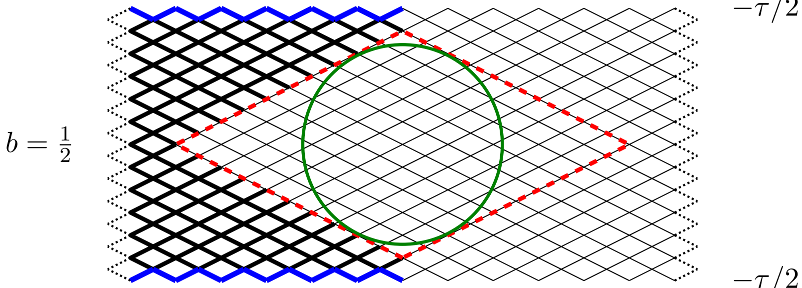


Six vertex model with domain wall boundary conditions

[Korepin 1982]

An Observation

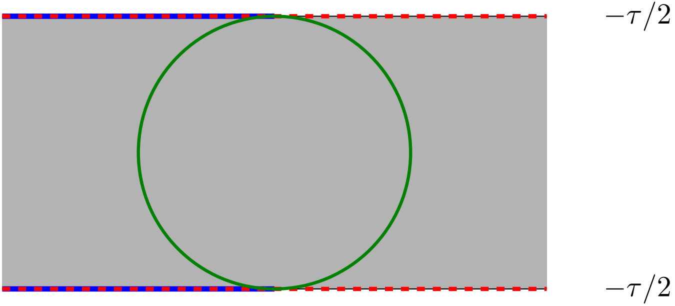
[JMS 2014]   [Allegra, Dubail, JMS, Viti 2016]



An Observation

[JMS 2014] [Allegra, Dubail, JMS, Viti 2016]

$b \to 0$



Hamiltonian (or Trotter) limit.

Relation through a transfer matrix (6-vertex model)

$$Z_n^{\mathrm{IK}}(b) = \langle \ldots \uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow \ldots | T(b)^{2n} | \ldots \uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow \ldots \rangle$$

partition function of the six vertex model with domain wall boundary conditions.

$$T(b) = 1 + bH_{XXZ} + O(b^2)$$

$$\lim_{N\to\infty} T(\tau/2n)^n = e^{\tau H_{XXZ}}$$

$$\mathcal{Z}(\tau) \;=\; \langle \Psi_0 | e^{\tau H} | \Psi_0 \rangle \;=\; \lim_{n\to\infty} Z_n^{\mathrm{IK}}(b = \frac{\tau}{2n})$$

Free fermions point $\Delta = 0$

$$H = \sum_{x\in\mathbb{Z}} \left(c^\dagger_{x+1} c_x + c^\dagger_x c_{x+1} \right) = \int \frac{dk}{2\pi} (\cos k) c^\dagger(k) c(k)$$

$$e^{(\frac{\tau}{2}+y)H} c^\dagger_x e^{(\frac{\tau}{2}-y)H} = \int \frac{dk}{2\pi} e^{-ikx+y\cos k+i\frac{\tau}{2}\sin k} c^\dagger(k)$$

Saddle point treatment: $x + iy \sin k + \frac{\tau}{2} \cos k = 0$, two solutions

- $x^2 + y^2 < \tau^2$: power law decay of correlations
- $x^2 + y^2 > \tau^2$: everything goes to zero

Remark: $\mathcal{Z}(\tau) = \langle \Psi_0 | e^{\tau H} | \Psi_0 \rangle = e^{\tau^2/8}$

Curved CFT approach inside the disk

[Allegra, Dubail, JMS & Viti 2016] Imaginary time propagator at short distances (up to some phases)

$$\langle c^\dagger(x+\delta x, y+\delta y)c(x,y)\rangle \sim \frac{1}{2\pi}\left[\frac{1}{\delta x + iv(x,y)\delta y} - \frac{1}{\delta x - iv(x,y)\delta y}\right]$$

This coincides with the propagator for the following action

$$\mathcal{S} \;=\; \frac{1}{2\pi}\int dz d\bar{z}\, e^{\sigma(x,y)}\left[\psi_{\rm R}^\dagger \overleftrightarrow{\partial}_{\bar{z}}\psi_{\rm R} + \psi_{\rm L}^\dagger \overleftrightarrow{\partial}_{z}\psi_{\rm L}\right],$$

provided

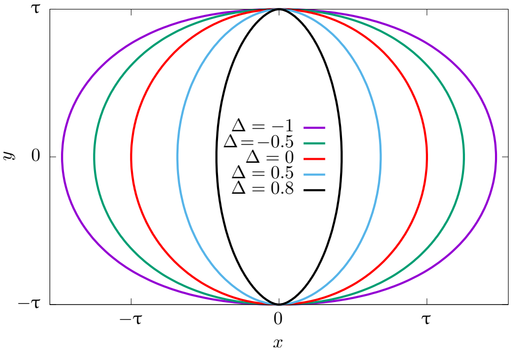
$$z(x,y) = \arccos\left(\frac{x}{\sqrt{(\tau/2)^2 - y^2}}\right) - i\,\text{arctanh}\frac{2y}{\tau}$$

$$e^{\sigma(x,y)} = \sqrt{(\tau/2)^2 - x^2 - y^2}$$

Interacting arctic curves ($\alpha = \pi/(\pi-\gamma)$)

$$\frac{x(s)}{\tau} = \frac{\alpha^2 \csc^2 \alpha s \left\{\cos(2\gamma+3s)(\cos s - \alpha \sin s \cot \alpha s) + \alpha \sin s \cos s \cot \alpha s + \cos^2 s - 2\right\} + 2}{\csc s \csc(\gamma+s)(\sin^2(\gamma+s)+\sin^2 s)}$$

$$\frac{y(s)}{\tau} = \frac{\left[2\alpha^2 \csc\gamma \sin^2 s \csc^2 \alpha s\{2\alpha \sin s \cot \alpha s \sin(\gamma+s) - \sin(\gamma+2s)\} - 1\right] + \sin^2 s}{\csc^2(\gamma+s)(\sin^2(\gamma+s)+\sin^2 s)}$$



Hamiltonian limit of [Colomo, Pronko 2009]

Related example: fermi gas in a harmonic potential

$$H = \int dx\, \psi^\dagger(x) \left[-\frac{1}{2}\partial_x^2 + \frac{x^2}{2} \right] \psi(x)$$

Modes $\psi_k^\dagger = \int_{\mathbb{R}} u_k(x)\psi^\dagger(x)\, dx$ given in terms of Hermite polynomials. Single particle energies $\epsilon_k = (k + 1/2)$, $k \in \mathbb{N}$

Density profile for N particles (Wigner Semicircle law)

$$\langle c^\dagger(x)c(x)\rangle \sim \frac{1}{\pi}\sqrt{L^2 - x^2} \qquad L = \sqrt{2N} \gg 1$$

Similar treatment (saddle point, etc)
Similar CFT interpretation
$|\Psi(x_1, \ldots, x_N)|^2$ coincides with eigenvalue pdf for GUE.

Izergin-Korepin partition function (interacting)

There is a remarkable exact determinant formula for Z_n^{IK}
[Izergin 1987, Izergin, Coker, Korepin 1992]

In the homogeneous limit it becomes a Hankel determinant:

$$Z_n^{\mathrm{IK}} = \frac{[\sin\epsilon]^{n^2}}{\prod_{k=0}^{n-1} k!^2} \det_{0\le i,j\le n-1} \left(\int_{-\infty}^{\infty} du\, u^{i+j} e^{-\epsilon u} \frac{1-e^{-\gamma u}}{1-e^{-\pi u}} \right)$$

where recall $b = \frac{\sin\epsilon}{\sin(\gamma+\epsilon)}$ and $\cos\gamma = \Delta$.

Can be rewritten as a Fredholm determinant [Slavnov 2003] (see also [Colomo Pronko 2003])

Hankel matrices and orthogonal polynomials

- Choose a scalar product $\langle f, g\rangle = \int dx f(x) g(x) w(x)$
- Let $\{p_k(x)\}_{k\geq 0}$ be a set of monic orthogonal polynomials for the scalar product , $\langle p_k, p_l\rangle = h_k \delta_{kl}$
- Consider the Hankel matrix A, with elements $A_{ij} = \langle x^{i+j}\rangle$

$$\det A = \prod_{k=0}^{n-1} h_k \quad , \quad (A^{-1})_{ij} = \left. \frac{\partial^{i+j} K_n(x,y)}{i! j! \partial x^i \partial y^j} \right|_{\substack{x=0 \\ y=0}} \text{ with}$$

$$K_n(x,y) = \sum_{k=0}^{n-1} \frac{p_k(x) p_k(y)}{h_k} = \frac{1}{h_{n-1}} \frac{p_n(x) p_{n-1}(y) - p_{n-1}(x) p_n(y)}{x-y}$$

Laguerre polynomials

$$w(x) = e^{-\epsilon x} \text{ on } \mathbb{R}_+ \quad , \quad \det(A) = \frac{\prod_{k=0}^{n-1} k!^2}{\epsilon^{n^2}}$$

$$Z_n = \left(\frac{\sin\epsilon}{\epsilon}\right)^{n^2} \times \frac{\det\limits_{0\le i,j\le n-1}\left(\int_{-\infty}^{\infty} du\, u^{i+j} e^{-\epsilon u} \frac{1-e^{-\gamma u}}{1-e^{-\pi u}}\right)}{\det\limits_{0\le i,j\le n-1}\left(\int_{-\infty}^{\infty} du\, u^{i+j} e^{-\epsilon u} \Theta(u)\right)}$$

Now use $\frac{\det A}{\det B} = \det(B^{-1}A) = \det(1 + B^{-1}(A-B))$ to get something well behaved in the Hamiltonian limit.

Result: exact fredholm determinant representation

$$\mathcal{Z}(\tau) \;=\; \langle \Psi_0 | e^{\tau H} | \Psi_0 \rangle \;=\; e^{-\frac{1}{24}(\tau \sin\gamma)^2} \det(I - V)$$

[JMS 2017]

$$V(x,y) = B(x,y)\,\omega(y)$$

$$B(x,y) = \frac{\sqrt{y} J_0(\sqrt{x}) J_0'(\sqrt{y}) - \sqrt{x} J_0(\sqrt{y}) J_0'(\sqrt{x})}{2(x-y)}$$

$$\omega(y) = \Theta(y) - \frac{1 - e^{-\gamma y/(2\tau \sin\gamma)}}{1 - e^{-\pi y/(2\tau \sin\gamma)}}$$

$$\log\det(I-V) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int_{\mathbb{R}^k} dx_1 \ldots dx_k V(x_1,x_2) \ldots V(x_k,x_1)$$

Asymptotics

Easiest: use [Zinn-Justin 2000] [Bleher, Fokin 2006]

$$\mathcal{Z}(\tau) \underset{\tau\to\infty}{\sim} \exp\left(\left[\frac{\pi^2}{(\pi-\gamma)^2} - 1\right]\frac{(\tau\sin\gamma)^2}{24}\right)\tau^{\kappa(\gamma)}O(1)$$

$$\kappa(\gamma) = \frac{1}{12} - \frac{(\pi-\gamma)^2}{6\pi\gamma}$$

Interpretation: free energy of the fluctuating region.

Back to real time

$$|\Psi(t)\rangle = e^{-itH}\,|\Psi_0\rangle \qquad , \qquad |\Psi_0\rangle = |\ldots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\ldots\rangle$$

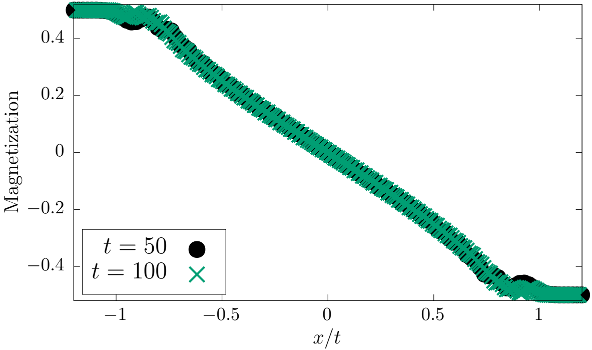
$$H = \sum_{x\in\mathbb{Z}} \left(S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2 + \Delta S_x^3 S_{x+1}^3\right)$$

Free fermion case ($\Delta = 0$) [Antal, Rácz, Rákos, and Schütz, 1999]
Interactions: Numerics [Gobert, Kollath, Schollwöck & Schütz 2005]

Back to real time

$$|\Psi(t)\rangle = e^{-itH}\,|\Psi_0\rangle \qquad , \qquad |\Psi_0\rangle = |\ldots \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \ldots\rangle$$

$$H = \sum_{x\in\mathbb{Z}} \left(S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2 + \Delta S_x^3 S_{x+1}^3\right)$$



Back to real time

Analytic continuation

- Return probability: $\tau = it$
- Correlations: $y = it$ and $\tau \to 0^+$

Continuation of the arctic curves should give the light cone:

Free fermions: $x^2 + y^2 = (\tau/2)^2 \quad \longrightarrow \quad x = \pm t$
Interactions: $\text{complicated} \quad \longrightarrow \quad x = \pm(\sin\gamma)t = \pm\sqrt{1-\Delta^2}\, t$

Analytic continuation

Numerical observations (huge precision, t up to 600 on laptop):

- Root of unity, $\gamma = \arccos\Delta = \frac{\pi p}{q}$

$$-\log\mathcal{R}(t) = \left(\frac{q^2}{(q-1)^2} - 1\right)\frac{(t\sin\gamma)^2}{12} + O(\log t)$$

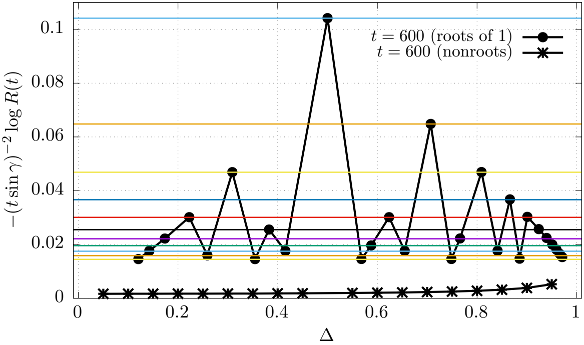
  Coincides with analytic continuation only when $p = 1$.

- non root of unity

$$-\log\mathcal{R}(t) = t\sin\gamma + O(\log t)$$

Analytic continuation

Numerical observations (huge precision, t up to 600 on laptop):



How about a proof using Riemann-Hilbert techniques?

[Its, Izergin, Korepin, Slavnov 1990]

Effective descriptions

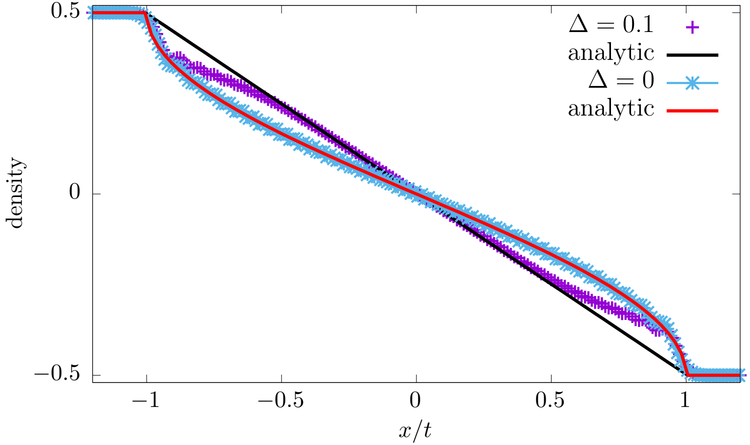
- Generalized hydrodynamics ($|\Delta| < 1$, ballistic)
  [Castro-Alvaredo, Doyon, Yoshimura 2016]
  [Bertini, Collura, De Nardis, Fagotti 2016]

  This particular quench (root of unity $\Delta = \cos \frac{\pi p}{q}$),

  $$S_x^3(x/t) = -\frac{q}{2\pi} \arcsin\left(\frac{\sin \frac{\pi}{q}}{\sin \frac{\pi p}{q}} \frac{x}{t}\right) \qquad (!)$$

  [De Luca, Collura, Viti 2018]

  Reproduces the front obtained by arctic curves machinery
  Density is nowhere continuous as a function of Δ, similar to return probability.



DMRG, $t = 80$ here [http://itensor.org]

What about entanglement entropy?

General wisdom for Entanglement scaling in 1+1d

- Ground state of a gapped Hamiltonian with local interactions.

  Area law: $S(\ell) \sim \ell^{d-1}$ [Srednicki 1993; Hastings, 2004]

- There can be mild (log) violations for critical systems

  1+1d CFT $S(L) \sim \frac{c}{3} \log L$
  [Holzhey, Larsen & Wilczek 1994; Calabrese & Cardy 2004]

  Systems with a Fermi surface $S(L) \sim L^{d-1} \log L$
  [Wolf 2006; Gioev & Klich 2006]

After a quantum quench (still critical)

- Local quench $S(t) \sim \frac{c}{3} \log t$. [Calabrese, Cardy 2007]
- Global quench $S(t) \sim t$. [Calabrese, Cardy 2005]

NB: Those are pure CFT calculations.

Chaotic systems: random circuits calculations [Nahum, Ruhman, Vijay & Haah 2017] also give $S(t) \sim t$, but have no notion of local quench.

EE after the quench studied here [Dubail, JMS, Viti & Calabrese 2017]

$$S_n(x,t) \sim \tfrac{1}{1-n} \log \epsilon^{\Delta_n} \langle \mathcal{T}_n(x, y = it) \rangle$$

Map to the upper half plane through $g(z) = e^{i(z+\pi/2)}$

$$\langle \mathcal{T}_n(z, \overline{z}) \rangle \; = \; \left(e^{\sigma(z,\bar{z})} \; \left| \frac{dg(z)}{dz} \right|^{-1} \; \mathrm{Im}\, g(z) \right)^{-\Delta_n}$$

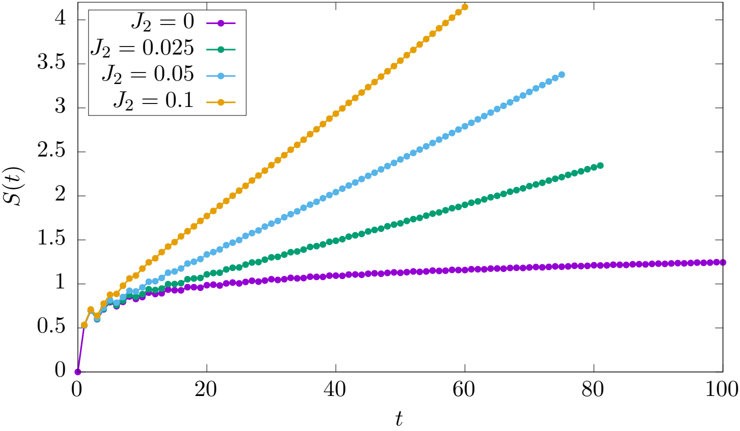
Careful that the UV cutoff $\epsilon \to \epsilon(x) = \epsilon_0 / \sin k_F(x)$ now depends on position.

$$S_n(x,t) = \frac{n+1}{12\,n} \ln \left[t \left(1 - (x/t)^2\right)^{3/2} \right]$$

Recovers the numerical guess made in [Eisler & Peschel 2014]

Caveat: Integrability and non Integrability

$$H = \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y \right) + J_2 \sum_j \left(\sigma_j^x \sigma_{j+2}^x + \sigma_j^y \sigma_{j+2}^y \right)$$



Conclusion

- The mysteries of the analytic continutation.
- Exact computations, valid at all times.
- For certain high energy states, the XXZ chain out of equilibrium can show exotic behavior.
- Entanglement growth: integrable vs non integrable.

Thank you!