Manifestations of conformal invariance in classical statistical mechanics

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Abstract

I try to give an idea how physicists guessed why the critical behavior of some important two-dimensional lattice models (Ising, Potts, dimers, etc) is related to the representation theory of certain Lie algebras, in particular the Virasoro algebra. This is the broad topic of 2d conformal field theory (CFT). I will illustrate some of these aspects, taking as examples dimer and Ising models.

I want to discuss the following claim, often made in physics literature, but poorly understood from a rigorous mathematical perspective: to understand critical statistical mechanical lattice models with a perfectly legitimate probabilistic interpretation, it is necessary to study representations of infinite dimensional Lie Algebras, the most prominent example being the Virasoro algebra (c is the central charge)

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \qquad [c, L_n] = 0$$

This is the celebrated result of [Belavin, Polyakov, Zamolodchikov 1983], classifying universality classes amounts to identifying irreducible representations of the algebra. The aim of these informal lectures is to give a partial idea of how physicists managed to guess this. My main starting examples will be the Ising and dimer models. There are two sides to the story

- Exact solution of specific models such as the Ising model. Lenz, Peierls, Kramers, Wannier, [Onsager 1944], Lieb, Baxter, etc. Lattice approach, try to take the scaling limit.
- Field theoretical approach based on conformal symmetry. Ideas by Polyakov (1970), conformal field theory [BPZ 1983], Cardy, etc.

1 Lattice fermions

Two-level systems— $\mathcal{H} = \mathbb{C}^2$, two basis states $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Wave functions are $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, (\alpha, \beta) \in \mathbb{C}^2$. Now introduce the annihilation $c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and creation $c^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ operators. Physicists like to use the "dagger" notation † to denote Hermitian conjugate, together with bra/ket notations, where e. g. the bra $\langle 0| = (|0\rangle)^\dagger = (0 \;\; 1)$. We have $c^\dagger\,|0\rangle = |1\rangle$, $c\,|1\rangle = |0\rangle$, $c^\dagger\,|1\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$, $c\,|0\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ (slight abuse of

notations for the last two, which we keep on doing in the following). Also, $c^\dagger c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $c^\dagger c + cc^\dagger = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Two two-level systems— $\mathcal{H} = (\mathbb{C}^2)^{\otimes 2}$. $\dim \mathcal{H} = 2^2 = 4$.

Spanned by $|00\rangle = |0\rangle \otimes |0\rangle$, $|01\rangle = |0\rangle \otimes |1\rangle$, $|10\rangle = |1\rangle \otimes |0\rangle$, $|11\rangle = |1\rangle \otimes |1\rangle$.

A collection of N two level systems— $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$, $\dim \mathcal{H} = 2^N$. Want a set of (Dirac) fermionic operators, that is, a set of $c_i, c_i^\dagger$ for $i = 1, \ldots, N$ that satisfy

$$c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij} I \tag{1}$$

$$c_i c_j = -c_j c_i \tag{2}$$

where $I = I_2 \otimes \ldots \otimes I_2$ is the identity operator. The relations (1) and (2) are usually called canonical anticommutation relations (CAR), since they involve the anticommutator $\{A, B\} = AB + BA$, instead of the commutator $[A, B] = AB - BA$. An explicit construction, originally due to [Jordan & Wigner, 1928] [1] is shown below:

$$c_1^\dagger = c^\dagger \otimes \underbrace{I_2 \otimes \ldots \otimes I_2}_{N-1 \text{ times}} \tag{3}$$

$$\vdots$$

$$c_k^\dagger = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{k-1 \text{ times}} \otimes c^\dagger \otimes \underbrace{I_2 \otimes \ldots \otimes I_2}_{N-k \text{ times}} \tag{4}$$

$$\vdots$$

$$c_N^\dagger = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{N-1 \text{ times}} \otimes c^\dagger \tag{5}$$

The chain of $k-1$ tensor products of $I_2 - 2c^\dagger c = (-1)^{c^\dagger c}$ in e. g. (4) is called a *Jordan-Wigner string*. One can readily check that the anticommutation relations (1), (2) are satisfied, using $(A \otimes B)(C \otimes D) = AC \otimes BD$ lots of times.

It is possible to write down essentially any state using fermionic operators acting on the vacuum $|\mathbf{0}\rangle = |0 \ldots 0\rangle = |0\rangle \otimes \ldots \otimes |0\rangle$. For example for $N = 5$, $|01101\rangle = +c_2^\dagger c_3^\dagger c_5^\dagger |\mathbf{0}\rangle$ (the sign is always positive provided the creation operators appear with increasing index from left to right). Further useful properties are stated in the red rounded box next page.

Free fermions Hamiltonian— A free (lattice) fermions Hamiltonian is a $2^N \times 2^N$ matrix that is quadratic in the fermions creation and annihilation operators (assume hermiticity here):

$$H = \sum_{i,j=1}^{N} \left(A_{ij} c_i^\dagger c_j + B_{ij} c_i^\dagger c_j^\dagger + B_{ij}^* c_j c_i \right) \tag{6}$$

[1] Physicist often use this result when studying one dimensional quantum magnets ("quantum spin chains"), which are modeled using Pauli matrices. The construction goes $\sigma_j^\alpha = I_2 \otimes \ldots \otimes \sigma^\alpha \otimes I_2 \ldots \otimes I_2$, for $\alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$, where $\sigma^\mathrm{x} = c^\dagger + c$, $\sigma^\mathrm{y} = -ic^\dagger + ic$, $\sigma^\mathrm{z} = 2c^\dagger c - I_2$ are the Pauli matrices. The "Pauli matrices acting on site j" are related to fermions through the *Jordan-Wigner transformation* $\sigma_j^\mathrm{z} = 2c_j^\dagger c_j - I$, $\sigma_j^x + i\sigma_j^y = 2c_j^\dagger \prod_{l=1}^{j-1} \left(I - 2c_l^\dagger c_l \right)$.

Useful fermionic formulas

- Anticommutation relations: $c_i^\dagger c_j = \delta_{ij} I - c_j c_i^\dagger \quad , \quad c_i c_j = -c_j c_i \quad , \quad c_i^\dagger c_j^\dagger = -c_j^\dagger c_i^\dagger$. In particular $c_i c_i = 0 = c_i^\dagger c_i^\dagger$.
- $c_i |\mathbf{0}\rangle = 0$ and $\langle \mathbf{0}| c_i^\dagger = (c_i |\mathbf{0}\rangle)^\dagger = 0$. Hence all c_i "annihilate" the vacuum $|\mathbf{0}\rangle$.
- Commutation relations for quadratic forms: $[c_i^\dagger c_j, c_k^\dagger c_l] = \delta_{jk} c_i^\dagger c_l - \delta_{il} c_k^\dagger c_j$. In particular, $[c_i^\dagger c_i, c_k^\dagger c_k] = 0$.
- Exponentiation: since $(c^\dagger c)^2 = c^\dagger c$, we have the relation $\exp\left(\alpha c_i^\dagger c_i\right) = 1 + (e^\alpha - 1) c_i^\dagger c_i$. Also, $e^{\alpha c_i^\dagger c_i} c_i^\dagger e^{-\alpha c_i^\dagger c_i} = e^\alpha c_i^\dagger$.

where A is a Hermitian matrix and B is a $N \times N$ matrix. This is of course a specific class of Hamiltonians, since we have at most $2N^2$ free parameters while the Hilbert space size is 2^N. In physics context, free really means quadratic. A wide class of systems can be mapped onto free fermions. Those include determinantal point processes (B can be made to vanish in those cases), the Ising model, etc. But not six vertex in general, or three state Potts, etc.

How to diagonalize a free fermions Hamiltonian? — Here we focus on the specific form

$$H = \sum_{i,j=1}^{N} A_{ij} c_i^\dagger c_j \tag{7}$$

for simplicity, but the procedure described below can be generalized to treat cases where B is a non zero matrix. Hamiltonians of the form (7) conserve the number of particles, which means applying it on n-particle state $c_{i_1}^\dagger \dots c_{i_n}^\dagger |\mathbf{0}\rangle$ returns a sum over n particle states (essentially, any fermion destroyed by c_j is immediately created back by $c_i^\dagger$). A is a hermitian $N \times N$ matrix, so can be diagonalized in an orthonormal basis. The corresponding eigenvalue equations read (assume no multiplicities for simplicity)

$$\sum_{j=1}^{N} A_{ij} u_{jk} = \epsilon_k u_{ik} \quad , \quad k = 1, \dots, N \tag{8}$$

The eigenvalues are the ϵ_k and the u_{jk} are orthonormal, meaning $\sum_{j=1}^N u_{jk}^* u_{jq} = \delta_{kq}$. Now introduce a new set of fermions as

$$f_k^\dagger = \sum_{j=1}^{N} u_{jk} c_j^\dagger \quad , \quad k = 1, \dots, N \qquad , \qquad f_k = (f_k^\dagger)^\dagger \tag{9}$$

Then it is easy to show $\{f_k, f_q^\dagger\} = \delta_{kq} I$ and $\{f_k, f_q\} = 0$, same as for real space fermions. So finally

$$\boxed{H = \sum_k \epsilon_k f_k^\dagger f_k} \tag{10}$$

Obtaining the spectrum is quite easy now. Obviously $H |\mathbf{0}\rangle = 0$. Using the anticommutation relations, $H f_k^\dagger |\mathbf{0}\rangle = \epsilon_k f_k^\dagger |\mathbf{0}\rangle$. Or $H f_{k_1}^\dagger f_{k_2}^\dagger |\mathbf{0}\rangle = (\epsilon_{k_1} + \epsilon_{k_2}) f_{k_1}^\dagger f_{k_2}^\dagger |\mathbf{0}\rangle$, $k_1 \neq k_2$. So the spectrum

of this system is

$$\sum_{i=1}^{n} \epsilon_{k_i} \quad , \quad \{k_1, \ldots, k_n\} \text{ subset of } \{1, \ldots, N\}$$

An eigenstate with smallest eigenvalue is obtained by filling all single particle energies that are the smallest, $|\mathbf{\Omega}\rangle = \prod_{k,\epsilon_k>0} f_k^\dagger |\mathbf{0}\rangle$ (irrespective of the order in which the product is taken).

2 Dimers on honeycomb and Ising on square

2.1 Dimers

Let us try and apply our technology for dimers on the honeycomb (drawn as a brickwall), uniform measure. This is one of the simplest example where our free fermion technology may be applied. Observations: a dimer configurations is uniquely determined by the occupancies along vertical edges, so we may ignore the rest.

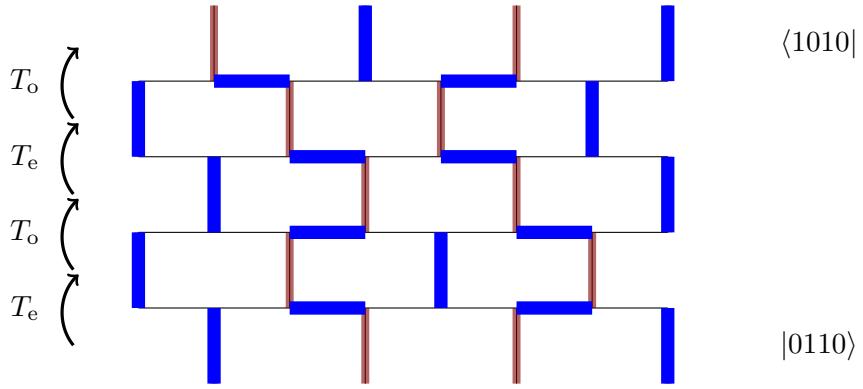

Figure 1: $(N = 4)\times(N' = 4)$ hexagonal lattice. An example of dimer covering (dimers are shown as thick blue lines). We see the occupation of the top and bottom vertical edges as imposed. In thinner dark red are drawn the vertical edges not occupied by dimers, those become particles (1) in the following, while real dimers are holes (0).

Transfer matrix method— Put a 0 on vertical edges occupied by a dimer (shown in thick blue), and a 1 otherwise (thiner darkred). We see the ones as a collection of particles propagating upwards. Then associate a vector to dimer occupancies along a given line. For example for $N = 4$ in the picture, to the bottom line configuration 0110 we associate the vector $|0110\rangle = c_2^\dagger c_3^\dagger |\mathbf{0}\rangle$ where recall $|\mathbf{0}\rangle = |0000\rangle$ is the vacuum.

Imagine I can find a $2^N \times 2^N$ matrix T, such that $\langle \mathcal{C}|T|\mathcal{C}'\rangle = 1$ if the configurations are compatible –meaning valid dimer covering– when stiched together and 0 otherwise. Then the partition function on the $N \times N'$ lattice is simply $Z = \langle \text{top}|T^{N'}|\text{bottom}\rangle$.

Proof. $\langle \text{top}|T^{N'}|\text{bottom}\rangle = \sum_{\mathcal{C}_1,\ldots,\mathcal{C}_{N'-1}} \langle \text{top}|T|\mathcal{C}_{N'-1}\rangle \langle \mathcal{C}_{N'-1}|T|\mathcal{C}_{N'-2}\rangle \ldots \langle \mathcal{C}_1|T|\text{bottom}\rangle$. Each element in the sum is one for valid configurations, zero otherwise.

It is also possible to just require the bottom and top boundaries to coincide (periodic boundary conditions). In this case $Z = \operatorname{Tr} T^{N'}$.

Here we actually need two transfer matrices, since the rule changes depending on the parity of the row considered. The bottom configuration is $|\text{bottom}\rangle = |0110\rangle$, and $T_e |0110\rangle = |0101\rangle + |0110\rangle + |0011\rangle$, and $T_o T_e |0110\rangle = 2|0101\rangle + |1001\rangle + 3|0110\rangle + 2|1010\rangle + |1100\rangle + |0011\rangle$. The partition function is $Z = \langle 1010|(T_o T_e)^2|0110\rangle = 20$ (or 14, forgot).

Transfer matrix in the one particle subspace— This is a $N \times N$ block, since the number of particles (number of 1) is conserved –in the zero particle sectors the transfer matrix acts as identity, $T|\mathbf{0}\rangle = |\mathbf{0}\rangle$. $(M_{\mathrm{e}})_{ij} = \delta_{ij} + \delta_{i+1,j}$, $(M_{\mathrm{o}})_{ij} = \delta_{ij} + \delta_{j+1,i}$. $(M^2)_{i,j} = (M_{\mathrm{e}}M_{\mathrm{o}}) =$ discrete laplacian $- \delta_{i,j}\delta_{i,N}$.

Transfer matrix as free fermions— Observations: (i) Ones propagate upwards to the left, or to the right. (ii) number of one is conserved (iii) two ones cannot jump to the same site. Labelling edges in a natural, way one sees the rules changes depending on parity of the row. We write $T^2 = T_{\mathrm{e}}T_{\mathrm{o}}$, and call it the transfer matrix. We have

$$\boxed{T^2 = \exp\left(\sum_{i,j} A_{ij} c_i^\dagger c_j\right)} \tag{11}$$

where A is actually the logarithm of the (tweaked at the right boundary) discrete $N \times N$ laplacian: $A = \log M, M_{ij} = 2\delta_{ij} + \delta_{i,j+1} + \delta_{i+1,j} - \delta_{i,j}\delta_{i,N}$. The logarithm is well defined since all eigenvalues of M are strictly positive.

Sketch of the Proof. From the construction of the transfer matrix in the one particle sector, we get $T^2 c_i^\dagger |\mathbf{0}\rangle = \sum_j M_{ij} c_j^\dagger |\mathbf{0}\rangle$. One needs to also determine the action on states $c_{i_1}^\dagger \dots c_{i_n}^\dagger |\mathbf{0}\rangle$, $i_1 < \dots < i_n$ in the n-particle sector, for $n = 2, \dots, N$. Assuming T^2 is invertible, and using $T^2 c_{i_1}^\dagger c_{i_2}^\dagger \dots c_{i_n}^\dagger |\mathbf{0}\rangle = T^2 c_{i_1}^\dagger T^{-2} T^2 c_{i_2}^\dagger T^{-2} \dots T^2 c_{i_n}^\dagger T^{-2} |\mathbf{0}\rangle$, a sufficient condition for T to be a correct transfer matrix is to satisfy $T^2 c_i^\dagger T^{-2} = M_{ij} c_j^\dagger$, and be the identity in the zero particle sector. Introducing "diagonalizing" fermions $f_k^\dagger$ as in the previous section yields $T^2 f_k^\dagger T^{-2} = \lambda_k f_k^\dagger$, where the λ_k are the (all positive) eigenvalues of M. Then

$$T = \exp\left(\sum_k \log(\lambda_k) f_k^\dagger f_k\right) \tag{12}$$

does the job, since $[f_k^\dagger f_k, f_q^\dagger f_q] = 0$, so $T^2 f_k^\dagger T^{-2} = e^{\log \lambda_k f_k^\dagger f_k} f_k^\dagger e^{-\log \lambda_k f_k^\dagger f_k} = \lambda_k f_k^\dagger$. Also T is the identity in the zero particle sector. Finally, expressing (12) in terms of the $c_i^\dagger$ gives (11).

Correlation functions— The quantity

$$\frac{\langle \mathbf{0}|T^2 c_2 c_2^\dagger T^2|\mathbf{0}\rangle}{\langle \mathbf{0}|T^4|\mathbf{0}\rangle} \tag{13}$$

gives the probability of having a vertical dimer in the second vertical edge (starting from left) in the middle in figure 1. In principle, any correlation function can be computed using this formalism, and we can say that the dimer operator at site j is $c_j c_j^\dagger$.

2.2 Ising model and Onsager's glory

$N \times N$ lattice, N is the horizontal length, M vertical, vertical coupling is J_v, horizontal coupling is J_h. Classical energy associated to a spin configuration

$$E(\sigma_{11}, \dots \sigma_{NN'}) = \sum_{x=1}^{N} \sum_{y=1}^{N'} J_h \sigma_{x,y}\sigma_{x+1,y} + J_v \sigma_{x,y}\sigma_{x,y+1}$$

We take here open boundary conditions, so $\sigma_{x,N'+1} = 0$ and $\sigma_{N+1,y} = 0$. [Onsager 1944] essentially cracked the lattice problem, and determined the transfer matrix using a more complicated

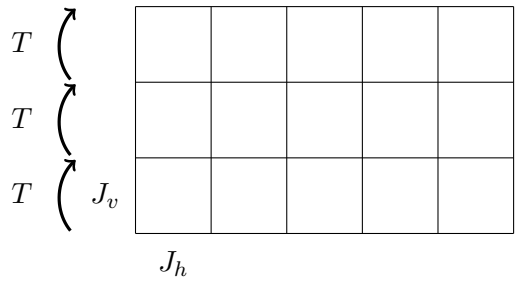

Figure 2: $(N = 6) \times (N' = 4)$ square lattice. The transfer matrix now acts on the vector space generated by the spin configuration of the N sites along a given row, not vertical edges as before. Also, there is just one transfer matrix now.

algebraic technique which uses what's known now as Onsager algebras. The solution was simplified in a series of subsequent works, in particular by Kaufman. Here we express it in the language of the review article [Schultz, Mattis, Lieb 1964], which makes use of free fermions. We give the final result without proof:

$$T = \rho^M \exp\left[\theta \sum_{j=1}^{N} \left(2c_j^\dagger c_j - 1\right)\right] \exp\left[\beta J_h \sum_{j=1}^{N-1} (c_j^\dagger + c_j)(c_{j+1} - c_{j+1}^\dagger)\right], \tag{14}$$

with $\rho \cosh\theta = e^{\beta J_v}$, $\rho \sinh\theta = e^{-\beta J_v}$. It is possible to diagonalize this transfer matrix using a generalization of the method explained on page 3. We do not elaborate further, since also the relation between spins and fermions is slightly more tricky. Let us just point out that (14) can be expressed as the exponential of a free fermions Hamiltonian. This follows from the Baker-Campbell-Hausdorff formula and the property $[\text{quadratic}, \text{quadratic}] = \text{quadratic}$.

3 Universal conformal spectrum

Recall the transfer matrix for dimers could be expressed as (11), where the $N \times N$ matrix A is the logarithm of a tweaked discrete laplacian. Diagonalization is rather straightforward. Introducing the non local Fourier-sine modes

$$f_k^\dagger = \sqrt{\frac{2}{N+1/2}} \sum_{j=1}^{N} \sin\left(\frac{k\pi j}{N+1/2}\right) c_j^\dagger \qquad , \qquad f_k = (f_k^\dagger)^\dagger \tag{15}$$

allows to express the transfer matrix as

$$T = \exp(-H) \qquad , \qquad H = \sum_k \epsilon_k f_k^\dagger f_k \qquad , \qquad \epsilon_k = -\log\left(2 + 2\cos\frac{k\pi}{N+1/2}\right) \tag{16}$$

The introduction of two minus signs might seem totally artificial, however it will be necessary to match physicist's conventions later on (minimizing energy is more natural in physics). Since each ϵ_k is associated to a single creator $f_k^\dagger$, we call it single particle energy. Also, since the transfer matrix is block diagonal with respect to particle number, it is fixed by boundary conditions. In the following, we set for simplicity the number of particle to be $N/2$, with N even.

3.1 Spectrum at low energy

We are interested in taking a scaling limit $N, N' \to \infty$, while keeping (say) the ratio N/N' fixed, and hopefully observe universal behavior. This is what we are after here, not explicit computations of observables in finite-size, and then take the scaling limit. In such a limit, one needs to apply very high powers of the tranfer matrix. Intuitively, this means that long range correlations will be dominated by the part of the spectrum corresponding to the biggest eigenvalues of the TM. Through (16), big eigenvalues of the TM translates to low eigenenergies of the Hamiltonian H.

Since we have $N/2$ particles and the single particle energies have all multiplicity one, there are $(N)!/(N/2)!^2$ possible eigenstates indexed by $\{k_1, \dots, k_{N/2}\} \subset \{1, \dots, N\}$:

$$f^\dagger_{k_1} \dots f^\dagger_{k_{N/2}} |\mathbf{0}\rangle \qquad \text{with energy} \qquad E(k_1, \dots, k_{N/2}) = \epsilon_{k_1} + \dots + \epsilon_{k_{N/2}}, \tag{17}$$

The corresponding eigenvalues of the TM are $\Lambda(k_1, \dots, k_{N/2}) = \exp(-E(k_1, \dots, k_{N/2}))$. Let us look at the biggest eigenvalue first, which means we need the $N/2$ smallest single particle energies. The corresponding eigenstate, the "ground state" is

$$|\mathbf{\Omega}_0\rangle = f^\dagger_1 f^\dagger_2 \dots f^\dagger_{N/2} |\mathbf{0}\rangle \overset{\text{notation}}{=} |\underbrace{\dots \bullet \bullet \bullet \bullet}_{N/2}|\underbrace{\circ \circ \circ \circ \dots}_{N/2}\rangle \tag{18}$$

with energy $E_0(N) = -\sum_{k=1}^{N/2} \log(2 + 2\cos\frac{k\pi}{N+1/2})$. Using the Euler-MacLaurin formula[2], its large N behavior is

$$E_0(N) = eN + e' - \frac{\pi}{96N} + O(1/N^2) \tag{19}$$

e and e' can be obtained exactly, but we do not care about them. Writing the energies as E_α in increasing order, we will see that for α finite

$$E_\alpha(N) = eN + e' + \frac{\pi}{N}\left(p_\alpha - \frac{1}{96}\right) + O(1/N^2) \tag{20}$$

where $p_\alpha \in \mathbb{N}$. The part proportional to $1/N$ in the previous equation is called conformal spectrum.

Remark. It is important to realize that the $1/N$ contributions to the ground state energy might depend on boundary conditions. Indeed here the matrix A we needed to diagonalize was the logarithm of a tweaked laplacian with matrix elements $2\delta_{i,j} + \delta_{i,j+1} + \delta_{i+1,j} - \delta_{i,j}\delta_{i,N}$. The last term comes from the mapping from dimers to particle configurations, it is in principle non negociable. Considering the logarithm of the untweaked laplacian instead also gives a perfectly legitimate stat mech models, however. The solution is very similar, and can be obtained by replacing all the denominators in $N + 1/2$ by $N + 1$ in the definitions of $f^\dagger_k$ and ϵ_k, and not changing anything else. In this case we find $E_\alpha(N) = eN + e' + \frac{\pi}{N}\left(p_\alpha - \frac{1}{24}\right) + O(1/N^2)$. **In the following and unless specified otherwise, we look at the (slightly simpler) plain laplacian, and set the ground state energy to zero. The $\frac{-\pi}{24N}$ and $\frac{-\pi}{96N}$ also have a CFT interpretation, but this is a longer story.**

3.2 Bosonization

We look here at low energy excitations –with same number of particles for simplicity– above the ground state, for large N. It is easy to construct the first few excited states, by considering "particle-hole" excitations above the ground state. For example the first one is obviously

$$|\mathbf{\Omega}_1\rangle = |\dots \bullet \bullet \bullet \circ| \bullet \circ \circ \circ \dots\rangle = f^\dagger_{N/2+1} f_{N/2} |\mathbf{\Omega}\rangle\,. \tag{21}$$

[2] $\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \frac{f(a)+f(b)}{2} + \frac{f'(b)-f'(a)}{12} + \dots$

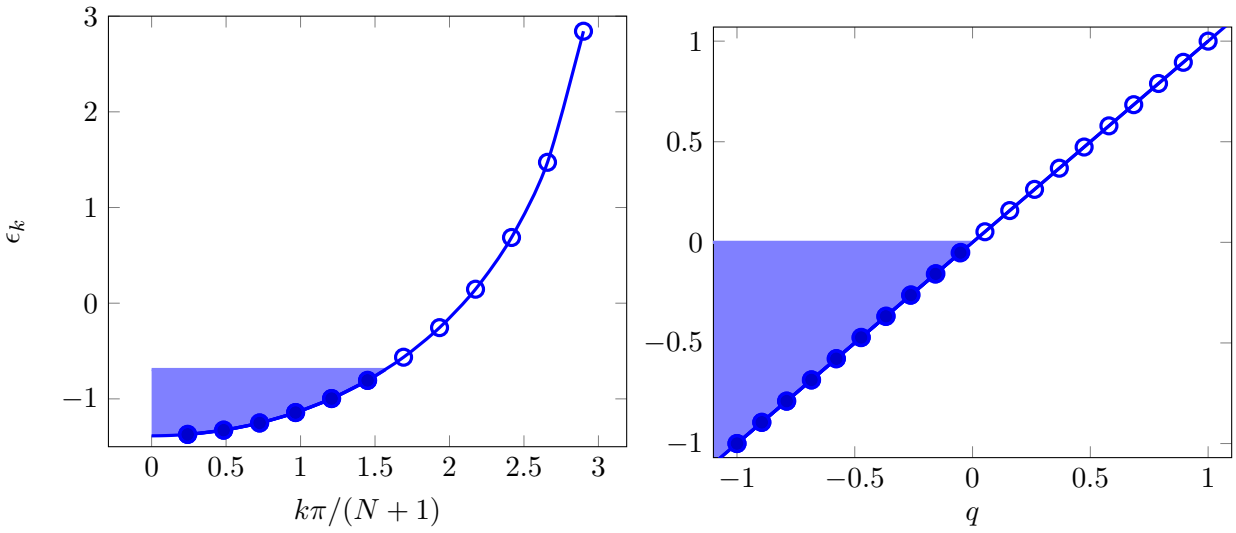

Figure 3: Left: Fermi see with $N/2 = 6$ particles and $N = 12$ allowed single particle energies. Right: in the limit $N \to \infty$, the Fermi see in the vicinity of the beach becomes, after rescaling, an infinitely deep Dirac see (we shift $q = k - N/2 - 1/2$). [Change the scale of the axis on the right]

Introducing $\varepsilon(K) = -\log(2 + 2\cos K)$, $\epsilon_k = \varepsilon(\frac{k\pi}{N+1/2})$, and the energy difference is $E_1 - E_0 = \varepsilon'(\pi/2)\frac{\pi}{N} + O(N^{-2})$, and it turns out $\varepsilon'(\pi/2) = 1$ (in general this is called Fermi speed, is noted v_F, and depends on the details of the lattice model). The second and third are

$$|\mathbf{\Omega}_2\rangle = |\ldots \bullet\bullet\bullet\circ|\circ\bullet\circ\circ\ldots\rangle \tag{22}$$

$$|\mathbf{\Omega}_3\rangle = |\ldots \bullet\bullet\circ\bullet|\bullet\circ\circ\circ\ldots\rangle\,, \tag{23}$$

with the same energy $E_{2,3} - E_0 = \frac{2\pi}{N} + O(N^{-2})$. Determining the E_α becomes a combinatorial problem. A nice way to obtain all the eigenstates from the ground-state is through a procedure known as **bosonization**. Introduce the modes [3]

$$a_n = \sum_{k=1}^{N} f_k^\dagger f_{k+n} \qquad , \qquad n \in \mathbb{Z}^* \tag{24}$$

with convention $f_q^\dagger = 0 = f_q$ if $q \notin \{1, \ldots, N\}$ to make life easier. One can check, for example, that $a_{-1}|\mathbf{\Omega}_0\rangle = |\mathbf{\Omega}_1\rangle$, $a_{-2}|\mathbf{\Omega}_0\rangle = |\mathbf{\Omega}_2\rangle - |\mathbf{\Omega}_3\rangle$, $(a_{-1})^2|\mathbf{\Omega}_0\rangle = |\mathbf{\Omega}_2\rangle + |\mathbf{\Omega}_3\rangle$, $a_{n>0}|\mathbf{\Omega}\rangle = 0$, etc.

Infinite dimensional bosonic algebra— The following is reasonably easy to prove. For any finite n, m, α, β, one can choose N sufficiently large, such that[4]

$$\langle\mathbf{\Omega}_\alpha|[a_n, a_m]|\mathbf{\Omega}_\beta\rangle = n\delta_{n+m,0}\delta_{\alpha,\beta} \tag{25}$$

For large N, notice also $a_n^\dagger = a_{-n}$. This means in the limit $N \to \infty$, it is natural to consider the infinite dimensional[5] algebra generated by the a_n. The Hilbert space of low lying excitations is

[3] Technically, we also need $Q = \sum_{k=1}^{N} : f_k^\dagger f_k := -\sum_{k=1}^{N/2} f_k f_k^\dagger + \sum_{k=1}^{N/2+1} f_k^\dagger f_k$ to take into account other sectors, in case we allow particle number to fluctuate.

[4] for example $[a_1, a_{-1}] = f_1^\dagger f_1 - f_N^\dagger f_N$, which acts only on the boundary "sites" $1, N$.

[5] An (bosonic) algebra of the form $[a_n, a_m] = n\delta_{n+m,0}$ has to be infinite dimensional (take trace), contrary to fermionic ones.

spanned by states of the form

$$\frac{1}{\sqrt{n_1^{m_1}\dots n_p^{m_p} m_1!\dots m_p!}}(a_{-n_1})^{m_1}\dots(a_{-n_p})^{m_p}\,|\mathbf{\Omega}_0\rangle \tag{26}$$

for non negative $m_1,\dots,m_p$ and $p\in\mathbb{N}$. One can check that they are orthonormal to each other, and also that they are eigenstates of $\tilde{H}$ with eigenvalue (up to corrections of order $1/N^2$)

$$E_\alpha - E_0 = \frac{\pi}{N}\left(m_1 n_1 + \dots + m_p n_p\right) \tag{27}$$

One can show that this procedure exhausts all possibilities for linearly independent eigenstates. What we have here is an exemple of a "conformal tower". The spectrum is discrete (in units of π/N), and each energy level M (eigenvalue $\frac{\pi M}{N}$) has a multiplicity given by the partition number of M, noted $p(M)$. As is well known from number theory, the generating function for this number is $\prod_{p=1}^{\infty}\frac{1}{1-\lambda^p}=\sum_{M=0}^{\infty}p(M)\lambda^M$. The large M behavior is given by the Ramanujan-Hardy formula $\log p(M)=\pi\sqrt{\frac{2M}{3}}+O(\log M)$, so multiplicities blow up very fast. The Hamiltonian can also be rewritten as

$$H - E_0 = \frac{\pi}{N}\left(\sum_{n=1}^{\infty} a_{-n}a_n\right) + O(N^{-2}) \tag{28}$$

The above (bosonization) procedure can be justified rigorously [Kac]. CFTs that have this as Hamiltonian are called free boson CFT, or free field, or free scalar field, or gaussian free field, or bosonic string, or Luttinger liquid, depending on context.

Relation to Virasoro— It is possible to construct a Virasoro algebra by using bosons:

$$L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} :a_{-j}a_{j+n}: \tag{29}$$

where $:a_x a_y: := a_x a_y$ if $x\le y$, and $a_y a_x$ otherwise. One can check that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}n(n^2-1)\delta_{n+m,0} \tag{30}$$

so the central charge is $c=1$. In that case the ground state is an eigenstate of L_0, the excited states are build from $|\mathbf{\Omega}_0\rangle$ by acting on it with the L_{-n}, choosing some order and sticking to it, similar to free boson. Of course, it is perfectly fine, if one wishes, to stay at the level of bosons, and not invoque Virasoro at all in studying dimer models.

3.3 A word on the Ising case

The transfer matrix is more complicated, but is still the exponential of a quadratic form. This is because (i) [quadratic, quadratic] = quadratic, see colored box (ii) Baker-Campbell-Hausdorf $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}$. In that case the relevant (Bogoliubov) transformation to diagonalize mixes creators and annihilators. At the critical point (and only at the critical point), we get after very long algebra a Hamiltonian of the form

$$H - E_0 = \frac{\pi v}{L}\left(\sum_{k=1}^{\infty} k\psi_{-k}\psi_k\right) \tag{31}$$

where v is some non universal parameter that depends on J_h, J_v. The ψ_k are called Majorana (or neutral) fermions. They satisfy the anticummutation relations $\{\psi_k, \psi_q\} = \delta_{kq}$, plus $\psi_k^\dagger = \psi_k$ different from the previous fermions[6], that are sometimes called Dirac fermions in the literature.

Let us mention that it is also possible to compute the degeneracies exactly in this model. One finds for large N, $\log p(M) = \pi\sqrt{\frac{M}{3}} + O(\log M)$. This is clearly a different beast from the one before, since the multiplicities grow less fast[7].

Relation to Virasoro— There is also a relation with Virasoro, which is given in this case by

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}'} k : \psi_{-k+n}\psi_k : \qquad , \qquad n \in \mathbb{Z}. \tag{32}$$

One can check that the central extension is $c = 1/2$ in that case.

4 Tentative relation to conformal field theory

Our aim is here is just to present some of the basic logic of CFT, which is non rigorous. There are of course many holes in the argument we present here, but the story I tell below hopefully helps getting an idea why the hell Virasoro. Some of the aspects presented here are discussed more thoroughly in [A mathematical introduction to conformal field theory, Schottenloher 2008]. For the physics side, see e.g. [Ginsparg 1988].

Quantum physics and representation theory— The question we want to address first is how some symmetry acts on the quantum states we have previously defined. Imagine for simplicity we have a connected Lie group, and we want to see the action of a certain group element $g \in G$ on a state $|\Psi\rangle$. This should generate another state $|\Psi'\rangle$. The important point is that the physical quantities in the underlying statmech model are the expectation values $\langle\Phi|\Psi\rangle$. We demand that these should be left invariant by the action of a ground element, meaning $\langle\Phi'|\Psi'\rangle = \langle\Phi|\Psi\rangle$, since those are the only truly physical observables in the underlying statmech model. Wigner proved that symmetries in that case are represented by unitary operators[8].

$$|\Psi'\rangle = U(g)\,|\Psi\rangle \qquad , \qquad U^\dagger = U^{-1}. \tag{33}$$

It is then natural to ask for U to be a homomorphism, $\forall g, h \in G, U(gh) = U(g)U(h)$ to once again preserve probabilities. Hence it is in principle very important to understand the representations of the Lie group (or Lie algebra, as we shall see later on).

Projective representations— There is a hole in our previous argument, it is actually not necessary to look at true representations of the underlying group. Just asking for $U(gh) = e^{i\theta(g,h)}U(g)U(h)$ also preserves probability. This means projective representations are more relevant general framework to study symmetries in quantum systems.

One might ask whether it is possible to see projective representations as (induced by) true representations, but in a larger group. The answer is yes, what one needs to consider is what is

[6] They can also be constructed from our previous fermions, even in finite dimension. For example, the set $\psi_{2j-1} = \frac{c_j + c_j^\dagger}{\sqrt{2}}$, $\psi_{2j} = \frac{c_j - c_j^\dagger}{i\sqrt{2}}$ is a set of Majorana fermions.

[7] The generating function in the identity sector is $\chi(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^{2n^2}}{\prod_{k=1}^{2n}(1-\lambda^k)} = 1+\lambda^2+\lambda^3+2\lambda^4+2\lambda^5+3\lambda^6+\dots$

[8] For non connected Lie group, anti unitary acting anti-linearly are also possible. The most famous of those is time-reversal, but let's not talk about it.

known as the central extension of the group. Hence, representations of the central extension of the group are the right objects to consider.

The conformal group— The set of Moebius transformations

$$w(z) = \frac{az+b}{cz+d} \quad , \quad a,b,c,d \in \mathbb{C} \quad , \quad ad-bc=1 \tag{34}$$

forms a group in the strictest mathematical sense. The transformation maps the Riemann sphere $\mathbb{C} \cup \{\infty\}$ to itself, and each transformation has a unique inverse

$$z(w) = \frac{dw-b}{-cw+a}. \tag{35}$$

One can show that the central extension of this group is trivial, so that's it. This group is called the group of "global conformal transformation" in Physics literature.

Lie algebras— A Lie group is a group that is also a differentiable manifold. It is nice, in the sense that the group operation are compatible with the smooth structure. For physicists, Lie group implement "continuous" symmetries. For example rotation symmetry in the plane is implemented by $SO(2)$. Connection to Lie algebras are provided by the exponential map: for example the translation implemented on some smooth real valued function can be obtained by Taylor expansion around x (we assume analyticity here)

$$f(x+a) = \sum_{k=0}^{\infty} \frac{a^k}{k!} (\partial_x)^k f(x) = e^{a\partial_x}.f(x) \tag{36}$$

so we say, for example, that the differential operator ∂_x generates translations. The generators typically satisfy some algebra. In physics, it is often much more convenient to work at the level of generators, and come back to the group with the exponential map if needed/possible [9]. One can have fun looking at other generators, for example $x\partial_x$ generates dilatations, since $e^{ax\partial_x} f(x) = f(e^a x)$. Also $e^{ax^2\partial_x}.f(x) = f(\frac{1}{1/x+a})$. In 2d it's similar. For example, one can show that $x\partial_x + y\partial_y$ generates dilatations, or that $x\partial_y - y\partial_x$, good old curl, generates rotations.

Conformal algebra— The Moebius transformations are not the only transformations that preserve angles in 2d; in fact, any function $f(x,y)$ satisfying the Cauchy-Riemann equations does. It is convenient to use complex variables, in which case all holomorphic functions are locally conformal. Such a set of conformal transformations does not form a group. [Take $f(z) = z^2$ for example. $f(1) = 1$ and $f(-1) = 1$ so 1 has two inverses. There are regularity issues also]. It only makes sense locally, at the level of algebra. Physicists refer to it as local conformal symmetry, and don't give a damn that it's not a group, and has no global meaning.

The underlying algebra turns out to be infinite dimensional. Using the notation $\partial_z + \partial_{\bar{z}} = \partial_x$, $i(\partial_z - \partial_{\bar{z}}) = \partial_y$, one can show that all local conformal transformations are generated by the $l_n + \bar{l}_n$, $-i(l_n - \bar{l}_n)$ which may be written as

$$l_n = -z^{n+1}\partial_z \quad , \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{37}$$

Some particular cases we mentioned before. For example, the $l_{-1} + \bar{l}_{-1}$ generate translations in the horizontal direction, $-i(l_{-1} + \bar{l}_{-1})$ generates translation in the vertical direction, $l_0 + \bar{l}_0$ generates dilatations, $-i(l_0 - \bar{l}_0)$ generates rotations, etc.

[9] One can always to that for a finite-dimensional simply connected Lie group, but this is not true in general for infinite dimensional algebras.

The "modes" in (37) satisfy the Witt algebra $[l_n, l_m] = (n-m)l_{n+m}$, same for $\bar{l}_n$, also $[l_n, \bar{l}_m] = 0$ (actually this is a subtle point, since we complexified the algebra, but ok).

It is important to note that the $l_{\pm 1}, l_0$ form a subalgebra. There is a group corresponding to this finite subalgebra, and guess what, it is the global conformal group. As mentioned before the central extension is trivial.

There is a unique non trivial central extension of the Witt algebra, it is called the Virasoro algebra. Since we assume the right thing to do is looking at projective representations of the Witt algebra, this is equivalent to looking at true representations at the Virasoro algebra. As a side note, there is no such thing as the Virasoro group, that would have the Virasoro algebra as a Lie algebra.

Lowest weight representation— From our dimer/Ising example, we have seen that the low energy spectrum of H (highest eigenvalues spectrum of the TM) is build as a "conformal tower" from a vacuum state $|\mathbf{\Omega}\rangle$. In terms of the Virasoro, the states in the tower are of the form

$$L_{-n_1}^{m_1} \dots L_{-n_p}^{m_p} |h\rangle \qquad , \qquad n_1, \dots, n_p > 0 \tag{38}$$

where $|h\rangle$ is the eigenstate of L_0 with minimal non negative eigenvalue h. The construction implies all states (38) are eigenstates of L_0. This lowest weight state is also annihilated by all L_n for $n > 0$.

$$\forall n > 0 \quad , \quad L_n |h\rangle = 0 \tag{39}$$

This is actually absolutely crucial: indeed, by using the commutation relations of the L_n, one can easily show $L_0 L_n |h\rangle = (L_n L_0 - n L_n) |h\rangle = (h-n) L_n |h\rangle$. So L_n lowers the eigenvalue by n. If the condition (39) is not fulfilled then the spectrum is not bounded from below (just act with many $L_{p>0}$), and any physical interpretation as dominant eigenvectors of a transfer matrix is lost. We are only interested in lowest weight representations of Virasoro, which have this property, and are deemed physical.

It is important to realize that nothing prevents the existence of other towers in general. There can be other states $|h'\rangle$ with $h' > h$ that are also annihilated by the $L_{n>0}$, and from $|h'\rangle$ one can build another conformal tower. The number of conformal towers need not be finite, not even in a countable set in general. Also, just because a tower can appear in a given CFT does not mean it will in the specific setup considered. A vast body of work went into better understanding such aspects, we do not discuss them here.

Unitary minimal models— As alluded to before, representation theory of Virasoro is incredibly rich, so from now on we are going to impose a bunch of other constraints, to see what we get. In the following we ask

1. The Hamiltonian is L_0, and we look at unitary representations, which implies $L_n^\dagger$ can be expressed in terms of the L_m. One can show that the only consistent choice is $L_{-n} = L_n^\dagger$. True for dimers and Ising.

2. Spectrum bounded from below. This seems reasonable from a transfer matrix point of view. Satisfied by all known critical models in statmech.

3. I look at irreducible representations.

4. I look in the region $0 < c < 1$. Satisfied by Ising ($c = 1/2$), not by dimers ($c = 1$).

Now this becomes pure representation theory of algebras, no physical intuition anymore. The main hero to solve this problem is the Kac determinant, it is discussed at length in CFT books,

let's just skip to the final result. The class of models that satisfy all four requirements is called the class of "unitary minimal models" $\mathcal{M}_m$. They are labeled by an integer $m \in \mathbb{N}, m \geq 3$ and have central charge

$$c = 1 - \frac{6}{m(m+1)}. \tag{40}$$

For a given value of m there are $m(m-1)/2$ possible towers with table of exponents (eigenvalues of L_0 on the "bottom of the tower state $|h_{rs}\rangle$")

$$h_{rs} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \quad , \quad 1 \leq r \leq m-1 \quad , \quad 1 \leq s \leq r. \tag{41}$$

called the Kac table. Ising is $m = 3$, and the exponents are $h_{11} = 0$, $h_{21} = 1/2$, $h_{22} = 1/16$. There is a correspondence between these exponents and a set of fields, called primary fields. I do not explain this here, but it implies that all critical exponents have to be in the table. For example the celebrated exponent for spin-spin correlations in Ising is $\Delta = 2 \times \frac{1}{16} = \frac{1}{8}$. There are underlying stat mech models for each unitary minimal model. For example the 3-state Potts model (Ising is 2-state Potts) has a critical point, which corresponds to $m = 5$ ($c = 4/5$). All critical exponents for this model are in the table (41) for $m = 5$, just multiply h_{pq} by 2. These statements have been checked numerically to high precision in countless models. This example is a highly nontrivial prediction of conformal field theory, since the critical point of this model as remained unaccessible to rigorous methods so far in three-state Potts.

For a central charge c, the multiplicity growth is given by $\log P_c(M) = \pi\sqrt{\frac{2cM}{3}} + O(\log M)$ in any of the towers. This is called the Cardy formula.

In the limit $c \to 1^-$, the number of allowed exponents/towers diverges. This is consistent with the observation of continuously varying exponents, which have been established in e.g the six vertex model and interacting dimer models.

What I did not talk about—

- There is much interesting physics/mathematics obtained by relaxing the unitarity constraint. Examples include percolation, loop models in general, Liouville, etc. The transfer matrix can also be sometimes non diagonalizable, and representation theory can become quite difficult.

- The Ising and dimer models I discussed are exactly solvable, using the mapping to free fermions that I discussed here. Those are a special case of a general family of models called (Yang-Baxter) integrable models. Computations in non free fermions integrable model (often referred to as interacting integrable models) are often much more involved.

- I did not explain why the Hamiltonian (with energy set to zero) on the strip is L_0. Actually, typical CFT arguments tell us that $H = \frac{\pi v_F}{N}\left(L_0 - \frac{c}{24}\right)$, so also predict the $1/N$ part of the ground state energy to be $\frac{\pi v_F}{N}\left(h - \frac{c}{24}\right)$. On a cylinder, there are two Fermi points (beaches) due to periodic boundary conditions, and the Hamiltonian is given by $\frac{\pi v_F}{N}\left(L_0 + \bar{L}_0 - \frac{c}{6}\right)$. One needs two copies of the Virasoro algebra in that case. Usually, one needs only one copy when studying systems defined in a simply connected domain with "boundaries", and this is dubbed chiral CFT or boundary CFT in physics literature.

- Using our algebraic objects, it is possible to define fields, roughly speaking generating functions for the modes of the Lie algebra. For example, the stress tensor is defined as $T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}$. The CFT point of view takes those fields as fundamental objects, and asks for consistency conditions, "operator product expansion", transformation under

conformal maps, conformal Ward identities, etc. There is also a distinction between primary operators and others, depending on how they transform under conformal maps. This is an important part of the story, since the fields themselves have interpretations as local observables in the statmech model. For the free boson CFT one can also define the free chiral field $\varphi(z) = i\sum_{n\neq 0}\frac{a_n}{nz^n} + q - ip\log z$, where the a_n are the bosonic modes from before, and $[q,p] = i$, with vacuum expectation value $\langle\varphi(z)\varphi(w)\rangle = -\log(z-w)$. Using this one, one can define vertex operators $V_\alpha(z) =: e^{i\alpha\varphi(z)} :$ with vacuum expectation value $\langle V_\alpha(z)V_\alpha(w)\rangle = (z-w)^{-\alpha^2}$.

- I did not discuss the computation of correlation functions, and how all those can be obtained from unitary minimal models and free boson CFT.
- Boundary conditions have to be discussed also of course.
- I only addressed the question of which towers can appear, not which ones do appear. This turns out to be related to the previous point.
- This topic is obviously related to SLE, which in particular provides a rigorous and more probabilistic perspective on conformal invariance.
- CFT is still an ongoing subject for research ...