Emptiness formation probability, Toeplitz determinants and conformal field theory

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Seminaire de physique mathematique — Saclay March 2013

JMS, arXiv:1303.5499

see also JMS and JD, arXiv:1303.3633

Outline

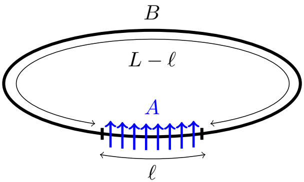
What is it?

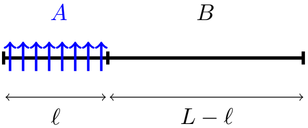
Take some spin-1/2 chain, e.g.

$$H = \sum_j \left(J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z \right) + h \sum_j \sigma_j^z$$

The ground-state is $|\psi\rangle$. Look at a subsystem A, with RDM $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$. The emptiness formation probability (EFP) is

$$\mathcal{P} = \mathrm{proba}(\text{all spins in } A \text{ are } \uparrow) = \langle \uparrow\uparrow \ldots \uparrow\uparrow | \rho_A | \uparrow\uparrow \ldots \uparrow\uparrow \rangle$$





$$\mathcal{E} = -\log \mathcal{P}$$

Physical motivations

- Introduced in the context of integrable systems [Korepin Izergin Essler & Uglov, Phys. Lett. A 1994], . . .

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- Fermion counting statistics

$$\chi(\lambda) = \left\langle e^{-\lambda \sum_j (1-\sigma_j^z)/2} \right\rangle = \sum_{m\geq 0} p_m e^{-\lambda m}$$

$\langle m^2 \rangle_c$ in principle accessible through quantum noise measurement [Cherng & Demler, NJP 2007]

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- Order parameter statistics $X(\lambda) = \left\langle e^{-\lambda \int dx\, \mathcal{O}(x)} \right\rangle$ [Lamacraft & Fendley, PRL 2008]

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- Quantum quenches, a la [Antal et al, PRE 1999]

This talk

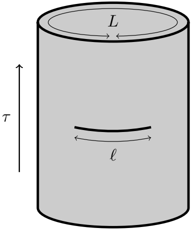
- Study of the EFP using field-theoretical techniques

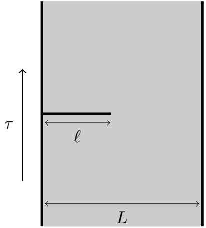
- Influence of the boundary conditions

- Conserved number of particles ($U(1)$), or not.

- Exact results in the free fermions limit (Toepliz determinants)

Magnetization string as a conformal boundary condition





$$\mathcal{P}_p = \frac{\mathcal{Z}_{\rm cyl}^{\rm (slit)}}{\mathcal{Z}_{\rm cyl}}$$

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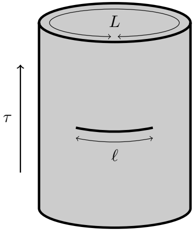

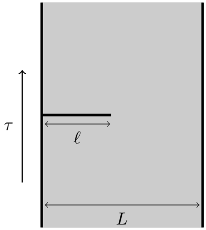

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Scaling in logarithmic form

$$\mathcal{E} = a_1 \times \ell + b \times \log(L\, f(\ell/L)) + a_0 + \dots$$

$b = \frac{c}{24} \sum_\alpha \left(\frac{\theta_\alpha}{\pi} - \frac{\pi}{\theta_\alpha} \right)$ due to sharp corners [Cardy & Peschel, NPB 1988]

Universality & semi-universality in the free energy ($\theta = 2\pi$)

[JMS and Jérôme Dubail, arXiv:1303.3633]

$$F = a_2 L^2 + a_1 L + b_0 \log L + a_0 + b_{-1} \frac{\log L}{L} + \frac{a_{-1}}{L} + o(1/L)$$

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What about short-length cutoffs? Make the substitution

$$L \to L + \epsilon$$

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- Semi-universal: invariant under $L \to L + \epsilon$
- Universal: invariant under $L \to aL + \epsilon$

$$b_{-1} = \xi \times (\text{universal term})$$

Where does the semi-universal term come from?

Boundary perturbation by the stress-tensor

$$S_{CFT} \longrightarrow S_{CFT} + \frac{\xi}{2\pi} \int_{\text{slit}} dx \, \langle T_{xx}(x) \rangle$$

ξ is the extrapolation length [Sorensen Chang Laflorencie & Affleck, JSM 2006] , [Dubail Read & Rezayi, PRB 2012]

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$$\Delta F = \frac{\xi}{\pi} \int_{\epsilon}^{\ell-\epsilon} dw \, \langle T(w) \rangle$$

The stress-tensor behaves as

$$\langle T(w) \rangle = \frac{\pi \, c}{12 \, \theta} \times \frac{g_{\text{geom}}}{L} \times \frac{1}{w^{2-2\pi/\theta}} + \ldots$$

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When $\theta = 2\pi$, $\langle T(w) \rangle \propto \frac{1}{w} + \dots$, and $\boxed{\Delta F \propto L^{-1} \log(\ell/\epsilon)}$

Scaling of the logarithmic EFP

$$\mathcal{E}_p \sim a_1\ell + \frac{c}{8}\log\left[\frac{L}{\pi}\sin\left(\frac{\pi\ell}{L}\right)\right] + a_0^{(p)} - \frac{\xi c}{8}\cot\left(\frac{\pi\ell}{L}\right)\frac{\log\ell}{L},$$

$$\mathcal{E}_o \sim a_1\ell - \frac{c}{16}\log\left[\frac{4L}{\pi}\frac{\tan^2\left(\frac{\pi\ell}{2L}\right)}{\sin\left(\frac{\pi\ell}{L}\right)}\right] + a_0^{(o)} + \frac{\xi c}{32}\times\frac{2-\cos\left(\frac{\pi\ell}{L}\right)}{\sin\left(\frac{\pi\ell}{L}\right)}\frac{\log\ell}{L}$$

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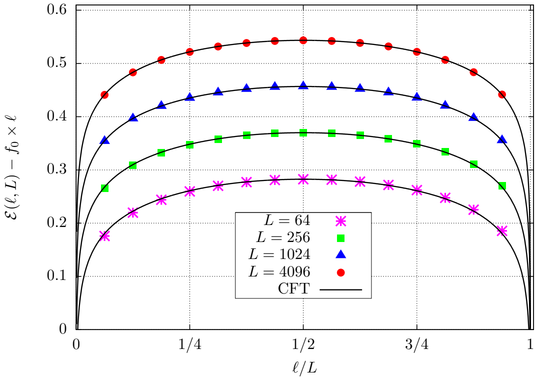

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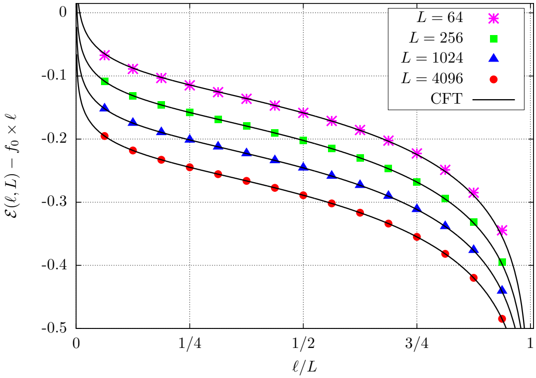
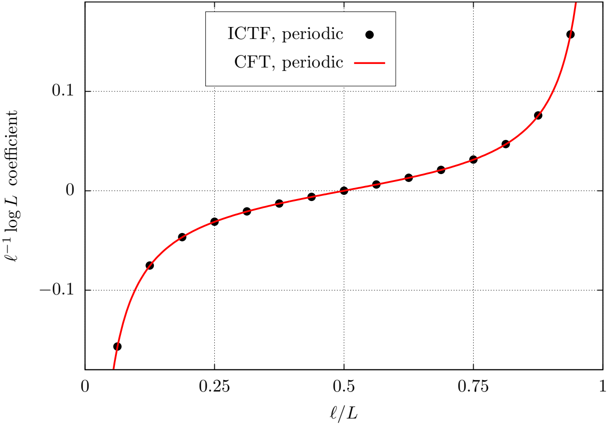


Logarithmic correction

Ising chain $H = -\sum_j \sigma_j^x \sigma_{j+1}^x - h \sum_j \sigma_j^z$.



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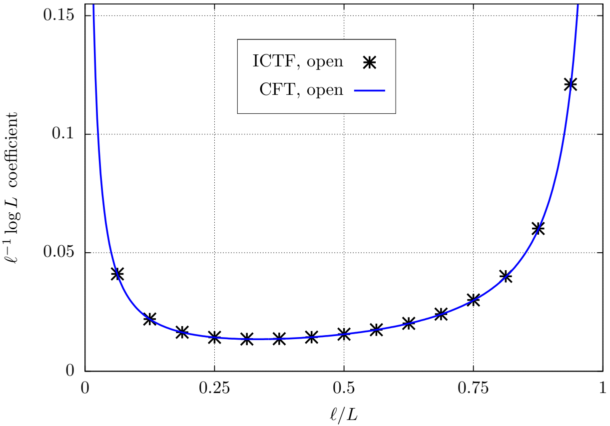

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Full counting statistics

$$\chi_\ell(\lambda) = \left\langle e^{-\lambda \sum_{j=1}^{z}(1-\sigma_j^z)/2} \right\rangle \to \left\langle e^{-i\lambda \int dx\, \psi(x)\overline{\psi}(x)} \right\rangle$$

$\psi(x)$ is a Majorana fermion field. The system is still cut into two, but the effect of $\psi(x)\overline{\psi}(x)$ is exactly marginal

[Oshikawa & Affleck, NBP 1997], [Fendley Fisher & Nayak, Ann. Phys 2009]

Result:

$$\begin{aligned} c &= \frac{1}{2} \to c_{\rm eff} = \frac{8}{\pi^2}\arctan^2\left[\tanh\frac{\lambda}{2}\right] \\ \xi &= \frac{1}{2} \to ? \quad \text{(Non universal)} \end{aligned}$$

1. Emptiness formation probability
   - Definition, physical motivations
   - Magnetization string as a conformal boundary condition
   - Universal and semi-universal terms

2. Connection with the theory of Toeplitz determinants
   - Known results: Onsager, Szegő and Fisher-Hartwig
   - Application to our problem

3. Non conformal case, and arctic circle
   - Imaginary time behavior
   - What about quantum quenches?

For the Ising chain

$$A_k = \frac{\delta_{k0}}{2} + \frac{1}{2L} \csc \left[\frac{\pi(k+1/2)}{L} \right]$$

The EFP is given by

$$\mathcal{P}_p = \det_{1 \leq i,j \leq \ell} \left(A_{i-j} \right) \tag{1}$$

$$\mathcal{P}_o = \det_{1 \leq i,j \leq \ell} \left(A_{i-j} + A_{i+j-1} \right) \tag{2}$$

Take the limit $L \to \infty$. This gives $A_k = \dfrac{\delta_{k0}}{2} + \dfrac{1}{2\pi(k+1/2)}$

- $\mathcal{P}_p$ is a Toeplitz determinant.
- $\mathcal{P}_o$ is a Toeplitz+Hankel determinant.

Fun with Toeplitz

$$
\begin{aligned}
D_N &= \det T_N(g) \\
&= \det_{1\le i,j\le N} \left(g_{i-j}\right) \\
&= \det_{1\le i,j\le N} \begin{pmatrix}
g_0 & g_1 & g_2 & g_3 & \cdots & g_{N-1} \\
g_{-1} & g_0 & g_1 & g_2 & \cdots & g_{N-2} \\
g_{-2} & g_{-1} & g_0 & g_1 & \cdots & g_{N-3} \\
g_{-3} & g_{-2} & g_{-1} & g_0 & \cdots & g_{N-4} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
g_{1-N} & g_{2-N} & g_{3-N} & g_{4-N} & \cdots & g_0
\end{pmatrix}
\end{aligned}
$$

Onsager-Ising spontaneous magnetization problem, monomer correlators, full counting statistics, entanglement entropy, Random matrix theory, ...

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\end{aligned}$$

Question: asymptotic behavior of $\det T_N(g)$ when $N \to \infty$.

Fun with Toeplitz (II)

First intuition: write down T_N in Fourier space. Introduce

$$g(\theta) = \sum_{k\in\mathbb{Z}} g_k e^{ik\theta} \quad , \quad g_k = [g]_k = \frac{1}{2\pi}\int_0^{2\pi} g(\theta) e^{-ik\theta}\, d\theta$$

$g(\theta)$ is called the symbol of the Toeplitz determinant. We have

$$U^\dagger T_N U \simeq \operatorname{diag}\left(g(0), g(\frac{2\pi}{N}), \ldots, g(2\pi - \frac{2\pi}{N})\right) \quad , \quad U_{jl} = e^{2\pi i j l/N}$$

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$$\det T_N(g) \approx \prod_{k=0}^{N-1} g(\frac{2k\pi}{N}) \approx \exp\left(\frac{N}{2\pi}\int_0^{2\pi} \log g(\theta)\,d\theta\right)$$

Toeplitz (III)

The Szegő strong limit theorem (SSLT)

Provided the symbol is sufficiently smooth, we have

$$\det T_N \sim \exp\left(\frac{N}{2\pi}\left[\log g\right]_0 + \sum_{k=1}^{\infty} k \left[\log g\right]_k \left[\log g\right]_{-k}\right)$$

$\log g$ needs to be well-defined $\Rightarrow$ g has winding number 0.

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Example: spontaneous magnetization in Ising (Kaufman-Onsager)

$$g(\theta) = \left(\frac{1-K_\text{o} e^{-i\theta}}{1-K_\text{o} e^{i\theta}}\right)^{1/2} \quad , \quad K_\text{o} = \left(\sinh[2\beta J_1] \sinh[2\beta J_2]\right)^{-1}$$

$$M = \left(1 - K_\text{o}^2\right)^{1/8} \sim (T - T_c)^{1/8}$$

Toeplitz (IV): Coulomb-Gas interpretation

$$D_N = \frac{1}{N!}\int \frac{d\theta_1}{2\pi} g(\theta_1) \int \frac{d\theta_2}{2\pi} g(\theta_2) \ldots \int \frac{d\theta_N}{2\pi} g(\theta_N) \prod_{j<k} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2$$

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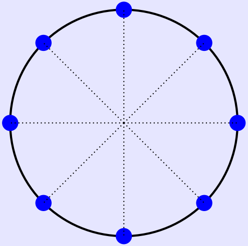
D_N is the partition function for a 2d Coulomb gas in an external potential $V(\theta) = -\log g(\theta)$:

$$D_N = \frac{1}{N!} \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \dots \frac{d\theta_N}{2\pi} \, e^{-E(\theta_1, \theta_2, \dots, \theta_N)}$$

$$E(\theta_1, \theta_2, \dots, \theta_N) = -2 \sum_{j<k} \log \left| e^{i\theta_j} - e^{i\theta_k} \right| + \sum_j V(\theta_j)$$

Toeplitz (V): Coulomb-Gas interpretation

Equilibrium positions, assuming $V(\theta) = 0$



$$\theta_j^{(0)} = \frac{2j\pi}{N}$$

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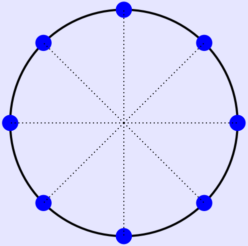
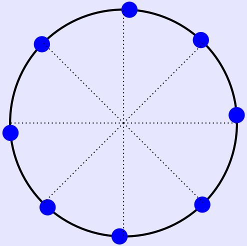


$$\theta_j^{(0)} = \frac{2j\pi}{N}$$

$$\begin{aligned} D_N &= (2\pi)^{-N} \int d\theta_1 \ldots d\theta_N \exp\left(-E_0(\theta_1, \ldots, \theta_N) + \textstyle\sum_j V(\theta_j)\right) \\ &\sim \exp\left(\textstyle\sum_j V(2j\pi/N)\right) \\ &\sim \exp\left(\frac{N}{2\pi} \int_0^{2\pi} V(\theta) d\theta\right) \end{aligned}$$

Toeplitz (V): Coulomb-Gas interpretation

Equilibrium positions, first order correction $V(\theta \neq 0)$



$$\theta_j = \theta_j^{(0)} + \frac{h(\theta_j^{(0)})}{N}$$

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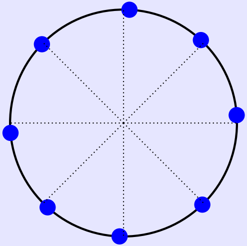


$$\theta_j = \theta_j^{(0)} + \frac{h(\theta_j^{(0)})}{N}$$

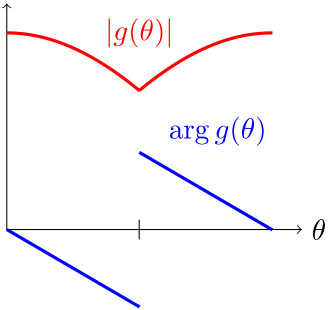
$$\delta E = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \, \frac{|h(\theta) - h(\phi)|^2}{|e^{i\theta} - e^{i\phi}|} + \frac{1}{2\pi}\int_0^{2\pi} d\phi \, V'(\phi) h(\phi)$$

Find $h(\phi)$ that minimizes δE. We get

$$\delta E_{min} = -\sum_{k=1}^{\infty} k([V]_k)^2$$

Fisher-Hartwig singularities (I)

For the EFP, we have: $$g(\theta) = \frac{1}{2} + \frac{1}{2}\text{sign}\,(\cos\theta)e^{-i\theta/2}$$



Singularity in the generating function at $\theta = \pi$.

Fisher-Hartwig singularities (II)

$$g(\theta) = f(\theta) \prod_{r=1}^{R} \exp(i\beta_r \arg[e^{i(\theta-\theta_r)}])$$

$$-\log D_N = \frac{N}{2\pi}[\log f]_0 - \left(\sum_{r=1}^{R} \beta_r^2\right) \log N + o(1)$$

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$$-\log D_N = \frac{N}{2\pi}[\log f]_0 - \left(\sum_{r=1}^{R} \beta_r^2\right) \log N + o(1)$$

- There are proof in several cases: [Basor, Ehrhardt, Böttcher, Tracy, Widom, Deift, Its, Krasovsky,. . .]
- Heuristic way: regularize the SSLT term $\sum_k k[\log g]_k[\log g]_{-k}$

Fisher-Hartwig singularities (III)

Ambiguity in the Fisher-Hartwig representation: to see that, make a shift $\beta_r \to \beta_r + n_r$, with $\sum n_r = 0$.

$$D_N = \det(T_N(g)) \sim (G[f])^N \sideset{}{'}\sum_{\{n_r\}} N^{\omega(\{\beta_r, n_r\})} E[g]$$

provided $f(\theta)$ is smooth [Deift Its & Krasovsky, Ann. Math. 2011]

Conjecture for the general structure [Kozlowki, 2008]

$$D_N = (G[f])^N \sideset{}{'}\sum_{\{n_r\}} N^{\omega(\{\beta_r, n_r\})} E[g, \{\beta_r, n_r\}] \left(1 + \sum_{i=1}^{\infty} \alpha^{(i)}_{\{\beta_r, n_r\}} N^{-i}\right)$$

Application to our problem

Here $\beta_1 = -\frac{1}{4}$, and one recovers the $\frac{c}{8} \log \ell = \frac{1}{16} \log \ell$ for PBC

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Here $\beta_1 = -\frac{1}{4}$, and one recovers the $\frac{c}{8}\log\ell = \frac{1}{16}\log\ell$ for PBC

Where does the $\ell^{-1}\log\ell$ come from?

Conjecture: it comes from the cusp at $\theta = \pi$ for the "regular part" $f(\theta)$ of the symbol. New parametrization:

$$g(z) = f(z)(1+z)^{-\nu(z)}(1+1/z)^{-\bar{\nu}(z)} \quad , \quad z = e^{i\theta}$$

Use $\nu(z)$ and $\bar{\nu}(z)$ to make $f(z)$ smooth:

$$\begin{aligned}\nu(z) &= -\frac{1}{4} - a\times(z+1) - a\times(z+1)^2 + \dots\\ \bar{\nu}(z) &= \frac{1}{4} + a\times(1+1/z) - a\times(1+1/z)^2 + \dots\end{aligned}$$

Application to our problem

Here $\beta_1 = -\frac{1}{4}$, and one recovers the $\frac{c}{8}\log\ell = \frac{1}{16}\log\ell$ for PBC

Following the Riemann-Hilbert analysis of [Kitanine, Kozlowski, Maillet, Slavnov & Terras, Comm. Math. Phys 2009] [Kozlowski, 2008], we get a contribution

$$-2\beta^2 a \times \ell^{-1}\log\ell = -\frac{1}{32\pi}\ell^{-1}\log\ell$$

We recover $\xi = 1/2$ for the LEFP Ising chain. Bonus:
$\xi = \frac{1}{2}\tanh\lambda$ for the full counting statistics generating function.

Remarks

- Better derivation of the $\ell^{-1} \log \ell$
- Toeplitz + Hankel case?
- Finite aspect ratio ℓ / L
- Other subleading corrections

1. Emptiness formation probability
   - Definition, physical motivations
   - Magnetization string as a conformal boundary condition
   - Universal and semi-universal terms

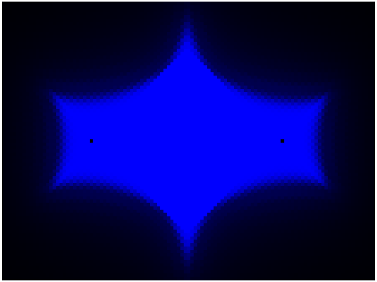
2. Connection with the theory of Toeplitz determinants
   - Known results: Onsager, Szegő and Fisher-Hartwig
   - Application to our problem

3. Non conformal case, and arctic circle
   - Imaginary time behavior
   - What about quantum quenches?

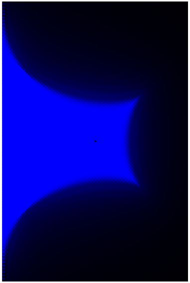
Imaginary time behavior

Now let us look at the imaginary time pictures

Imaginary time behavior



Infinite



Semi-infinite

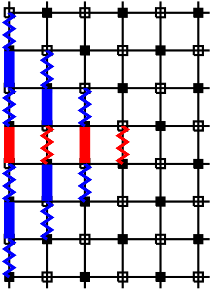
All degrees of freedom are frozen in a region of area $\propto \ell^2$. Hence

$$\mathcal{E} = a_2\ell^2 + a_1 + b\log\ell + o(1)$$

Agrees with exact results [Kitanine Maillet Slavnov & Terras, JPA 2001 – JPA 2002] [Kozlowski, JSM 2008] for the leading term in XXZ.

Imaginary time behavior (II)

- Same phenomenon as for the "arctic circle" in dimer models.



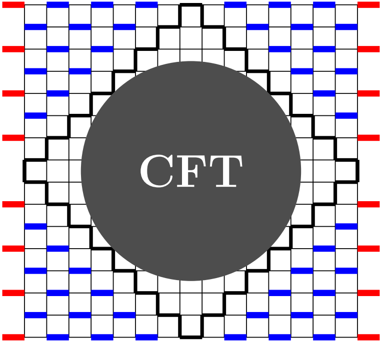
In dimer language, $|\uparrow\uparrow\uparrow\uparrow\rangle = |1010\rangle$.

- Shape is non universal, but the logarithms seem universal
- In the fluctuating region, possibly [work in progress]

$$S = \frac{\kappa_0}{4\pi} \int \kappa(x,\tau) (\nabla\varphi)^2 \, dx d\tau$$

Imaginary time behavior (III)

The arctic circle as an emptiness formation probability



$$Z = \langle \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \,|\, T^{16} \,|\, \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \rangle$$

The Antal quench

Prepare an XX chain in

$$|\psi(0)\rangle = |\ldots \downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow \ldots\rangle$$

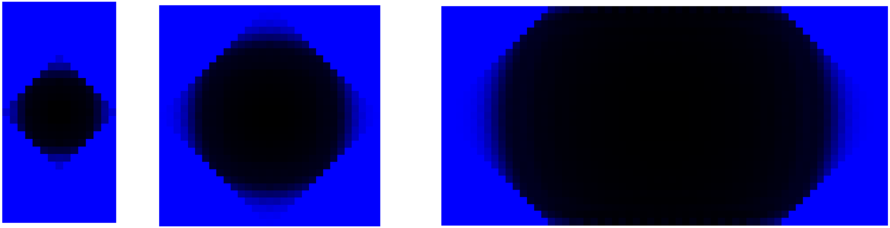
Evolve with the critical Hamiltonian $H = H_{XX}$.
[Antal Racz Rakos Schütz, PRE 1999],...

$$|\psi(t)\rangle = e^{iHt}|\psi(0)\rangle$$

- Magnetization profile
- Stationary behavior
- XXZ [Sebetta & Misguich, in preparation]

Imaginary time Loschmidt echo

$$\mathcal{L}(\tau) = \langle \psi(0) \,|\, e^{-\tau H} \,|\, \psi(0) \rangle$$



$\tau = L/2$ $\quad$ $\tau = L$ $\quad$ $\tau = 2L$

For $\tau < L$, the imaginary time Loschmidt echo grows as

$$\mathcal{L}(\tau) \sim \exp\left(\alpha \tau^2\right)$$

Analytic continuation $\tau \to it$:

$$\boxed{\mathcal{L}(t) \sim \exp\left(-\alpha t^2\right)}$$

Loschmidt echo (exact result)

In terms of fermions,

$$H = \sum_{i,j} t_{ij}\, c_i^\dagger c_j + h.c = \sum_k \epsilon(k) d_k^\dagger d_k$$

- PBC: For $t < L$, $\mathcal{L}(t)$ is a $L \times L$ Toeplitz determinant. The symbol is

$$g(\theta) = e^{-i\epsilon(\theta)}$$

  Apply the Szegő theorem:

$$\mathcal{L}(t) = \exp\left(-\alpha t^2\right) \qquad , \qquad \alpha = \sum_k k[\epsilon]_k^2$$

- OBC: Toeplitz+Hankel. Substitution $\alpha \to \alpha/2$.
- Correction is exponentially small $O(e^{-A_t/L})$.

Conclusion

- $U(1)$ vs non $U(1)$.
- Universal logarithm in the $U(1)$ case.
- Other semi-universal terms?
- Better understanding of the connection with quenches: density profile, guess the field theory outside of the frozen region, ...

Thank you!