Entanglement in simple 2d critical wave functions

Jean-Marie Stéphan[1]

[1]University of Virginia, Charlottesville

Condensed Matter Seminar — Waterloo July 2013

[JMS, Misguich & Pasquier, Phys. Rev. B. (2011)]
[JMS, Ju, Fendley & Melko, New. J. Phys. (2013)]

Outline

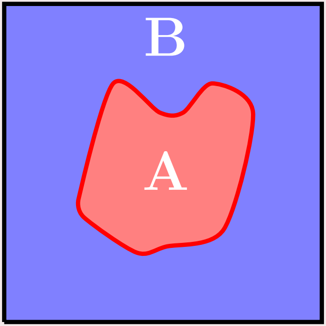
1. Introduction
   - Rényi entanglement entropy
   - The wave functions (Rokhsar-Kivelson)
   - Free gaussian field

2. Universal subleading terms
   - Schmidt decomposition
   - Results
   - Phase transition scenario

3. Some related problems
   - Other universality classes
   - Classical Mutual information

Entanglement entropy (1/2)

Bipartition

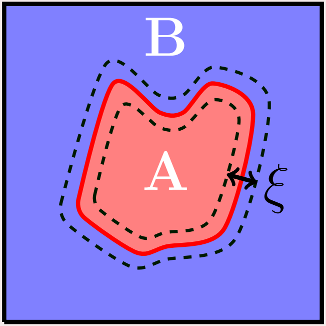


[von Neumann, 1955]

- $|\psi\rangle$ ground state of $H_{A\cup B}$
- $\rho = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$
- $S_n = \frac{1}{1-n} \log\left(\mathrm{Tr}\, \rho^n\right)$

Entanglement entropy (1/2)

Bipartition



[von Neumann, 1955]

- $|\psi\rangle$ ground state of $H_{A\cup B}$
- $\rho = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$
- $S_n = \frac{1}{1-n} \log\left(\mathrm{Tr}\, \rho^n\right)$

Extended quantum system: *Boundary law*

- correlation length ξ, dimension d.
- $S_n(L) = a_n L^{d-1} + o(L^{d-1})$

Entanglement entropy (2/2)

Why studying this quantity?

- How to store efficiently quantum states in a computer?
- Tool to distinguich between subtly different phases of matter.
- Replica trick: Twist, Swap, Switch.

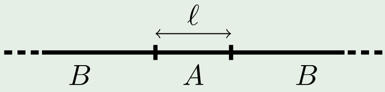
Entanglement entropy (2/2)

Why studying this quantity?

- How to store efficiently quantum states in a computer?
- Tool to distinguich between subtly different phases of matter.
- Replica trick: Twist, Swap, Switch.

Classic results

- $1d$ critical systems: $S \sim \frac{c}{6}\left(1+\frac{1}{n}\right)\log \ell$ [Holzhey et al, NPB 1994 — Vidal et al, PRL 2003 — Calabrese & Cardy, JSM 2004]



Entanglement entropy (2/2)

Why studying this quantity?

- How to store efficiently quantum states in a computer?
- Tool to distinguich between subtly different phases of matter.
- Replica trick: Twist, Swap, Switch.

Classic results

- $1d$ critical systems: $S \sim \frac{c}{6}\left(1+\frac{1}{n}\right)\log \ell$ [Holzhey et al, NPB 1994 — Vidal et al, PRL 2003 — Calabrese & Cardy, JSM 2004]
- Topological order in gapped systems: $S_n = aL + S_{\rm topo} + o(1)$ [Kitaev & Preskill, PRL 2006 — Levin & Wen, PRL 2006]

Entanglement entropy (2/2)

Why studying this quantity?

- How to store efficiently quantum states in a computer?
- Tool to distinguich between subtly different phases of matter.
- Replica trick: Twist, Swap, Switch.

Classic results

- $1d$ critical systems: $S \sim \frac{c}{6}\left(1+\frac{1}{n}\right)\log \ell$ [Holzhey et al, NPB 1994 — Vidal et al, PRL 2003 — Calabrese & Cardy, JSM 2004]
- Topological order in gapped systems: $S_n = aL + S_{\mathrm{topo}} + o(1)$ [Kitaev & Preskill, PRL 2006 — Levin & Wen, PRL 2006]

Issues

- Difficult to compute in dimension $d > 1$
- What about experiments? [Cardy, PRL 2011]

The wave functions

- Take some classical statistical model $Z = \sum_c e^{-\beta E(c)}$
- Construct some Hilbert space $|c\rangle$.
- Orthogonality $\langle c|c'\rangle = \delta_{c,c'}$
- $|\psi\rangle = \frac{1}{\sqrt{Z}} \sum_c e^{-\beta E(c)/2} |c\rangle$

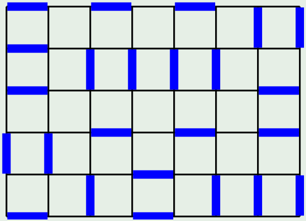
[Rokhsar & Kivelson, PRL 1988] [Henley, J. Phys. Cond. Mat 2004]

The wave functions

- Take some classical statistical model $Z = \sum_c e^{-\beta E(c)}$
- Construct some Hilbert space $|c\rangle$.
- Orthogonality $\langle c|c'\rangle = \delta_{c,c'}$
- $|\psi\rangle = \frac{1}{\sqrt{Z}} \sum_c e^{-\beta E(c)/2} |c\rangle$

[Rokhsar & Kivelson, PRL 1988] [Henley, J. Phys. Cond. Mat 2004]

A possible choice of classical model



Resonating valence bonds (RVB)

[Anderson, Mat. Res. Bull. 1973]

- Nearest neighbors $SU(2)$ RVB. Same wave function, but

$$\left| \textcolor{blue}{\rule{1em}{0.2em}} \right\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$$

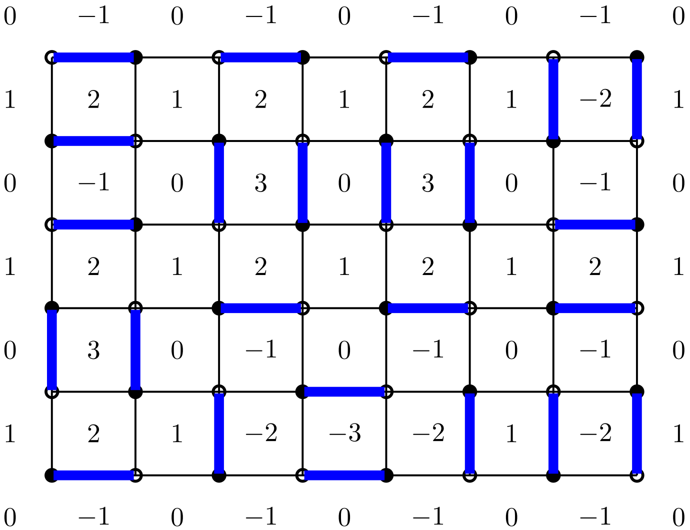
- Valence bond configurations are not orthogonal: $\langle c|c'\rangle \neq \delta_{cc'}$ [Sutherland, PRB 1988]

- Exponentially decaying spin-spin correlations

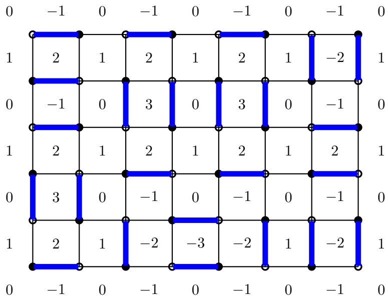
- Algebraic decay of dimer-dimer correlations: $C_{dd}(r) \sim r^{-\alpha}$, with $\alpha \simeq 1.2$ [Albuquerque & Alet, PRB 2010], [Tang, Sandvik & Henley, PRB 2011].

- $\alpha = 2$ for pure dimers [Fisher & Stephenson, PR 1963],

Free field description of dimers (1/2)



Free field description of dimers (1/2)



$$h = h + 4$$
(compactification)

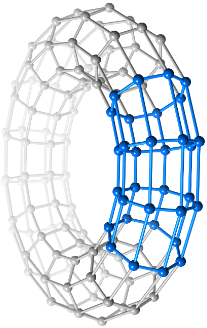
Assume Gaussian fluctuations: $S[h] = \frac{\kappa}{4\pi}\int dx\, dy\, (\nabla h)^2 \;+\; \text{Irrelevant}$

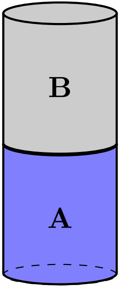
$$\mathcal{Z} = \int [\mathcal{D}h] \exp\left(-S[h]\right)$$

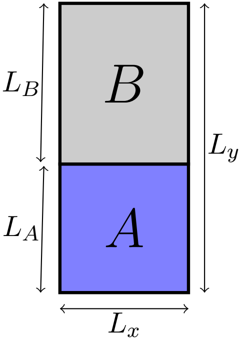
Free field description of dimers (2/2)

- Exponent of dimer-dimer correlations related to the stiffness: $\kappa = \frac{1}{\alpha}$.
- Euclidean version of the Luttinger liquid. $\frac{1}{2\kappa}$ is the Luttinger parameter.
- This is a conformal field theory (CFT) with central charge $c = 1$.
- Works for bipartite lattices: square, honeycomb, etc. Also vertex models.
- RK version: conformal quantum critical points (fined-tuned) [Ardonne, Fendley & Fradkin, Ann. Phys 2004]

Quantum to classical mapping (geometries)





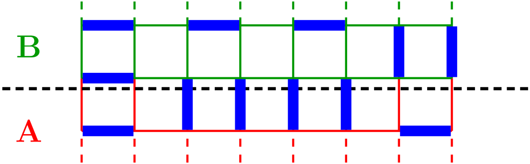


$$\tau = i\frac{L_y}{L_x} \qquad , \qquad y = \frac{L_A}{L_y}$$

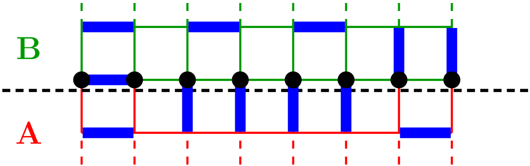
$S_n = aL_x + s_n$ expected (torus/cylinder) [Hsu et al, PRB 2009]

$S_n = aL_x + l_n \log L_x$ expected (strip) [Fradkin & Moore, PRL 2006]

Quantum to classical mapping (Schmidt decomposition)

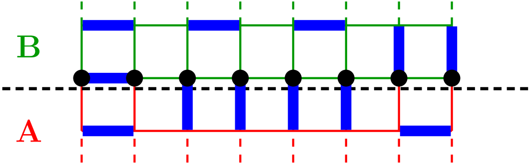


Quantum to classical mapping (Schmidt decomposition)



Boundary configuration $|i\rangle = |\sigma_1, \sigma_2, \ldots \sigma_L\rangle$

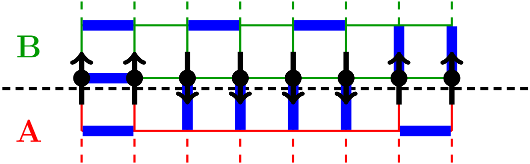
Quantum to classical mapping (Schmidt decomposition)



Boundary configuration $|i\rangle = |\sigma_1, \sigma_2, \ldots \sigma_L\rangle$

Rule: $\sigma = \begin{cases} \downarrow & , \quad \text{site occupied by a dimer in A} \\ \uparrow & , \quad \text{site occupied by a dimer in B} \end{cases}$

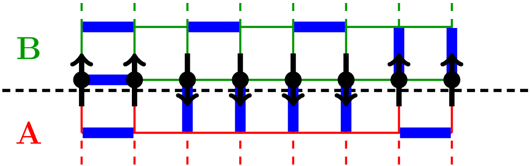
Quantum to classical mapping (Schmidt decomposition)



Boundary configuration $|i\rangle = |\sigma_1, \sigma_2, \dots \sigma_L\rangle$

Rule: $\sigma = \begin{cases} \downarrow & , \quad \text{site occupied by a dimer in A} \\ \uparrow & , \quad \text{site occupied by a dimer in B} \end{cases}$

Quantum to classical mapping (Schmidt decomposition)



Boundary configuration $|i\rangle = |\sigma_1, \sigma_2, \ldots \sigma_L\rangle$

Rule: $\sigma = \begin{cases} \downarrow & , \quad \text{site occupied by a dimer in A} \\ \uparrow & , \quad \text{site occupied by a dimer in B} \end{cases}$

$$|\psi\rangle = \sum_i \sqrt{p_i} |\psi_A^i\rangle |\psi_B^i\rangle \qquad p_i = \frac{Z_A^i Z_B^i}{Z} \quad \text{(classical probabilities)}$$

$$\boxed{S_n = \frac{1}{1-n} \log \left(\sum_i p_i^n \right)}$$

Works because

[Furukawa & Misguich, PRB 2007]

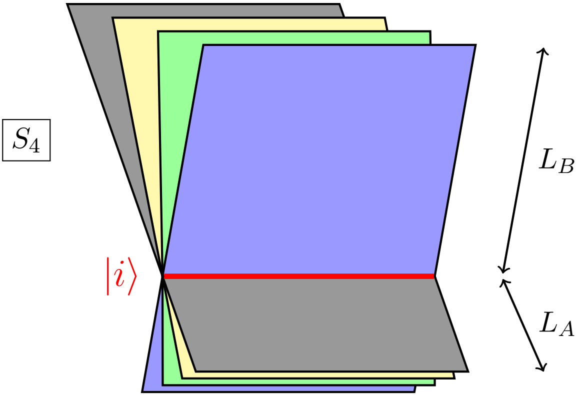
[JMS, Furukawa, Misguich & Pasquier, PRB 2009]

- Orthogonality of dimer configurations: $\langle c|c'\rangle = \delta_{cc'}$

- Hardcore constraints

- Interactions only between bond variables sharing a common site

Replica trick and classical book picture



$$S_n = \frac{1}{1-n} \log \left[\frac{\mathcal{Z}_{\rm book}}{\left(\mathcal{Z}_{\rm sheet}\right)^n} \right]$$

Infinite cylinder limit

$$\mathrm{Tr}\,\rho^n = \sum_{\phi} p(\phi)^n$$

$$p(\phi)^n \propto \exp(-S_\kappa(\phi))^n = \exp(-nS_\kappa(\phi)) = \exp(-S_{n\kappa}(\phi))$$

$$\boxed{p_\kappa(\phi)^n \propto p_{n\kappa}(\phi)}$$

Infinite cylinder limit

$$\mathrm{Tr}\,\rho^n = \sum_\phi p(\phi)^n$$

$$p(\phi)^n \propto \exp(-S_\kappa(\phi))^n = \exp(-nS_\kappa(\phi)) = \exp(-S_{n\kappa}(\phi))$$

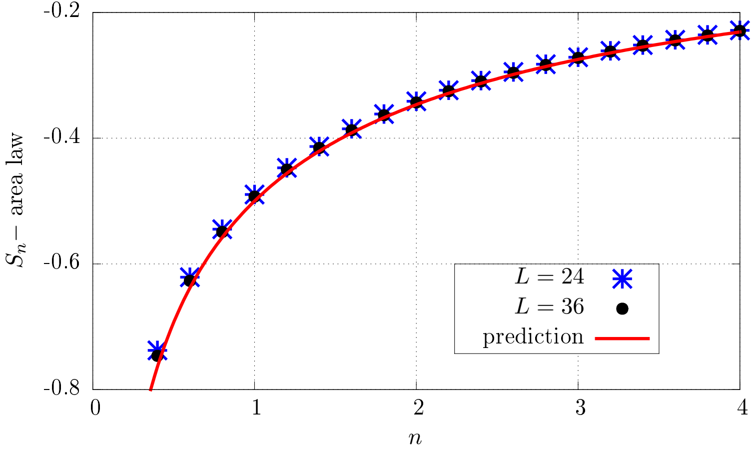
$$\boxed{p_\kappa(\phi)^n \propto p_{n\kappa}(\phi)}$$

Close to the boundary, the stiffness is modified to $\kappa \longrightarrow n\kappa$. We get:

$$s_n = \frac{1}{1-n}\left[\log\left(\frac{\mathcal{Z}_{n\kappa}}{Z^D_{n\kappa}}\right) - n\log\left(\frac{\mathcal{Z}_\kappa}{\mathcal{Z}^D_\kappa}\right)\right]$$

Infinite cylinder limit

$$s_n = \log \frac{2}{\alpha} - \frac{1}{2} \frac{\log n}{n-1}$$



Finite cylinder

$$\boxed{s_n(y) = \log\left[\frac{\eta(2y\tau)\eta(2(1-y)\tau)}{\Theta\left(\alpha\tau M_{2n-1}[y]\right)}\right] + \mathrm{cst}(n,\alpha,\tau)}$$

η is the Dedekind Eta function

$$\eta(\tau) = \exp\left(\frac{i\pi\tau}{12}\right)\prod_{k=1}^{\infty}\left(1-\exp\left(2i\pi k\tau\right)\right)$$

Θ is the Riemann Theta function:

$$\Theta(F_N) = \sum_{\mathbf{k}\in\mathbb{Z}^N}\exp\left(i\pi\sum_{j,l}k_j F_{jl} k_l\right)$$

Finite cylinder

$$\boxed{s_n(y) = \log\left[\frac{\eta(2y\tau)\eta(2(1-y)\tau)}{\Theta\left(\alpha\tau M_{2n-1}[y]\right)}\right] + \mathrm{cst}(n,\alpha,\tau)}$$

The matrix takes a simple form. For example for $n = 2$ we have

$$M_3(y) = \begin{pmatrix} 1 & y & y \\ y & 1 & y \\ y & y & 2y \end{pmatrix}$$

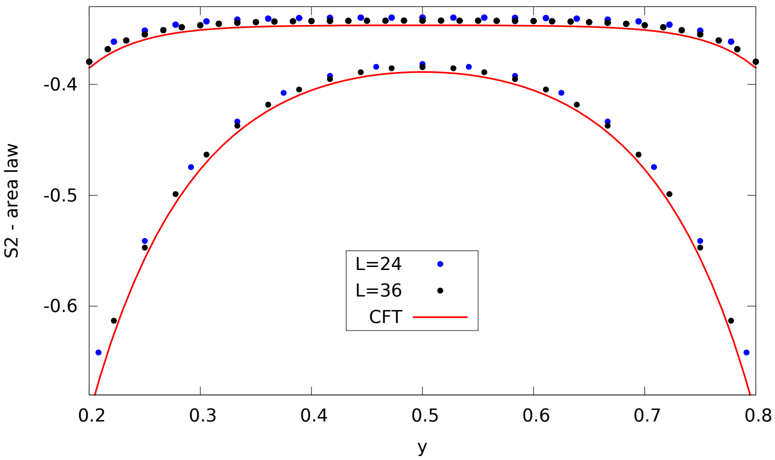
slight generalization of

[Oshikawa, arXiv:1007.3739]

[Hsu & Fradkin, JSM (2010)]

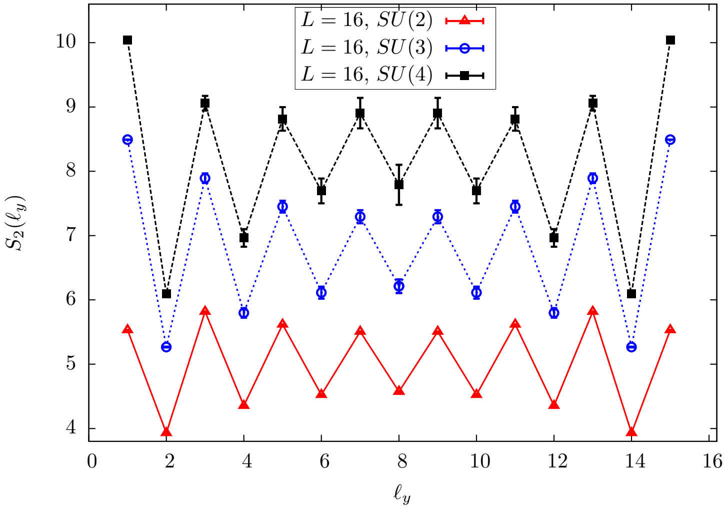
[JMS Furukawa Misguich & Pasquier, PRB (2009)]

Numerical checks, example of the honeycomb lattice



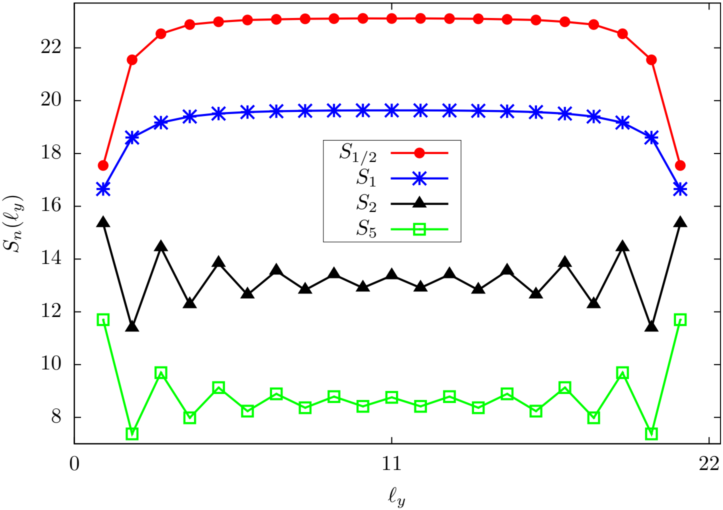
Is this so simple?

[Ju, Kallin, Fendley, Hastings & Melko, PRB 2012]



Is this so simple?

[JMS, Ju, Fendley & Melko, NJP 2013]



Phase transition (1/2)

- Vertex operators in the action (d integer)

$$V_d = \cos\left(\frac{\pi d}{2} h\right)$$

- Irrelevant if $d^2 > 2\kappa$. Otherwise locks the field to a flat configuration with degeneracy d. [Coleman, PRB 1975]

Phase transition (1/2)

- Vertex operators in the action (d integer)

$$V_d = \cos\left(\frac{\pi d}{2} h\right)$$

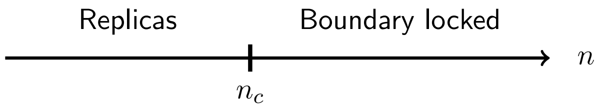
- Irrelevant if $d^2 > 2\kappa$. Otherwise locks the field to a flat configuration with degeneracy d. [Coleman, PRB 1975]

However, $\kappa \to n\kappa$ near the boundary in the book.

$$\Rightarrow \text{Phase transition at } n_c = d^2/(2\kappa)$$

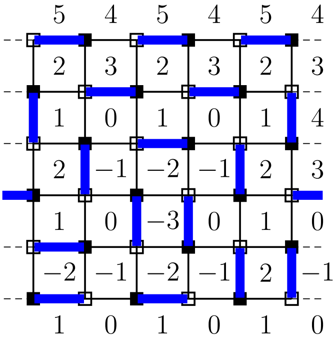
Phase transition (2/2)

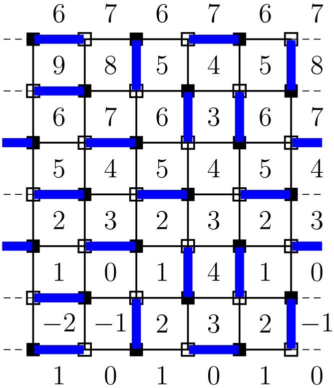
- Square lattice $d = 1$, $n_c = 1$
- Honeycomb lattice $d = 3, n_c = 9$.
- Unclear what is n_c for square lattice RVB, but $n_c < 2$.



In the locked phase, we have $2n$ "half-sheets".

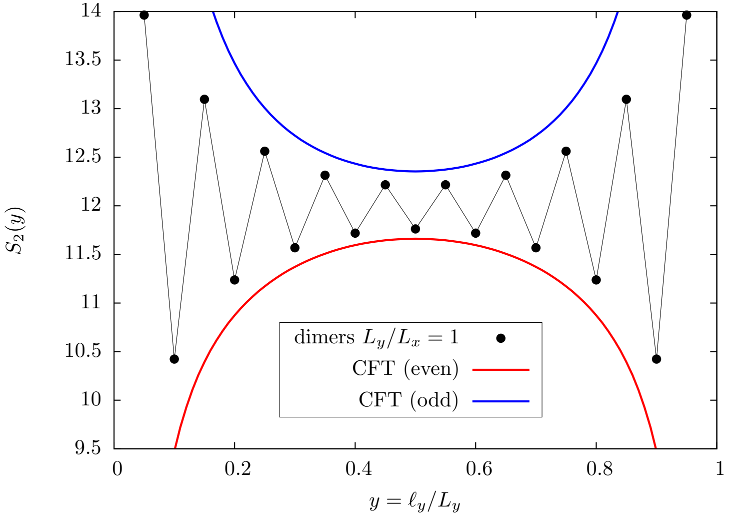
$$s_n = \frac{n}{1-n} \log \left(\frac{\mathcal{Z}(L_A)\mathcal{Z}(L_B)}{\mathcal{Z}(L_A + L_B)} \right)$$



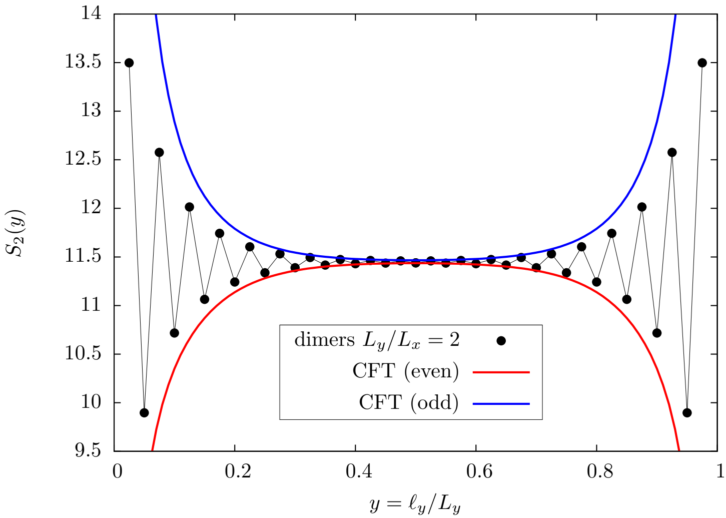


Subtleties in the boundary conditions

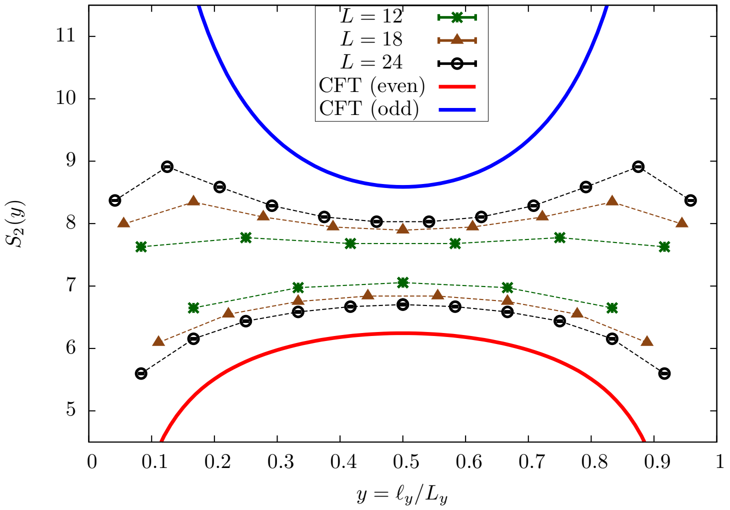
Some checks


$S_2(y)$
14
13.5
13
12.5
12
11.5
11
10.5
10
9.5
0
0.2
0.4
0.6
0.8
1
$y = \ell_y/L_y$
dimers $L_y/L_x = 1$
CFT (even)
CFT (odd)

Some checks


14
13.5
13
12.5
12
11.5
11
10.5
10
9.5
0
0.2
0.4
0.6
0.8
1
$S_2(y)$
$y = \ell_y/L_y$
dimers $L_y/L_x = 2$
CFT (even)
CFT (odd)

Some checks

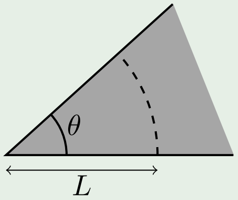


$L = 12$
$L = 18$
$L = 24$
CFT (even)
CFT (odd)
11
10
9
8
7
6
5
$S_2(y)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$y = \ell_y / L_y$

Geometries with corners(1/2)

Why do we expect logarithms?

The Cardy-Peschel formula



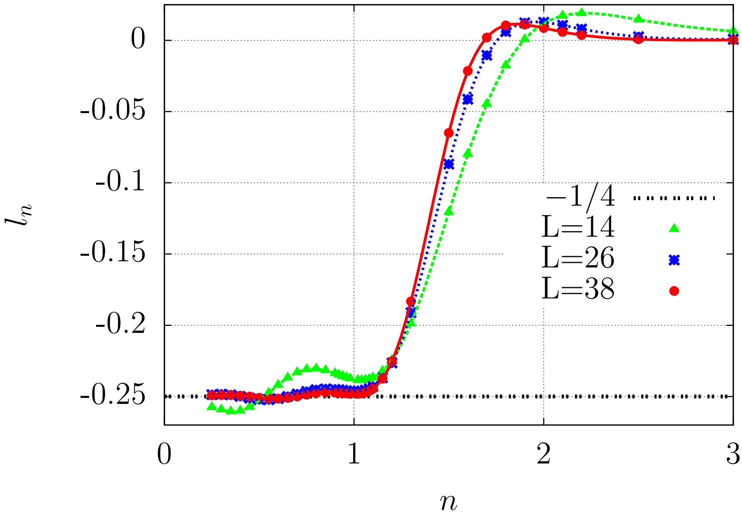
$$\Delta F = \frac{c}{24}\left(\frac{\theta}{\pi} - \frac{\pi}{\theta}\right)\log L$$

[Cardy & Peschel, NPB 1988]

Here

$$l_n = \begin{cases} -\frac{1}{4}\log L & , \quad n < n_c \\ \frac{n}{n-1}\left(\frac{1}{4} - \frac{1}{2\alpha}\right)\log L & , \quad n > n_c \end{cases}$$

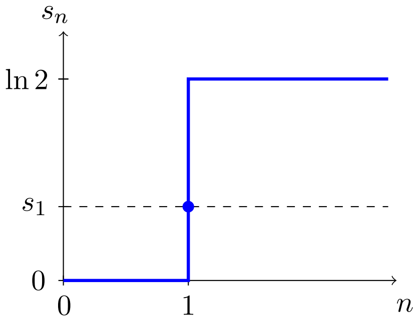
Geometries with corners(2/2)



Other models with critical points (Example of Ising)

[JMS Misguich & Pasquier, PRB 2010]

Transition at $n_c = 1$. $s_1 = 0.254392(1) = ?$

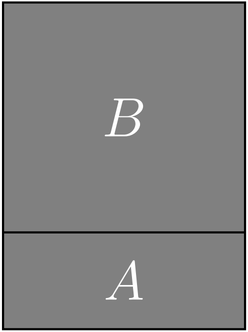


Similar logarithmic terms [Zaletel, Badarson & Moore, PRL 2011]

Classical Mutual Information (1/2)

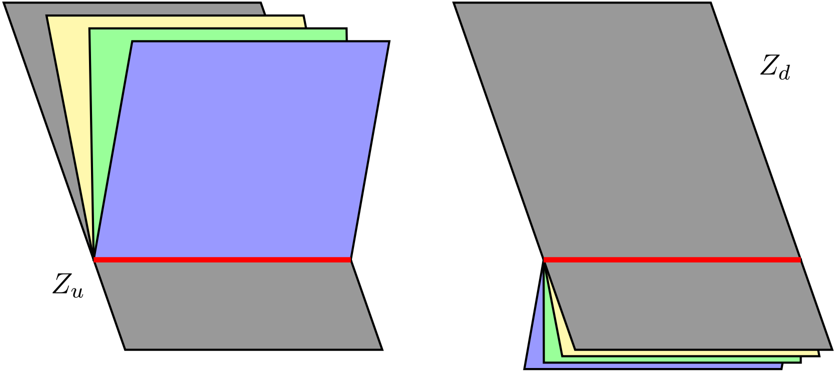
$$S_n^{\rm cl} = \frac{1}{1-n}\log\left(\mathrm{Tr}\,\rho_{\rm cl}^n\right) \qquad , \qquad \rho_{\rm cl} = \sum_{\mathcal{C}} e^{-\beta E(\mathcal{C})}|\mathcal{C}\rangle\langle\mathcal{C}|$$

$$I_n(A,B) = \frac{1}{2}\left(S_n^{\rm cl}(A) + S_n^{\rm cl}(B) - S_n^{\rm cl}(A\cup B)\right)$$



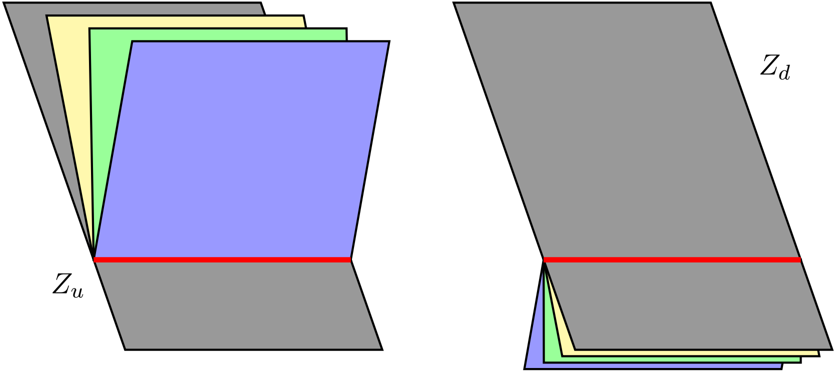
[Iaconis, Inglis, Kallin & Melko, PRB 2013]

- Detects phase transitions
- Monte Carlo



$$I_n = \frac{1/2}{1-n} \log \left[\frac{Z_u Z_d}{(Z_{\text{sheet}})^n} \right]$$

Classical Mutual Information (2/2)



$$I_n = \frac{1/2}{1-n} \log \left[\frac{Z_u Z_d}{\left(Z_{\rm sheet}\right)^n} \right]$$

If $L_A = L_B$, then $\boxed{I_n^{cl} = \frac{1}{2} I_{\frac{n+1}{2}}}$ [Rahmani & Cherng, arXiv:1304.4160]

Conclusion

- Universal terms in the Rényi entropy. Comparison CFT/numerics.

- Phase transitions, rich behavior. Rényi index distinguiches between compteting orders in the wave function.

- What about more realistic 2+1 wave functions? [Inglis & Melko, arXiv:1305.1069]

- Can be applied to $\mathbb{Z}_2$ topological phases (dimers on the triangular) [JMS, Misguich & Pasquier, JSM 2012]