Free fermions at the edge of interacting systems

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New trends in integrable systems
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Based mostly on
[JMS, SciPost Physics 2019]
[JMS, J. Stat. Mech 2017]

Outline

$$H = \int_{\mathbb{R}} dx\, \psi^\dagger(x) \left[-\frac{\partial^2}{\partial x^2} + x^2 \right] \psi(x)$$

$$\{\psi(x)^\dagger, \psi(y)\} = \delta(x-y) \quad , \quad \{\psi(x)^\dagger, \psi(y)^\dagger\} = \{\psi(x), \psi(y)\} = 0$$

- Well known model in a confining potential, relevant for cold atomic systems.
- Exactly solvable, free fermions.
- Single particle states can be obtained in explicit form [Hermite].

$$H = \int_{\mathbb{R}} dx\, \psi^\dagger(x) \left[-\frac{\partial^2}{\partial x^2} + x \right] \psi(x)$$

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Introduce the modes $\chi^\dagger(\lambda)=\int dx\,u(\lambda,x)\,\psi^\dagger(x)$. Then

$$H=\int_{\mathbb{R}} d\lambda\epsilon(\lambda)\chi^\dagger(\lambda)\chi(\lambda)$$

provided

$$\left[-\frac{\partial^2}{\partial x^2}+x\right]u(\lambda,x)=\epsilon(\lambda)u(\lambda,x)$$

The "single particle" wave functions are Airy functions

$$u(\lambda,x)=\mathrm{Ai}(x+\lambda)\quad,\quad\epsilon(\lambda)=-\lambda\quad,\quad\mathrm{Ai}(x)=\int_{\mathbb{R}}\frac{dq}{2\pi}e^{i(qx+q^3/3)}.$$

Remarks

[Pokrovsky & Talapov 1979, Prähofer & Spohn 2000, Spohn 2005]

- The potential does not confine to a given region of space.

- The ground state is a Dirac sea, with propagator

$$\langle \psi^\dagger(x)\psi(y)\rangle = \int_0^\infty d\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$

- Particle number is infinite: on the interval $[-a,\infty)$ it diverges as $\frac{2}{3\pi}a^{3/2}$.

- Full counting statistics (FCS) for charge $Q(x) = \psi^\dagger(x)\psi(x)$

$$\Upsilon_s(\alpha) = \langle e^{\alpha \int_s^\infty dx\, Q(x)}\rangle$$

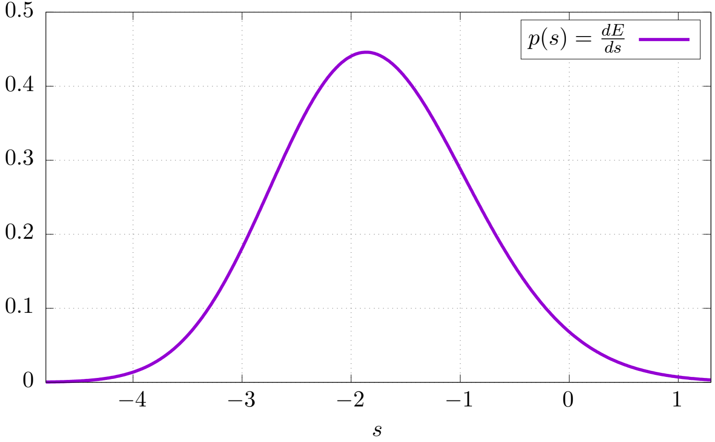
Full counting statistics in the ground state

FCS can be computed exactly

$$
\begin{aligned}
\Upsilon_s(\alpha) &= 1 + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \int_{A^n} dx_1 \dots dx_n \, \langle Q(x_1) \dots Q(x_n) \rangle \\
&\overset{\text{Wick}}{=} 1 + \sum_{n=1}^{\infty} \frac{(e^{\alpha}-1)^n}{n!} \int_{A^n} dx_1 \dots dx_n \, \langle : Q(x_1) \dots Q(x_n) : \rangle \\
&= 1 + \sum_{n=1}^{\infty} \frac{(e^{\alpha}-1)^n}{n!} \int_{A^n} dx_1 \dots dx_n \det_{1 \le i,j \le n} (\langle \psi^{\dagger}(x_i) \psi(x_j) \rangle) \\
&\overset{\text{def}}{=} \det_s (I + (e^{\alpha}-1) G_{\text{Airy}})
\end{aligned}
$$

The quantity $E(s) = \lim_{\alpha \to -\infty} \Upsilon_s(\alpha) = \det_s(I - G_{\text{Airy}})$ is the probability that the interval $A = [s, \infty)$ is empty of fermions.
This is called the Tracy-Widom distribution [Tracy & Widom, 1993].

Tracy-Widom distribution



Back to the harmonic trap: semiclassical solution

$$H = \int_{\mathbb{R}} dx\, \psi^\dagger(x) \left[-\frac{\partial^2}{\partial x^2} + x^2 - \mu \right] \psi(x)$$

Assume separation of scales. Around some point x_0, the single particle ground state looks like the projector onto

$$-\frac{d^2}{(d\delta x)^2} < \mu - x_0^2 \qquad \longrightarrow \qquad k^2 < \mu - x_0^2 = k(x_0)$$

This is a disk in phase (k, x_0) space. Thinking in Fourier, the projection acts as a filter with sine-kernel response function

$$\langle \psi^\dagger(x_0 + \delta x) \psi(x_0 + \delta y) \rangle = \frac{\sin\left[k(x_0)(\delta x - \delta y)\right]}{\pi(\delta x - \delta y)}$$

Connection to random matrices

$$|\varphi(x_1,\ldots,x_N)|^2 \propto \prod_{i<j}(x_i - x_j)^2 e^{-\sum_j x_j^2}$$

which is the joint eigenvalue pdf for GUE. See [Eisler 2013; Calabrese, Majumdar & Le Doussal 2014, Dean, Le Doussal, Majumdar & Schehr 2014, . . .]

Density $\langle \psi^\dagger(x_0)\psi(x_0)\rangle = \frac{k(x_0)}{\pi} = \frac{1}{\pi}\sqrt{\mu - x_0^2}$ [Wigner semicircle law].
Here $\mu = \sqrt{2N}$, where N is particle number.

What about the edge scaling close to $\mu = 2N$? From semiclassics ([Praehoffer & Spohn 2000; Spohn 2006]):

$$k^2 + (\sqrt{\mu} + x)^2 \leq \mu$$

$$\Rightarrow\ k^2 + 2\sqrt{\mu}x \cancel{+ x^2} \leq 0$$

$$-\frac{d^2}{dx^2} + 2\sqrt{\mu}x \leq 0$$

With $x = (4\mu)^{-1/6}y$, we get

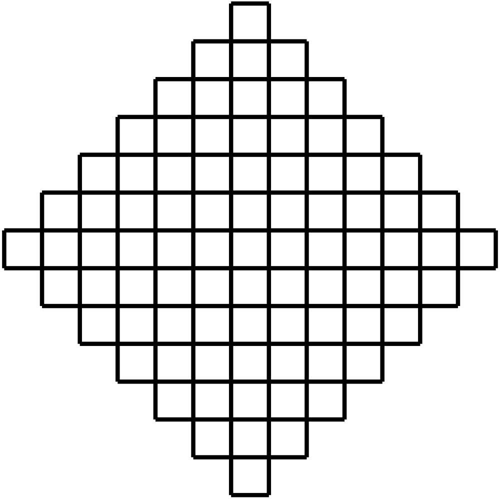
$$-\frac{d^2}{dy^2} + y \leq 0$$

Hence, Airy kernel scaling at the edge, and Tracy-Widom follows.

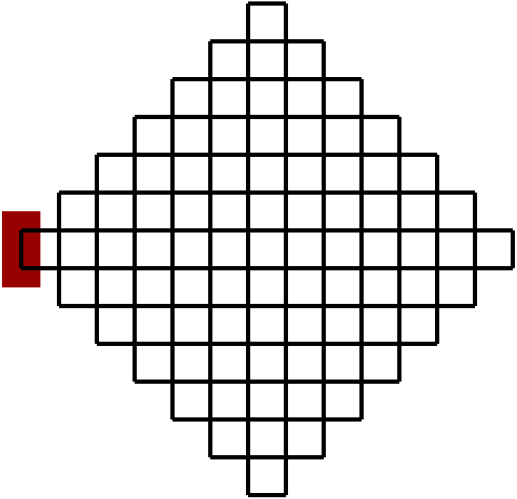
Semiclassically, we go from a disk $k^2 + x^2 \leq \mu$ to a parabolic region $q^2 + y \leq 0$.

Another example: dimers

Dimer coverings on the Aztec diamond



Dimer coverings on the Aztec diamond



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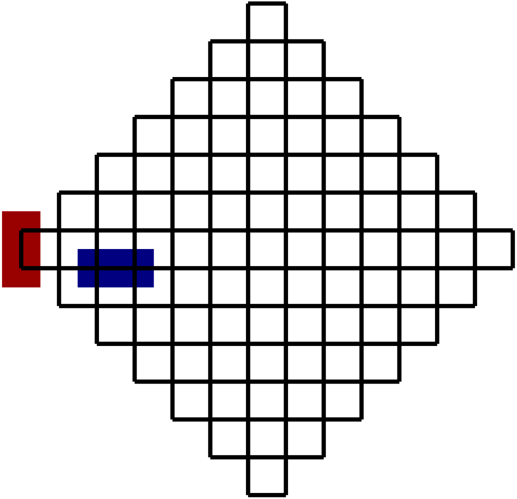

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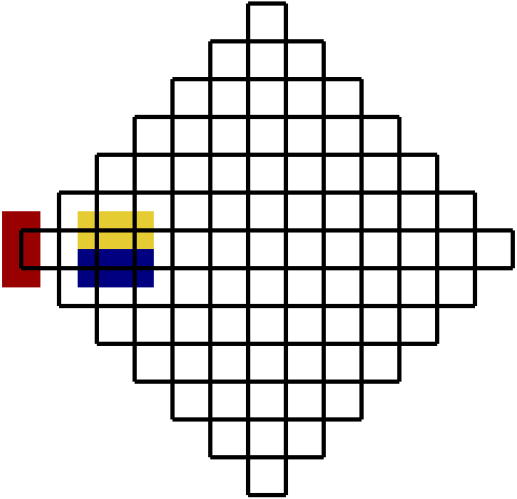

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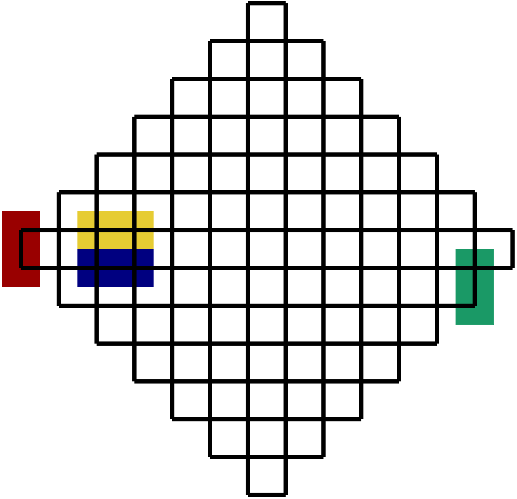

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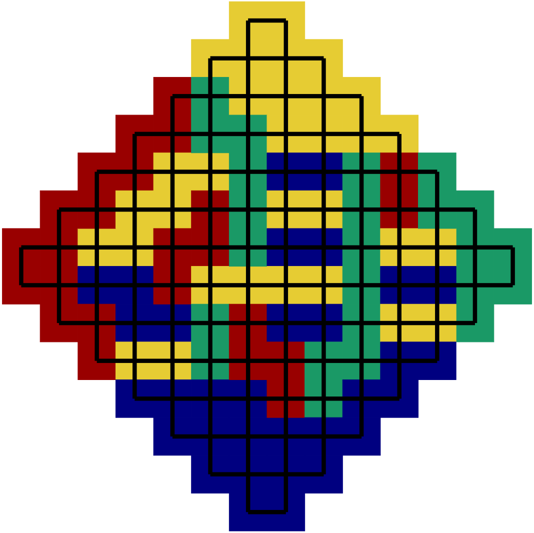
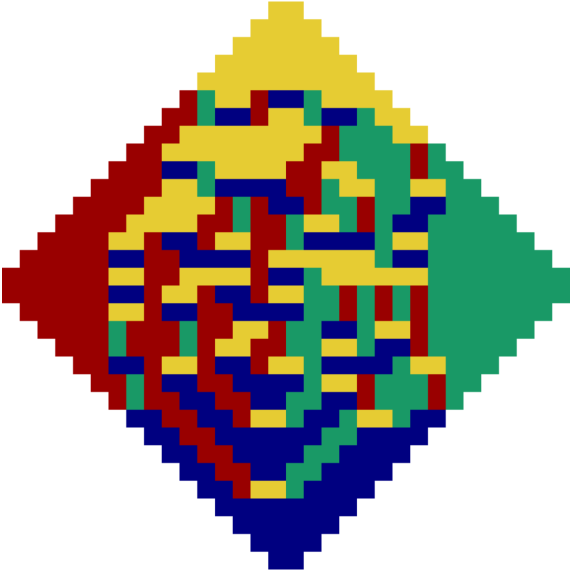
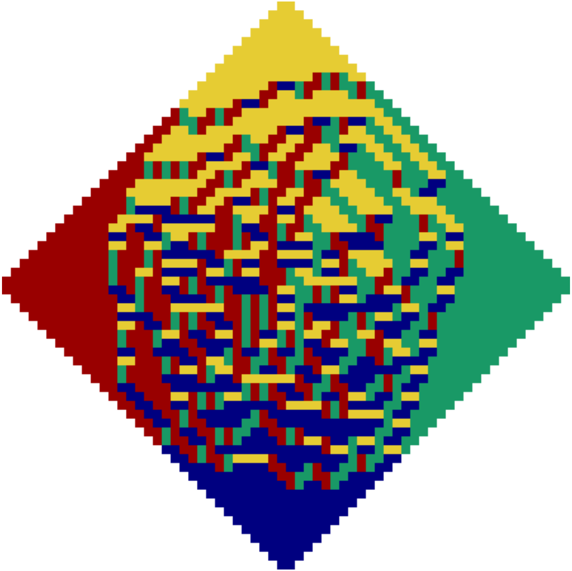
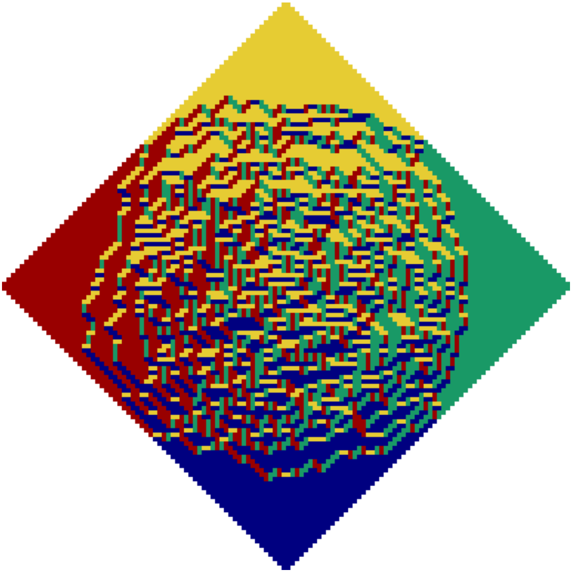
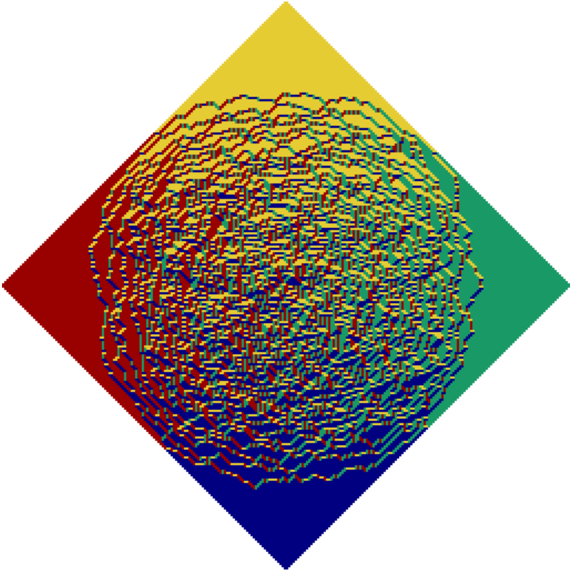
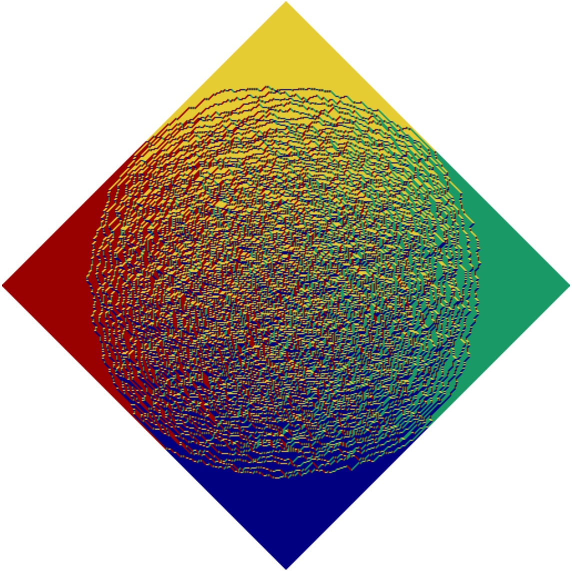


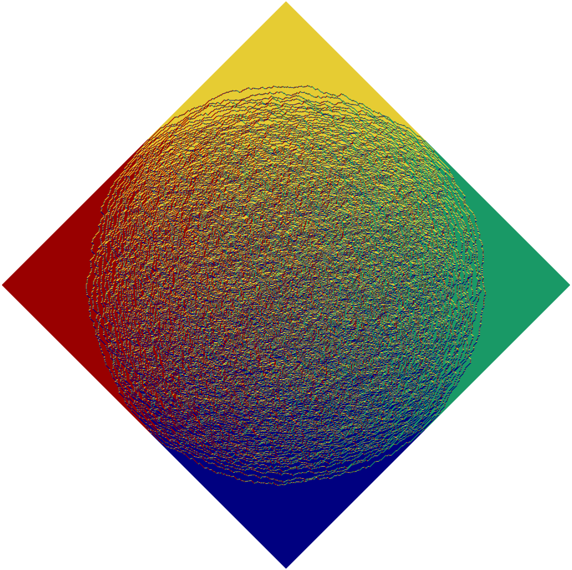


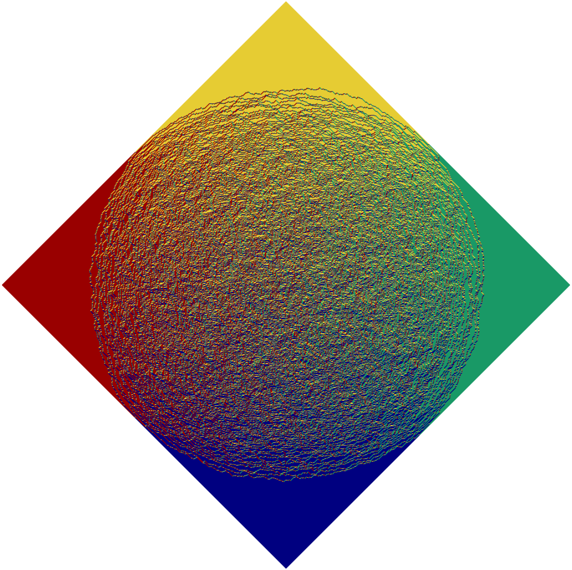




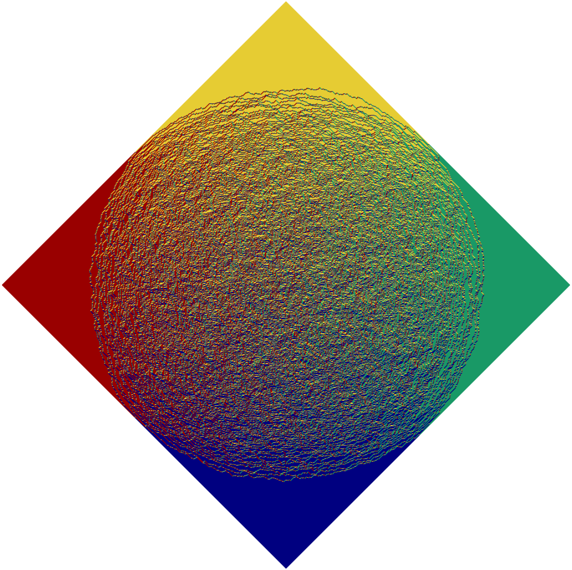








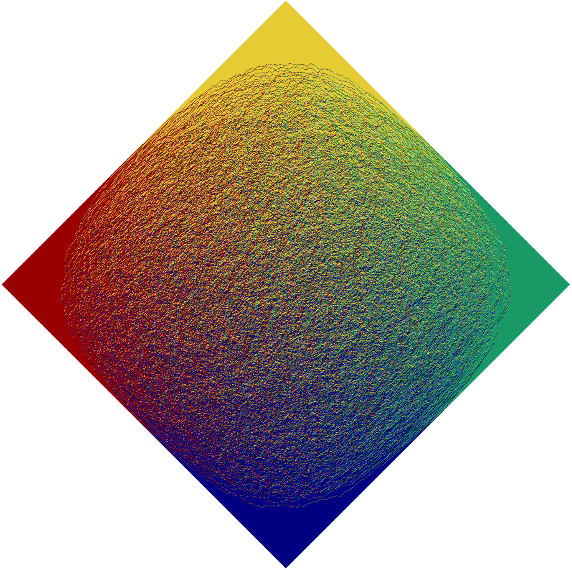
Arctic circle theorem [Jockusch, Propp and Shor 1998]



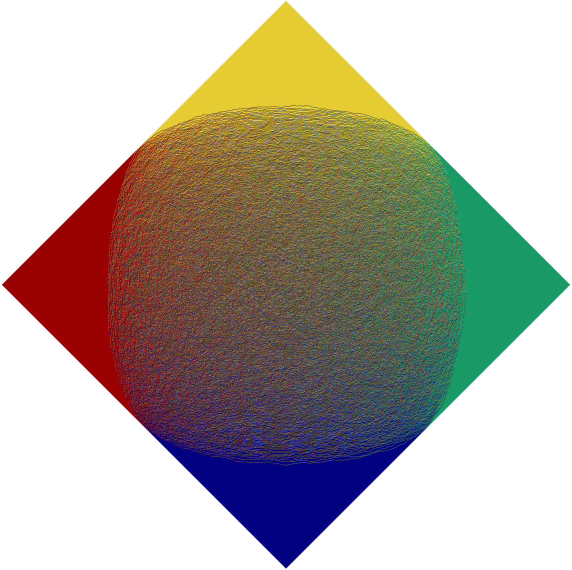
$N^{1/3}$ scaling near the edge: rightmost particle follows the Tracy-Widom distribution [Johansson 2005]

What about interactions?

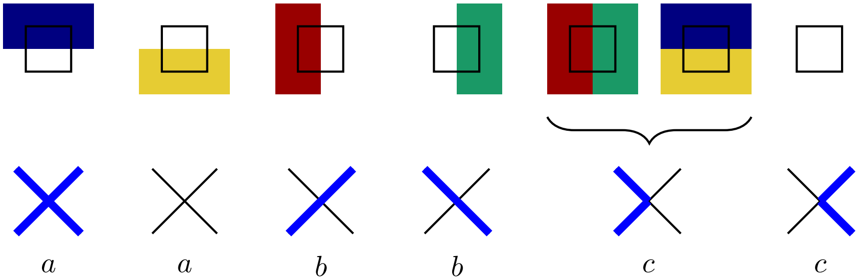
Interacting dimers (repulsive)



Interacting dimers (attractive)



Can also do six vertex model



$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} = 1 - e^{\lambda}$$

- There is still a limit shape. Conjecture for the arctic curve in six vertex [Colomo & Pronko]
- T-W scaling is difficult to establish with interactions, but widely believed to be true.
- Proof for a different model, e. g. ASEP with step initial conditions [Tracy & Widom, 2009].

Why should the edge be free?

Bethe equations (e.g. six vertex model/XXZ spin chain)

$$\left[e^{-\nu}\frac{\sinh\left(\lambda_j+i\gamma/2\right)}{\sinh\left(\lambda_j-i\gamma/2\right)}\right]^L=\prod_{k\neq j}^{N}\frac{\sinh(\lambda_j-\lambda_k+i\gamma)}{\sinh(\lambda_j-\lambda_k-i\gamma)}$$

- For $\gamma=\pi/2$, the rhs simplifies to $\prod_{k\neq j}(-1)$. This is the free fermion point.
- Away from free fermions and for fixed N/L the Bethe equations are not well understood. The Bethe roots densify on an unknown curve in the complex plane.
- However, close to the edge $N/L\to 0$, and the solutions become free fermions-like again.

Lieb Liniger in a harmonic trap (repulsive)

$$H = \int dx \Psi^\dagger(x) \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \mu + x^2 \right] \Psi(x) + \frac{\hbar g}{2} \Psi^{\dagger 2}(x) \Psi(x)^2$$

Look at an $\hbar \to 0$ limit (still interacting).

The trapping potential breaks integrability, but assuming LDA, the density profile may be computed using thermodynamic Bethe ansatz (TBA). [Dunjko, Laurent & Olshanii 2001; Gangardt & Shlyapnikov 2003; Brun & Dubail 2018]

TBA for the homogeneous ground state ($V(x) = 0$):

$$\begin{aligned} \rho(k,\mu) - \int_{-k_F}^{k_F} \frac{dq}{2\pi} V(k,q)\rho(q,\mu) &= \frac{1}{2\pi} \\ \epsilon(k,\mu) - \int_{-k_F}^{k_F} \frac{dq}{2\pi} V(k,q)\epsilon(q,\mu) &= k^2 - \mu \end{aligned}$$

Ask $\epsilon(k_F, \mu) = 0$. The kernel V is known explicitly, and vanishes for free fermions, in which case we recover $k_F(\mu) = \sqrt{\mu}$.

LDA just tells us to replace $\mu \to \mu(x) = \mu - x^2$.

Edge scaling

Claim: can look at the edge scaling using TBA considerations. In that case interactions only provide extra subleading corrections, compared to free fermions.

So TW scaling still holds, we find $-\frac{d^2}{dx^2} + 2\sqrt{\mu}x \leq 0$, which exactly the same result irrespective of interactions for the harmonic trap.

Physically, interacting particles are diluted near the edge, they just renormalize to free fermions. In terms of Luttinger parameter, $K(x)$ varies with position, but $K(x_\mathrm{e}) = 1$.

XXZ in a varying magnetic field

$$\sum_{x\in\mathbb{Z}+1/2}\left(S^{\mathrm{x}}_x S^{\mathrm{x}}_{x+1}+S^{\mathrm{y}}_x S^{\mathrm{y}}_{x+1}+\Delta S^{\mathrm{z}}_x S^{\mathrm{z}}_{x+1}+h(x/R)S^{\mathrm{z}}_x\right)$$

The edge is at $x_{\mathrm{e}}=Rh^{-1}(1+\Delta)$. We find the edge scaling

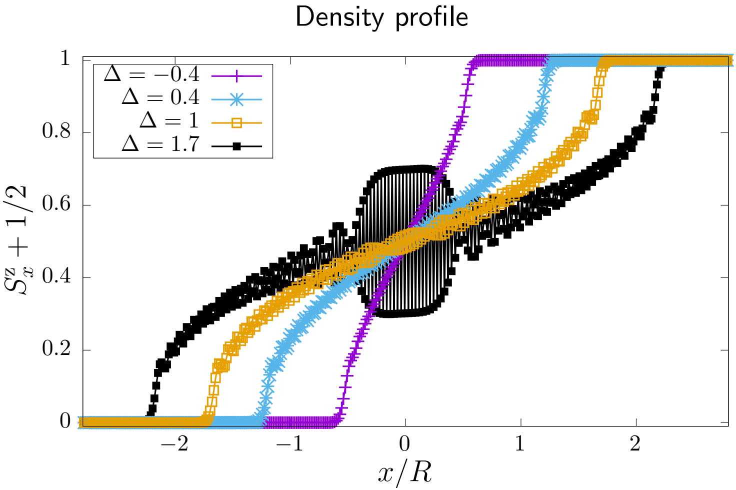
$$-\frac{d^2}{dx^2}+2\frac{h'(h^{-1}(1+\Delta))}{R}x<0$$

Tracy-Widom scaling occurs on a scale ℓ_Δ now, with

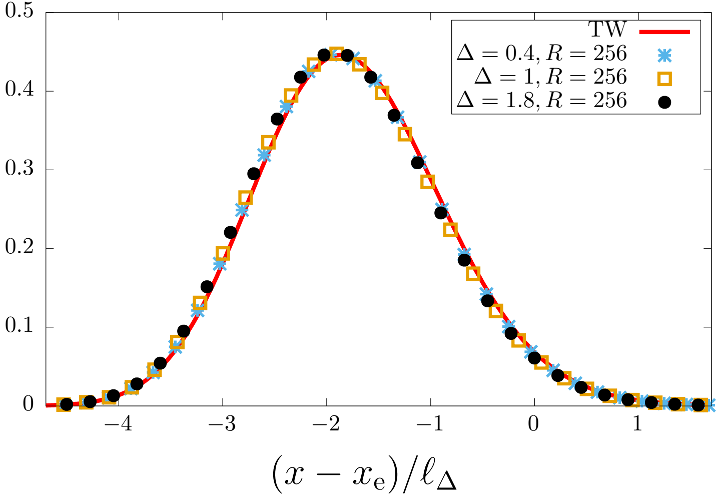
$$\ell_\Delta=\left[\frac{R}{2h'(h^{-1}(1+\Delta))}\right]^{1/3}$$

Variance now depends on interactions, through Δ.

Numerical checks



Distribution of the rightmost down spin from the emptiness formation probability $E_x = \langle \prod_{j=x}^{\infty} \rangle \frac{(1+2S_j^z)}{2}$



An exception: Calogero-Sutherland

$$H = \sum_{j=1}^{N} \left(-\frac{\partial^2}{\partial x_j^2} + x_j^2 \right) + \sum_{i \neq j} \frac{\beta(\beta/2 - 1)}{(x_i - x_j)^2}$$

Luttinger parameter is known to be $K = 2/\beta$, constant, everywhere in the domain. So the edge is not free for $\beta \neq 2$.

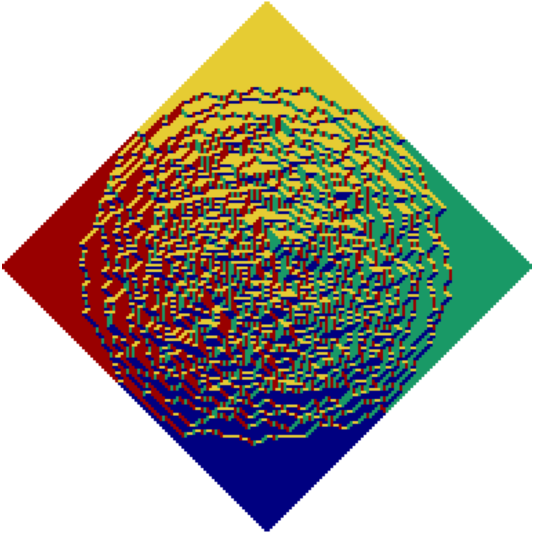
Ground state wave function

$$|\varphi(x_1, \ldots, x_N)|^2 \propto \prod_{i<j} (x_i - x_j)^\beta e^{-\sum_j x_j^2}$$

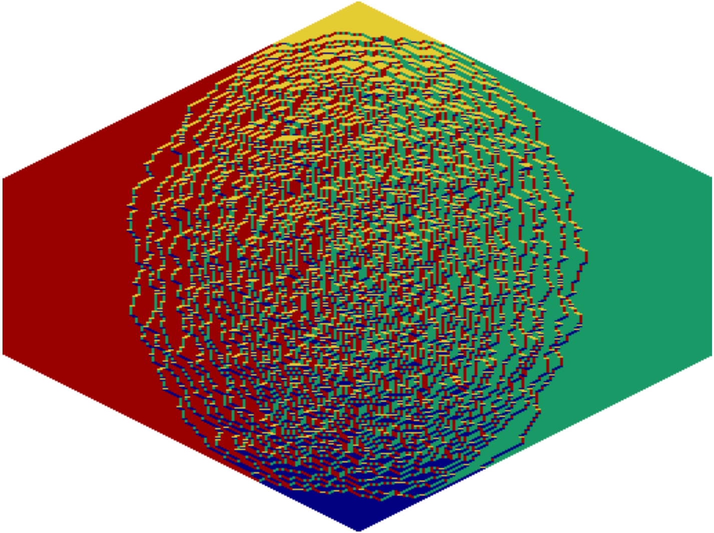
which is exactly the eigenvalue pdf for β-ensemble in random matrix theory, which leads to β-deformed Tracy-Widom.

Out of equilibrium setups

The Hamiltonian (or Trotter) limit



The Hamiltonian (or Trotter) limit



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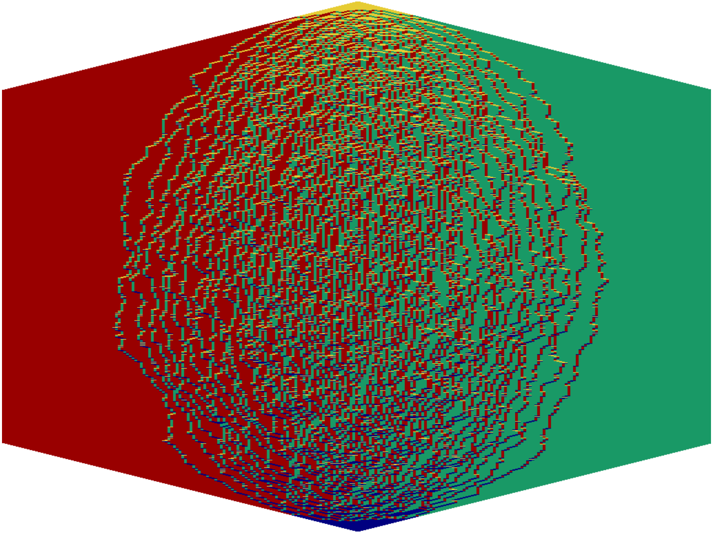
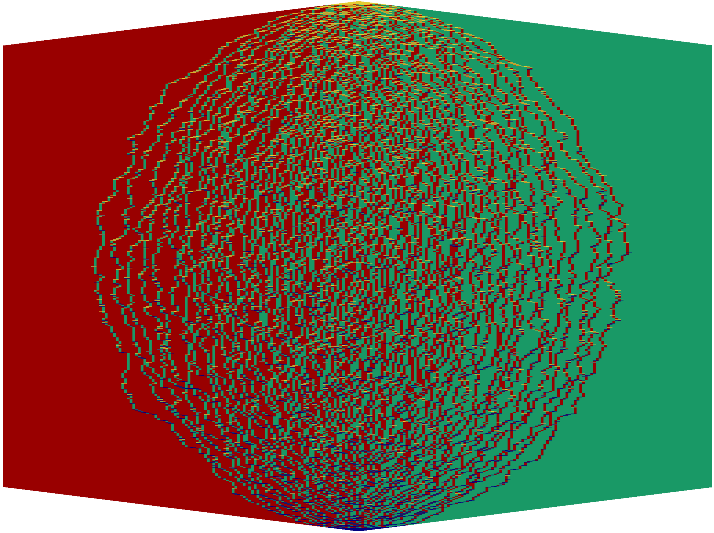


The Hamiltonian (or Trotter) limit



The Hamiltonian (or Trotter) limit

In six vertex language, set $a = 1$ first, and choose a value of Δ.

Then set $b = R/N$, vertical spacing $\Delta y = 1/N$ and take the limit $N \to \infty$.

The partition function becomes

$$Z(R) = \langle\psi|e^{RH_{\mathrm{XXZ}}}|\psi\rangle$$

where $|\psi\rangle = |\ldots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\ldots\rangle$

Six vertex model with domain wall boundary conditions

Introduced by [Korepin 1982]. Determinant formula [Izergin 1987]

$$Z = \frac{[\sin\epsilon]^{N^2}}{\prod_{k=0}^{N-1} k!^2} \det_{0\leq i,j\leq N-1} \left(\int_{-\infty}^{\infty} du\, u^{i+j} e^{-\epsilon u} \frac{1-e^{-\gamma u}}{1-e^{-\pi u}} \right)$$

Then take the Hamiltonian limit [JMS, J. Stat. Mech 2017]

$$\boxed{\mathcal{Z}(\tau) = \langle e^{\tau H} \rangle = e^{-\frac{1}{24}(\tau\sin\gamma)^2} \det(I - V)}$$

$$V(x,y) = B_0(x,y)\,\omega(y)$$

$$B_\alpha(x,y) = \frac{\sqrt{y}J_\alpha(\sqrt{x})J'_\alpha(\sqrt{y}) - \sqrt{x}J_\alpha(\sqrt{y})J'_\alpha(\sqrt{x})}{2(x-y)}$$

$$\omega(y) = \Theta(y) - \frac{1-e^{-\gamma y/(2\tau\sin\gamma)}}{1-e^{-\pi y/(2\tau\sin\gamma)}}$$

Asymptotics

$$\cos\gamma = \Delta < 1$$

Easiest: use [Zinn-Justin 2000] [Bleher, Fokin 2006]

$$-\log \mathcal{Z}(\tau) = \left[\frac{\pi^2}{(\pi-\gamma)^2} - 1\right] \frac{(\tau \sin\gamma)^2}{24} + O(\log\tau)$$

Interpretation: free energy of the fluctuating region.

The Wick rotation (conjectures)

For $t \in \mathbb{R}$, $\mathcal{R}(t) = |Z(it)|^2$ is a (quantum) return probability.

- Root of unity, $\gamma = \frac{\pi p}{q}$

$$-\log \mathcal{R}(t) = \left(\frac{q^2}{(q-1)^2} - 1 \right) \frac{(t \sin \gamma)^2}{12} + O(\log t)$$

  Coincides with analytic continuation of the asymptotic result only when $p = 1$.

- non root of unity

$$-\log \mathcal{R}(t) = t \sin \gamma + O(\log t)$$

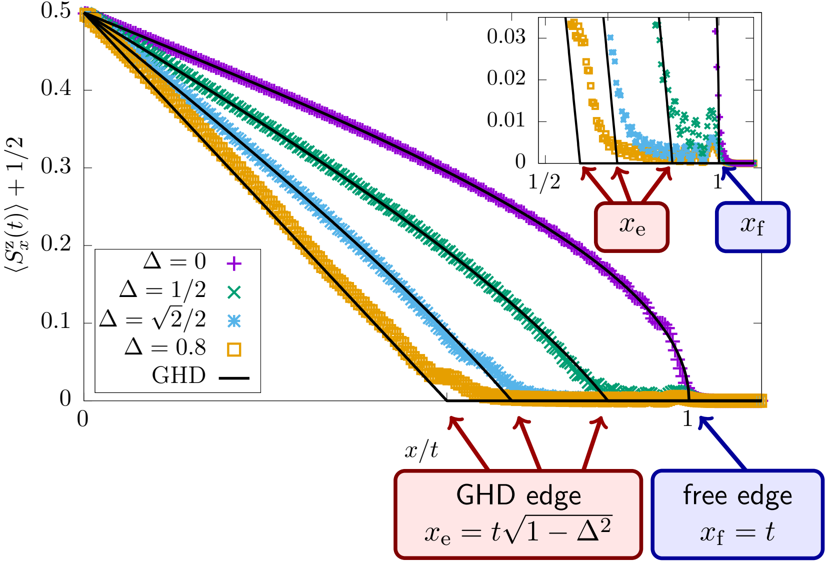
$$|\Psi(t)\rangle = e^{-itH}\,|\Psi_0\rangle \qquad , \qquad |\Psi_0\rangle = |\ldots \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \ldots\rangle$$

$$H = \sum_{x\in\mathbb{Z}} \left(S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2 + \Delta S_x^3 S_{x+1}^3\right)$$

Free fermion case $(\Delta = 0)$ [Antal, Rácz, Rákos, and Schütz, 1999, ...]
Interactions: Numerics [Gobert, Kollath, Schollwöck & Schütz 2005,...]

Generalized hydrodynamics (GHD) framework $(|\Delta| < 1$, ballistic)
[Castro-Alvaredo, Doyon, Yoshimura 2016] [Bertini, Collura, De Nardis, Fagotti 2016]

This quench: light cone $x_{\mathrm{e}}(t) = t\sqrt{1-\Delta^2}$ [JMS, 2017]
Full density profile from GHD [De Luca, Collura, Viti 2017]



Contrary to previous situations density profile is linear near the light cone, compared to previous root behavior.

Tracy-Widom edge for $\Delta = 0$ (free fermions) [Eisler & Racz 2013]

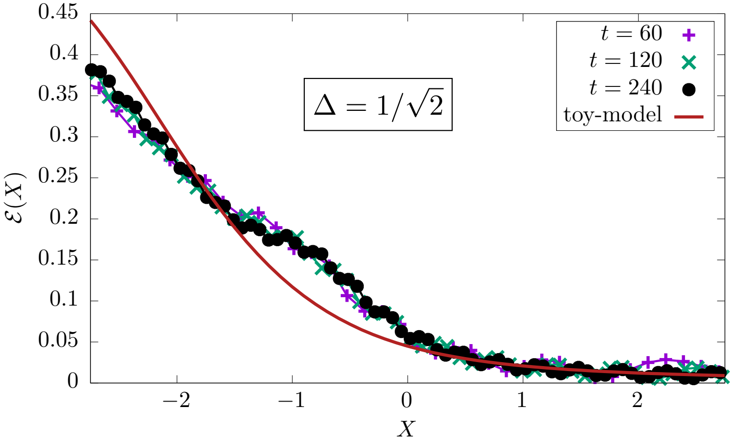
From GHD considerations in the bulk, [De Luca, Collura & Viti 2017] guessed a new (diffusive) kernel for the edge. In our language, this reads free fermions $k + x/t < 0$.

Fastest quasi-particle goes as $x \simeq t$ [Sabetta & Misguich 2013]. [Bulchandani & Karrasch 2018] observed $t^{1/3}$ scaling near this other front, which they interpreted as signature of Tracy-Widom.

Subleading corrections to GHD are generically diffusive in the bulk [De Nardis, Bernard & Doyon 2018]

Rescaled density profiles

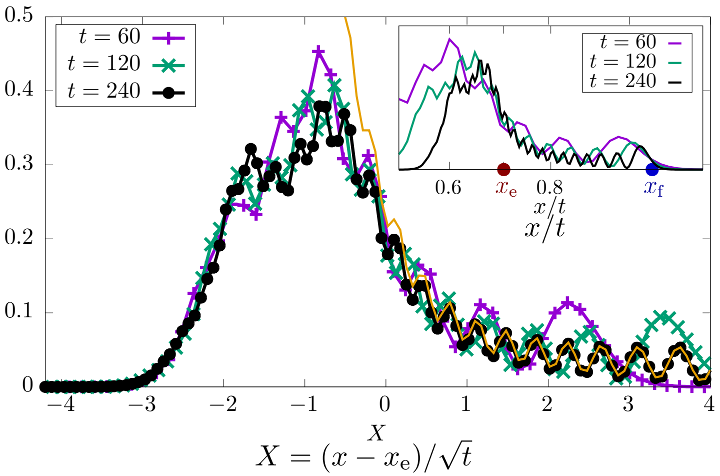
$X = \frac{x - t\sqrt{1-\Delta^2}}{\sqrt{t}}$ Rescaled density decays as $\rho(X \gg 1) \sim \frac{1}{4\pi^2 X}$



$X = (x - x_{\mathrm{e}})/\sqrt{t}$

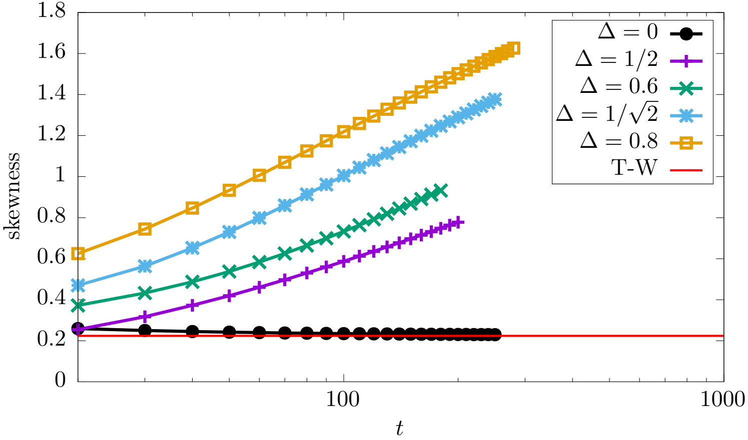
Rescaled distribution of the last particle (fat tails)

$\boxed{\Delta = 1/\sqrt{2}}$



Skewness

$$\mathrm{sk} = \frac{\langle (x - \langle x \rangle)^3 \rangle}{\langle (x - \langle x \rangle)^2 \rangle^{3/2}}. \qquad \mathrm{sk} \simeq 0.22408 \text{ for Tracy-Widom.}$$



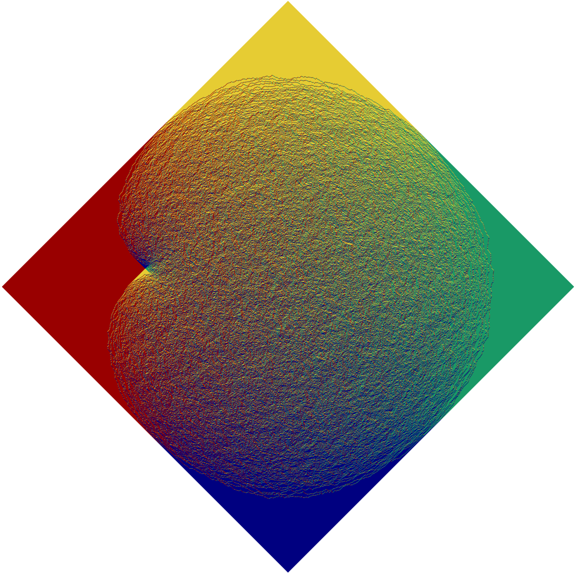
- Not clear if it's free fermions at the edge in that case.

- The tail on the right of $x_{\mathrm{e}} = t\sqrt{1-\Delta^2}$ is very long. Skewness appears to diverge.

- Hard cutoff at $x_{\mathrm{LR}} = t$, due to Lieb-Robinson type bounds. $t^{1/3}$ behavior near $x = t$ is the right tail of the quantum delocalization of the rightmost particle. Not Tracy-Widom.

Exact computations?

L-shaped partition function and Emptiness



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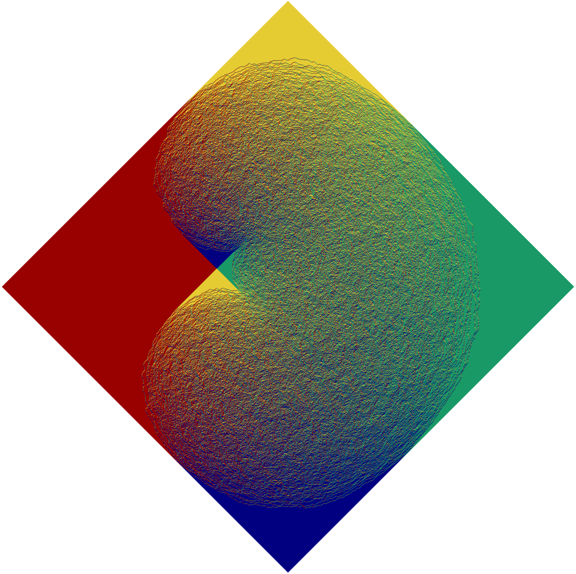

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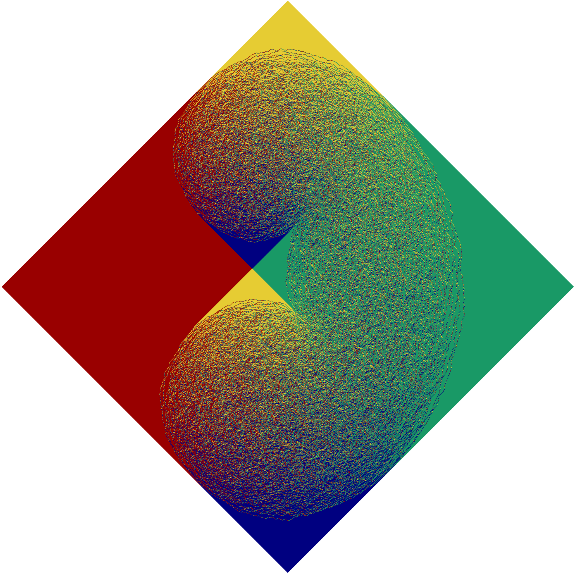

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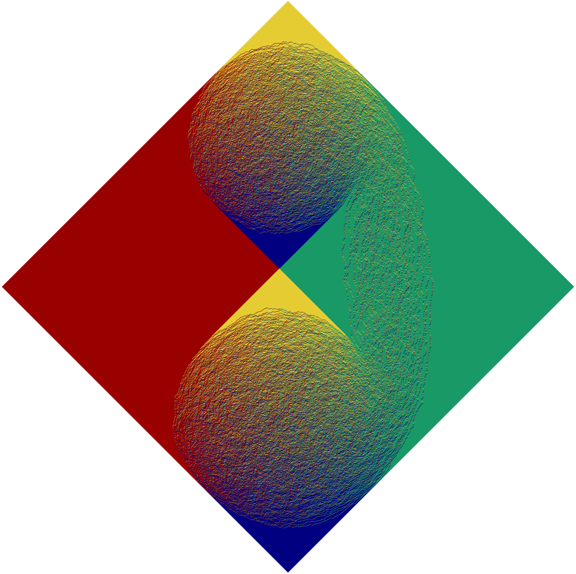
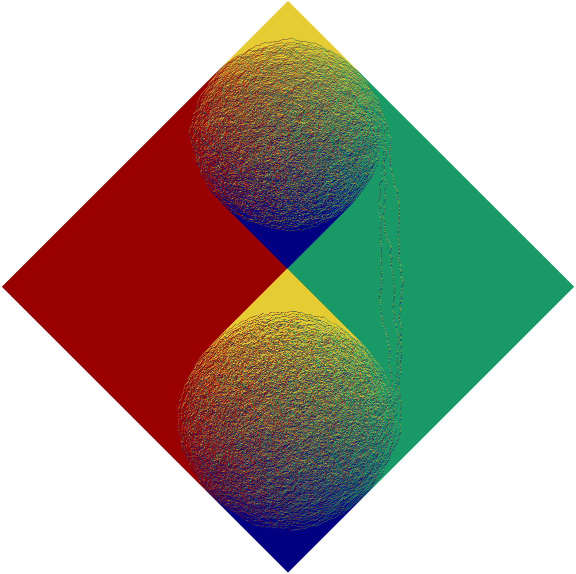
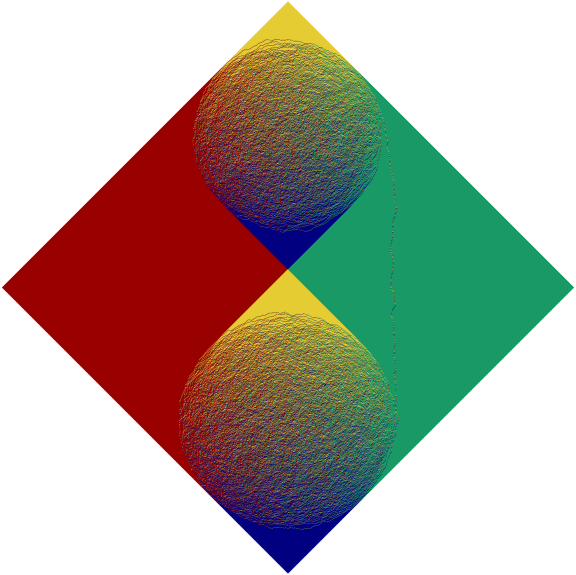


L-shaped partition function and Emptiness



L-shaped partition function and Emptiness



L-shaped partition function and Emptiness

$$E_N(r,s) = \frac{Z_N(r,s)}{Z_N(0,0)}$$

- $E_N(a,b,\Delta|r,s)$ can be computed exactly. [Colomo & Pronko 2007]

- Thermodynamic limit: $E_N \sim e^{-N^2 f(r/N, s/N)}$

  The function f is not known. However it vanishes on the arctic curve, which is known exactly. [Colomo & Pronko 2009]

Time-dependent Emptiness formation probability

$$\tilde{E}_R(x,y) = \lim_{N\to\infty} E_N(a, b = \tfrac{R}{N}, \Delta | r = \tfrac{N-x+Ny/R}{2}, s = \tfrac{N+x+Ny/R}{2})$$

Problem: The Colomo Pronko formula is a multiple integral type formula, and the number of integrals explodes in the Hamiltonian limit.

However, it is possible to write it as [Colomo unpublished]

$$\oint_C \frac{dw_1}{2i\pi} \dots \oint_C \frac{dw_x}{2i\pi} \prod_{1\le j<k\le x} \phi_{j,k} \det(\dots)\det(\dots)$$

where there are x integrals. This is more suitable for Hamiltonian limit.

Summary

- The edge of several inhomogeneous interacting systems 'renormalizes' to free fermions. The most typical example in that case is Tracy-Widom.

- Quantum quench problems provide us with new edge universality classes, which are worth exploring.

Thank you!