Fermionic limit shapes

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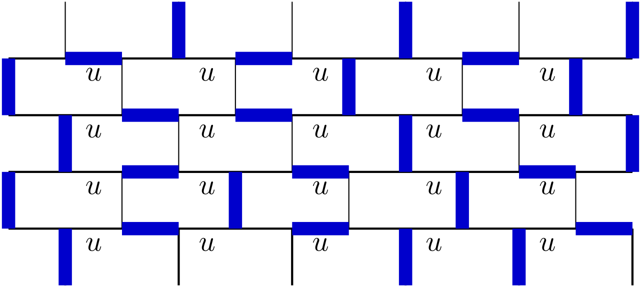
Mathematical Physics Seminar, Montreal 2021

Based on [Saverio Bocini & JMS, arXiv:2007.06621]

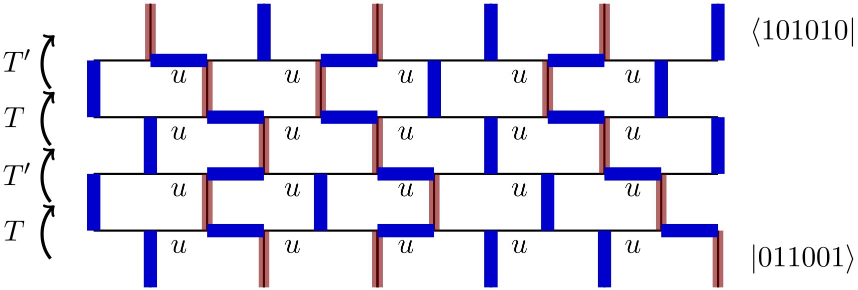
Outline

Dimers on a brickwall (honeycomb) lattice



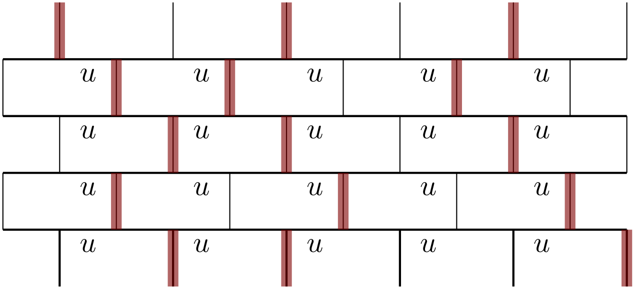
- Dimers (in blue) cover the whole lattice: each site is occupied by exactly one dimer.
- Weight $u > 0$ for some horizontal dimers, 1 for the others.
- $\mathbb{P}(\text{configuration shown in the picture}) = u^4/Z$.

Dimers on a brickwall (honeycomb) lattice



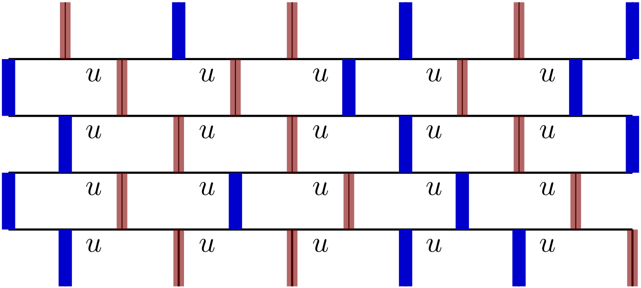
Mapping to particle configurations: vertical dimers are holes '0', while empty vertical edges are particles '1' shown in red.

Dimers on a brickwall (honeycomb) lattice



Can reconstruct the dimer configuration from the particle configuration.

Dimers on a brickwall (honeycomb) lattice



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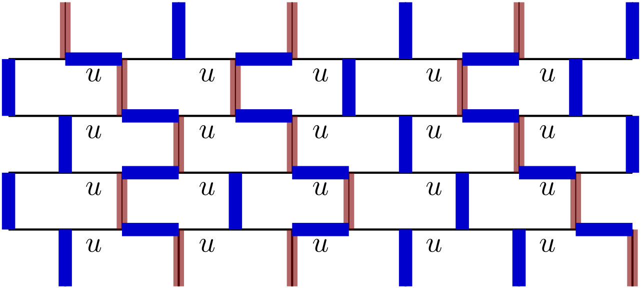


Can reconstruct the dimer configuration from the particle configuration.

Mapping to fermions (Jordan-Wigner)

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad c^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|011001\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle$$

$$c_j^\dagger = \underbrace{s \otimes \ldots \otimes s}_{j-1} \otimes \; c^\dagger \otimes \underbrace{I_2 \otimes \ldots \otimes I_2}_{L-j} \quad , \quad c_j = (c_j^\dagger)^\dagger$$

$$\boxed{c_i c_j^\dagger = \delta_{ij} I - c_j^\dagger c_i \quad , \quad c_i c_j = -c_j c_i}$$

Dimer configurations in terms of ordered fermionic operators, e.g.

$$|110101\rangle = c_1^\dagger c_2^\dagger c_4^\dagger c_6^\dagger \, |\mathbf{0}\rangle$$

where $|\mathbf{0}\rangle = |000000\rangle$ is called the vacuum.

Transfer matrix as free fermions

T, T' constructed such that $Z = \langle 101010 | T'TT'T | 011001 \rangle$ in the previous picture. [Onsager, Lieb, Baxter, ...]

$\mathcal{T} = T'T$ satisfies $\mathcal{T}\,|\mathbf{0}\rangle = |\mathbf{0}\rangle$, and

$$\mathcal{T} c_i^\dagger = \left(u c_{i-1}^\dagger + (1+u^2) c_i^\dagger + u c_{i+1}^\dagger \right) \mathcal{T} = \left(\sum_j A_{ij} c_j^\dagger \right) \mathcal{T}$$

$$\mathcal{T} = T'T = \exp\left(\sum_{i,j} B_{ij} c_i^\dagger c_j \right) \quad , \quad B = \log A$$

Broader picture

Related to many other topics in Mathematical Physics and Probability Theory: Schur functions, determinantal point processes, non-intersecting lattice paths, six vertex model, free fields, conformal field theory . . .

From the perspective of integrability, write $\mathcal{T} = \mathcal{T}(u)$. Then

$$\mathcal{T}(u)\mathcal{T}(v) = \mathcal{T}(v)\mathcal{T}(u)$$

Transfer matrix as free fermions (2)

Infinite lattice: by translation invariance, and introducing $c^\dagger(k) = \sum_{x\in\mathbb{Z}} e^{\mathrm{i}kx} c_x^\dagger$, $\mathcal{T}$ reads in momentum space

$$\mathcal{T} = \exp\left(\int_{-\pi}^{\pi} \frac{dk}{2\pi} \varepsilon(k) c^\dagger(k) c(k)\right) \qquad \text{or} \qquad \mathcal{T} c^\dagger(k) = e^{\varepsilon(k)} c^\dagger(k) \mathcal{T}$$

$\mathcal{T} = e^H$, where H is a quadratic Hamiltonian with dispersion $\varepsilon(k)$.

For dimers, we have

$$\varepsilon(k) = \log\left[(1+ue^{ik})(1+ue^{-ik})\right] = \log\left[1+u^2+2u\cos k\right]$$

Fermions anticommute, why is this supposed to be positive?

For dimers it follows from the relation

$$\mathcal{T} c_j^\dagger = \left(u c_{j-1}^\dagger + (1+u^2) c_j^\dagger + u c_{j+1}^\dagger \right) \mathcal{T}$$

which implies the fermions never change order. Using this one can show $\langle \phi | \mathcal{T} | \psi \rangle \geq 0$ for all particle configurations $|\phi\rangle, |\psi\rangle$.

Fermions anticommute, why is this supposed to be positive?

Positive dispersions can be classified, since this problem is related to the notion of total positivity for matrices [Edrei 1952, Thoma 1964]

The only positive dispersions are linear combinations of the

$$1 \quad , \quad e^{\mathrm{i}k} \quad , \quad e^{-\mathrm{i}k}$$

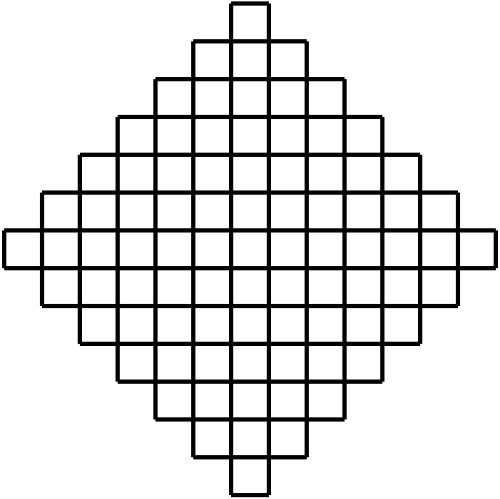
with positive coeffcients, and the

$$\log(1+\alpha e^{ik}) \quad , \quad \log(1+\beta e^{-ik}) \quad , \quad \log\frac{1}{1-\gamma e^{ik}} \quad , \quad \log\frac{1}{1-\delta e^{-ik}}$$

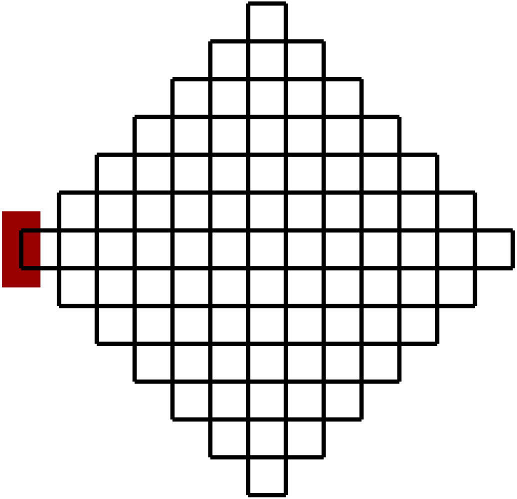
with positive integer coefficients, and $\alpha, \beta, \gamma, \delta \geq 0$.

Limit shapes

Uniform dimer coverings on the Aztec diamond



Uniform dimer coverings on the Aztec diamond



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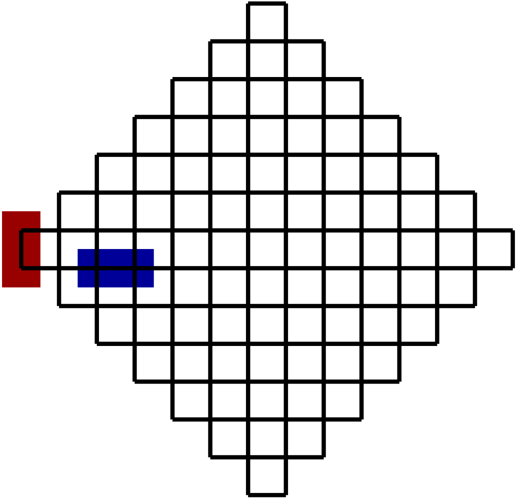

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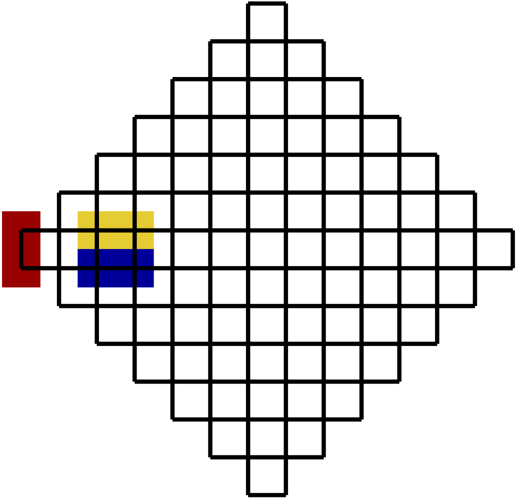

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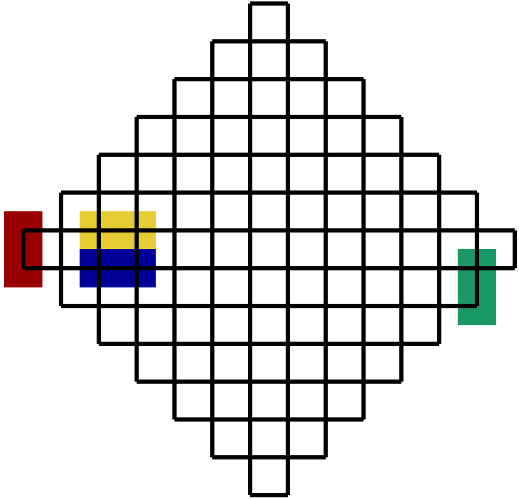

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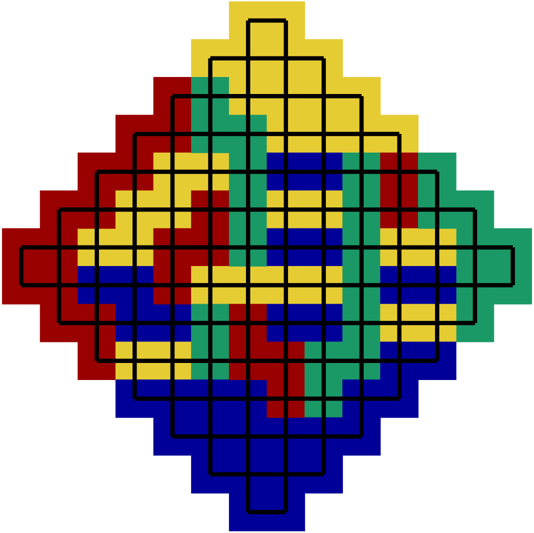
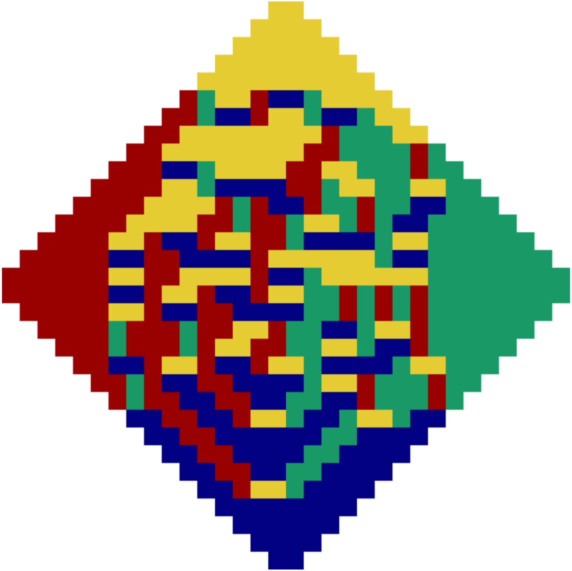
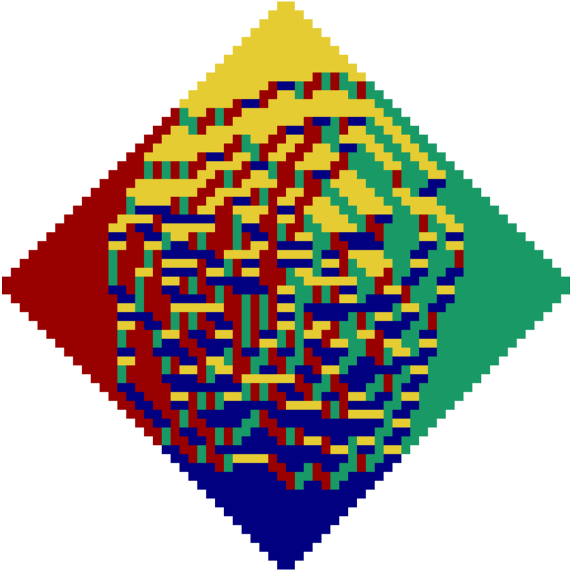
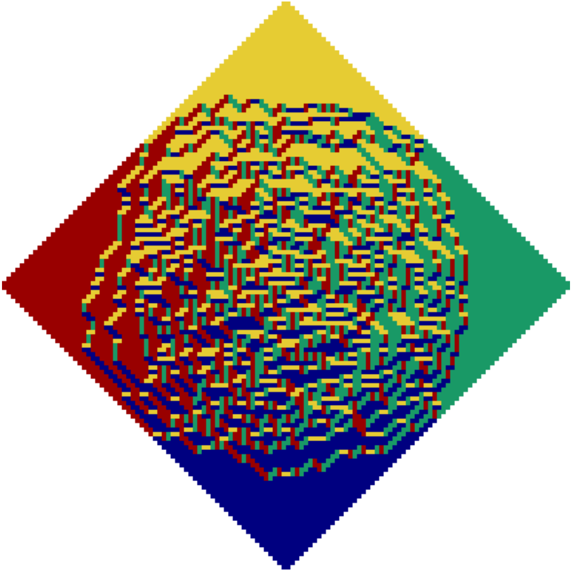
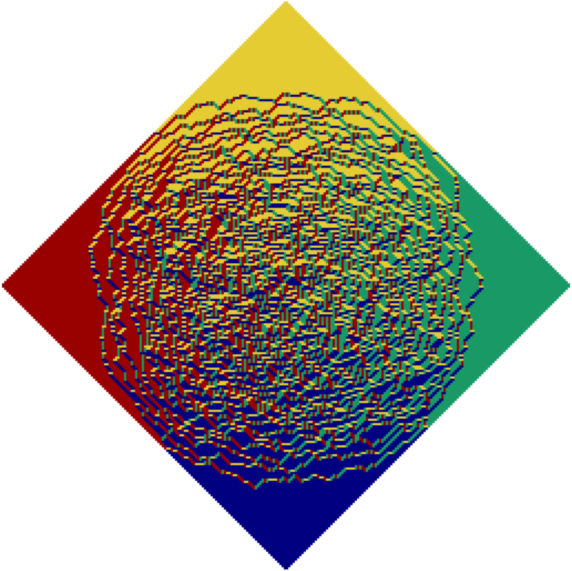
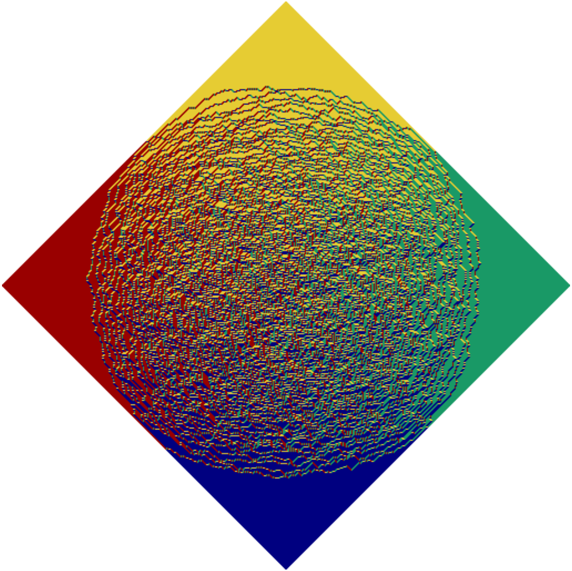


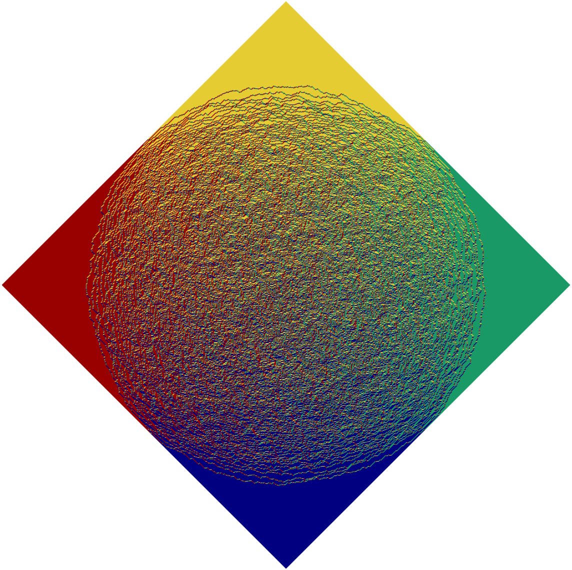


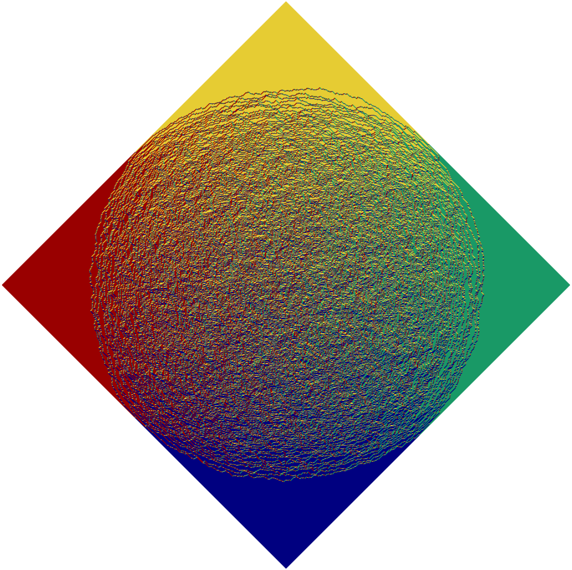




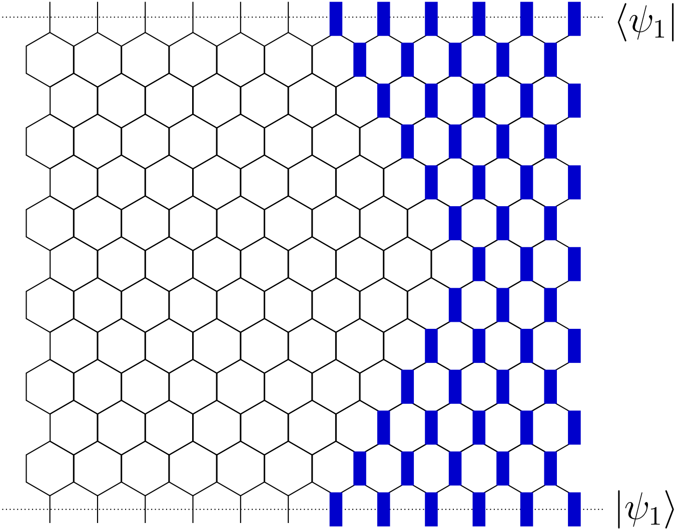




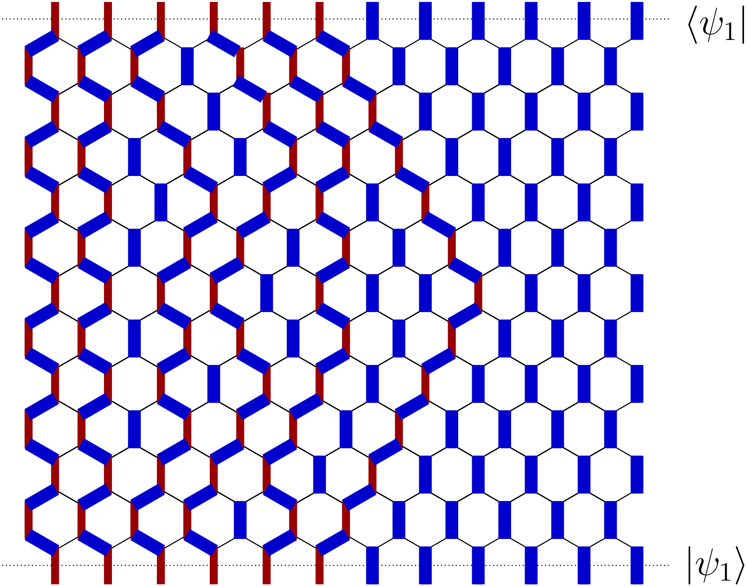




Arctic circle theorem [Jockusch, Propp and Shor 1998]



Domain wall: $|\psi_1\rangle = |111111000000\rangle$


$\langle\psi_1|$
$|\psi_1\rangle$

Average density profile

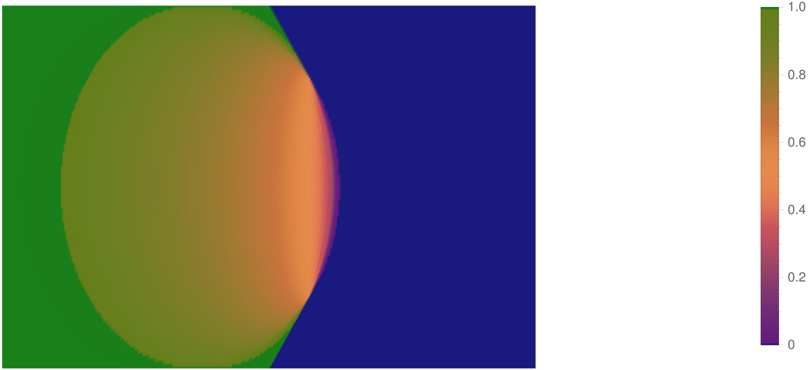
Recall $\mathcal{T} = e^H$. Top/bottom boundaries at $y = \pm R$.

Using the transfer matrix formalism

$$\langle n_x(y) \rangle = \frac{\langle \psi_1 | e^{(R-y)H} c_x^\dagger c_x e^{(R+y)H} | \psi_1 \rangle}{\langle \psi_1 | e^{2RH} | \psi_1 \rangle}.$$

Exact formulas are sometimes possible: Wick's theorem buys you a ratio of semi-infinite determinants.

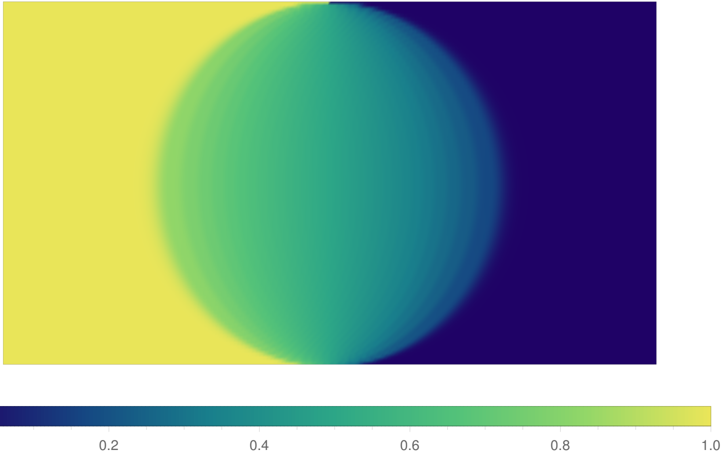
Average density profile (dimers $u = 1/2$)



Density is frozen (to 1 or 0) outside an "arctic" ellipse in the limit $R \to \infty$ with fixed $X = x/R$, $Y = y/R$.

Average density profile for $\varepsilon(k) = \cos k$

Previously studied in relation to growth models [Prähofer, Spohn 2000]



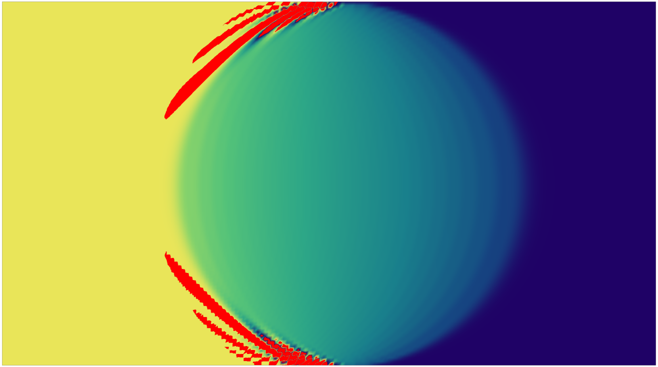
Density is frozen (to 1 or 0) outside an "arctic" circle.

What about dispersions such as

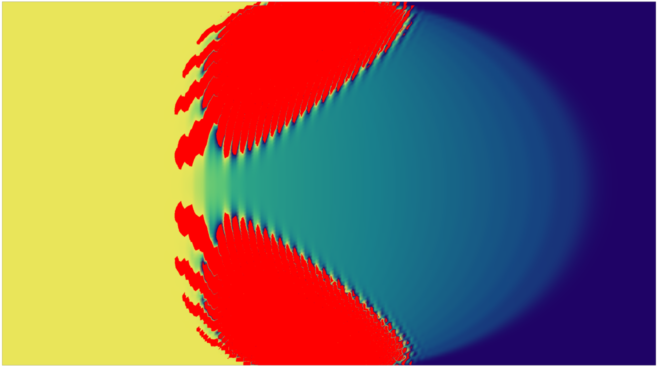
$$\varepsilon(k) = \cos k + \alpha \cos(2k)$$

which are not guaranteed to be positive?

$\alpha = \frac{1}{15}$, new "crazy regions" in red with density not in $[0,1]$.



$\alpha = \frac{1}{4}$, new "crazy regions" in red with density not in $[0, 1]$.



Sign issues

$$e^{\tau H}|\psi_1\rangle = \sum_{\mathcal{C}} a_{\mathcal{C}}(\tau)|\mathcal{C}\rangle \qquad , \qquad a_{\mathcal{C}}(\tau) = \langle \mathcal{C}|e^{\tau H}|\psi_1\rangle$$

$$a_{\mathcal{C}}(\tau) = \sum_{m=0}^{\infty} \frac{\tau^m}{m!}\langle \mathcal{C}|H^m|\psi_1\rangle .$$

$$\begin{aligned} H|\psi_1\rangle &= H|..1111100000..\rangle \\ &= |..1111010000..\rangle + \alpha|..1111001000..\rangle - \alpha|..1110110000..\rangle \end{aligned}$$

so for sufficiently small τ, some $a_{\mathcal{C}}(\tau)$ are negative. Hence

$$\mathbb{P}(\mathcal{C}, y) = \frac{a_{\mathcal{C}}(R-y)a_{\mathcal{C}}(R+y)}{\sum_{\mathcal{C}} a_{\mathcal{C}}(R-y)a_{\mathcal{C}}(R+y)}$$

can be negative (if $y \neq 0$).

More general wall states

$|\psi_n\rangle$ for $n \in \{1, 2, 3 \ldots\}$

$$|\psi_1\rangle = |\ldots 111111111000000 \ldots\rangle$$

$$|\psi_2\rangle = |\ldots 101010101000000 \ldots\rangle$$

$$|\psi_3\rangle = |\ldots 100100100100000 \ldots\rangle$$

One fermion every n-th site, then no fermions.

A new exact formula (using standard techniques)

$$n_x(y) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi+\mathrm{i}\eta}^{\pi+\mathrm{i}\eta} \frac{dq}{2\pi} \frac{e^{\Phi_n(k,x,y)-\Phi_n(q,x,y)} e^{\Omega_n(k)+\Omega_n(q)}}{1-e^{-\mathrm{i}n(k-q)}}$$

$$\Phi_n(k,x,y) = -\mathrm{i}kx - y\varepsilon(k) + \mathrm{i}R\tilde{\varepsilon}_n(nk),$$

$$\Omega_n(k) = R\left[\varepsilon(k) - \varepsilon_n(nk)\right]$$

$$\varepsilon_n(k) = \frac{1}{2R} \log\left(\frac{1}{n}\sum_{p=0}^{n-1} e^{2R\varepsilon\left(\frac{k+2p\pi}{n}\right)}\right)$$

$\tilde{\varepsilon}_n$ denotes the periodic Hilbert transform of ε_n.

This formula works only for the initial states $|\psi_n\rangle$.

Consider initial states of the form $|\psi\rangle = c^\dagger_{s(1)} \dots c^\dagger_{s(l)} |\mathbf{0}\rangle$.

$$K_{ij} = \frac{\langle\psi|e^{\tau_1 H} c_i^\dagger e^{\tau_2 H} c_j e^{\tau_3 H}|\psi\rangle}{\langle\psi|e^{(\tau_1+\tau_2+\tau_3)H}|\psi\rangle} \underset{\text{Wick}}{=} \frac{\det\begin{pmatrix} 0 & u \\ v & M \end{pmatrix}}{\det\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}}$$

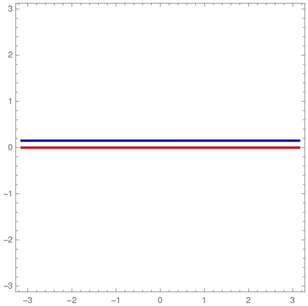
$$M_{ab} = \langle 0|c_{s(a)} e^{(\tau_1+\tau_2+\tau_3)H} c^\dagger_{s(b)}|0\rangle = \int \frac{dk}{2\pi} e^{-\mathrm{i}k(s(a)-s(b))} e^{(\tau_1+\tau_2+\tau_3)\varepsilon(k)}$$

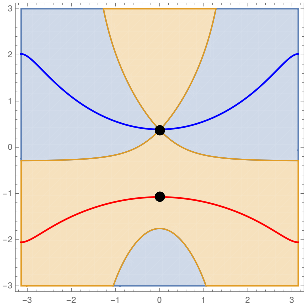
u a is a l-line vector with elements $\langle 0|c_a e^{\tau_1 H} c_i^\dagger|0\rangle$ and v a $l-$column vector with elements $\langle 0|c_j e^{\tau_3 H} c_b^\dagger|0\rangle$

For initial states $|\psi_n\rangle$, $s(a)-s(b) = s(a-b)$, so M is a Toeplitz matrix, which can be inverted.

Saddle point analysis for $n = 1$ ($\alpha = 0$)

Known positive case, fluctuating region



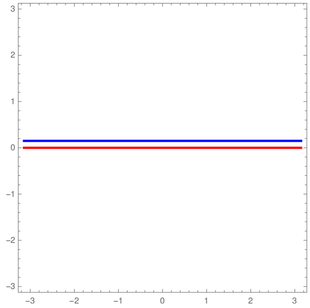


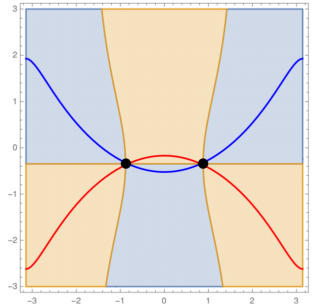
Blue region $\mathrm{Re}\,\varphi(q) < \mathrm{Re}\,\varphi(z_+)$. Orange $\mathrm{Re}\,\varphi(k) > \mathrm{Re}\,\varphi(z_+)$.

With the deformation shown $\mathrm{Re}(\varphi(k) - \varphi(q)) < \mathrm{cst} < 0$, so integrant is exponentially small.

Saddle point analysis for $n = 1$ $(\alpha = 0)$

Known positive case, fluctuating region



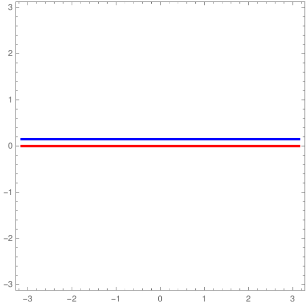


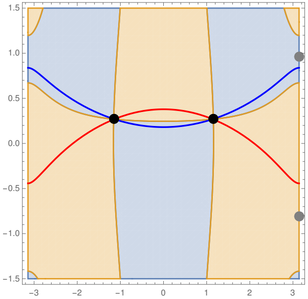
Blue region $\operatorname{Re}\varphi(q) < \operatorname{Re}\varphi(z_+)$. Orange $\operatorname{Re}\varphi(k) > \operatorname{Re}\varphi(z_+)$.

With the deformation shown $\operatorname{Re}(\varphi(k) - \varphi(q)) < \text{cst} < 0$, so integrant is exponentially small + residue contribution.

Saddle point analysis for $n = 1$ $(\alpha > 0)$

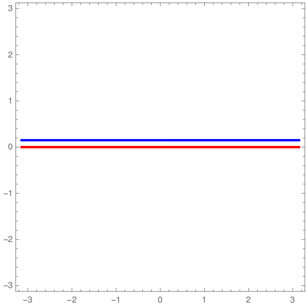
Four saddle points. Still normal

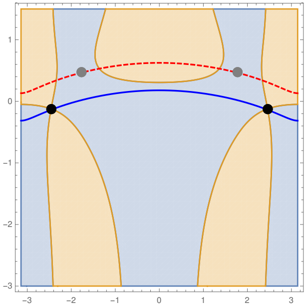




Saddle point analysis for $n = 1$ $(\alpha > 0)$

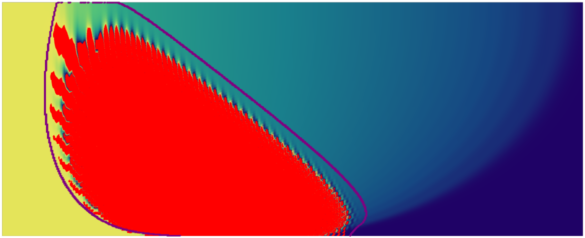
Four saddle points. New crazy region





Cannot do a similar deformation. Can show exponential blow-up.

Comparison to simulations in finite size



$\alpha = 1/4$. Violet curve is the boundary of the crazy region.

Dilution argument

Minus signs occur when one fermion hop around another, e.g.

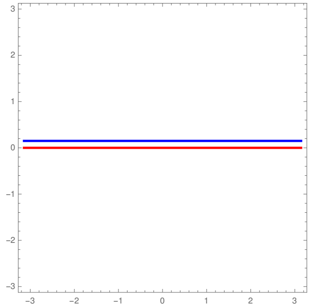
$$|10101\overset{\curvearrowright}{1}0111\rangle$$

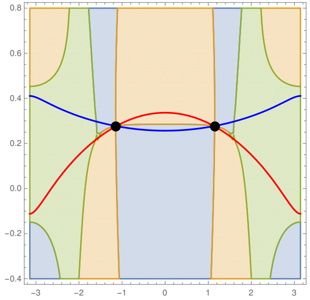
so if one thinks of density as reasonably smooth, minus signs are only generated in regions with high –but not too high– densities.

Makes sense to look at lower density boundary conditions, such as $|\psi_2\rangle$, $|\psi_3\rangle$, etc.

Saddle point analysis for $n = 2$

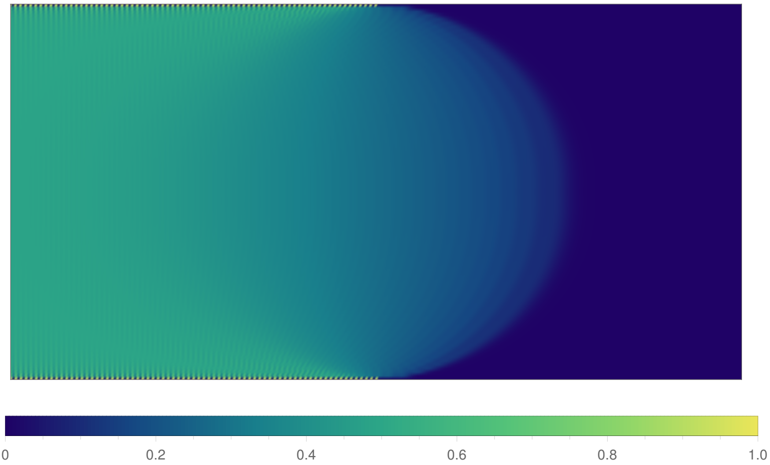
$\alpha > 0$. Only two relevant saddle points.





Can show there are no crazy regions for $R \to \infty$, and compute the density profile exactly. Sign problem disappears for $n \geq 2$!

Density profile (simulations in finite size)



No sign of crazy region, even for finite R.

Discussion/conclusion

- Always positive for $y = 0$. Edge behavior is interesting [Betea, Bouttier, Walsh 2020] related to higher order Tracy-Widom behavior [Di Francesco, Ginsparg, Zinn-Justin 1995] [Akemann, Atkin 2012] [Le Doussal, Majumdar, Schehr 2018].
- There are many (weaker) forms of positivity.
- Similar story in the presence of several bands.
- Presumably similar story in the presence of interactions (add higher order charges to the XXZ Hamiltonian).

Thank you!